Lexicographic Beliefs and Assumption[*]

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1   Introduction

Lexicographic beliefs (henceforth ℓ-beliefs) have become a relatively standard tool, both for studying refinements and for providing epistemic characterizations of solution concepts.[1] The appeal of ℓ-beliefs is that they can be used to address a tension between being certain that an opponent is rational and having full-support beliefs about opponents' actions. To clarify, suppose that, if Bob is rational, he will not play specific actions. Can Ann be certain that Bob is rational, and at the same time be cautious and assign non-zero probability to all of Bob's actions? The answer is no if Ann has standard probabilistic beliefs. Suppose instead that Ann has ℓ-beliefs. That is, she has a vector $(\mu_0, ..., \mu_{n-1})$ of probabilities over the relevant space of uncertainty, S_b (Bob's strategy space) and uses them lexicographically to determine her preferences over her own strategies: Ann first ranks her strategies using μ_0; if that leads to more than one best reply for Ann, she uses μ_1 to rank them, and so on. If the union of the the supports of the probabilities μ_i is all of S_b, then Ann's beliefs have, in a sense, full support. At the same time, Ann can still be confident in Bob's rationality, for example in the sense that the primary hypothesis μ_0 assigns positive probability only to strategies of Bob that are rational.

There are two notions of ℓ-beliefs that have been studied and used in the literature: lexicographic conditional probability systems (henceforth LCPSs) in which, loosely speaking, the supports of the different beliefs (i.e., the μ_i's) are disjoint, and the more general class

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[1]See, for example, Blume, Brandenburger and Dekel (1991b), Brandenburger (1992), Stahl (1995), Mailath, Samuelson and Swinkels (1997), Rajan (1998), Asheim (2002), Govindan and Klumpp (2003), Brandenburger, Friedenberg and Keisler (2008), Keisler and Lee (2010), Lee (2013b), Yang (2013), and Catonini and De Vito (2014) amongst many others.

of lexicographic probability systems (LPSs) in which this disjointedness condition is not imposed. In particular, LCPSs are used by Brandenburger, Friedenberg and Keisler (2008, henceforth, BFK) to provide an epistemic characterization of iterated admissibility—thereby answering a long-standing open question.

However, there are reasons not to find the restriction to LCPSs appealing. First, while Blume, Brandenburger and Dekel (1991a) provide an axiom that characterizes LCPS's within the class of LPS's, their axiom has a flavor of reverse-engineering: it says no more than the probabilities in the LPS have disjoint support; it offers no further normative or other appeal. Indeed, the interpretation of LPSs is quite natural and intuitive. The probability μ_0 is the player's primary hypothesis, in the sense that she is (almost fully) confident in it. The probability μ_1 is her secondary hypothesis: she is willing to entertain it as an alternative assumption, but considers it "infinitely" less plausible than μ_0; and so on. There is no reason that primary and secondary hypotheses must have disjoint supports. For instance, one may be confident that a coin is fair, but entertain the secondary hypothesis that it is biased towards falling on heads.[2] Second, the marginal of an LCPS need not be an LPS. For example, assume two players are playing the game in Figure 1.1, where the pairs of actions A, B for each constitute a zero-sum matching pennies game, (A, C) and (C, A) give $(-2, 3)$ and $(3, -2)$ respectively and anything else gives $(-4, -4)$. Consider the ℓ-belief over this game where μ_0 is that the players are playing the equilibrium of the matching pennies game while μ_1 is that they are playing the Pareto superior outcome that requires correlation of (A, C) and (C, A) with probability one half each. The marginal of the LCPS on one player's actions has the first belief being that A and B are equally likely while the second belief is that A and C are equally likely, which is clearly not an LCPS. Thus, if one takes a small-worlds approach in which the beliefs we use to study a particular game are the marginals of some belief on a larger space, then the beliefs in the game need not be an LCPS (even if one were to assume that the overall belief is an LCPS). For these reasons we find LPSs more suitable for the study of refinements than LCPSs.

	A	B	C
A	-1,1	1,-1	-2,3
B	1,-1	-1,1	-4,-4
C	3,-2	-4,-4	-4,-4

Figure 1.1: The marginal of an LCPS may not be an LCPS

The question then arises whether BFK's characterization of iterated admissibility requires

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[2]Of course one may instead have the secondary hypothesis that the coin will fall on an edge, which would have disjoint support, but that does not seem like the only story one could tell.

the use of LCPSs. This paper shows that their result *does* hold when using the more general notion of LPSs. This involves two steps.

First, BFK define what it means for one player to "be certain" that another is rational using what they call *assumption*. On a finite space, an event E is *assumed* if it is "infinitely more likely" than its complement. This can be formalized in terms of preferences as follows:

> (*) whenever a player prefers an act x to an act y conditional on E (loosely speaking, if she were to be informed of E), she also prefers x to y unconditionally (i.e., without this information).[3]

In the usual case where the player has one level of beliefs, this corresponds exactly to probability-1 belief. (See Section 3 for precise statements.) BFK show that, with LCPS beliefs, condition (*) is (essentially) equivalent to the following:

> there is a belief level j such that:

> (BFK-i) for all $i \leq j$ we have $\mu_i(E) = 1$ and
>
> (BFK-ii) for all $i > j$ we have $\mu_i(E) = 0$.

However, the equivalence between the preference-based condition (*) and its ℓ-belief counterpart, conditions (BFK-i) and (BFK-ii), *only* holds if the player's beliefs are represented by an LCPS. We illustrate this in Examples 3.1-3.3, which also show that the problem lies with condition (BFK-ii) above.

Our main result, Theorem 4.1, characterizes condition (*) in terms of (unrestricted) LPSs. In particular, it provides a precise weakening of condition (BFK-ii) required for the equivalence. Intuitively, condition (BFK-ii) implies:

> (†) the payoffs at states in E "do not matter" as far as the probabilities of level greater than j are concerned.

With LCPSs, (†) implies that the probability of E is zero at such levels. But, when we allow for unrestricted LPSs, (†) is also satisfied whenever, for all $i > j$, the restriction of μ_i to E is a linear combination of the lower-level probabilities $\mu_0, \ldots, \mu_j$ (restricted to E). That is, this linear combination property is the modification of (BFK-ii) required to characterize (*) with LPSs.

Second, BFK provide an epistemic characterization of iterated admissibility (and self-admissible sets, or SASs; see Definition 5.4) in terms of mutual assumption of ℓ-rationality,

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[3]We emphasize that, as in Savage, there is no real "information" in our static setting; this is just suggestive language.

using the LCPS formulation of "assumption," i.e., conditions (BFK-i) and (BFK-ii). We show that, when players' beliefs are represented by unrestricted LPSs, the very same epistemic conditions continue to characterize iterated admissibility (and SASs), provided we use our LPS formulation of assumption.[4]

Sections 2-3 introduce the framework and basic definitions. Section 4 provides the behavioral characterization of assumption for LPSs. Section 5 applies this characterization to the epistemic characterizations of iterated admissibility and SASs. Section 6 discusses the closely related work of Lee (2013a,b). The Appendix provides proofs not included in the body.

2 Preliminaries

Let $(\Omega, \mathcal{S})$ be a Polish space, where $\mathcal{S}$ is the Borel σ-algebra on Ω. Write $\mathcal{P}(\Omega)$ for the set of probability measures on Ω and endow $\mathcal{P}(\Omega)$ with the topology of weak convergence, so that it is also a Polish space.

A **lexicographic probability system** (**LPS**) on Ω will be some $\sigma = (\mu_0, \ldots, \mu_{n-1})$ where each $\mu_i \in \mathcal{P}(\Omega)$. An LPS $\sigma = (\mu_0, \ldots, \mu_n)$ has **full support** if $\bigcup_{i=0}^{n} \operatorname{supp} \mu_i = \Omega$.

Let $\mathcal{A}$ be the set of all measurable functions from Ω to $[0, 1]$. A particular function $x \in \mathcal{A}$ is an act. For $c \in [0, 1]$, write $\overrightarrow{c}$ for the constant act associated with c, i.e. $\overrightarrow{c}(\Omega) = \{c\}$. Given acts $x, y \in \mathcal{A}$ and a Borel subset E in Ω, write $(x_E, y_{\Omega \backslash E})$ for the act z with

$$
z(\omega) = \begin{cases} x(\omega) & \text{if } \omega \in E \\ y(\omega) & \text{if } \omega \in \Omega \backslash E. \end{cases}
$$

When $\Omega = \{\omega_0, \omega_1, \ldots, \omega_K\}$, write $(x_0, x_1, \ldots, x_K)$ for an act x with $x(\omega_k) = x_k$. In this case, we also write $\mu = (\mu(\omega_1), \ldots, \mu(\omega_K))$ for some $\mu \in \mathcal{P}(\Omega)$.

