Additional Material on "Hierarchies of Conditional Beliefs and Interactive Epistemology in Dynamic Games"

Pierpaolo Battigalli
Princeton University
and
European University Institute

Marciano Siniscalchi
Princeton University

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Proof of Lemma 3.6

Notice first that φ_0, the identity on Σ, can be written as $(\varphi_{01}, \varphi_{02})$, where φ_{0i} is the identity on Σ_i, for $i = 1, 2$. Fix $t_i \in T_i$ as in the above statement and let $\mu = g_i(t_i)$, $\mu' = g'_i(\varphi_i(t_i))$; then, by definition, one can find two CPSs $\mu_i \in \Delta^{\mathcal{B}_i}(\Sigma_i)$, $\mu_j \in \Delta^{\mathcal{B}_j}(\Sigma_j \times T_j)$ such that, for all $B = B_1 \times B_2 \in \mathcal{B}$, $\mu(.|B \times T_j) = \mu_i(.|B_i) \otimes \mu_j(.|B_j \times T_j)$. We construct two marginal CPSs $\mu'_i \in \Delta^{\mathcal{B}_i}(\Sigma_i)$, $\mu'_j \in \Delta^{\mathcal{B}_j}(\Sigma_j \times T'_j)$ such that, for all $B = B_1 \times B_2 \in \mathcal{B}$, $\mu'(.|B \times T'_j) = \mu'_i(.|B_i) \otimes \mu'_j(.|B_j \times T'_j)$.

To this end, recall that $\widehat{\varphi_{-i}} : \Delta^{\mathcal{B}}(\Sigma \times T_j) \to \Delta^{\mathcal{B}}(\Sigma \times T'_j)$ is defined by

$$\widehat{\varphi_{-i}}(\mu)(M|B \times T'_j) = \mu(\varphi_{-i}^{-1}(M)|B \times T_j) \quad \forall M \in \mathcal{A}'$$

(where $\mathcal{A}'$ denotes the Borel σ–algebra generated by the product topology on $\Sigma \times T'_j$.)

This suggests the following definitions: for all $B_i \in \mathcal{B}_i$ and measurable $M_i \subset \Sigma_i$, let

$$\mu'_i(M_i|\Sigma_i(h')) = \mu_i(\varphi_{0i}^{-1}(M_i)|\Sigma_i(h));$$

*Email: battigal@datacomm.iue.it, marciano@princeton.edu.

also, for all $B_j \in \mathcal{B}_j$ and measurable $M_j \subset \Sigma_j \times T'_j$ let

$$\mu'_j(M_j|\Sigma_j(h') \times T'_j) = \mu_j((\varphi_{0j}, \varphi_j)^{-1}(M_j)|\Sigma_j(h') \times T_j)$$

These assignments yield CPSs $\mu'_i \in \Delta^{\mathcal{B}_i}(\Sigma_i)$, $\mu'_j \in \Delta^{\mathcal{B}_j}(\Sigma_j \times T'_j)$.

Now fix a rectangular event $E = E_i \times E_j \in \mathcal{A}'$ such that $E_i \subseteq \Sigma_i$ and $E_j \subseteq \Sigma_j \times T'_j$. Notice that, since φ_{-i} maps $\Sigma_i \times \Sigma_j \times T_j$ to $\Sigma_i \times \Sigma_j \times T'_j$ coordinate–by–coordinate,

$$\varphi_{-i}^{-1}(E_i \times E_j) = \varphi_{0i}^{-1}(E_i) \times (\varphi_{0j}, \varphi_j)^{-1}(E_j)$$

which is thus also a rectangular event in $\mathcal{A}$, the Borel σ–algebra on $\Sigma \times T_j$. Since by assumption μ is independent, for any $B = B_1 \times B_2 \in \mathcal{B}$,

$$\mu(\varphi_{-i}^{-1}(E)|B \times T_j) = \mu_i(\varphi_{0i}^{-1}(E_i)|B_i) \cdot \mu_j((\varphi_{0j}, \varphi_j)^{-1}(E_j)|B_j \times T_j)$$

