Foundations for Structural Preferences

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Abstract

The analysis of key game-theoretic concepts such as sequential rationality or backward- and forward-induction hinges on assumptions about players' actions and beliefs at information sets that are not actually reached during game play, and that players themselves do not expect to reach. However, it is not obvious how to elicit intended actions and conditional beliefs at such information sets. In Siniscalchi (2016a) I address this concern by introducing a novel optimality criterion, *structural rationality*, which implies sequential rationality but allows for the incentive-compatible elicitation of beliefs and intended actions. The present paper complements the analysis by providing an axiomatic foundation for structural preferences.

Keywords: conditional probability systems, sequential rationality, structural rationality.

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1 Introduction

The prevalent notion of rationality in dynamic games, *sequential rationality*, is problematic from the perspective of single-person choice theory. Sequential rationality requires that a player (plan to) choose conditionally optimal actions at every information set, including those that she does not expect to reach, and that are indeed not reached in the course of game play. If an information set I is not reached, one cannot observe a player's action at I, or attempt to elicit her conditional beliefs. In addition, it is not obvious how to provide incentives to a sequentially rational player *ex-ante*, so as to induce her to truthfully reveal what she plans to do, or what she would believe, at an information set she does not expect to reach. Therefore, sequential rationality entails restrictions on intended behavior and beliefs that are not testable. Other key game-theoretic concepts such as backward and forward induction also involve assumptions on intended behavior and beliefs off the predicted path of play. Testing such assumptions is just as challenging.

In Siniscalchi (2016a, SP henceforth), I suggest that these difficulties are a consequence of taking sequential rationality—conditional expected-payoff maximization—as the optimality criterion. To overcome these issues, I propose a novel optimality criterion for dynamic games, called *structural rationality*. I show that it implies sequential rationality, but still allows for the incentive-compatible elicitation of intended actions and conditional beliefs throughout the game. Furthermore, structural rationality is consistent with experimental evidence showing that (a) subjects behave differently when a dynamic game is presented as a tree or as a "reduced-form" matrix, but (b) they respond qualitatively similarly when playing the dynamic game directly, or by committing to extensive-form strategies ahead of time. In Siniscalchi (2016b), I demonstrate how structural preferences can be used in the epistemic analysis of forward induction (cf. Battigalli and Siniscalchi, 2002). The main message that emerges from the analysis is that, if players are structurally rational, then epistemic conditions can be interpreted as testable behavioral assumptions, and are thus consistent with a choice-theoretic view of dynamic game theory.

This paper provides an axiomatic characterization of structural preferences: while SP defines structural preferences via their functional representation, the present paper highlights the behavioral properties that distinguish them from other decision rules, and that allow for the unique identification of tastes (i.e., utilities, or payoffs) and conditional beliefs. The analysis is motivated mainly by the application to dynamic games, but applies equally to the analysis of dynamic decision problems. It is set in the decision framework of Anscombe and Aumann (1963), and allows the state space to have arbitrary cardinality.

A central idea in the axiomatization of structural preferences is that of a *negligible event*. Savage (1954) deems an event N *null* if, whenever two prospects f and g yield the same consequences at states outside N, the individual is indifferent between them. Thus, a null event is decision-theoretically irrelevant. If preferences are consistent with expected-utility maximization (EU), an event is null if and only if it has zero probability. The notion of negligibility is more nuanced: N is negligible if, whenever f yields a strictly better consequence than g in every state outside N, the individual strictly prefers f to g. In other words, if f is statewise better than g outside N, it does not matter what consequences f and g yield at states in N. This allows for the possibility that, if f is *not* statewise better than g (for instance, if f and g are equal there), the consequences assigned at states in N determine the individual's ranking of f vs. g. Under mild assumptions that, in particular, are satisfied by both EU and structural preferences, a null event is negligible. In addition, for EU preferences, there is no distinction between null and negligible events. However, for structural preferences, there is a distinction: an event may be negligible, but not null. Yet, an event is negligible if and only if it has zero prior probability. More generally, N is negligible for the preferences the individual holds at an information set I if and only if it has zero probability at I. Thus, negligible events are essential to formalize the idea that an individual cares about unexpected (i.e., zero-probability) events.

A second central idea is that preferences are shaped by, or adapted to, the extensive form of the dynamic game (or decision tree) of interest. For example, structural preferences do not satisfy the standard Anscombe and Aumann (1963) or Fishburn (1970) continuity axiom. However, Axiom 8 implies that, in particular, if at an information set I no strategy profile consistent with I is deemed negligible, then preferences conditional on reaching I *must* be continuous. This approach allows the identification of beliefs conditional upon every information set in the game. However, it does *not* allow one to define beliefs conditional upon arbitrary events. This is by design: as argued in SP, structural preferences are an extensive-form construct. By way of contrast, lexicographic preferences (Blume, Branderburger, and Dekel, 1991b) are defined with reference to a game in matrix (strategic) form.

This paper is organized as follows. Section 2 introduces the formal setup. Section 3 defines structural preferences, as well as necessary ancillary concepts. Section 4 introduces and discusses the axioms for structural preferences, and states the main characterization result. Section 5 concludes. All proofs are in the Appendix.

2 Setup

I now describe the decision environment faced by players in a dynamic game. Subsection 2.1 introduces extensive game forms. Subsection 2.2 describes the domain of each player's preferences, which includes the set of Anscombe and Aumann (1963)–style acts that depend upon coplayers' strategies as well as, possibly, additional sources of uncertainty.

2.1 Extensive Game Forms

Structural preferences are defined in SP for general extensive games with possibly imperfect information, defined essentially as in Osborne and Rubinstein (1994, Def. 200.1, pp. 200-201; OR henceforth). Fix a finite set $\mathcal{N}$ of players. An extensive game form is described in OR essentially by listing histories, i.e., sequence of action choices. Partial histories where player i moves are partitioned into information sets; player i's information partition is denoted $\mathcal{I}_i$.

Payoffs or outcomes are attached to terminal histories.

However, for the purposes of the axiomatic analysis of players' preferences, only certain derived objects are required; see SP for details.

1. Player i's **strategies** are mappings from information sets $I \in \mathscr{I}_i$ to actions. S_i denotes the set of strategies for player i; as usual, $S_{-i} = \prod_{j \neq i} S_j$ and $S = \prod_{i \in \mathscr{N}} S_i$.

2. For every $i \in \mathscr{N}$ and $I \in \mathscr{I}_i$, $S(I)$ is the set of strategies that **reach information set** I. By perfect recall, $S(I) = S_i(I) \times S_{-i}(I)$, where $S_i(I) = \text{proj}_{S_i} S(I)$ and $S_{-i}(I) = \text{proj}_{S_{-i}} S(I)$.

3. For every $i \in \mathscr{N}$, $\mathscr{I}_i(s_i)$ is the collection of information sets that are **allowed by** s_i; thus, $I \in \mathscr{I}_i(s_i)$ iff $s_i \in S_i(I)$.

4. For every $s \in S$, $\zeta(s)$ is the **terminal history induced by** s.

Throughout the remainder of the paper, fix a distinguished player $i \in \mathscr{N}$. I omit the subscript i from often-used notation, when this cannot cause confusion.

2.2 Choice domain

Fix a convex set X of material outcomes; for instance, X may be the set of simple lotteries on some prize space Y, as in Anscombe and Aumann (1963).

The state space for player i comprises her coplayers' strategies, as well as possible additional uncertainty. Such uncertainty may represent the unobserved realization of an randomizing device, as in the elicitation game analyzed in SP. It may also represent *incomplete information*: for instance, the unobserved, common value of an object being auctioned, or private signals received by coplayers. Finally, it may represent coplayers' *epistemic types*, as e.g. in Battigalli and Siniscalchi (1999).

Formally, consider a set W, endowed with a sigma-algebra $\mathscr{W}$. The domain of player i's overall uncertainty is $\Omega = S_{-i} \times W$, endowed with the product sigma-algebra $\Sigma = 2^{S_{-i}} \times \mathscr{W}$.

The distinguished player i is characterized by a preference relation $\succcurlyeq$ on the set of **acts** $f:\Omega\to X$, denoted $\mathscr{A}$.

In SP, attention is largely confined to acts associated with a strategy $s_i\in S_i$. Fix an *outcome function* $\xi : Z\times W\to X$; the interpretation is that, when terminal history z is reached and the realization of player i's additional uncertainty is $w\in W$, then player i's material outcome is $\xi_i(z,w)$. Also consider a strategy s_i of i. Then, for every state $\omega=(s_{-i},w)$, the profile (s_i,s_{-i}) induces terminal history $\zeta(s_i,s_{-i})$, and, given the realization w of concomitant uncertainty, this leads to outcome $\xi(\zeta(s_i,s_{-i}),w)$. This determines an act $f^{s_i}:\Omega\to X$. The axiomatic analysis in the present paper considers arbitrary acts, not just those associated with strategies.

The class of **conditioning events** for player i is defined by

$$\mathscr{F}=\{\Omega\}\cup\{S_{-i}(I)\times W\ :\ I\in\mathscr{I}_i\}. \tag{1}$$

Observe that Ω is always a conditioning event, even if there is no information set $I\in\mathscr{I}_i$ such that $S_{-i}(I)\times W=\Omega$. This is convenient (though not essential) to relate structural preferences to ex-ante expected-payoff maximization.

It is also convenient to introduce the following notation: for each $I\in\mathscr{I}_i$,

$$[I]=S_{-i}(I)\times W. \tag{2}$$

For example, with this notation, $\mathscr{F}=\{\Omega\}\cup\{[I]:I\in\mathscr{I}_i\}$.

3 Conditional Beliefs and Structural Preferences

I now introduce the structural-preference representation. For interpretation and additional analysis of the definitions in this section, see SP.

3.1 Conditional Probability Systems

Following Ben-Porath (1997); Battigalli and Siniscalchi (1999, 2002), player i's beliefs are represented using conditional probability systems (Rényi, 1955). For a measurable space $(Y, \mathscr{Y})$, $pr(\mathscr{Y})$ denotes the set of *finitely* additive probability measures on $\mathscr{Y}$.