Given an LPS $\sigma = (\mu_0, \ldots, \mu_{n-1})$ on Ω, define a preference relation $\succsim^\sigma$ on $\mathcal{A}$ where $x \succsim^\sigma y$ if and only if

$$
\left(\int_\Omega x(\omega) \mathrm{d}\mu_i(\omega) \right)_{i=0}^{n-1} \geq^L \left(\int_\Omega y(\omega) \mathrm{d}\mu_i(\omega) \right)_{i=0}^{n-1}.
$$

Write $\succ^\sigma$ for the associated strict preference relation. Given a Borel set E, define the conditional preference given E in the usual way, i.e., $x \succsim_E^\sigma y$ if for some $z \in \mathcal{A}$, $(x_E, z_{\Omega \backslash E}) \succsim^\sigma (y_E, z_{\Omega \backslash E})$. (The choice of z does not affect the conditional preference relation.[5]) Write $\succ_E^\sigma$ for the associated strict preference relation and $\sim_E^\sigma$ for the associated indifference relation.

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[4]The proofs of these epistemic results follow BFK closely; the only significant modification is in establishing measurability. See the Appendix.

[5]This follows from the fact that $\succsim^\sigma$ satisfies independence.

3 Assumption

Assumption is defined in terms of the preference relation $\succsim^\sigma$ associated with the LPS σ. The informal idea is that an event E is assumed if states in E "determine" strict preferences.

Definition 3.1 (BFK, Definition A.3) *Say a set E is **assumed** under $\succsim^\sigma$ if E is Borel and the following hold:*

Non-Triviality: *for each open set U with $E \cap U \neq \emptyset$, there are $x, y \in \mathcal{A}$ with $x \succ^\sigma_{E \cap U} y$,*

Strict Determination: *for all $x, y \in \mathcal{A}$, $x \succ^\sigma_E y$ implies $x \succ^\sigma y$.*

Non-Triviality states that every "part" of E can potentially determine strict preferences. Strict Determination states that, if x is strictly preferred to y conditional on E, then x is also unconditionally strictly preferred to y, regardless of the outcomes x and y may deliver outside of E.

BFK restrict attention to LPSs that, loosely speaking, consist of measures having disjoint supports (cf. their Definition 4.1). Refer to these as lexicographic *conditional* probability systems (LCPS's); this terminology is due to Blume, Brandenburger and Dekel (1991*a*).

Proposition A.2 and Lemma B.1 of BFK show that, given a full-support LCPS $\sigma = (\mu_0, \ldots, \mu_{n-1})$, the preference $\succsim^\sigma$ assumes an event E if and only if there exists some $j \in \{0, \ldots, n-1\}$ such that the following three **BFK conditions** hold:

(BFK-i) $\mu_i(E) = 1$ for all $i \leq j$,

(BFK-ii) $\mu_i(E) = 0$ for all $i > j$, and

(BFK-iii) $E \subseteq \bigcup_{i \leq j} \operatorname{supp} \mu_i$.

One direction of this equivalence result holds for all full-support LPSs: If there is some j that satisfies Conditions (BFK-i)–(BFK-iii), then $\succsim^\sigma$ assumes E, even if σ does *not* have disjoint supports. However, the converse need not hold for all full-support LPSs. In particular, we will argue that the problem arises from Condition (BFK-ii).

Notice that Condition (BFK-ii) holds trivially for $j = n-1$, whether or not E is assumed under $\succsim^\sigma$. But, if Ω is finite and $E \subsetneq \Omega$ is assumed by a full-support LPS, then Conditions (BFK-i) and (BFK-iii) can only hold for some $j < n - 1$.[6] So, if all three conditions hold for *some* j, it must be the case that $j < n - 1$. We now provide examples where Condition (BFK-ii) can hold *only* for $j = n - 1$. So, to obtain a general characterization of assumption for all full-support LPSs, we relax Condition (BFK-ii).

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[6]Suppose that $j = n-1$. Then, by (BFK-i), $\operatorname{supp} \mu_i \subseteq E$ for all i, so $\Omega = \bigcup_i \operatorname{supp} \mu_i \subseteq E$, contradiction.

Example 3.1 *Take $\Omega = \{\omega_0, \omega_1\}$ and consider an LPS $\sigma = (\mu_0, \mu_1)$ with $\mu_0 = (1, 0)$ and $\mu_1 = (\frac{1}{2}, \frac{1}{2})$. Let $E = \{\omega_0\}$. If $x \succ^\sigma_{\{\omega_0\}} y$, then $x(\omega_0) > y(\omega_0)$. It follows that $x \succ^\sigma y$, so E is assumed under $\succsim^\sigma$. Note that Conditions (BFK-i) and (BFK-iii) hold only for $j = 0$, while Condition (BFK-ii) holds only for $j = 1$. Thus, one direction of BFKs characterization fails.*

We now establish a more general fact, which also illustrates the first step in the proof of Theorem 4.1: for any full-support LPS $\sigma = (\mu_0, \dots, \mu_{n-1})$ on $\Omega = \{\omega_0, \omega_1\}$, if the event $E = \{\omega_0\}$ is assumed under $\succsim^\sigma$, then $\mu_0(E) = 1$. Hence, if E is assumed, Conditions (BFK-i) and (BFK-iii) hold for $j = 0$, but in general Condition (BFK-ii) may hold only (trivially) for $j = n - 1$.

To prove this fact, suppose that E is assumed under $\succsim^\sigma$, but $\mu_0(E) < 1$. Let x be an act with $x = (x_0, 0)$ and $x_0 > 0$; let $y = (0, 1)$. Since $\mu_i(\{\omega_0\}) > 0$ for some $i = 0, \dots, n - 1$, $x \succ^\sigma_{\{\omega_0\}} y$. But, since $\mu_0(\{\omega_1\}) > 0$, for $x(\omega_0)$ sufficiently small, $y \succ^\sigma x$; this yields a contradiction. ∎

In Example 3.1, the definition of assumption implies that Conditions (BFK-i) and (BFK-iii) hold for $j = 0$. The next example illustrates that, with more than two states, Conditions (BFK-i) and (BFK-iii) may not hold for $j = 0$, but rather for some $j = 1, \dots, n - 2$.

Example 3.2 *Let $\Omega = \{\omega_0, \omega_1, \omega_2\}$ and $E = \{\omega_0, \omega_1\}$. Consider the LPS $\sigma = (\mu_0, \mu_1, \mu_2)$ such that $\mu_0 = (1, 0, 0)$, $\mu_1 = (\frac{1}{2}, \frac{1}{2}, 0)$ and $\mu_2 = (0, \frac{1}{2}, \frac{1}{2})$. We claim that $\succsim^\sigma$ assumes E. Non-Triviality is immediate. (For any open set U with $E \cap U \neq \emptyset$, $\overrightarrow{1} \succ^\sigma_{E \cap U} \overrightarrow{0}$.) For Strict Determination, suppose $x \succ^\sigma_E y$. It must be the case that $x(\omega_0) \geq y(\omega_0)$; if not, $y \succ^\sigma_E x$. Hence there are two possibilities: either (i) $x(\omega_0) > y(\omega_0)$, or (ii) $x(\omega_0) = y(\omega_0)$ and $x(\omega_1) > y(\omega_1)$; if not, $y \succsim^\sigma_E x$. In either case, $x \succ^\sigma y$, so $\succsim^\sigma$ assumes E. Notice that Conditions (BFK-i) and (BFK-iii) do not hold for $j = 0$, but do hold for $j = 1$. On the other hand, Condition (BFK-ii) holds only for $j = 2$.*

We now establish a more general fact, which illustrates the second step in the proof of Theorem 4.1: if an event is assumed, then Conditions (BFK-i) and (BFK-iii) hold for some j. Consider an arbitrary full-support LPS $\sigma = (\mu_0, \dots, \mu_{n-1})$ on $\Omega = \{\omega_0, \omega_1, \omega_2\}$, and suppose that $\succsim^\sigma$ assumes $E = \{\omega_0, \omega_1\}$. The argument given in Example 3.1 still implies that $\mu_0(E) = 1$. We now argue that, if the support of μ_0 does not contain E, i.e., Condition (BFK-iii) fails for $j = 0$, then $\mu_1(E) = 1$. For simplicity, continue to assume that $\mu_0(\omega_0) = 1$. Now suppose that $\mu_1(\omega_0) + \mu_1(\omega_1) < 1$. Consider acts x, y such that $x(\omega_0) = y(\omega_0) = 0$, $x(\omega_1) > 0 = y(\omega_1)$, and $x(\omega_2) = 0 < 1 = y(\omega_2)$. By full support of the LPS, $x \succ^\sigma_E y$. However, $\int_\Omega x \mathrm{d}\mu_0 = x(\omega_0) = y(\omega_0) = \int_\Omega y \mathrm{d}\mu_0$ but, for $x(\omega_1)$ sufficiently small, $\int_\Omega x \mathrm{d}\mu_1 < \int_\Omega y \mathrm{d}\mu_1$. Hence, $\mu_1(E) = 1$. We can repeat this argument if $\mu_2(E) < 1$, etc.; due to full support, we

will eventually reach a j such that E is contained in $\bigcup_{i=0}^{j} \operatorname{supp} \mu_i$. For this j, Conditions (BFK-i) and (BFK-iii) both hold.