Since $\mu = g_i(t_i)$ and $\widehat{\varphi_{-i}} \circ g_i = g'_i \circ \varphi_i$, the LHS equals $\mu'(E|B \times T'_j)$. But then, applying the definitions above,

$$\mu'(E|B \times T'_j) = \mu'_i(E_i|B_i) \times \mu'_j(E_j|B_j \times T'_j).$$

Hence, $\mu'(.|B \times T'_j)$ and $\mu'_i(.|B_i) \otimes \mu'_j(.|B_j \times T'_j)$ agree on the algebra of rectangles in $\Sigma_i \times (\Sigma_j \times T'_j)$. By standard arguments, they must agree on $\mathcal{A}'$. Therefore $\mu' \in I\Delta^{\mathcal{B}}(\Sigma_i, \Sigma_j \times T'_j)$.

The argument just given shows that, whenever $g_i(t_i) \in I\Delta^{\mathcal{B}}(\Sigma_i, \Sigma_j \times T_j)$, $\widehat{\varphi_{-i}} \circ g_i(t_i) \in I\Delta^{\mathcal{B}}(\Sigma_i, \Sigma_j \times T'_j)$. By the definition of type–morphism, this implies that $g'_i \circ \varphi_i(t_i) \in I\Delta^{\mathcal{B}}(\Sigma_i, \Sigma_j \times T'_j)$, as needed. ■

Proof of Proposition 5.6, part (b)

The preliminary Lemma below shows that, if (rationality and) CCOR given $\mathcal{R}$, the collection of relevant histories, is possible, then the backward induction profile survives arbitrarily many iterations of the $\Sigma^n_{\mathcal{R}}$ procedure. That is, backward induction is "consistent with" rationality and CCOR given $\mathcal{R}$.

Lemma 0.1. *Fix a game of complete and perfect information with no ties among different terminal nodes. Let σ^B_j, $j = 1, 2$, be the (unique) backward induction strategy for Player j. Then, if $R \cap CCOR_{\mathcal{R}} \neq \emptyset$, for every $n \geq 1$, $(\sigma^B_1, \sigma^B_2) \in \Sigma^n_{\mathcal{R}}$.*

Proof: Fix a player $i \in \{1, 2\}$ and let $j \neq i$. For any history $h \in \mathcal{H}$, we can find a strategy $\sigma^h_j \in \Sigma_j$ such that $\sigma^h_j \in \Sigma_j(h)$ and $\sigma^h_j(h') = \sigma^B_j(h')$ for all $h' \in \mathcal{H}$

such that either h is a subhistory of h' (that is, for all h' weakly following h) or h' is not a subhistory of h (that is, for all h' which are not on the unique path to h.)

Observe that, whenever h is a subhistory of h' and at each history h'' such that $h \subset h'' \subset h'$, $\sigma_j^{h'}(h'') = \sigma_j^B(h'')$ (i.e. whenever h' can be reached from h by a sequence of Player j's backward induction moves[1],) $\sigma_j^{h'} = \sigma_j^h$ by construction. In particular, $\sigma_j^h = \sigma_j^B$ for all $h \in \mathcal{H}$ on the backward induction path.

Then it is immediate to see that the collection $\mu_i^B = \{\mu_i^B(\cdot|\Sigma_j(h))\}_{h\in\mathcal{H}}$ with $\mu_i^B(\{\sigma_j^h\}|\Sigma_j(h)) = 1$ for all $h \in \mathcal{H}$ is a well–defined CPS. Also, clearly $\sigma_i^B \in r_i(\mu_j^B)$ (i.e. σ_i^B is a sequential best reply against μ_i^B.) This immediately implies that $(\sigma_1^B, \sigma_2^B) \in \Sigma_{\mathcal{R}}^1$. Now assume that $(\sigma_1^B, \sigma_2^B) \in \Sigma_{\mathcal{R}}^n$; by part (a) of Proposition 5.6, $h \in \mathcal{R}$ implies that h is on the backward induction path, and at any such history $\mu_i^B(\Sigma_{j,\mathcal{R}}^n|\Sigma_j(h)) \geq \mu_i^B(\{\sigma_j^B\}|\Sigma_j(h)) = 1$. Thus, $\mu_i^B \in \Lambda_{i,\mathcal{R}}(\Sigma_{j,\mathcal{R}}^n)$ for $i = 1, 2$ and $j \neq i$, which establishes the induction step. ■

Again, consider a game of perfect and complete information with no ties between payoffs at terminal nodes. Let $\mathcal{H}^n$ be the set of histories h such that the longest continuation of h has length n, that is, let

- $\mathcal{H}^0 = \mathcal{Z}$
- $\mathcal{H}^{n+1} = \left\{h \in \mathcal{H} : \forall a \in A(h), (h, a) \in \bigcup_{k=0}^{n} \mathcal{H}^k\right\}$,

where $A(h)$ is the set of feasible actions (or action profiles) at h.