Definition 1 *A **conditional probability system** (CPS) for player i is a collection* $\mu \equiv \big(\mu(\cdot|F)\big)_{F\in\mathscr{F}}$ *such that:*

(1) for every $F \in \mathscr{F}$, $\mu(\cdot|F) \in pr(\Sigma)$ *and* $\mu(F|F) = 1$;

(2) for every $E \in \Sigma$ *and* $F, G \in \mathscr{F}$ *such that* $E \subseteq F \subseteq G$,

$$\mu(E|G) = \mu(E|F) \cdot \mu(F|G); \tag{3}$$

Thus, the set $\mathscr{F}$ defined in Eq. (1) is the collection of conditioning events for player i. The characterizing feature of a CPS is the assumption that the chain rule of conditioning, Equation 3, holds even conditional upon events that have zero ex-ante probability.

A CPS μ for player i induces a plausibility ranking over conditioning events, as follows.

Definition 2 *Fix a CPS* μ *on* $(\Sigma, \mathscr{F})$. *For all* $D, E \in \mathscr{F}$, D ***is at least as plausible as*** E ($D \triangleright E$) *if there are* $F_1, \ldots, F_L \in \mathscr{E}$ *such that* $F_1 = E$, $F_L = D$, *and for all* $\ell = 1, \ldots, L-1$, $\mu(F_{\ell+1}|F_\ell) > 0$.

The central notion of a basis can now be introduced.

Definition 3 *Fix a CPS* μ *on* $(\Sigma, \mathscr{F})$. *A **basis** for* μ *is a collection* $(p_C)_{C\in\mathscr{F}} \subset pr(\Sigma)$ *such that*

(1) for every $C, D \in \mathscr{F}$, $p_C = p_D$ *if and only if both* $C \triangleright D$ *and* $D \triangleright C$;

(2) for every $C \in \mathscr{F}$, $p_C(\cup\{D \in \mathscr{F} : C \triangleright D, D \triangleright C\}) = 1$;

(3) for every $C \in \mathscr{F}$, $p_C(C) > 0$ *and, for every* $E \in \Sigma$, $\mu(E \cap C|C) = \frac{p_C(E\cap C)}{p_C(C)}$.

In other words, consider an equivalence class $\mathscr{C} \subseteq \mathscr{F}$ of equally plausible events—that is, an equivalence class for the symmetric part of the relation $\triangleright$. Then there is a probability measure

p that assigns positive probability to each element $C \in \mathscr{C}$, and generates the conditional belief $\mu(\cdot|C)$ by updating. Furthermore, p assigns probability one to the union of all events in $\mathscr{C}$.

It is shown in SP that, if a basis exists, it is unique. Furthermore, SP identifies a property, Consistency, that fully characterizes CPSs that admit a basis.

Notation: The set of CPS for player i is denoted by $cpr(\Sigma, \mathscr{F})$. Denote by $B_0(\Sigma)$ the set of Σ-measurable real functions with finite range[1], and by $B(\Sigma)$ its sup-norm closure. For any probability charge $\pi \in pr(\Sigma)$ and function $a \in B(\Sigma)$, let $\mathrm{E}_\pi[a] = \int_{\Omega_i} a \, d\pi$, the standard Dunford integral of a with respect to π; when no confusion can arise, I will sometimes omit the square brackets.

3.2 Structural Preferences

It is finally possible to formalize the notion of *structural preference* introduced in SP.

Definition 4 *Fix a utility function $u : X \to \mathbb{R}$ and a CPS μ for i that admits a basis $\mathbf{p} = (p_F)_{F \in \mathscr{F}}$. For any pair of acts $f, g \in \mathscr{A}$, f is (weakly)* ***structurally preferred*** *to g given u and μ, written $f \succcurlyeq^{u,\mu} g$, iff for every $F \in \mathscr{F}$ such that $\mathrm{E}_{p_F} u \circ f < \mathrm{E}_{p_F} u \circ g$, there is $G \in \mathscr{F}$ such that $G \triangleright F$ and $\mathrm{E}_{p_G} u \circ f > \mathrm{E}_{p_G} u \circ g$.*

4 Behavioral analysis

The axioms characterizing structural preferences weaken the standard Anscombe and Aumann (1963)–Fishburn (1970) axioms for subjective expected utility. The central axioms characterizing expected-utility maximization, namely Independence and Monotonicity, are maintained. However, as noted in the Introduction, structural preferences may be incomplete and discontinuous: Figure 1 illustrates this point.

[1] Recall that, while S_{-i} is finite, the set W, and hence the state space Ω_i, need not be. Hence the need to define the set of simple functions explicitly.

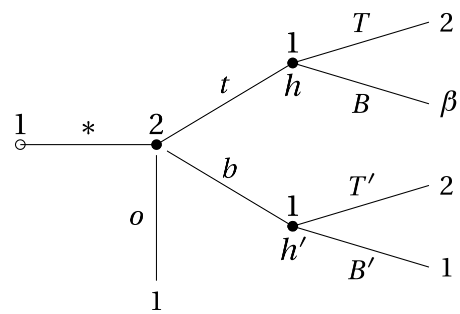

Figure 1: Failures of completeness and continuity; player 2's payoffs are omitted

Take the point of view of player 1 and let $\Omega = S_2 = \{o, t, b\}$; observe that $\mathscr{F} = \{\Omega, \{t\}, \{b\}\}$. Let player 1's CPS be such that $\mu(\{o\}|\Omega) = 1$; by definition, one must also have $\mu(\{t\}|\{t\}) = \mu(\{b\}|\{b\}) = 1$. Its basis is $\mathbf{p} = (p_\Omega, p_{\{t\}}, p_{\{b\}})$, with $p_F = \mu(\cdot|F)$ for all $F \in \mathscr{F}$.

Consider 1's strategies TB' and BT' (the notation omits the trivial action $*$ at the initial node), with payoffs as in the figure and $\beta = 1$. These strategies are not ranked by the structural-preference criterion: they attain the same expected payoff given p_Ω, but TB' has a strictly higher expected payoff given $p_{\{t\}}$, whereas BT' does better given $p_{\{b\}}$.

As for continuity, consider the strategies TB' and BT', with payoffs as in the figure, and the CPS ν such that $\nu(\{t\}|\Omega) = \nu(\{t\}|\{t\}) = 1 = \nu(\{b\}|\{b\})$. The basis for ν is now $\mathbf{q} = (q_\Omega, q_{\{t\}}, q_{\{b\}})$ with $q_\Omega = q_{\{t\}} = \mu(\cdot|\Omega)$ and $q_{\{b\}} = \mu(\cdot|\{b\})$ because $\Omega \triangleright \{t\}$ and $\{t\} \triangleright \Omega$. If $\beta < 2$, then the expected payoff of TB' given q_Ω is strictly greater than that of BT', so TB' is strictly structurally preferred to BT'. However, if $\beta = 2$, the ex-ante expected payoff is the same, but given $q_{\{b\}}$, strategy BT' does strictly better: thus, BT' is strictly preferred when $\beta = 2$.

To sum up, completeness and continuity must be relaxed and/or restricted to specific classes of acts. In turn, this requires changes in other axioms.

Axiom 1 (Preorder) *(i) Reflexivity: for all $f \in \mathscr{A}$, $f \sim f$; (ii) Transitivity: for all $f, g, k \in \mathscr{A}$, if $f \succcurlyeq g$ and $g \succcurlyeq k$, then $f \succcurlyeq k$.*

Axiom 2 (Prize Completeness) *For all prizes $x, y \in X$, either $x \succcurlyeq y$ or $y \succcurlyeq x$ (or both).*

Axiom 3 (Monotonicity) *For all acts* $f,g \in \mathscr{A}$*: if* $f(\omega) \succcurlyeq g(\omega)$ *for all* $\omega \in \Omega$*, then* $f \succcurlyeq g$*.*

Axiom 4 (Independence) *For all acts* $f,g,k \in \mathscr{A}$*,* $f \succcurlyeq g$ *if and only if* $\alpha f + (1-\alpha)k \succcurlyeq \alpha g + (1-\alpha)k$

As in the case of atemporal expected-utility preferences, Axiom 4 implies Savage's Sure-Thing Principle (Postulate P2).

Remark 1 *Assume Axioms 1–4. For all pairs* $f,g,k,k' \in \mathscr{A}$*, and all events* $E \in \Sigma$*:* $f\,E\,k \succcurlyeq g\,E\,k$ *if and only if* $f\,E\,k' \succcurlyeq g\,E\,k'$*.*

Hence, one can define *conditional preferences* following Savage (1954), by modifying pairs so that their act components coincide outside of the conditioning event:

Definition 5 *For all event* $E \in \Sigma$*, player* i*'s* ***conditional preference*** $\succcurlyeq_E$ ***given*** E *is defined as follows: for every* f,g*,* $f \succcurlyeq_E g$ *if, for some* $k \in \mathscr{A}$*,* $f\,E\,k \succcurlyeq_F g\,E\,k$*.*

Remark 1 ensures that any $k \in \mathscr{A}$ will yield the same conditional ranking of the strategies f,g. Note that $\succcurlyeq_\Omega$ is simply the prior preference $\succcurlyeq$.

The remaining axioms involve conditional preferences, as per Def. 5. I emphasize that even these axioms should be interpreted as assumptions on the *prior* preference $\succcurlyeq$; conditional preferences are solely a convenient way to formalize them.