To sum up, the examples suggest that the definition of assumption implies that Conditions (BFK-i) and (BFK-iii) hold for some j. In Example 3.1, where $\Omega = \{\omega_0, \omega_1\}$, the condition that $\mu_0(E) = 1$ was not just necessary, but also sufficent for $E = \{\omega_0\}$ to be assumed.[7] This may suggest that Conditions (BFK-i) and (BFK-iii) may be sufficient. The next example shows that this is not the case.[8]

Example 3.3 *Let $\Omega = \{\omega_0, \omega_1, \omega_2\}$ and $E = \{\omega_0, \omega_1\}$. Consider the LPS $\sigma = (\mu_0, \mu_1, \mu_2)$ such that $\mu_0 = (\frac{1}{2}, \frac{1}{2}, 0)$, $\mu_1 = (0, 0, 1)$, and $\mu_2 = (1, 0, 0)$. Conditions (BFK-i) and (BFK-iii) hold (only) for $j = 0$. But, E is not assumed under $\succsim^\sigma$. Consider acts x, y with $x(\omega_0) = 1$, $x(\omega_1) = x(\omega_2) = 0$, and $y = 1 - x$. Then $x \succ_E^\sigma y$ and $y \succ^\sigma x$, contradicting Strict Determination.*

In light of Examples 3.1-3.3, we amend Condition (BFK-ii). To understand how this is done, suppose that, for some j, BFKs Conditions (BFK-i), (BFK-ii), and (BFK-iii) hold. Observe an immediate implication of Condition (BFK-ii): if two acts x, y agree on the complement of E (that is, they assign the same outcomes at states $\omega \notin E$), and their expected utilities are the same for all $i = 0, \ldots, j$, then their expected utilities are the same for *all* $j = 0, \ldots, n - 1$. Now notice that, in the first paragraphs of Examples 3.1 and 3.2, the property just stated also holds, even though Condition (BFK-ii) does not hold. On the other hand, in Example 3.3 the property just stated fails. This suggests that, if attention is restricted to the event E, the additional measures $\mu_{j+1}, \ldots, \mu_{n-1}$ must be "redundant."

The main methodological contribution of this paper is to show that indeed this is the appropriate formulation of condition (BFK-ii) for general LPS's. To formalize the notion of redundancy, we draw insight from Theorem 3.1 in Blume, Brandenburger and Dekel (1991*a*) and require that the measures $\mu_{j+1}, \ldots, \mu_{n-1}$ be *linear combinations* of $\mu_0, \ldots, \mu_j$, when restricted to events in E.

4   Characterization

We provide our main result: a characterization of assumption in terms of an LPS.

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[7]Consider a full-support LPS σ. Non-Triviality is immediate. For Strict Determination, consider acts x, y. If $x \succ_E^\sigma y$, then $x(\omega_0) > y(\omega_0)$ because $E = \{\omega_0\}$, so also $x \succ^\sigma y$ because $\mu_0(E) = 1$. Hence, E is assumed under $\succsim^\sigma$.

[8]This point is implicit in the work of BFK; for completeness, we provide an example that uses an LPS that does not have disjoint supports.

Definition 4.1 *Fix a full support LPS $\sigma = (\mu_0, \ldots, \mu_{n-1})$. Say a set $E \subseteq \Omega$ is **assumed under σ at level j** if E is Borel and*

(i) *$\mu_i(E) = 1$ for all $i \leq j$,*

(ii) *for each $k > j$, there exists $(\alpha_0^k, \ldots, \alpha_j^k) \in \mathbb{R}^{j+1}$ so that, for each Borel $F \subseteq E$, $\mu_k(F) = \sum_{i=0}^j \alpha_i^k \mu_i(F)$, and*

(iii) *$E \subseteq \bigcup_{i \leq j} \operatorname{supp} \mu_i$.*

*Say a set $E \subseteq \Omega$ is **assumed under σ** if it is assumed under σ at some level j.*

Conditions (BFK-i)-(BFK-iii), imply Conditions (i)-(iii). When σ is an LCPS, the above conditions are equivalent to the conditions in BFK. This will be a consequence of our characterization result below plus BFK's characterization.

Notice that condition (ii) of Definition 4.1 requires linear—not convex—combinations. This is necessary. For instance, in Example 3.2, $\mu_2(F) = -\frac{1}{2}\mu_0(F) + \mu_1(F)$ for all $F \subseteq \{\omega_0, \omega_1\}$. There is no $\alpha_0^2, \alpha_1^2 \geq 0$, so that $\mu_2(F) = \alpha_0^2 \mu_0(F) + \alpha_1^2 \mu_1(F)$ for all $F \subseteq \{\omega_0, \omega_1\}$.

Theorem 4.1 *Fix some full-support LPS σ. A set $E \subseteq \Omega$ is assumed under $\succsim^\sigma$ if and only it is assumed under σ.*

We first prove that Definition 4.1 is sufficent, i.e., if E is assumed under σ, then it is assumed under $\succsim^\sigma$. We need two preliminary results.

Remark 4.1 *Fix some $\sigma = (\mu_0, \ldots, \mu_{n-1})$ and a Borel set E. There is some i with $\mu_i(E) > 0$ if and only if there are $x, y \in \mathcal{A}$ with $x \succ_E^\sigma y$.*

Proof. If $\mu_i(E) = 0$ for each i, then $x \sim_E^\sigma y$. Conversely, if $\mu_i(E) > 0$ for some i, then $\overrightarrow{1} \succ_E^\sigma \overrightarrow{0}$. ∎

Remark 4.2 *Fix some full-support LPS $\sigma = (\mu_0, \ldots, \mu_{n-1})$ with $E \subseteq \Omega$ assumed under σ at level j. Then, for each open set U with $E \cap U \neq \emptyset$, $\mu_i(U) = \mu_i(E \cap U) > 0$ for some $i \leq j$.*

Proof. Fix some open set U with $E \cap U \neq \emptyset$. By Condition (iii) of Definition 4.1, for each $\omega \in E \cap U$, there is some $i \leq j$ with $\omega \in \operatorname{supp} \mu_i$. Since U is an open neighborhood of ω, $\mu_i(U) > 0$. By Condition (i) of Definition 4.1, $\mu_i(E \cap U) = \mu_i(U) > 0$. ∎

Proof of Theorem 4.1, sufficiency. Suppose E is assumed under σ at level j. Nontriviality follows from Remark 4.2 and Remark 4.1. We focus on Strict Determination.

Assume $x \succ_E^\sigma y$. Then, there exists some $k = 0, \ldots, n-1$ so that

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(a) $\int_E (x - y) \mathrm{d}\mu_i = 0$ for all $i \le k - 1$ and

(b) $\int_E (x - y) \mathrm{d}\mu_k > 0$.

It suffices to show that $k \le j$; if so, then by part (i) of Definition 4.1, it follows that $x \succ^\sigma y$.

Suppose, contra hypothesis, $k > j$. Then, by part (ii) of Definition 4.1, there exists $(\alpha_0^k, \ldots, \alpha_j^k) \in \mathbb{R}^{j+1}$ so that

$$\int_E (x - y) \mathrm{d}\mu_k = \sum_{i=0}^{j} \alpha_i^k \int_E (x - y) \mathrm{d}\mu_i = 0,$$

where the second equality follows from (a). But this contradicts (b). ∎

We now turn to the proof of necessity: if E is assumed under $\succsim^\sigma$, then it is assumed under σ. The proof consists of four main steps. Suppose that E is assumed under $\succsim^\sigma$. First, we show that condition (i) must hold for $j = 0$, as in Example 3.1. Second, we show that there exists j such that conditions (i) and (iii) hold, as in Example 3.2. Third, we show that there exists j such that, in addition, the measures $\mu_{j+1}, \ldots, \mu_{n-1}$ are 'redundant' in the sense discussed on page 7. Fourth, and finally, we show that condition (ii) characterizes this notion of redundance. The four Lemmata to come correspond to these four steps.

Lemma 4.1 *Fix an LPS $\sigma = (\mu_0, \ldots, \mu_{n-1})$. If E is assumed under $\succsim^\sigma$, then $\mu_0(E) = 1$.*

Proof. Suppose that E is assumed under $\succsim^\sigma$. We break the proof into two steps. First, we show that $\mu_0(E) > 0$. Then, we use this fact to show that $\mu_0(E) = 1$.

Step 1: Construct acts x, y so that $x = (\overrightarrow{1}_E, \overrightarrow{0}_{\Omega \backslash E})$ and $y(\omega) = 1 - x(\omega)$. By Non-Triviality (Remark 4.1), there exists some i so that $\mu_i(E) > 0$. Thus, $x \succ_E^\sigma y$. By Strict Determination, $x \succ^\sigma y$, from which it follows that $\mu_0(E) > 0$: If $\mu_0(E) = 0$, then $y \succ^\sigma x$.