The following Lemma implies part (b) of Proposition 5.6, as required.

Lemma 0.2. *Consider a game of perfect and complete information with no ties between payoffs at terminal nodes and let $\mathcal{R}$ be the set of its relevant histories. Suppose $R \cap CCOR_{\mathcal{R}} \neq \emptyset$. Then $\forall n \geq 1$, $\forall h \in \mathcal{H}^n$, $\forall \sigma \in \Sigma(h) \cap \Sigma_{\mathcal{R}}^n$, if player i is active at h, σ_i prescribes the backward induction action at h*

Proof: The base step ($n = 1$) is obvious. Thus, suppose the claim holds for $n \geq 1$ and fix $h \in \mathcal{H}^{n+1}$, $\sigma = (s, \theta) \in \Sigma(h) \cap \Sigma_{\mathcal{R}}^{n+1}$. Let Player i be active at h. Since $\Sigma_{\mathcal{R}}^{n+1} \subset \Sigma_{\phi}^1$, either at h Player i has a dominant continuation strategy, or $h \in \mathcal{R}$. In the first case, σ_i chooses the dominant continuation strategy at h, and of course this coincides with the backward induction prescription. Otherwise, by

[1]This does not preclude the possibility that the other player may have to deviate from her backward induction strategy for h' to be reached starting from h.

part (a) of Proposition 5.6, h is on the backward induction path. Therefore, the backward induction strategy profile reaches h: $(\sigma_1^B, \sigma_2^B) \in \Sigma(h)$.

For any profile $(\sigma_i', \sigma_j) \in \Sigma(h) \cap \Sigma_{\mathcal{R}}^n$, the induction hypothesis implies that the induced continuation path beginning with the history $(h, \sigma_i'(h))$ is the path prescribed in that subgame by the backward induction profile. Thus, if μ is the CPS justifying σ_i, $h \in \mathcal{R}$ implies $\mu(\Sigma_{j,\mathcal{R}}^n \cap \Sigma_j(h)|\Sigma_j(h)) = 1$, and one has $E_\mu[U_i(\sigma_i', \cdot)|h] = U_i(\sigma_i', \sigma_j^B)$ (recall that $\sigma_j^B \in \Sigma_j(h)$.)

By definition, $E_\mu[U_i(\sigma_i, \cdot)|h] \geq E_\mu[U_i(\sigma_i', \cdot)|h]$ for all $\sigma_i' \in \Sigma_i(h)$, and hence *a fortiori* $U_i(\sigma_i, \sigma_j^B) = E_\mu[U_i(\sigma_i, \cdot)|h] \geq E_\mu[U_i(\sigma_i', \cdot)|h] = U_i(\sigma_i', \sigma_j^B)$ for all $\sigma_i' \in \Sigma_i(h) \cap \Sigma_{i,\mathcal{R}}^n$. Indeed, since the game is assumed to be generic, $U_i(\sigma_i, \sigma_j^B) > U_i(\sigma_i', \sigma_j^B)$ whenever (σ_i, σ_j^B) and (σ_i', σ_j^B) reach distinct terminal nodes.

Equivalently, $\sigma_i(h)$ is the *unique* payoff–maximizing action under the assumption of backward induction continuation, when Player i's choices are restricted to the set of actions specified by strategies in $\Sigma_i(h) \cap \Sigma_{i,\mathcal{R}}^n$ at h. But by Lemma 0.1, and since $\sigma_i^B \in \Sigma_i(h)$, this set includes the backward induction choice $\sigma_i^B(h)$. By definition, the latter is (uniquely) optimal, under the assumption of backward induction continuation, among *all* actions available at h, hence *a fortiori* in the restricted set we consider here. This immediately implies that $\sigma_i(h) = \sigma_i^B(h)$, and the proof is complete. ■

Proof of Proposition 5.7

Lemma 0.3. $[h] \cap R_i \subset \beta_{i,h}(R_i)$.