Axiom 5 (Nondegeneracy) *For all* $F \in \mathscr{F}$*, there exist* $x,y \in X$ *such that not* $x \succcurlyeq_F y$*.*

Axiom 6 (Prize Continuity) *For all* $F \in \mathscr{F}$ *and prizes* $x,y,z \in X$*: if* $x \succ_F y$ *and* $y \succ_F z$*, then there exist* $\alpha,\beta \in (0,1)$ *such that* $\alpha x + (1-\alpha)z \succ_F y$ *and* $y \succ_F \beta x + (1-\beta)z$*.*

Prize continuity is in the spirit of Blume, Brandenburger, and Dekel (1991a), except that the conditioning events correspond to information sets in the game. Conditional non-degeneracy is also a substantive requirement, because it implies that all conditioning events "matter" for preferences (even though some may be assigned zero ex-ante probability in the structural-preference representation).

The final four axioms, which are novel, involve a novel notion of "unlikely" events. To motivate it, begin with Savage (1954)'s notion of *null events*: an event N is **Savage-null** for $\succcurlyeq_E$, $E \in \Sigma$, if, for all $f, g \in \mathscr{A}$, $f(\omega) = g(\omega)$ for all $\omega \notin N$ implies $f \sim_E g$. Thus, an event N is null if the outcomes delivered at states in N do not affect preferences. It is immediate to see that, with expected-utility preferences, an event is null if and only if it has zero probability. Now return to the game in Figure 1, with beliefs given by the CPS ν and the basis $\mathbf{q}$ defined above. Recall that, with payoffs as in the figure and $\beta = 2$, the strategy BT' is structurally strictly preferred to TB'; this holds despite the fact that $\nu(\{o, b\}|\Omega) = q_\Omega = 0$ and both TB' and BT' yield the same payoff $\beta = 2$ in state t. Therefore, with structural preferences, an event may have zero prior (or conditional) probability, and yet not be null.

However, there is a sense in which the event $\{o, b\}$ is "not very relevant" for preferences: if $\beta < 2$, then TB' is structurally strictly preferred to BT', even though BT' does just as well as TB' in state o, and strictly better in state b. Indeed, this remains true even if one considers arbitrary acts $f, g \in \mathscr{A}$ (rather than acts induced by strategies in the game). So long as the prize that f yields in state t—the only state not in $\{o, b\}$—is strictly better than the one delivered by g, the ex-ante expected utility of f will be strictly higher than that of g; by Definition 4, this implies that f is strictly structurally preferred. Intuitively, prizes assigned at states in $\{o, b\}$ only matter if there is a tie in state $\{t\}$: modifying prizes on $\{o, b\}$ does not change a strict preference in state $\{t\}$. This leads to the following definition.

Definition 6 *Fix an event $E \in \Sigma$. An event $N \in \Sigma$ is **negligible** given E if, for all $f, g \in \mathscr{A}$, $f(\omega) \succ_E g(\omega)$ for all $\omega \notin N$ implies $f \succ_E g$.*

For EU and structural preferences, any null event is negligible, but, as was just demonstrated, the converse does not hold. Importantly, under the preceding axioms, the union of two negligible events is negligible (cf. Lemma 8 part 3 in the Appendix).

If an event F is *not* negligible given E, then it has a "first-order" effect upon the individual's preferences conditional on E. We can interpret this as indicating that, in the eyes of the individual, F is "at least as plausible" as $E \setminus F$, or (equivalently) as E itself. It may be the case that E is negligible given F, in which case F may be deemed strictly more plausible than E; otherwise, E and F are equally plausible.

The following definition builds upon this intuition. First, it identifies sequences of conditioning events (that is, elements of $\mathscr{F}$) that are ordered in terms of plausibility. Second, given an event $E \subset S_{-i} \times W$, it identifies the strategies consistent with E that are most plausible.

Definition 7 *An* ***n-sequence*** *is an ordered list* $F_1, \ldots, F_L \in \mathscr{F}$ *such that, for every* $\ell = 1, \ldots, L-1$, $F_{\ell+1}$ *is not negligible given* F_ℓ.

The ***(strategic) support*** *of an event* $E \in \Sigma$ *is* $\sigma(E) = \bigcup \big\{ \{s_{-i}\} \times W : \{s_{-i}\} \times W$ *is not negligible given* $E \big\}$.

Thus, in an n-sequence $(F_1, \ldots, F_L)$, $F_{\ell+1}$ is at least as plausible as F_ℓ; note that the converse may or may not hold (that is, two or more consecutive elements of an n-sequence may be pairwise equally plausible). The notion of μ-sequence in SP is closely related to that of n-sequence; indeed, a key step in the proof of Theorem 1 is to show that the two coincide.

The notion of strategic support is easiest to interpret if there is no additional uncertainty. In this case, $\sigma(E)$ is just the collection of all strategy profiles in E that the individual deems plausible. One important observation is that any two strategies in the support of E should be interpreted as being equally plausible. Otherwise, one of them would be negligible given E, and hence not part of the support.[2] All other strategy profiles in E are negligible, hence implausible, given E.

The key novel axioms in this paper can now be stated. The first two require that, for every n-sequence $F_1, \ldots, F_L$, preferences on acts that agree outside the strategic support of $\cup_\ell F_\ell$ are consistent with EU. The intuition is as follows. Assume that there is no additional uncer-

[2] Formally, this follows because, if $s_{-i}, s'_{-i} \in E$ and s_{-i} is less plausible than s'_{-i}, i.e., if it is negligible given $\{s_{-i}, s'_{-i}\}$, then it is also negligible given E: see Lemma 8 part 2 in the Appendix.

tainty for simplicity. Structural preferences should deviate from EU only insofar as events of different degrees of plausibility are involved. But, by definition, the support of an event is a collection of strategy profiles that are equally plausible. Hence, preferences conditional upon such supports should satisfy standard properties, including completeness and continuity.

In addition, the axioms are not imposed upon arbitrary conditioning events, but only on events that can be obtained as unions of n-sequences. This conveys a second essential intuition: structural preferences are determined by behavior conditional upon unions of *conditioning events*. Therefore, structural preferences are defined in the context of a specific extensive game form.

Axiom 7 (Non-Negligible Completeness) *For all n-sequences $F_1,\dots,F_L$, $\succcurlyeq_{\sigma(\cup_\ell F_\ell)}$ is complete.*

Axiom 8 (Non-Negligible Continuity) *For all n-sequences $F_1,\dots,F_L$, if $e \succ_{\sigma(\cup_\ell F_\ell)} f$ and $f \succ_{\sigma(\cup_\ell F_\ell)} g$, then there exist $\alpha,\beta \in (0,1)$ such that $\alpha e+(1-\alpha)g \succ_{\sigma(\cup_\ell F_\ell)} f$ and $f \succ_{\sigma(\cup_\ell F_\ell)} \beta e+(1-\beta)g$.*

The next axiom relates null and negligible events. In general, a negligible event N is not Savage-null, because preferences may be affected in case of ties on the complement of N. Thus, N acts as a "tie-breaker." Axiom 9 restricts this further: negligible events matter only if they break ties *conditional upon reaching some information set* in the game. Thus, again, the extensive-form structure of the game plays a direct role in shaping preferences.

Axiom 9 *For all $N \in \Sigma$, if N is negligible given every $F \in \mathscr{F}$, then it is Savage-null for $\succcurlyeq$.*

The final axiom captures the intuition that ex-ante preferences reflect a trade-off between preferences conditional upon distinct events. It is useful to consider EU preferences as a starting point. It may well be the case that f is weakly preferred to g ex-ante, but there is some positive-probability event F conditional upon which f is strictly worse than g. Of course, in this case f must do strictly better than g conditional upon $\Omega \setminus F$, so as to "compensate" for the fact that $f \prec_F g$. The same holds for structural preferences, if F has positive prior probability.

However, the intuition that structural preferences take unexpected events into account suggests that some form of compensation should be allowed in case F has zero prior probability. Axiom 10 imposes restrictions on the kind of compensation that structural preferences allow.

First of all, the event F must be an element of $\mathscr{F}$—it must correspond to some information set in the game. Once again, the extensive form shapes structural preferences. Second, if F has zero ex-ante probability, the compensating event may be just as plausible as F, or strictly more plausible than F. (In both cases, the compensating event may well have zero prior probability). Importantly, relative plausibility is determined using n-sequences. This is a further channel through which the extensive-form structure determines preferences.

Axiom 10 (Conditional Compensation) *For all $f, g \in \mathscr{A}$ and $F \in \mathscr{F}$: if $f \succcurlyeq g$ and $f \prec_F g$, then there exists an n-sequence $F_1, \ldots, F_L$ such that either (i) $F = F_L$ and $f \succcurlyeq_{\sigma(\cup_\ell F_\ell)} g$, or (ii) $F = F_1$ and $f \succ_{\sigma(\cup_\ell F_\ell)} g$.*

To see how Axiom 10 captures the noted intuition, recall that F_L is the most plausible conditioning event in the n-sequence $F_1, \ldots, F_L$, and F_1 is the least plausible. Thus, case (i) corresponds to a situation in which $f \prec_{F_L} g$, but by considering other conditioning events $F_{L-1}, F_{L-2}, \ldots, F_m$ which are just as plausible as F_L, a weak preference for f results. Case (ii) instead corresponds to compensation via events $F_L, F_{L-1}, \ldots, F_m$ that are strictly more plausible than $F_1 = F$. Finally, the actual conditioning event in cases (i) and (ii) is the strategic support of the n-sequence $F_1, \ldots, F_L$. This has the advantage of (a) concisely expressing the notion that, for some m, only the most plausible events $F_m, \ldots, F_L$ are considered, and (b) eliminating knife-edge cases in which the compensating event is a negligible subset of $F_m \cup \ldots \cup F_L$.

The main result of this paper can now be stated.