Step 2: Suppose $\mu_0(E) \in (0, 1)$ and fix some number $Y \in (\mu_0(E), 1]$. Construct acts x, y with

$$x(\omega) = \begin{cases} \mu_0(\Omega \backslash E) & \text{if } \omega \in \operatorname{supp} \mu_0 \cap E \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad y(\omega) = \begin{cases} Y & \text{if } \omega \in \operatorname{supp} \mu_0 \backslash E \\ 0 & \text{otherwise.} \end{cases}$$

By Step 1, $\mu_0(E) = \mu_0(\operatorname{supp} \mu_0 \cap E) > 0$ and, by hypothesis, $\mu_0(\Omega \backslash E) > 0$. Thus, $x \succ_E^\sigma y$. But, since $\mu_0(\Omega \backslash E) > 0$,

$$\mu_0(\Omega \backslash E)\mu_0(\operatorname{supp} \mu_0 \cap E) = \mu_0(\Omega \backslash E)\mu_0(E) < \mu_0(\Omega \backslash E)Y = \mu_0(\operatorname{supp} \mu_0 \backslash E)Y$$

and so $y \succ^\sigma x$. This contradicts the fact that E is assumed under $\succsim^\sigma$. ∎

Lemma 4.2 *Fix a full-support LPS $\sigma = (\mu_0, \ldots, \mu_{n-1})$. If $E \subseteq \Omega$ is assumed under $\succsim^\sigma$, then there is some $j = 0, \ldots, n-1$ so that*

(1) $\mu_i(E) = 1$ for all $i \leq j$, and

(2) $E \subseteq \bigcup_{i \leq j} \operatorname{supp} \mu_i$.

Proof. Fix some $E \subseteq \Omega$ that is assumed under $\succsim^\sigma$. By Lemma 4.1, $\mu_0(E) = 1$. We will show that, if (a) $\mu_i(E) = 1$ for all $i \leq k$ and (b) $E \backslash \bigcup_{i \leq k} \operatorname{supp} \mu_i \neq \emptyset$, then $\mu_{k+1}(E) = 1$. By full-support of the LPS, this gives that there exists some j satisfying conditions (1)-(2) in the statement of the Lemma.

Throughout, suppose that there exists k satisfying (a)-(b), but $\mu_{k+1}(\Omega \backslash E) > 0$. It will be convenient to define $F = E \backslash \bigcup_{i \leq k} \operatorname{supp} \mu_i$. Note that $U = (\Omega \backslash \bigcup_{i \leq k} \operatorname{supp} \mu_i)$ is an open set and $F = E \cap (\Omega \backslash \bigcup_{i \leq k} \operatorname{supp} \mu_i) = E \cap U$. By (b), $F = E \cap U \neq \emptyset$. Thus, it follows from Non-Triviality (Remark 4.1) that there exists some $l \geq k+1$ so that $\mu_l(F) > 0$. We make use of this fact below.

First we show that $\mu_{k+1}(F) > 0$. Then we use this fact to show that $\mu_{k+1}(E)$ cannot be strictly less than 1.

	F	$E \backslash F$	$\Omega \backslash E$
x	1	1	0
y	0	1	1

Table 1: Illustration of Acts

Step 1: Construct acts x, y so that $x = (\overrightarrow{1}_E, \overrightarrow{0}_{\Omega \backslash E})$ and $y = (\overrightarrow{0}_F, \overrightarrow{1}_{\Omega \backslash F})$; these are illustrated in Table 1. Since $\mu_l(F) > 0$ for some l and $F \subseteq E$, it follows that $x \succ^\sigma_E y$. From Strict Determination then, $x \succ^\sigma y$.

For each $i \leq k$, $\mu_i(E \backslash F) = 1$ and so, for each $i \leq k$, $\int x d\mu_i = \int y d\mu_i$. If $\mu_{k+1}(F) = 0$, then using the hypothesis that $\mu_{k+1}(\Omega \backslash E) > 0$, it would follow that $\int_\Omega y d\mu_{k+1} > \int_\Omega x d\mu_{k+1}$ and so $y \succ^\sigma x$. This contradicts the earlier claim that $x \succ^\sigma y$ and so $\mu_{k+1}(F) > 0$.

Step 2: Since $\mu_{k+1}(F) \leq \mu_{k+1}(E) < 1$, there exists $Y \in (\mu_{k+1}(F), 1]$. Define acts x, y so that

$$x(\omega) = \begin{cases} \mu_{k+1}(\operatorname{supp} \mu_{k+1} \backslash E) & \text{if } \omega \in \operatorname{supp} \mu_{k+1} \cap F \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad y(\omega) = \begin{cases} Y & \text{if } \omega \in \operatorname{supp} \mu_{k+1} \backslash E \\ 0 & \text{otherwise.} \end{cases}$$

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Note that $\mu_{k+1}(\operatorname{supp}\mu_{k+1}\backslash E) = \mu_{k+1}(\Omega\backslash E) > 0$ and, by Step 1, $\mu_{k+1}(\operatorname{supp}\mu_{k+1}\cap F) = \mu_{j+1}(F) > 0$. This gives that $x \succ_E^\sigma y$ and, so, by strict determination, $x \succ^\sigma y$.

But, $y \succ^\sigma x$: For $i \le k$, $\mu_i(E\backslash F) = 1$ and so $\int_\Omega x d\mu_i = \int_\Omega y d\mu_i = 0$. Moreover,

$$\int_\Omega y d\mu_{k+1} = Y\mu_{k+1}(\operatorname{supp}\mu_{k+1}\backslash E) > \mu_{k+1}(F)\mu_{k+1}(\operatorname{supp}\mu_{k+1}\backslash E) = \int_\Omega x d\mu_{k+1},$$

since $\mu_{k+1}(\operatorname{supp}\mu_{k+1}\backslash E) > 0$. This contradicts $x \succ^\sigma y$. ∎

Lemma 4.3 *Fix a full-support LPS $\sigma = (\mu_0, \ldots, \mu_{n-1})$. If $E \subseteq \Omega$ is assumed under $\succsim^\sigma$, then there is some $j = 0, \ldots, n-1$ so that*

(1) $\mu_i(E) = 1$ for all $i \le j$,

(2) $E \subseteq \bigcup_{i \le j}\operatorname{supp}\mu_i$, and

(3) if $\int_E(x-y)d\mu_i = 0$ for all $i \le j$, then $\int_E(x-y)d\mu_i = 0$ for all $i = 0, \ldots, n-1$.

Proof. Fix a full-support LPS $\sigma = (\mu_0, \ldots, \mu_{n-1})$ on Ω, so that E is assumed under $\succsim^\sigma$. By Lemma 4.2, there exists some k so that:

$(k.a)$ $\mu_i(E) = 1$ for $i \le k$, and

$(k.b)$ $E \subseteq \bigcup_{i \le k}\operatorname{supp}\mu_i$.

We will suppose further that

$(k.c)$ there exists some x, y so that

- $\int_E(x-y)d\mu_i = 0$ for all $i \le k$, and
- $\int_E(x-y)d\mu_i > 0$ for some $i > k$.

We will show that $\mu_{k+1}(E) = 1$, so that $(k+1).a$-$(k+1).b$ also hold. Repeatedly applying this argument gives that there exists some j satisfying conditions (1)-(3) of the Lemma.

Note that throughout we fix x, y satisfying $(k.c)$. We can and do take this choice of x, y to satisfy $x \succ_E^\sigma y$. In this case, we can choose $l - 1 \ge k$ so that

- $\int_E(x-y)d\mu_i = 0$ for all $i \le l-1$, and
- $\int_E(x-y)d\mu_l > 0$.

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We will show that, if $\mu_{k+1}(E) < 1$, then we can use x, y to construct acts $\hat{x}$ and $\hat{z}$ so that $\hat{x} \succ_E^\sigma \hat{z}$ and $\hat{z} \succ^\sigma \hat{x}$. This contradicts Strict Determination.

For each $\rho \in (0, 1)$, let $z[\rho]$ be the act with $z[\rho](\omega) = \rho x(\omega) + (1 - \rho)y(\omega)$ for all $\omega \in \Omega$. Note that for any $i = 0, \ldots, n - 1$,

$$\int_E (x - z[\rho])d\mu_i = (1 - \rho)\int_E (x - y)d\mu_i.$$

So, for each $\rho \in (0, 1)$, (i) $\int_E (x - z[\rho])d\mu_i = 0$ for all $i \leq l - 1$, and (ii) $\int_E (x - z[\rho])d\mu_l > 0$. It follows that, for each $\rho \in (0, 1)$, $x \succ_E^\sigma z[\rho]$.