Proof. Suppose $(\sigma, \tau_1, \tau_2) \in [h] \cap R_i$. Then (σ_i, τ_i) satisfies conditions (1), (2) and (3) of Definition 5.2 and $\sigma_i \in \Sigma_i(h)$. The latter fact and condition (1) imply

$$g_{i,h}(\tau_i)\left(\{\sigma_i\} \times \Sigma_j \times T_j\right) = 1.$$

Since $(\sigma, \tau_1, \tau_2) \in R_i$,

$$\{\sigma_i\} \times \Sigma_j \times T_j \subset \left\{(\sigma', \tau_j') : (\sigma_i', \tau_i) \text{ satisfies (1),(2),(3)}\right\} = [R_i]_{\tau_i}.$$

Therefore

$$g_{i,h}(\tau_i)\left([R_i]_{\tau_i}\right) = 1$$

and $(\sigma, \tau_1, \tau_2) \in \beta_{i,h}(R_i)$. ■

Let $R_{i,h}^1 = R_i$ and $R_{i,h}^{n+1} = R_{i,h}^n \cap \beta_{i,h}(R_{j,h}^n)$. (This definition of $R_{i,h}^{n+1}$ is equivalent to the definition given in the proof of Proposition 5.5.) It is easily verified that

$$R_i \cap CCOR_{i,h} = \bigcap_{n \geq 1} R_{i,h}^n.$$

It is then sufficient to show that for all $n \geq 2$

$$[h] \cap R_{i,h}^n = [h] \cap R_i \cap \beta_{i,h}\left(\bigcap_{k=0}^{n-2} \beta_h^k(R)\right), \; i = 1, 2, \tag{0.1}$$

where the iterated operator β_h^k is defined in the usual way, with $\beta_h^0(E) := E$.

Base Step. Using the definition of $R_{i,h}^2$, Lemma 0.3 and conjunction,

$$[h] \cap R_{i,h}^2 = [h] \cap R_i \cap \beta_{i,h}(R_j) =$$

$$[h] \cap R_i \cap \beta_{i,h}(R_i) \cap \beta_{i,h}(R_j) = [h] \cap R_i \cap \beta_{i,h}(R).$$

Induction Step. Assume that eq. 0.1 holds. We have to show that

$$[h] \cap R_{i,h}^{n+1} = [h] \cap R_i \cap \beta_{i,h}\left(\bigcap_{k=0}^{n-1} \beta_h^k(R)\right), \; i = 1, 2.$$

Using the definition of $R_{i,h}^{n+1}$, eq. 0.1, positive introspection (i.e. $\beta_{i,h}(E) \subset \beta_{i,h}(\beta_{i,h}(E))$), conjunction and monotonicity (in this order), we obtain

$$[h] \cap R_{i,h}^{n+1} = [h] \cap R_{i,h}^n \cap \beta_{i,h}(R_{j,h}^n) =$$

$$[h] \cap R_i \cap \beta_{i,h}\left(\bigcap_{k=0}^{n-2} \beta_h^k(R)\right) \cap \beta_{i,h}\left(R_j \cap \beta_{j,h}\left(\bigcap_{k=0}^{n-2} \beta_h^k(R)\right)\right) =$$

$$[h] \cap R_i \cap \beta_{i,h}\left(\bigcap_{k=0}^{n-2} \beta_h^k(R)\right) \cap \beta_{i,h}\left(\beta_h\left(\bigcap_{k=0}^{n-2} \beta_h^k(R)\right)\right) \cap \beta_{i,h}(R_j) =$$

$$[h] \cap R_i \cap \beta_{i,h}\left(\bigcap_{k=0}^{n-2} \beta_h^k(R)\right) \cap \beta_{i,h}\left(\bigcap_{k=1}^{n-1} \beta_h^k(R)\right) \cap \beta_{i,h}(R_j) =$$

$$[h] \cap R_i \cap \beta_{i,h}\left(\bigcap_{k=0}^{n-1} \beta_h^k(R)\right).$$

■