Theorem 1 *Let $\succcurlyeq$ be a preference on $\mathscr{A}$. The following statements are equivalent:*

1. *$\succcurlyeq$ satisfies Axioms 1–10;*

2. *there is a non-constant, affine function* $u : X \to \mathbb{R}$*, and a CPS* μ *that admits a basis such that, for all acts* $f, g \in \mathscr{A}$*,* $f \succcurlyeq g$ *if and only if* $f \succcurlyeq^{u,\mu} g$*.*

Furthermore, in (2), u *is unique up to positive affine transformations, and* μ *is unique.*

Theorem 1 has the usual structure of characterization theorems: the preference $\succcurlyeq$ satisfies certain axioms if and only if it admits the representation of interest. However, it turns out that Axioms 1–9, sans Axiom 10, are enough to obtain a unique utility u and CPS μ that admits a basis, such that the preference is *consistent* with the structural preference induced by u and μ. That is, under Axioms 1–9, the preference $\succcurlyeq$ may differ from $\succcurlyeq^{u,\mu}$ only in that it is *finer* (i.e., it may rank more acts); furthermore, it still uniquely identifies tastes and conditional beliefs.

Theorem 2 *If* $\succcurlyeq$ *satisfies Axioms 1–9, then there exists a non-constant, affine function* $u : X \to \mathbb{R}$ *and a CPS* $\mu \in cpr(\Omega_i, \mathscr{F})$ *such that, for every pair of acts* $f, g \in \mathscr{A}$*,*

$$f \succcurlyeq^{u,\mu} g \Longrightarrow f \succcurlyeq g \qquad \text{and} \qquad f \succ^{u,\mu} g \Longrightarrow f \succ g.$$

Furthermore, u *is unique up to positive affine transformations, and* μ *is unique.*

This is useful because SP shows that, if the act f^{s_i} induced by a strategy s_i is maximal with respect to structural preferences, then s_i is sequentially rational. Now suppose that $\succcurlyeq$ satisfies Axioms 1–9, though not necessarily Axiom 10. Suppose further that f^{s_i} is maximal for $\succcurlyeq$: there is no strategy t_i such that $f^{t_i} \succ f^{s_i}$. Then, in particular, there is no strategy t_i such that $f^{t_i} \succ^{u,\mu} f^{s_i}$. The result in SP then implies that s_i must be sequentially rational. Thus, *Axioms 1–9 are enough to imply sequentially rational behavior.*

At a technical level, Theorem 2 shows that Axiom 10 has a relatively limited scope in the characterization of sequential preferences.

[**Note:** Insert example of preference that satisfies Axioms 1–9 but not Axiom 10, and preference that satisfies Axiom 10 but violates one or more of Axioms 1–9.]

5 Discussion and conclusions

[**Note:** To be written]

A Main result: Preliminaries

I recall certain key definitions and results from SP that will be invoked in the proofs of both sufficiency and necessity.

An ordered list $F_1,\ldots,F_L \in \mathscr{F}$ is a μ-**sequence** if $\mu(F_{\ell+1}|F_\ell) > 0$ for all $\ell = 1,\ldots,L-1$. The following Remark lists immediate consequences of the definition of μ-sequence:

Remark 2 *Fix a CPS $\mu \in cpr(\Sigma,\mathscr{F})$ with plausibility ranking $\triangleright$.*

1. *If $F_1,\ldots,F_L$ is a μ-sequence and $1 \le \ell \le m \le L$, then $F_\ell,\ldots,F_m$ is a μ-sequence.*
2. *If $F_1,\ldots,F_L$ and $G_1,\ldots,G_M$ are μ-sequences, and $\mu(G_1|F_L) > 0$, then $F_1,\ldots,F_L,G_1,\ldots,G_M$ is a μ-sequence.*
3. *$F \triangleright G$ iff there is a μ-sequence $F_1,\ldots,F_L$ such that $F_1 = G$ and $F_L = F$.*
4. *if $F_1,\ldots,F_L$ is a μ-sequence and $1 \le \ell \le m \le L$, then $F_m \triangleright F_\ell$.*
5. *if $\mathscr{C}$ is an equivalence class of $\triangleright$, then for any $F \in \mathscr{C}$ there is a μ-sequence $G_1,\ldots,G_M$ such that $\mathscr{C} = \{G_1,\ldots,G_M\}$ and $G_1 = G_M = F$.*

Proof: (1) and (2) are immediate; (3) restates the definition of $\triangleright$, and (4) follows from (1) and (3). For (5), let $\{F_1,\ldots,F_L\}$ be an enumeration of $\mathscr{C}$ with $F_L = F$. Since in particular $F_L \triangleright F_1 \triangleright F_2 \triangleright \ldots \triangleright F_L$, for every $\ell = L,\ldots,2$ there is a μ-sequence $F^\ell_1,\ldots,F^\ell_{L(\ell)}$ with $F^\ell_1 = F_\ell$ and $F^\ell_{L(\ell)} = F_{\ell-1}$. Furthermore, there is a μ-sequence $F^1_1,\ldots,F^1_{L(1)}$ such that $F^1_1 = F_1$ and $F^1_{L(1)} = F_L$. Since $F^\ell_{L(\ell)} = F^{\ell-1}_1$ for all $\ell = L,\ldots,2$, repeated applications of part (3) shows that

$$F^L_1,\ldots,F^L_{L(L)},F^{L-1}_1,\ldots,F^2_{L(2)},F^1_1,\ldots,F^1_{L(1)}$$

is a μ-sequence that contains $\{F_1,\ldots,F_L\}$. Furthermore, $F_1^L = F_L = F^1_{L(1)}$, so for every $\ell = 1,\ldots,L$ and $m = 1,\ldots,L(\ell)$, $F^\ell_m \triangleright F_L$ and $F_L \triangleright F^\ell_m$: that is, $F^\ell_m \in \{F_1,\ldots,F_L\}$. Hence, the above displayed equation provides the required μ-sequence $G_1,\ldots,G_M$, with $G_1 = G_M = F_L = F$. ■

Lemma 1 (SP, Lemma 1) *Let μ be a CPS for player $i \in N$ that admits a basis $\mathbf{p} = (p_F)_{F\in\mathscr{F}}$. Denote by $\triangleright$ the plausibility relation induced by μ.*

1. *For all $E, F \in \mathscr{F}$, $p_F(E) > 0$ implies $E \triangleright F$.*
2. *For all $E, F \in \mathscr{F}$, if $E \triangleright F$ and $p_F \neq p_E$, then $p_E(F) = 0$.*

Given a CPS μ, let

$$\mathscr{F}_\mu = \{\cup_\ell F_\ell : F_1,\ldots,F_L \text{ is a } \mu\text{-sequence }\}.$$

Theorem 3 (cf. SP, Theorem 1) *Let $\mu \in cpr(\Sigma,\mathscr{F})$ be a CPS for player i. The following are equivalent:*

1. *μ admits a unique basis;*
2. *there is a uniqe CPS $\nu \in cpr(\Sigma,\mathscr{F}_\mu)$ such that $\nu(\cdot|F) = \mu(\cdot|F)$ for all $F \in \mathscr{F}$.*

If $\mathbf{p} = (p_F)_{F\in\mathscr{F}}$ is a basis for μ, and $\nu \in cpr(\Sigma,\mathscr{F}_\mu)$ satisfies $\nu(\cdot|F) = \mu(\cdot|F)$ for all $F \in \mathscr{F}$, then, for every $F \in \mathscr{F}$, $p_F = \nu(\cdot|\cup\{G : F \triangleright G, G \triangleright F\})$.

The following facts about μ-sequences will also be useful.

Lemma 2 *Fix a CPS $\mu \in cpr(\Sigma,\mathscr{F})$ with basis $\mathbf{p} = (p_F)_{F\in\mathscr{F}}$. Let $F_1,\ldots,F_L \in \mathscr{F}$ be a μ-sequence. Let $m = \min\{\bar\ell \in \{1,\ldots,L\} : \forall \ell = \bar\ell,\ldots,L-1,\ \mu(F_\ell|F_{\ell+1}) > 0\}$. Also let $\nu \in cpr(\Sigma,\mathscr{F}_\mu)$ be the CPS in part 2 of Theorem 3.*

1. *For every $\ell = 1,\ldots,L$, $F_\ell \triangleright F_L$, $p_{F_\ell} = p_{F_L}$, and $p_{F_L}(F_\ell) > 0$ if and only if $\ell \geq m$.*

2. *For all* $n=1,\dots,L$, $\mu(F_n|\cup_\ell F_\ell)>0$ *iff* $n\ge m$; *and for all* $E\in\Sigma$, $\nu(E|\cup_\ell F_\ell)=\frac{p_{F_L}(E\cap\cup_\ell F_\ell)}{p_{F_L}(\cup_\ell F_\ell)}$.

An index m as in the above statement certainly exists, as $\bar\ell=L$ trivially belongs to the set on the right-hand side.

Proof: By Remark 2 part 4, $F_{\ell+1}\triangleright F_\ell$ for all $\ell=1,\dots,L-1$, and so $F_L\triangleright F_\ell$ by transitivity of $\triangleright$.

(1): from the definition of m, for $\ell\ge m$, $F_\ell\triangleright F_{\ell+1}$, and so $F_\ell\triangleright F_L$ by transitivity of $\triangleright$. Thus, both $F_\ell\triangleright F_L$ and $F_L\triangleright F_\ell$; by part (1) of Def. 3, $p_{F_\ell}=p_{F_L}$ for all $\ell\ge m$. By part (3), $p_{F_L}(F_\ell)=p_{F_\ell}(F_\ell)>0$. It remains to be shown that these properties fail for $\ell<m$.

By contradiction, suppose $F_n\triangleright F_L$ for some $n<m$. The definition of μ-sequence and of the relation $\triangleright$ imply that $F_\ell\triangleright F_n$ for $\ell>n$, so by transitivity of $\triangleright$, $F_\ell\triangleright F_L$ for all $\ell\ge n$. In particular, $F_{m-1}\triangleright F_L$. Again, by the definition of μ-sequence, $F_L\triangleright F_{m-1}$; then by Def. 3 part (1), $p_{F_{m-1}}=p_{F_L}=p_{F_m}$; and by part (3) and the definition of μ-sequence, $p_{F_{m-1}}(F_{m-1})>0$ and

$$0<\mu(F_{m-1}\cap F_m|F_{m-1})=\frac{p_{F_{m-1}}(F_{m-1}\cap F_m)}{p_{F_{m-1}}(F_{m-1})}.$$

Thus, $p_{F_{m-1}}(F_{m-1}\cap F_m)>0$. But then, again by part (3) of Def. 3, $p_{F_m}(F_m)>0$ and

$$\mu(F_{m-1}\cap F_m|F_m)=\frac{p_{F_m}(F_{m-1}\cap F_m)}{p_{F_m}(F_m)}=\frac{p_{F_{m-1}}(F_{m-1}\cap F_m)}{p_{F_m}(F_m)}>0,$$

which contradicts the definition of m.