Construct acts $\hat{x} = (x_E, \overrightarrow{0}_{\Omega \backslash E})$ and $\hat{z}[\rho] = (z[\rho]_E, \overrightarrow{1}_{\Omega \backslash E})$. Certainly, for each $\rho \in (0, 1)$, $\hat{x} \succ_E^\sigma \hat{z}[\rho]$. We will next show that, if $\mu_{k+1}(\Omega \backslash E) > 0$, then there is some $\rho \in (0, 1)$ so that $\hat{z}[\rho] \succ^\sigma \hat{x}$. To do so, first note that, since $\mu_i(E) = 1$ for all $i \leq k$, it follows that, for each $\rho \in (0, 1)$ and each $i \leq k$, $\int_\Omega (\hat{z}[\rho] - \hat{x})d\mu_i = 0$. Next note that, for each $\rho \in (0, 1)$,

$$\int_\Omega (\hat{z}[\rho] - \hat{x})d\mu_{k+1} = (1 - \rho)\int_E (y - x)d\mu_{k+1} + \mu_{k+1}(\Omega \backslash E).$$

If $\mu_{k+1}(\Omega \backslash E) > 0$, there exists $\rho^* \in (0, 1)$ large enough so $\int_\Omega (\hat{z}[\rho^*] - \hat{x})d\mu_{j+1} > 0$ and so $\hat{z}[\rho^*] \succ^\sigma \hat{x}$. ∎

We now provide the fourth and final part of the proof.

Lemma 4.4 *Fix a full-support LPS $\sigma = (\mu_0, \ldots, \mu_{n-1})$. Suppose that $E \subseteq \Omega$ is Borel and, for some $j = 0, \ldots, n - 1$,*

(1) $\mu_i(E) = 1$ for all $i \leq j$,

(2) if $\int_E (x - y)d\mu_i = 0$ for all $i \leq j$, then $\int_E (x - y)d\mu_i = 0$ for all $i = 0, \ldots, n - 1$.

(3) $E \subseteq \bigcup_{i \leq j} \text{supp}\, \mu_i$, and

Then E is assumed under σ.

Proof. Take j so that conditions (1)-(3) hold. This implies that, for j, conditions (i) and (iii) in Definition 4.1 hold. We must show that condition (ii) in Definition 4.1 holds as well: that is, for each $k > j$, there exists $(\alpha_0^k, \ldots, \alpha_j^k) \in \mathbb{R}^{j+1}$ so that, for any Borel $F \subseteq E$, $\mu_k(F) = \sum_{i=0}^j \alpha_i^k \mu_i(F)$.

Let $\mathcal{B}$ denote the vector space of bounded Borel-measurable functions $b : \Omega \to \mathbb{R}$. For each $i = 1, \ldots, j, k$, define linear functionals $T_1, \ldots, T_j, T_k$ on $\mathcal{B}$ by $T_i(b) = \int_E b\, d\mu_i$. By condition (2), if $x, y \in \mathcal{A}$ with $T_i(x - y) = 0$ for all $i \leq j$, then $T_k(x - y) = 0$. Now, note that

$\mathcal{B}$ is the set of all functions of the form $\gamma(x - y)$ for $\gamma \in \mathbb{R}_{++}$ and $x, y \in \mathcal{A}$. So, for each $b \in \mathcal{B}$, $T_i(b) = 0$ for all $i \leq j$ implies that $T_k(b) = 0$. Hence, by the Theorem of the Alternative (see Aliprantis and Border, 2007, Corollary 5.92), there exists $(\alpha_0^k, \ldots, \alpha_j^k) \in \mathbb{R}^{j+1}$ with $T_k = \sum_{i=0}^{j} \alpha_i^k T_i$.

For any $F \subseteq E$ Borel, it follows that

$$\mu_k(F) = \int_E (\overrightarrow{1}_F, \overrightarrow{0}_{\Omega \backslash F}) d\mu_k = \sum_{i=0}^{j} \alpha_i^k \int_E (\overrightarrow{1}_F, \overrightarrow{0}_{\Omega \backslash F}) d\mu_i = \sum_{i=0}^{j} \alpha_i^k \mu_k(F),$$

as desired. ∎

Proof of Theorem 4.1, necessity.   Immediate from Lemmata 4.3 and 4.4. ∎

5   Application: SAS and IA

This section applies the LPS-based characterization of assumption to BFK's game-theoretic analysis. We consider type structures where types map to arbitrary LPSs, rather than LCPSs. We formalize (lexicographic) rationality and mutual or common assumption thereof. We then show that self-admissible sets and iterated admissibility capture the behavioral implications of these epistemic conditions in arbitrary and, respectively, complete type structures.

As in BFK, we restrict attention to two-player games. Fix a game $\langle S_a, S_b, \pi_a, \pi_b \rangle$ where S_a (resp. S_b) is a finite strategy set for Ann (resp. Bob) and π_a (resp. π_b) is a payoff function.

5.1   Solution Concepts

The following definitions are standard.

Definition 5.1 *Fix $X \times Y \subseteq S_a \times S_b$. A strategy $s_a \in X$ is **weakly dominated with respect to** $X \times Y$ if there exists $\sigma_a \in \mathcal{M}(S_a)$, with $\sigma_a(X) = 1$, such that $\pi_a(\sigma_a, s_b) \geq \pi_a(s_a, s_b)$ for every $s_b \in Y$, and $\pi_a(\sigma_a, s_b) > \pi_a(s_a, s_b)$ for some $s_b \in Y$. Otherwise, say s_a is **admissible with respect to** $X \times Y$. If s_a is admissible with respect to $S_a \times S_b$, simply say that s_a is **admissible**.*

Definition 5.2 *Set $S_a^0 = S_a$ and $S_b^0 = S_b$. Define inductively*

$$S_a^{m+1} = \{s_a \in S_a^m : s_a \text{ is admissible with respect to } S_a^m \times S_b^m\};$$

*and, likewise, define S_b^{m+1}. A strategy $s_a \in S_a^m$ is called m-**admissible**. A strategy $s_a \in \bigcap_{m=0}^{\infty} S_a^m$ is called **iteratively admissible (IA)**.*

The following definitions are due to BFK.

Definition 5.3 *Say r_a **supports** s_a if there exists some $\sigma_a \in \mathcal{P}(S_a)$ with $r_a \in \operatorname{supp} \sigma_a$ and $\pi_a(\sigma_a, s_b) = \pi_a(s_a, s_b)$ for all $s_b \in S_b$. Write $\operatorname{su}(s_a)$ for the set of $r_a \in S_a$ that support s_a.*

Definition 5.4 *Fix $Q_a \times Q_b \subseteq S_a \times S_b$. The set $Q_a \times Q_b$ is a **self-admissible set (SAS)** if:*

(a) *each $s_a \in Q_a$ is admissible,*

(b) *each $s_a \in Q_a$ is admissible with respect to $S_a \times Q_b$,*

(c) *for any $s_a \in Q_a$, if $r_a \in \operatorname{su}(s_a)$ then $r_a \in Q_a$,*

and likewise for each $s_b \in Q_b$.

5.2 Epistemic Analysis

For each $n \in \mathbb{N}$, write $\mathcal{N}_n(\Omega)$ for the set of LPS's of length n and write $\mathcal{N}(\Omega) = \bigcup_{n \in \mathbb{N}} \mathcal{N}_n(\Omega)$ for the set of LPS. Write $\mathcal{N}^+(\Omega)$ for the set of $\sigma \in \mathcal{N}$ that have full support. Define a metric on $\mathcal{N}(\Omega)$ as follows: The distance between two sequences of measures $(\mu_0, \ldots, \mu_{n-1})$ and $(\nu_0, \ldots, \nu_{n-1})$ of the same length is the maximum of the Prohorov distances between μ_i and ν_i for all $i < n$. The distance between two sequences of measures of different lengths is 1. With this, $\mathcal{N}(\Omega)$ is a Polish space and, by Corollary C.1 in BFK, $\mathcal{N}^+(\Omega)$ is Borel.

Definition 5.5 *An (S_a, S_b)-**based type structure** is a structure*

$$\langle S_a, S_b, T_a, T_b, \lambda_a, \lambda_b \rangle,$$

*where T_a and T_b are nonempty Polish **type spaces**, and $\lambda_a : T_a \to \mathcal{N}(S_b \times T_b)$ and $\lambda_b : T_b \to \mathcal{N}(S_a \times T_a)$ are Borel measurable **belief maps**. A type structure is **complete** if $\mathcal{N}^+(S_b \times T_b) \subsetneq \operatorname{range} \lambda_a$ and $\mathcal{N}^+(S_a \times T_a) \subsetneq \operatorname{range} \lambda_b$.[9]*

Type structures are our basic representation of interactive LPS-based beliefs. This differs from BFK's Definition 7.1 in that it does not require that types be mapped to LCPSs (or limits of LCPSs). A type structure induces a set of **states**, i.e., $S_a \times T_a \times S_b \times T_b$.

In the remainder of this subsection, we fix a (S_a, S_b)-based type structure $\langle S_a, S_b, T_a, T_b, \lambda_a, \lambda_b \rangle$. All definitions have counterparts with a and b reversed.

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[9]We write $\operatorname{range} \lambda_a$ for the range of the function λ_a.