Thus, not $F_n\triangleright F_L$. By part (1) of Def. 3, $p_{F_n}\ne p_{F_L}$. Finally, if $p_{F_L}(F_n)>0$, then Lemma 1 part 1 implies $F_n\triangleright F_L$, contradiction: thus, $p_{F_L}(F_n)=0$.

(2): by Remark 2 part 5, there is a μ-sequence $F_{L+1},\dots,F_{L+M}$ such that $\{F_{L+1},\dots,F_{L+M}\}$ is the equivalence class of $\triangleright$ that contains F_L—and hence, by part (1) of this Lemma, $F_m,\dots,F_{L-1}$ as well—and $F_{L+1}=F_L$. By Remark 2 part 2, $F_1,\dots,F_{L+M}$ is a μ-sequence.

By construction, $\{F_m,\dots,F_{L+M}\}=\{F_{L+1},\dots,F_{L+M}\}$. Since $\{F_{L+1},\dots,F_{L+M}\}=\{G\in\mathscr{F}:G\triangleright F_L, F_L\triangleright G\}$, by Theorem 3, $\nu(\cdot|\cup_{\ell=m}^{L+M}F_\ell)=p_{F_L}$. Hence, by part (3) of Definition 3, $\nu(F_n|\cup_{\ell=m}^{L+M}F_\ell)=p_{F_L}(F_n)>0$ for $n=m,\dots,L+M$.

I claim that $\nu(\cup_{\ell=m}^{L+M}F_\ell|\cup_{\ell=1}^{L+M}F_\ell)>0$. By contradiction, suppose this is not the case; let $n_0\in\{1,\dots,m\}$ be such that $\nu(\cup_{\ell=n_0}^{L+M}F_\ell|\cup_{\ell=1}^{L+M}F_\ell)=0$ and $\nu(F_{n_0-1}|\cup_{\ell=1}^{L+M}F_\ell)>0$. One such n_0 must exist, because by assumption $\nu(\cup_{\ell=m}^{L+M} F_\ell | \cup_{\ell=1}^{L+M} F_\ell) = 0$, and clearly $\nu(\cup_{\ell=1}^{L+M} F_\ell | \cup_{\ell=1}^{L+M} F_\ell) = 1$. By the chain rule, since $F_1, \dots, F_L$ is a μ-sequence,

$$0 < \mu(F_{n_0} \cap F_{n_0-1} | F_{n_0-1}) = \frac{\nu(F_{n_0} \cap F_{n_0-1} | \cup_{\ell=1}^{L+M} F_\ell)}{\nu(F_{n_0-1} | \cup_{\ell=1}^{L+M} F_\ell)},$$

so $\nu(F_{n_0} | \cup_{\ell=1}^{L+M} F_\ell) \geq \nu(F_{n_0} \cap F_{n_0-1} | \cup_{\ell=1}^{L+M} F_\ell) > 0$: but this contradicts the definition of n_0, which proves the claim.

By the chain rule, conclude that $\nu(F_n | \cup_{\ell=1}^{L+M} F_\ell) > 0$ for $n = m, \dots, L$. Then also $\nu(\cup_{\ell=1}^{L} F_\ell | \cup_{\ell=1}^{L+M} F_\ell) > 0$, and so a further application of the chain rule yields $\nu(F_n | \cup_{\ell=1}^{L} F_\ell) > 0$ for $n = m, \dots, L$.

Finally, consider the first claim. Suppose that $\nu(F_{n_1} | \cup_{\ell=1}^{L} F_\ell) > 0$ for some $n_1 < m$. I claim that then $\nu(F_n | \cup_{\ell=1}^{L} F_\ell) > 0$ for all $n = n_1, \dots, m-1$. The claim is true by assumption fo $n = n_1$. Inductively, assume it is true for some $n - 1 \geq n_1$. Since $F_1, \dots, F_L$ is a μ-sequence, by the chain rule

$$0 < \mu(F_n \cap F_{n-1} | F_{n-1}) = \frac{\nu(F_n \cap F_{n-1} | \cup_{\ell=1}^{L} F_\ell)}{\nu(F_{n-1} | \cup_{\ell=1}^{L} F_\ell)},$$

so $\nu(F_n | \cup_{\ell=1}^{L} F_\ell) > 0$. Hence, in particular, $\nu(F_{m-1} | \cup_{\ell=1}^{L} F_\ell) > 0$. Again by the chain rule and the definition of μ-sequence,

$$0 < \mu(F_m \cap F_{m-1} | F_{m-1}) = \frac{\nu(F_m \cap F_{m-1} | \cup_{\ell=1}^{L} F_\ell)}{\nu(F_{m-1} | \cup_{\ell=1}^{L} F_\ell)},$$

so $\nu(F_m \cap F_{m-1} | \cup_{\ell=1}^{L} F_\ell) > 0$; since, as was just shown, $\nu(F_m | \cup_{\ell=1}^{L}) > 0$, the chain rule implies that

$$\mu(F_{m-1} | F_m) = \mu(F_{m-1} \cap F_m | F_m) = \frac{\nu(F_m \cap F_{m-1} | \cup_{\ell=1}^{L} F_\ell)}{\nu(F_m | \cup_{\ell=1}^{L} F_\ell)} > 0.$$

But then $F_{m-1} \triangleright F_m$, so by transitivity $F_{m-1} \triangleright F_L$, which contradicts part 1 of this Lemma. Therefore, $\nu(F_n | \cup_{\ell=1}^{L} F_\ell) = 0$ for $n = 1, \dots, m-1$.

For the second claim, it is enough to consider a measurable $E \subseteq \cup_\ell F_\ell$. Since, as was just shown, $\nu(F_\ell | \cup_{\ell=1}^{L} F_\ell) = 0$ for $\ell = 1, \dots, m-1$, $\nu(E | \cup_\ell F_\ell) = \nu(E \cap \cup_{\ell=m}^{L} F_\ell | \cup_\ell F_\ell)$. By the chain rule, $\nu(E \cap \cup_{\ell=m}^{L} F_\ell | \cup_\ell F_\ell) = \nu(E \cap \cup_{\ell=m}^{L} F_\ell | \cup_{\ell=m}^{L} F_\ell) \nu(\cup_{\ell=m}^{L} F_\ell | \cup_{\ell=1}^{L} F_\ell)$. Again because $\nu(F_\ell | \cup_{\ell=1}^{L} F_\ell) = 0$ for $\ell = 1, \dots, m-1$, $1 \leq \nu(\cup_{\ell=m}^{L} F_\ell | \cup_{\ell=1}^{L} F_\ell) + \nu(\cup_{\ell=1}^{m-1} F_\ell | \cup_\ell F_\ell) = \nu(\cup_{\ell=m}^{L} F_\ell | \cup_{\ell=1}^{L} F_\ell)$, so $\nu(\cup_{\ell=m}^{L} F_\ell | \cup_{\ell=1}^{L} F_\ell) = 1$, and $\nu(E\cap\cup_{\ell=m}^{L}F_\ell|\cup_\ell F_\ell)=\nu(E\cap\cup_{\ell=m}^{L}F_\ell|\cup_{\ell=m}^{L}F_\ell)$. Finally, since by definition $F_{L+1},\ldots,F_{L+M}$ is the equivalence class of $\triangleright$ containing F_L, Theorem 3 and part (3) of Def. 3 imply that $\nu(F_L|\cup_{\ell=m}^{L+M}F_\ell)=\nu(F_L|\cup_{\ell=L+1}^{L+M}F_\ell)=p_{F_L}(F_L)>0$, so $\nu(\cup_{\ell=m}^{L}F_\ell|\cup_{\ell=m}^{L+M}F_\ell)>0$. Therefore, by the chain rule and, again, Theorem 3,

$$\nu(E\cap\cup_{\ell=m}^{L}F_\ell|\cup_\ell F_\ell)=\nu(E\cap\cup_{\ell=m}^{L}F_\ell|\cup_{\ell=m}^{L}F_\ell)=\frac{\nu(E\cap\cup_{\ell=m}^{L}F_\ell|\cup_{\ell=m}^{L+M}F_\ell)}{\nu(\cup_{\ell=m}^{L}F_\ell|\cup_{\ell=m}^{L+M}F_\ell)}=\frac{p_{F_L}(E\cap\cup_{\ell=m}^{L}F_\ell)}{p_{F_L}(\cup_{\ell=m}^{L}F_\ell)}.$$

Since, by part (1) of this Lemma, $p_{F_L}(F_\ell)=0$ for $\ell<m$, for any event $G\in\Sigma$ one has $p_{F_L}(G\cap[\cup_{\ell=1}^{L}F_\ell])=p_{F_L}(G\cap[\cup_{\ell=m}^{L}F_\ell])+p_{F_L}(G\cap[\cup_{\ell=1}^{L}F_\ell\setminus\cup_{\ell=m}^{L}F_\ell])=p_{F_L}(G\cap[\cup_{\ell=m}^{L}F_\ell])$. Taking $G=E$ and $G=\cup_{\ell=1}^{L}F_\ell$ yields the claim. ■

For every $s_{-i}\in S_{-i}$, let $[s_{-i}]=\{s_{-i}\}\times W$. Fix a CPS $\mu\in cpr(\Sigma,\mathscr{F})$ with basis $\mathbf{p}=(p_F)_{F\in\mathscr{F}}$, and let $\nu\in cpr(\Sigma,\mathscr{F}_\mu)$ be as in condition 2 of Theorem 3. Define the *μ-support* of a μ-sequence $F_1,\ldots,F_L$ as $\sigma_\mu(\cup_\ell F_\ell)=\cup\{[s_{-i}]:\nu([s_{-i}]|\cup_\ell F_\ell)>0\}$.

Note that, by Lemma 2, equivalently $\sigma_\mu(\cup_\ell F_\ell)=\cup\{[s_{-i}]:p_{F_L}([s_{-i}]\cap\cup_\ell F_\ell)>0\}$.

Also observe that, by Remark 2 part 5, every equivalence class $\mathscr{C}$ for $\triangleright$ can be written as $\mathscr{C}=\cup_{\ell=1}^{L}F_\ell$ for some μ-sequence $F_1,\ldots,F_L$. The definition of σ_μ only depends upon the union $\cup\mathscr{C}=\cup_\ell F_\ell$, so one can write $\sigma_\mu(\cup\mathscr{C})$ without any ambiguity. Indeed in such case $\sigma_\mu(\cup\mathscr{C})=\cup\{[s_{-i}]:p_C([s_{-i}]>0\}$, where $C\in\mathscr{C}$ can be chosen arbitrarily.

I temporarily distinguish between the μ-support $\sigma_\mu(\cdot)$ and the support $\sigma(\cdot)$ introduced in Definition 6; however, the characterization of negligible events in both the necessity and sufficiency part of the proof immediately implies that $\sigma_\mu=\sigma$.

Lemma 3 *Fix a CPS $\mu\in cpr(\Sigma,\mathscr{F})$ that admits a basis $\mathbf{p}=(p_F)_{F\in\mathscr{F}}$. Then*

1. *For all distinct equivalence classes $\mathscr{C},\mathscr{D}$ of $\triangleright$, $\sigma_\mu(\cup\mathscr{C})\cap\sigma_\mu(\cup\mathscr{D})=\emptyset$.*

2. *Fix a μ-sequence $F_1,\ldots,F_L$, and let m be as in Lemma 2. Then $\sigma_\mu(\cup_{\ell=1}^{L}F_\ell)=\sigma_\mu(\cup_{\ell=m}^{L}F_\ell)$, and $\sigma_\mu(\cup_\ell F_\ell)\subseteq\sigma_\mu(\cup\mathscr{C})$, where $\mathscr{C}=\{G\in\mathscr{F}:G\triangleright F_L,F_L\triangleright G\}$. Furthermore, for all other equivalence classes $\mathscr{D}\neq\mathscr{C}$ of $\triangleright$, $\sigma_\mu(\cup_\ell F_\ell)\cap\sigma_\mu(\cup\mathscr{D})=\emptyset$, so $p_D(\cup_\ell F_\ell)=0$ for all $D\in\mathscr{D}$.*

Proof: (1): fix $C \in \mathscr{C}$ and $D \in \mathscr{D}$ arbitrarily. Consider $s_{-i} \in S_{-i}$. If $[s_{-i}] \in \sigma_\mu(\cup\mathscr{C}) \cap \sigma_\mu(\cup\mathscr{D})$, then $p_C([s_{-i}]) > 0$ and $p_D([s_{-i}]) > 0$. Since, by part (2) of Def. 3, $p_C(\cup\{F : F \triangleright C, C \triangleright F\}) = p_C(\cup\mathscr{C}) = 1$, it must be the case that $[s_{-i}] \cap (\cup\mathscr{C}) \neq \emptyset$. Since every $F \in \mathscr{C}$ is of the form $F = S_{-i}(I) \times W$ for some $I \in \mathscr{I}_i$, $[s_{-i}] \subseteq \cup\mathscr{C}$. Similarly, $[s_{-i}] \in \cup\mathscr{D}$. Let $F \in \mathscr{C}$, $G \in \mathscr{D}$ such that $[s_{-i}] \subseteq F$ and $[s_{-i}] \subseteq G$. Then $p_C(G) \geq p_C([s_{-i}]) > 0$ and $p_D(F) \geq p_D([s_{-i}]) > 0$. By Lemma 1 part (1), $G \triangleright C$ and $F \triangleright D$. But since C, F and, respectively, D, G are in the same equivalence class, also $C \triangleright F$ and $D \triangleright G$, so by transitivity $D \triangleright C$ and $C \triangleright D$, which contradicts the fact that $C \in \mathscr{C}$, $D \in \mathscr{D}$, and $\mathscr{C}$ and $\mathscr{D}$ are distinct equivalence classes.

(2): From Lemma 2 part 1, $\ell < m$ implies $p_{F_L}(F_\ell) = 0$. Therefore, $p_{F_L}([s_{-i}] \cap \cup_{\ell=1}^{L} F_\ell) \leq p_{F_L}([s_{-i}] \cap \cup_{\ell=1}^{m-1} F_\ell) + p_{F_L}([s_{-i}] \cap \cup_{\ell=m}^{L} F_\ell) = p_{F_L}([s_{-i}] \cap \cup_{\ell=m}^{L} F_\ell)$, i.e., $p_{F_L}([s_{-i}] \cap \cup_{\ell=1}^{L} F_\ell) = p_{F_L}([s_{-i}] \cap \cup_{\ell=m}^{L} F_\ell)$. By definition, this implies that $\sigma_\mu(\cup_{\ell=1}^{L} F_\ell) = \sigma_\mu(\cup_{\ell=m}^{L} F_\ell)$.

If $\mathscr{C}$ is the equivalence class for $\triangleright$ that contains F_L, $\sigma_\mu(\cup\mathscr{C}) = \cup\{[s_{-i}] : p_{F_L}([s_{-i}]) > 0\}$. Hence, $\sigma_\mu(\cup_{\ell=1}^{L} F_\ell) = \sigma_\mu(\cup_{\ell=m}^{L} F_\ell) = \cup\{[s_{-i}] : p_{F_L}([s_{-i}] \cap \cup_{\ell=1}^{L}) > 0\} \subseteq \cup\{[s_{-i}] : p_{F_L}([s_{-i}]) > 0\} = \sigma_\mu(\cup\mathscr{C})$.

Now let $\mathscr{D}$ be another equivalence class of $\triangleright$. By part 1 of this Lemma, $\sigma_\mu(\cup\mathscr{C}) \cap \sigma_\mu(\cup\mathscr{D}) = \emptyset$. It follows that $\sigma_\mu(\cup_\ell F_\ell) \cap \sigma_\mu(\cup\mathscr{D}) = \emptyset$ for any $\mathscr{D} \neq \mathscr{C}$. The last claim follows from the observation that $p_D(\sigma_\mu(\cup\mathscr{D})) = \sum_{s_{-i}:p_D([s_{-i}]>0)} p_D([s_{-i}]) = \sum_{s_{-i} \in S_{-i}} p_D([s_{-i}]) - \sum_{s_{-i}:p_D([s_{-i}])=0} p_D([s_{-i}]) = p_D(\Omega) - 0 = 1$ for any $D \in \mathscr{D}$. ■

B Characterization: necessity

Assume throughout that μ admits a basis $\mathbf{p} = (p_F)_{F \in \mathscr{F}}$, and $\succcurlyeq$ is as in Def. 4. By Theorem 3, μ admits a unique extension to $\mathscr{F}_\mu$; for notational simplicity, this will be referred to by μ as well. Furthermore, by Theorem 3, for all $F \in \mathscr{F}$, $p_F = \mu(\cdot | \cup\{G : F \triangleright G, G \triangleright F\})$.

This lemma characterizes negligibility for a conditioning event in terms of the basis $\mathbf{p}$.

Lemma 4 *Consider $N \in \Sigma$ and a μ-sequence $F_1, \ldots, F_L$. Then $N \in \Sigma$ is negligible given $\cup_\ell F_\ell$ iff $p_{F_L}(N \cap (\cup_\ell F_\ell)) = 0$. In particular, N is negligible given $F \in \mathscr{F}$ iff $\mu(N|F) = 0$.*

Proof:

Suppose $p_{F_L}(N \cap \cup_\ell F_\ell) = 0$. If $f, g \in \mathscr{A}$ satify $f(\omega) \succ g(\omega)$ for $\omega \notin N$, in particular this holds for $\omega \in (\cup_\ell F_\ell) \setminus N$. Since $p_{F_L}((\cup_\ell F_\ell) \setminus N) = p_{F_L}(\cup_\ell F_\ell) - p_{F_L}(N \cap (\cup_\ell F_\ell)) = p_{F_L}(\cup_\ell F_\ell)$, and $p_{F_L}(\cup_\ell F_\ell) \geq p_{F_L}(F_L) > 0$ by part (3) of Def. 3, $\int u \circ f(\cup_\ell F_\ell) g \, d p_{F_L} > \int u \circ g \, d p_{F_L}$.

Now consider another conditioning event $G \in \mathscr{F}$. If $p_G(\cup_\ell F_\ell) = 0$, then $\int u \circ f(\cup_\ell F_\ell) g \, d p_G = \int_{\Omega \setminus (\cup_\ell F_\ell)} u \circ f(\cup_\ell F_\ell) g \, d p_G = \int_{\Omega \setminus (\cup_\ell F_\ell)} u \circ g \, d p_G = \int u \circ g \, d p_G$. If instead $p_G(\cup_\ell F_\ell) > 0$, in particular $p_G(F_n) > 0$ for some $n \in \{1, \ldots, L\}$. By Lemma 1 part 1, $F_n \triangleright G$, so part 4 of Remark 2 and transitivity of $\triangleright$ imply $F_L \triangleright G$.