Definition 5.6 *A strategy s_a is **optimal** under $\sigma = (\mu_0, \ldots, \mu_{n-1})$ if $\sigma \in \mathcal{N}(S_b \times T_b)$ and*

$$\left(\pi_a(s_a, \operatorname{marg}_{S_b} \mu_i(s_b))\right)_{i=0}^{n-1} \geq^L \left(\pi_a(r_a, \operatorname{marg}_{S_b} \mu_i(s_b))\right)_{i=0}^{n-1}$$

for all $r_a \in S_a$.[10]

Here, $\operatorname{marg}_{S_b} \mu_i$ denotes the marginal on S_b of the measure μ_i. In words, Ann will prefer strategy s_a to strategy r_a if the associated sequence of expected payoffs under s_a is lexicographically greater than the sequence under r_a. (If σ is a length-one LPS (μ_0), we will sometimes say that s_a is optimal under the measure μ_0 if it is optimal under (μ_0).)

We now formalize the epistemic assumptions of interest as restrictions on strategy-type pairs.

Definition 5.7 *A strategy-type pair $(s_a, t_a) \in S_a \times T_a$ is **rational** if $\lambda_a(t_a)$ is a full-support LPS and s_a is optimal under $\lambda_a(t_a)$.*

Next, for $E \subseteq S_b \times T_b$, set

$$A_a(E) = \{t_a \in T_a : E \text{ is assumed under } \lambda_a(t_a)\}.$$

In words, $A_a(E)$ is the set of types $t_a \in T_a$ with associated LPS's $\lambda_a(t_a)$ that assume the event E (in $S_b \times T_b$). Note, if $E \subseteq S_b \times T_b$ is not Borel, then $A_a(E) = \emptyset$.

For finite m, define sets R_a^m as follows. Let R_a^1 be the set of all rational $(s_a, t_a) \in S_a \times T_a$. Inductively, set

$$R_a^{m+1} = R_a^m \cap [S_a \times A_a(R_b^m)].$$

Definition 5.8 *If $(s_a, t_a, s_b, t_b) \in R_a^{m+1} \times R_b^{m+1}$, say there is **rationality and mth-order assumption of rationality ($Rm AR$)** at this state. If $(s_a, t_a, s_b, t_b) \in \bigcap_{m=1}^{\infty} R_a^m \times \bigcap_{m=1}^{\infty} R_b^m$, say there is **rationality and common assumption of rationality (RCAR)** at this state.*

5.3   Results

We begin with the characterization of RCAR in terms of SAS's.

Theorem 5.1

(1) For each type structure, $\operatorname{proj}_{S_a} \bigcap_m R_a^m \times \operatorname{proj}_{S_b} \bigcap_m R_b^m$ is an SAS.

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[10]If $x = (x_0, \ldots, x_{n-1})$ and $y = (y_0, \ldots, y_{n-1})$, then $x \geq^L y$ if and only if $y_j > x_j$ implies $x_k > y_k$ for some $k < j$.

(2) For each SAS $Q_a \times Q_b$, there exists some type structure so that $\operatorname{proj}_{S_a} \bigcap_m R_a^m \times \operatorname{proj}_{S_b} \bigcap_m R_b^m = Q_a \times Q_b$.

Part (2) is Theorem 8.1(ii) in BFK. They show this by constructing an LCPS-based type structure which is, a fortiori, an LPS-based type structure. Part (1) is an analogue of Theorem 8.1(i) in BFK. In contrast to BFK, we allow for arbitrary LPS-based type structures and employ our characterization of assumption.

Now turn to the analysis of RmAR in complete type structures.

Theorem 5.2 *Fix a complete type structure. For each m, $\operatorname{proj}_{S_a} R_a^m \times \operatorname{proj}_{S_b} R_b^m = S_a^m \times S_b^m$.*

Theorem 5.2 is an analogue of Theorem 9.1 in BFK. Again, we allow for an LPS-based notion of a complete type structure and employ our characterization of assumption.

We do not provide self-contained proofs of these theorems. The structure of the arguments follows the proofs of Theorems 8.1-9.1 in BFK. We discuss the required modifications in the Appendix.

6 Discussion: Related Literature

Lee (2013a,b) also extends the results of BFK to LPSs. His elegant approach is different from but complementary to ours. His starting point is that the same lexicographic preference relation may be represented by more than one LPS. (See Blume, Brandenburger and Dekel (1991a, page 66)). Lee (2013a) shows that a lexicographic preference relation $\succsim$ assumes an event E if and only if Conditions (BFK-i)–(BFK-iii) hold for *some* LPS σ for which $\succsim^\sigma = \succsim$. That is, instead of providing conditions that a *given* LPS must satisfy for the corresponding preference relation to assume an event E, he provides conditions that must be satisfied by at least one of the many LPSs that represent the same preferences. The Corollary below shows that such a result also follows from our Theorem 4.1. Lee (2013b) then uses this characterization to analyze RCAR.

Corollary 6.1 *Fix a full-support LPS σ. A set $E \subseteq \Omega$ is assumed under $\succsim^\sigma$ if and only if there is some LPS ρ satisfying Conditions (BFK-i)–(BFK-iii) such that $\succsim^\sigma = \succsim^\rho$.*

Proof. Fix the LPS $\sigma = (\mu_0, \ldots, \mu_{n-1})$. Since Conditions (BFK-i)–(BFK-iii) imply the conditions in Definition 4.1, the "if" direction is immediate from Theorem 4.1. Suppose $\succsim^\sigma$ assumes E. By Theorem 4.1, σ assumes E at some level j. For every $k = j+1, \ldots, n-1$,

define a measure μ_k by letting

$$\nu_k(F) = \begin{cases} \mu_k(F) & \text{if } \mu_k(E) = 1 \\ \left[\mu_k(F) - \sum_{i=0}^{j} \alpha_i^k \mu_i(F)\right] / [1 - \mu_k(E)] & \text{if } \mu_k(E) < 1. \end{cases}$$

for every Borel $F \subseteq \Omega$, where $(\alpha_i^k)_{i=0}^{j}$ are the weights appearing in condition (ii) of Definition 4.1. Next, define the LPS $\rho' = (\mu_0, \ldots, \mu_j, \nu_{j+1}, \ldots, \nu_{n-1})$. Finally, define the LPS ρ by deleting from ρ' the measures ν_k for which $k \geq j + 1$ and $\mu_k(E) = 1$.

We claim that $\succsim^\rho = \succsim^{\rho'}$. To see this, consider a Borel set $F \subseteq \Omega$ and an index $k > j$ for which $\mu_k(E) = 1$. By condition (ii) of Definition 4.1, $\mu_k(F) = \sum_{i=0}^{k} \alpha_i^k \mu_i(F)$ for $F \subseteq E$. Furthermore, for $F \not\subseteq E$, $\mu_k(F) = \mu_k(F \cap E) = \sum_{i=0}^{k} \alpha_i^k \mu_i(F \cap E) = \sum_{i=0}^{k} \alpha_i^k \mu_i(F)$, because $\mu_i(E) = 1$ for $i = 0, \ldots, j$ by condition (i) of Definition 4.1. Hence, μ_k is a linear combination of $\mu_0, \ldots, \mu_i$, so it is redundant and can be dropped from ρ' without changing preferences.

Moreover, $\succsim^\sigma = \succsim^{\rho'}$. To see this, note that, for every $k > j$ such that $\nu_k \neq \mu_k$, ν_k is a linear combination of μ_k (with weight $\frac{1}{1-\mu_k(E)}$) and $\mu_0, \ldots, \mu_j$ (with weights $-\frac{\alpha_i^k}{1-\mu_k(E)}$, $i = 0, \ldots, j$). Hence, for every $k = 0, \ldots, n - 1$, the k-th level measure in ρ' is either equal to μ_k, or else is a linear combination of measures in σ. The claim follows.

It follows that $\succsim^\sigma = \succsim^\rho$. By construction, if $\mu_k(E) < 1$, $\nu_k(E) = 0$, because μ_k satisfies Condition (ii). This implies that ρ satisfies Conditions (BFK-i)–(BFK-iii). ∎

Appendix A   Theorems 5.1-5.2

The proofs of Theorems 8.1 and 9.1 in BFK rely on three results concerning the properties of assumption for LCPS-based type structures. (See Lemma D.1, Property 6.3, and Lemma C.4 in BFK.) The statement and proofs of these results rely on BFK's characterization of assumption for LCPS's. We have seen that the characterization does not apply to arbitrary LPS's. To address this, we state and prove analogous properties in our setting. (See Lemmata A.1, A.2, and A.3.)