To conclude the proof of this direction, if $G \triangleright F_L$, then there are two subcases: (i) $p_G(\cup_\ell F_\ell) = 0$, or (ii) $p_G(\cup_\ell F_\ell) > 0$, in which case $F_L \triangleright G$ and so $p_G = p_{F_L}$ by part (1) of Def. 3. In both subcases, $\int u \circ f(\cup_\ell F_\ell) g \, d p_G \geq \int u \circ g \, d p_G$. Therefore, not $g \succcurlyeq f(\cup_\ell F_\ell) g$, i.e., not $g \succcurlyeq_{\cup_\ell F_\ell} f$. Furthermore, if $\int u \circ f(\cup_\ell F_\ell) g \, d p_G < \int u \circ g \, d p_G$ for some $G \in \mathscr{F}$, it must be the case that $p_G(\cup_\ell F_\ell) > 0$. But then $F_L \triangleright G$ and $\int u \circ f(\cup_\ell F_\ell) g \, d p_{F_L} > \int u \circ g \, d p_{F_L}$, so $f(\cup_\ell F_\ell) g \succcurlyeq g$, i.e., $f \succcurlyeq_{\cup_\ell F_\ell} g$. Thus, $f \succ_{\cup_\ell F_\ell} g$. This implies that N is negligible given $\cup_\ell F_\ell$.

For the converse, it is enough to consider the case $N \subseteq \cup_\ell F_\ell$: for general $N \in \Sigma$, if $N_1 = N \cap (\cup_\ell F_\ell)$ and $N_2 = N \setminus (\cup_\ell F_\ell)$, then $p_{F_L}(N \cap (\cup_\ell F_\ell)) = p_{F_L}(N_1 \cap (\cup_\ell F_\ell)) + p_{F_L}(N_2 \cap (\cup_\ell F_\ell)) = p_{F_L}(N_1 \cap (\cup_\ell F_\ell))$, and $N_1 \subseteq \cup_\ell F_\ell$.

Suppose that $p_{F_L}(N) > 0$. As argued above, $p_{F_L}(\cup_\ell F_\ell) > 0$. Choose x, y such that $x \succ y$, and

$$\alpha \in \left(0, \frac{p_{F_L}(N)}{p_{F_L}(\cup_\ell F_\ell)}\right).$$

Let $f = \alpha x + (1-\alpha) y$ and $g = x N y$; note that, for $\omega \notin N$, $f(\omega) = \alpha x + (1-\alpha) y \succ y = g(\omega)$.

However,

$$\int u\circ f(\cup_\ell F_\ell)g\,d\,p_{F_L} = p_{F_L}(\cup_\ell F_\ell)[\alpha u(x)+(1-\alpha)u(y)]+[1-p_{F_L}(\cup_\ell F_\ell)]u(y) <$$
$$< p_{F_L}(\cup_\ell F_\ell)\left[\frac{p_{F_L}(N)}{p_{F_L}(\cup_\ell F_\ell)}u(x)+\left(1-\frac{p_{F_L}(N)}{p_{F_L}(\cup_\ell F_\ell)}\right)u(y)\right]+(1-p_{F_L}(\cup_\ell F_\ell))u(y) =$$
$$= p_{F_L}(N)u(x)+[1-p_{F_L}(N)]u(y) = \int u\circ g\,d\,p_{F_L};$$

the inequality follows from the choice of α, and is strict because $p_{F_L}(\cup_\ell F_\ell) > 0$. Furthermore, consider $G \in \mathscr{F}$ such that $G \triangleright F_L$. If $p_G = p_{F_L}$, then $\int u\circ f(\cup_\ell F_\ell)g\,d\,p_G < \int u\circ g\,d\,p_G$. Suppose instead that $p_G \neq p_{F_L}$. Suppose that $p_G(F_\ell) > 0$ for some ℓ. By Lemma 1 part 1, $F_\ell \triangleright G$. The assumption that $G \triangleright F_L$ implies by transitivity that $F_\ell \triangleright F_L$. By Remark 2 part 4, $F_L \triangleright F_\ell$, so part (1) of Def. 3 implies that $p_{F_L} = p_{F_\ell}$. By a similar argument, the assumption that $G \triangleright F_L$ implies by transitivity that $G \triangleright F_\ell$, so $p_G = p_{F_\ell}$. But then $p_G = p_{F_L}$, contradiction. Therefore, $p_G(F_\ell) = 0$ for all ℓ, so $p_G(\cup_\ell F_\ell) = 0$ and hence $\int u\circ f(\cup_\ell F_\ell)g\,d\,p_G = \int u\circ g\,d\,p_G$. Therefore, it is not the case that $f(\cup_\ell F_\ell)g \succcurlyeq g$, i.e. $f \succcurlyeq_{\cup_\ell F_\ell} g$, so a fortiori it is not the case that $f \succ_{\cup_\ell F_\ell} g$. Therefore, N is not negligible given $\cup_\ell F_\ell$.

The last claim follows by noting that any $F \in \mathscr{F}$ is a degenerate μ-sequence of length $L = 1$, so N is negligible given F iff $p_F(N\cap F) = 0$. By part (1) of Def. 3, $p_F(F) > 0$ and $\mu(N\cap F|F) = p_F(N\cap F)/p_F(F)$. Hence, the claim holds for all $N \subseteq F$. For general N, the claim holds because $\mu(N|F) = \mu(N\cap F|F)$. ■

It follows that a n-sequence in the sense of Definition 6 is a μ-sequence, and conversely. Finally, for any μ-sequence $F_1,\dots,F_L$, $\sigma(\cup_\ell F_\ell) = \cup\{[s_{-i}] : p_{F_L}([s_{-i}]\cap(\cup_\ell F_\ell)) > 0\} = \sigma_\mu(\cup_\ell F_\ell)$, where σ_μ is defined in Appendix A.

Throughout the remainder of this Section, I will not distinguish between n-sequences and μ-sequences, or between σ and σ_μ.

In particular, Lemma 3 implies

Lemma 5 *For every* μ*-sequence* $F_1,\dots,F_L$, $\succcurlyeq_{\sigma(\cup_\ell F_\ell)}$ *is an EU preference relation, represented by* u *and* $q \in pr(\Sigma)$*, where* $q(E) = \frac{p_{F_L}(E\cap(\cup_\ell F_\ell))}{p_{F_L}(\cup_\ell F_\ell)}$.

Notice that the measure q in this lemma is exactly $\mu(\cdot|\cup_\ell F_\ell)$, by Lemma 2 part 2.

Proof: Consider two acts $f,g \in \mathscr{A}$. By Lemma 3 part 2, if $G \in \mathscr{F}$ is such that either not $F_L \triangleright G$ or not $G \triangleright F_L$, then $\int u\circ f\sigma(\cup_\ell F_\ell)g\,dp_G = \int u\circ g\,dp_G$, because $p_G(\sigma(\cup_\ell F_\ell)) = 0$ and $f\sigma(\cup_\ell F_\ell)g$ agrees with g outside the event $\sigma(\cup_\ell F_\ell)$.

Therefore, if $\int u\circ f\sigma(\cup_\ell F_\ell)g\,dp_{F_L} > \int u\circ g\,dp_{F_L}$, there is no $G \in \mathscr{F}_L$ such that $G \triangleright F_L$ such that $\int u\circ f\sigma(\cup_\ell F_\ell)g\,dp_G < \int u\circ g\,dp_G$; thus, not $g \succcurlyeq_{\sigma(F)} f$. On the other hand, there is no $G \in \mathscr{F}$ such that $\int u\circ f\sigma(\cup_\ell F_\ell)g\,dp_G < \int u\circ g\,dp_G$, so trivially $f \succcurlyeq_{\sigma(\cup_\ell F_\ell)} g$. Therefore, $f \succ_{\sigma(\cup_\ell F_\ell)} g$.

Similarly, if $\int u\circ f\sigma(\cup_\ell F_\ell)g\,dp_{F_L} < \int u\circ g\,dp_{F_L}$, then $f \prec_{\sigma(\cup_\ell F_\ell)} g$.

If instead $\int u\circ f\sigma(\cup_\ell F_\ell)g\,dp_{F_L} = \int u\circ g\,dp_{F_L}$, then $\int u\circ f\sigma(\cup_\ell F_\ell)g\,dp_G = \int u\circ g\,dp_G$ for all $G \in \mathscr{F}$, and therefore $f \sim_{\sigma(\cup_\ell F_\ell)} g$.

Thus, $f \succcurlyeq_{\sigma(\cup_\ell F_\ell)} g$ iff $\int u\circ f\sigma(\cup_\ell F_\ell)g\,dp_{F_L} \geq \int u\circ g\,dp_{F_L}$. Since by Def. 3 $p_{F_L}(F_L) > 0$, equivalently $f \succcurlyeq_{\sigma(\cup_\ell F_\ell)} g$ iff $\int u\circ f\,dq \geq \int u\circ g\,dq$, where q is as in the statement of this Lemma. ■

It is now possible to **verify that Axioms 1–10 hold**. Assume that $\succcurlyeq$ is defined via Definition 5. The reflexivity part of Axiom 1, as well as Axioms 2 (Prize Competeness), 3 (Monotonicity), and 4 (Independence) are straightforward to verify.