Lemma A.1 *Let* $\lambda_a(t_a) = (\mu_0, \ldots, \mu_n)$ *be a full-support LPS. Suppose* t_a *assumes* $E \subseteq S_b \times T_b$. *Then, there exist some* j *so that*

$$\bigcup_{i \leq j} \text{supp marg}_{S_b} \mu_i = \text{proj}_{S_b} E.$$

Note that if $S_b \times T_b$ is finite and $\lambda_a(t_a) = (\mu_0, \ldots, \mu_{n-1})$ assumes E at level j, then $E = \bigcup_{i \leq j} \text{supp } \mu_i$. If $S_b \times T_b$ is infinite, the same may not hold. Lemma A.1 establishes an analogue for the marginal LPS (marg) on the finite set S_b and, correspondingly, the project

of E on S_b. Lemma shows that, if $\Omega = S_b \times T_b$ is infinite, a similar statement holds if we consider the marginal LPS $(\text{marg}_{S_b}\mu_0, \ldots, \text{marg}_{S_b}\mu_{n-1})$ and, correspondingly, the projection of E on S_b.

Lemma A.2 *If a full-support LPS $\sigma \in \mathcal{N}^+(S_b \times T_b)$ assumes $R_b^1, R_b^2, \ldots$, then it assumes $\bigcap_m R_b^m$.*

Note that Lemma A.2 will be a consequence of a conjunction property of assumption.

Lemma A.3 *The sets R_a^m and R_b^m are Borel.*

To prove Theorem 5.1, it is enough to replace Lemma D.1 and Property 6.3 in BFK's proof with Lemma A.1 and Lemma A.2. To prove Theorem 5.2, two changes to BFK's proof are needed. First, replace Lemma D.1 and Lemma C.4 in BFK with Lemma A.1 and Lemma A.3. Second, modify the proof of Lemma E.3 in BFK, for the case where $m \geq 2$: Skip the construction that ensures that $\mu_i(U) = 0$ for all i. (That particular construction does not work for arbitrary LPS's. Fortunately, it is not needed in our setting.)

We now prove Lemmata A.1, A.2, and A.3.

Proof of Lemma A.1. Suppose t_a assumes $E \subseteq S_b \times T_b$ at level j. If $s_b \in \text{proj}_{S_b}E$, then there exists $i \leq j$ such that $\mu_i(\{s_b\} \times T_b) > 0$. (See Remark 4.2.) It follows that, if $s_b \in \text{proj}_{S_b}E$, $s_b \in \text{supp}\,\text{marg}_{S_b}\mu_i$. Conversely, if $s_b \notin \text{proj}_{S_b}E$, then $E \cap (\{s_b\} \times T_b) = \emptyset$. Since each $\mu_i(E) = 1$ for $i \leq j$, it follows that $\mu_i(\{s_b\} \times T_b) = 0$ for $i \leq j$, i.e., $s_b \notin \bigcup_{i \leq j} \text{supp}\,\text{marg}_{S_b}\mu_i$. $\blacksquare$

Lemma A.4 *Fix Borel sets $E_1, E_2, \ldots$, with $E_{m+1} \subseteq E_m$. If a full-support LPS $\sigma = (\mu_0, \ldots, \mu_n)$ assumes each of $E_1, E_2, \ldots$, then it assumes $\bigcap_m E_m$.*

Proof. For each $m = 1, 2, \ldots$, there exists some $j[m] \in \{0, \ldots, n\}$ so that σ assumes E_m at level $j[m]$. Let $j = \min\{j[m] : m \geq 1\}$. Let M be some m with $j = j[M]$. We show that $\bigcap_m E_m$ is assumed under σ at level $j = j[M]$.

For Condition (i) note that, for each $i \leq j$, $\mu_i(E_m) = 1$ for all m. So, by continuity, $\mu_i(\bigcap_m E_m) = 1$. For Condition (iii) note that $\bigcap_m E_m \subseteq E_{M[j]} \subseteq \bigcup_{i \leq j} \text{supp}\,\mu_i$. For Condition (ii), note that each Borel $F \subseteq \bigcap_m E_m$ is also a subset of $E_{M[j]}$. Thus, Condition (ii) applied to $\bigcap_m E_m$ follows from Condition (ii) applied to $E_{M[j]}$. $\blacksquare$

Proof of Lemma A.2. Immediate from Lemma A.4. $\blacksquare$

We now turn to the proof of Lemma A.3. This is more involved. We will break the proof into several Lemmata. The first Lemma is standard (and so the proof is omitted).

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Lemma A.5 *Fix some strategy $s_a \in S_a$.*

(1) The set of $\mu \in \mathcal{P}(S_b)$ so that s_a is optimal under μ is closed.

(2) The set of $\mu \in \mathcal{P}(S_b)$ so that s_a is strictly optimal under μ is open.

Lemma A.6 *The sets R_a^1 and R_b^1 are Borel.*

Proof. For each $s_a \in S_a$, define $\mathbb{O}[s_a, n]$ to be

$$\mathbb{O}[s_a, n] = \{\sigma \in \mathcal{N}_n(S_b \times T_b) : s_a \text{ is optimal under } \sigma\}.$$

Note, that

$$R_a^1 = \bigcup_{s_a \in S_a} \bigcup_{n \in \mathbb{N}^0} \left[\{s_a\} \times \left((\lambda_a)^{-1}(\mathbb{O}[s_a, n]) \cap (\lambda_a)^{-1}(\mathcal{N}_n^+(S_b \times T_b)) \right) \right].$$

Since λ_a is measurable and $\mathcal{N}_n^+(S_b \times T_b)$ is Borel (insert cite), it suffices to show that each $\mathbb{O}[s_a, n]$ is measurable.

Write $O[s_a]$ for the set of $\mu \in \mathcal{P}(S_b)$ under which s_a is optimal, $O^s[s_a]$ for the set of $\mu \in \mathcal{P}(S_b)$ under which s_a is strictly optimal, and $O^w[s_a] = O[s_a] \backslash O^s[s_a]$. By Lemma A.5, $O^w[s_a]$, $O^s[s_a]$, and $O[s_a]$ are Borel. Note that

$$\mathbb{O}[s_a, n] = (O^s[s_a] \times \mathcal{N}_{n-1}(S_b \times T_b)) \cup (O^w[s_a] \times O^s[s_a] \times \mathcal{N}_{n-2}(S_b \times T_b)) \cup \cdots \cup (O^w[s_a] \times O^w[s_a] \times \cdots \times O[s_a]),$$

so that $\mathbb{O}[s_a, n]$ is Borel. ∎

Given a Borel set $E \subseteq \Omega$, write $\mathcal{S}_E$ for the set of $F \subseteq E$ that are Borel. Of course, $\mathcal{S}_E \subseteq \mathcal{S}$. Moreover, $\mathcal{S}_E$ is the Borel σ-algebra on E. (See Aliprantis and Border, 2007, Lemma 4.20.)

Lemma A.7 *Fix some $n \in \mathbb{N}^0$ and some $j = 0, \ldots, n$. If $E \in \mathcal{S}(\Omega)$, then*

$$\{\sigma \in \mathcal{N}_n(\Omega) : E \text{ is assumed under } \sigma \text{ at level } j\}$$

is Borel.

A Corollary of Lemma A.7 is:

Corollary A.1 *If $E \in \mathcal{S}$, then*

$$\{\sigma \in \mathcal{N}(\Omega) : E \text{ is assumed under } \sigma\}$$

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is Borel.

To prove Lemma A.7, we will make use of a number of auxiliary results.

Lemma A.8 *Fix a Borel $E \subseteq \Omega$. There exists a countable algebra $\mathcal{F}_E$ on E that generates $\mathcal{S}_E$.*

Proof. Since E is a subset of a second countable space, it is second countable. Thus, there exists a countable subbase $\{U^1, U^2, \ldots\}$ that generates $\mathcal{S}_E$. Let $\mathcal{F}_E$ be the algebra generated by $\{U^1, U^2, \ldots\}$. By Rao and Rao (1983, Corollary 1.1.14), $\mathcal{F}_E$ is countable. Moreover, it generates $\mathcal{S}_E$. ■

In what follows, we write $\mathcal{F}_E$ for a countable algebra on E that generates $\mathcal{S}_E$.

Lemma A.9 *Fix an LPS $\sigma = (\mu_0, \ldots, \mu_n)$. Fix also some $j = 0, \ldots, n-1$ and $k > j$. Then, the following are equivalent:*

(1) *There exists $\alpha \in \mathbb{R}^{j+1}$ with $\mu_k(F) = \sum_{i=0}^{j} \alpha_i \mu_i(F)$ for all $F \in \mathcal{S}_E$.*

(2) *There exists an integer $M \geq 1$ such that, for all integers $m \geq 1$, there exists $\beta^m = (\beta_0^m, \ldots, \beta_j^m) \in \mathbb{Q}^{j+1} \cap [-M, M]^{j+1}$ with $|\mu_k(F) - \sum_{i=0}^{j} \beta_i^m \mu_i(F)| \leq \frac{1}{m}$ for all $F \in \mathcal{S}_E$.*

(3) *There exists an integer $M \geq 1$ such that, for all integers $m \geq 1$, there exists $\beta^m = (\beta_0^m, \ldots, \beta_j^m) \in \mathbb{Q}^{j+1} \cap [-M, M]^{j+1}$ with $|\mu_k(F) - \sum_{i=0}^{j} \beta_i^m \mu_i(F)| \leq \frac{1}{m}$ for all $F \in \mathcal{F}_E$.*

Proof. Suppose part (1) holds. If $\sum_{i=0}^{j} \mu_i(E) = 0$, then $\mu_k(F) = \sum_{i=0}^{j} \alpha_i^k \mu_i(F) = 0$ for every $F \subseteq E$ Borel. In this case, take $M = 1$ and $\beta = (0, \ldots, 0) \in \mathbb{Q}^{j+1} \cap [-1, 1]^{j+1}$.

Thus, we focus on the case where $\sum_{i=0}^{j} \mu_i(E) > 0$. In this case, for each $m \geq 1$, we can choose $\varepsilon^m \in (0, \frac{1}{m \sum_{i=0}^{j} \mu_i(E)}]$ and $\beta^m \in \mathbb{Q}^{j+1}$ such that $\max_i |\beta_i^m - \alpha_i| \leq \varepsilon^m$. By construction, $\beta^m \to \alpha$, and so the sequence $(\beta^m)_m$ is bounded. This implies that there exists $M \geq 0$ such that $\beta^m \in [-M, M]^{j+1}$ for all m. Moreover, for each $m \geq 1$ and each $F \subseteq E$ Borel,

$$
\begin{aligned}
\left| \mu_k(F) - \sum_{i=0}^{j} \beta_i^m \mu_i(F) \right| &= \left| \sum_{i=0}^{j} \alpha_i \mu_i(F) - \sum_{i=0}^{j} \beta_i^m \mu_i(F) \right| \\
&= \left| \sum_{i=0}^{j} (\alpha_i - \beta_i^m) \mu_i(F) \right| \\
&\leq |\alpha_i - \beta_i^m| \sum_{i=0}^{j} \mu_i(F) \\
&\leq \varepsilon^m \sum_{i=0}^{j} \mu_i(E) \\
&\leq \frac{1}{m}.
\end{aligned}
$$

This establishes part (2), which in turn establishes part (3).

Next, suppose part (3) holds, i.e., there exist an integer $M \geq 1$ and a sequence $(\beta^m)_m$ such that, for every $m \geq 1$, $\beta^m \in \mathbb{Q}^{j+1} \cap [-M, M]^{j+1}$ and $|\mu_k(F) - \sum_{i=0}^{j} \beta_i^m \mu_i(F)| \leq \frac{1}{m}$ for all $F \in \mathcal{F}_E$. Let $\mathcal{M}$ be the collection of all $F \in \mathcal{B}_E$ for which $|\mu_k(F) - \sum_{i=0}^{j} \beta_i^m \mu_i(F)| \leq \frac{1}{m}$ holds for all $m \geq 1$. We will show that $\mathcal{S}_E \subseteq \mathcal{M}$, thereby establishing part (2).

By Lemma A.8, $\mathcal{S}_E$ is the σ-algebra generated by $\mathcal{F}_E$. So, by the Monotone Class Lemma Aliprantis and Border (2007, Lemma 4.13), $\mathcal{S}_E$ is the smallest monotone class containing $\mathcal{F}_E$. As such, to show $\mathcal{S}_E \subseteq \mathcal{M}$, it suffices to show that $\mathcal{M}$ is a monotone class containing $\mathcal{F}_E$.

The fact that $\mathcal{M}$ contains $\mathcal{F}_E$ follows from part (3). To see that $\mathcal{M}$ is a monotone class, consider a monotonically increasing (resp. decreasing) sequence (F^n) of elements of $\mathcal{M}$. Then $F \equiv \bigcup_n F^n$ (resp. $F \equiv \bigcap_n F^n$) are Borel and, by continuity of the measures $\mu_0, \ldots, \mu_j, \mu_k$, $\lim_{n \to \infty} \mu_i(F^n) = \mu_i(F)$ for $i = 0, \ldots, j, k$. Therefore, $\lim_{n \to \infty} |\mu_k(F^n) - \sum_{i=0}^{j} \beta_i^m \mu_i(F^n)| = |\mu_k(F) - \sum_{i=0}^{j} \beta_i^m \mu_i(F)|$, and so $|\mu_k(F) - \sum_{i=0}^{j} \beta_i^m \mu_i(F)| \leq \frac{1}{m}$. Thus, $\mathcal{M}$ is a monotone class containing $\mathcal{F}_E$.

Finally, suppose part (2) holds. Since $\beta^m \in \mathbb{Q}^{j+1} \cap [-M, M]^{j+1}$, there exists a convergent subsequence $(\beta^{m(\ell)})_\ell$; let $\beta = (\beta_0, \ldots, \beta_j)$ be its limit. By assumption, for each $m(\ell) \geq 1$ and each $F \in \mathcal{B}_E$ $|\mu_k(F) - \sum_{i=0}^{j} \beta_i^{m(\ell)} \mu_i(F)| \leq \frac{1}{m(\ell)}$. It follows that, for each $F \in \mathcal{B}_E$, $|\mu_k(F) - \sum_{i=0}^{j} \beta_i \mu_i(F)| = 0$. This establishes (1). $\blacksquare$

Lemma A.10 *Fix some $n \in \mathbb{N}^0$ and some $j = 0, \ldots, n-1$. If $E \in \mathcal{S}$, then*

$$\bigcap_{k=j+1,\ldots,n} \bigcup_{\alpha^k \in \mathbb{R}^{j+1}} \bigcap_{F \in \mathcal{S}_E} \{\sigma \in \mathcal{N}_n(\Omega) : \mu_k(F) = \sum_{i=0}^{j} \alpha_i^k \mu_i(F)\}$$

is Borel.

Proof. It suffices to show that the set

$$A^k[2] := \bigcup_{\alpha^k \in \mathbb{R}^{j+1}} \bigcap_{F \in \mathcal{S}_E} \{\sigma \in \mathcal{N}_n(\Omega) : \mu_k(F) = \sum_{i=0}^{j} \alpha_i^k \mu_i(F)\}$$

is Borel. Note, by Lemma A.9, $A^k[2] = X^k$ where

$$X^k := \bigcup_{M \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} \bigcup_{\alpha \in \mathbb{Q}^{j+1} \cap [-M,M]^{j+1}} \bigcap_{F \in \mathcal{F}_E} \{\sigma \in \mathcal{N}_n(\Omega) : |\mu_k(F) - \sum_{i=0}^{j} \alpha_i \mu_i(F)| \leq \frac{1}{m}\}.$$

Note that, in the definition of X^k, each of the unions and intersections are taken over countable sets. (Use Lemma A.8 to conclude that $\mathcal{F}_E$ is countable.) Thus, to show that the set $A^k[2]$ is Borel, it suffices to show that, for each $M \geq 1$, $\alpha \in \mathbb{Q}^{j+1} \cap [-M, M]^{j+1}$, $m \in \mathbb{N}$,

and $F \in \mathcal{F}_E$ the set

$$\{\sigma \in \mathcal{N}_n(\Omega) : |\mu_k(F) - \sum\nolimits_{i=0}^{j} \alpha_i \mu_i(F)| \leq \frac{1}{m}\}$$

is Borel. To show this set is Borel, it suffices to show that the map $F : \mathcal{N}_n(\Omega) \to \mathbb{R}$ defined by

$$F(\mu_0, \ldots, \mu_n) = |\mu_k(F) - \sum\nolimits_{i=0}^{j} \alpha_i \mu_i(F)|$$

is measurable.

Note that F is measurable if and only if G is measurable, where

$$G(\mu_0, \ldots, \mu_n) = \mu_k(F) - \sum\nolimits_{i=0}^{j} \alpha_i \mu_i(F).$$

(See Aliprantis and Border, 2007, Theorem 4.27.) Define maps $g_i : \mathcal{N}_n(\Omega) \to \mathbb{R}$ where $g_i(\mu_0, \ldots, \mu_n) = \mu_i(F)$. For each i, g_i is measurable. (See Aliprantis and Border (2007, Lemma 15.16)) With this $G = g_k - \sum_{i=0}^{j} \alpha_i g_i$ is measurable Aliprantis and Border (2007, Theorem 4.27), as desired. ∎

Proof of Lemma A.7. Write

$$A[1] = \bigcap\nolimits_{i=0}^{j} \{\sigma \in \mathcal{N}_n(\Omega) : \mu_i(E) = 1\}$$

and

$$A[3] = \{\sigma \in \mathcal{N}_n(\Omega) : E \subseteq \bigcup\nolimits_{i \leq j} \operatorname{supp} \mu_i\}.$$

Repeating the argument in the proof of Lemma C.3 in Brandenburger, Friedenberg and Keisler (2008), each of the sets $A[1]$ and $A[3]$ are Borel.

Now define

$$A = \{\sigma \in \mathcal{N}_n(\Omega) : E \text{ is assumed under } \sigma \text{ at level } j\}.$$

If $j = n$, then $A = A[1] \cap A[3]$, so A is Borel. Otherwise $A = A[1] \cap A[3] \cap \bigcap_{k=j+1}^{n} A^k[2]$; by Lemma A.10, A is Borel. ∎

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