For Transitivity, suppose that $f \succcurlyeq g$ and $g \succcurlyeq h$. Let $F \in \mathscr{F}$ be such that $\mathrm{E}_{p_F} u\circ f < \mathrm{E}_{p_F} u\circ h$ (if there is no such F, then by definition $f \succcurlyeq h$ and there is nothing to prove). I show that there is $G \triangleright F$ such that $\mathrm{E}_{p_G} u\circ f > \mathrm{E}_{p_G} u\circ h$. Let $G_1 = F$. Either $\mathrm{E}_{p_{G_1}} u\circ f < \mathrm{E}_{p_{G_1}} u\circ g$, or $\mathrm{E}_{p_{G_1}} u\circ g < \mathrm{E}_{p_{G_1}} u\circ h$ (or both). Suppose $\mathrm{E}_{p_{G_1}} u\circ f < \mathrm{E}_{p_{G_1}} u\circ g$: then, since $f \succcurlyeq g$, there is $G_2 \in \mathscr{F}$ such that $G_2 \triangleright G_1$ and $\mathrm{E}_{p_{G_2}} u\circ f > \mathrm{E}_{p_{G_2}} u\circ g$. If also $\mathrm{E}_{p_{G_2}} u\circ g \geq \mathrm{E}_{p_{G_2}} u\circ h$, then $\mathrm{E}_{p_{G_2}} u\circ f > \mathrm{E}_{p_{G_2}} u\circ h$, and one can take $G = G_2$. Otherwise, $\mathrm{E}_{p_{G_2}} u\circ g < \mathrm{E}_{p_{G_2}} u\circ h$, and the assumption that $g \succcurlyeq h$ implies that there is $G_3 \in \mathscr{F}$ such that $G_3 \triangleright G_2$ and $\mathrm{E}_{p_{G_3}} u\circ g > \mathrm{E}_{p_{G_3}} u\circ h$. Again, if also $\mathrm{E}_{p_{G_3}} u\circ f \geq \mathrm{E}_{p_{G_3}} u\circ g$, then one can take $G = G_3$; otherwise, continue as above. Since the precedence relation is acyclic, at each iteration n a different $G_n \in \mathscr{F}$ is identified. Since there are finitely many conditioning events, the process must stop. Finally, if the process stops in round n, event G_n is such that $\mathrm{E}_{p_{G_n}} u \circ f > \mathrm{E}_{p_{G_n}} u \circ h$. A similar iteration results if, in the first step, $\mathrm{E}_{p_{G_1}} u \circ g < \mathrm{E}_{p_{G_1}} u \circ h$. Thus, in any case, a suitable $G \in \mathscr{F}$ can be found. Thus, $f \succcurlyeq h$, so Axiom 1 holds.

For Axioms 5 and 6, fix $F \in \mathscr{F}$. Then, by Def. 3, $p_F(F) > 0$. Now consider $x, y \in X$. If $u(x) = u(y)$, then $\mathrm{E}_{p_G} u \circ xFy = u(y) = \mathrm{E}_{p_G} u(y)$ for all $G \in \mathscr{F}$, and so by definition $xFy \sim y$, i.e., $x \sim_F y$. If $u(x) > u(y)$, then $\mathrm{E}_{p_G} u \circ xFy \geq u(y) = \mathrm{E}_{p_G} u(y)$ for all $G \in \mathscr{F}$; furthermore, $\mathrm{E}_{p_F} u \circ xFy > \mathrm{E}_{p_F} u(y)$. Hence $xFy \succ x$, so $x \succ_F y$. Similarly, $u(x) < u(y)$ implies $x \prec_F y$. Therefore, $x \succcurlyeq_F y$ iff $u(x) \geq u(y)$. Since u is a non-constant, affine utility function, Axioms 5 and 6 hold.

For Axioms 7 and 8, Lemma 5 shows that, for every n-sequence $F_1, \ldots, F_N$, $\succcurlyeq_{\sigma(\cup_n F_n)}$ is an EU preference relation; hence, it is complete and Archimedean, so the Axioms hold.

For Axiom 9, suppose that N is negligible given every $G \in \mathscr{F}$. Fix $F \in \mathscr{F}$. By Lemma 4, $p_G(N \cap G) = 0$ for all $G \in \mathscr{F}$; hence, if $G \triangleright F$ and $F \triangleright G$, then by Def. 3 part (1) $p_G = p_F$, and so $p_F(N \cap G) = p_G(N \cap G) = 0$. By part (2) of the same definition, $p_F(\cup\{G : G \triangleright F, F \triangleright G\}) = 1$. Therefore, $p_F(N) = p_F(N \cap \cup\{G : G \triangleright F, F \triangleright G\}) \leq \sum_{G: G \triangleright F, F \triangleright G} p_F(N \cap G) = 0$. Therefore, if $f, g \in \mathscr{A}$ satisfy $f(\omega) = g(\omega)$ for every $\omega \notin N$, then $\int u \circ f \, dp_F = \int_{\Omega \setminus N} u \circ f \, dp_F = \int_{\Omega \setminus N} u \circ g \, dp_F = \int u \circ g \, dp_F$. Since this is true for all $F \in \mathscr{F}$, $f \sim g$. Hence, N is Savage-null for $\succcurlyeq$.

Finally, for Axiom 10, suppose that $f \succcurlyeq g$, $F \in \mathscr{F}$, and $f \prec_F g$. If $\mathrm{E}_{p_F} u \circ f \geq \mathrm{E}_{p_F} u \circ g$, let $F_1, \ldots, F_L$ be a μ-sequence (hence, an n-sequence) such that $\{F_1, \ldots, F_L\}$ is the equivalence class of $\triangleright$ containing F, with $F_L = F$: one such μ-sequence exists by Remark 2 part 5. Then, by Lemma 5 and the fact that $p_{F_L}(\cup_\ell F_\ell) = 1$ by part (2) of Def. 3, $\mathrm{E}_{p_F} u \circ f \geq \mathrm{E}_{p_F} u \circ g$ implies $f \succcurlyeq_{\sigma(\cup_\ell F_\ell)} g$. Thus, (i) in Axiom 10 holds. Suppose instead that $\mathrm{E}_{p_F} u \circ f < \mathrm{E}_{p_F} u \circ g$. Since $f \succcurlyeq g$, there must be $E \in \mathscr{F}$ with $E \triangleright F$ and $\mathrm{E}_{p_E} u \circ f > \mathrm{E}_{p_E} u \circ g$. By Remark 2 part 3, there is a μ-sequence (hence an n-sequence) $F_1, \ldots, F_L$ with $F_1 = F$ and $F_L = E$. By Remark 2 part 5, there is a μ-sequence $F_{L+1}, \ldots, F_{L+M}$ with $F_{L+1} = F_L$ such that $\{F_{L+1}, \ldots, F_{L+M}\}$ is the equivalence class of $\triangleright$ that contains F_L. Notice that this implies that $F_{L+M} \triangleright F_L = E$ and $E = F_L \triangleright F_{L+M}$, so by part

(1) of Def. 3 $p_{F_{L+M}} = p_E$. By part 2 of the same Remark, $F_1, \dots, F_{L+M}$ is also a μ-sequence, hence an n-sequence. Moreover, $p_{L+M}(\cup_{\ell=1}^{L+M} F_\ell) \geq p_{L+M}(\cup_{\ell=L+1}^{L+M} F_\ell) = 1$ by part (2) of Def. 3. Hence, by Lemma 5, $\mathrm{E}_{p_{F_{L+M}}} u \circ f = \mathrm{E}_{p_E} u \circ f > \mathrm{E}_{p_E} u \circ g = \mathrm{E}_{p_{F_{L+M}}} u \circ g$ implies $f \succ_{\sigma(\cup_{\ell=1}^{L+M} F_\ell)} g$. Thus, (ii) in the Axiom holds.

C Sufficiency: Preliminaries

Begin by proving Remark 1; the proof is essentially standard, but it is provided here to emphasize that it does not rely on completeness of preferences.

Proof: Fix $f, g, k, k', E \in \mathscr{A}$ as in the Remark. By Independence (Axiom 4),

$$f E k \succcurlyeq g E k \quad \Leftrightarrow \quad \frac{1}{2} f E k + \frac{1}{2} k' \succcurlyeq \frac{1}{2} g E k + \frac{1}{2} k',$$

and similarly

$$f E k' \succcurlyeq g E k' \quad \Leftrightarrow \quad \frac{1}{2} f E k' + \frac{1}{2} k \succcurlyeq \frac{1}{2} g E k' + \frac{1}{2} k.$$

Now observe that $\frac{1}{2} f E k + \frac{1}{2} k' = \frac{1}{2} f E k' + \frac{1}{2} k$: in every state $\omega \in E$

$$\frac{1}{2}(f E k)_{\text{act}}(\omega) + \frac{1}{2} k'(\omega) = \frac{1}{2} f(\omega) + \frac{1}{2} x = \frac{1}{2} f E k'(\omega) + \frac{1}{2} k(\omega),$$

and in every state $\omega \notin E$

$$\frac{1}{2} f E k)(\omega) + \frac{1}{2} k'(\omega) = \frac{1}{2} k_{\text{act}}(\omega) + \frac{1}{2} k'(\omega) = \frac{1}{2} k(\omega) + \frac{1}{2} k'(\omega) = \frac{1}{2} k(\omega) + \frac{1}{2} k'(\omega) = \frac{1}{2} k(\omega) + \frac{1}{2} f E k'(\omega).$$

Similarly $\frac{1}{2} g E k + \frac{1}{2} k' = \frac{1}{2} g E k' + \frac{1}{2} k$. The claim follows. ■

It is routine to verify that, if $\succcurlyeq$ satisfies Axioms 1–4 and conditional preferences are defined as in Definition 5, they satisfy the conditional versions of these Axioms .[3] This fact will be used

[3]In particular, for Axiom 2, consider $x, y \in X$ and $E \in \Sigma$. By Axiom 2, either $x \succcurlyeq y$ or $y \succcurlyeq x$. If $x \succcurlyeq y$, then by Monotonicity (Axiom 3), $x E y \succcurlyeq y$, so $x \succcurlyeq_E y$. Otherwise, $x \preccurlyeq_E y$.

without further notice. The following is another immediate consequence of the definition of conditional preferences and Monotonicity (Axiom 3) of $\succcurlyeq$. It states that, for the preference $\succcurlyeq_F$, $\Omega \setminus F$ is Savage-null.

Observation 1 (Null Complement) Fix an event $E \in \Sigma$ and acts $f, g \in \mathscr{A}$. If $f(\omega) = g(\omega)$ for all $\omega \in E$, then $f \sim_E g$.

This implies that negligibility has the following equivalent characterization.

Remark 3 (Negligible events) *Assume Axioms 1–5. For all $F, N \in \Sigma$, N is negligible given F if and only if, for all $f, g \in \mathscr{A}$ with $f(\omega) \succ g(\omega)$ for $\omega \in F \setminus N$ implies $f \succ_F g$.*

That is, it is sufficient to restrict attention to states in F.

Proof: (If): assume that the property in the Remark holds. Consider f, g such that $f(\omega) \succ g(\omega)$ for all $\omega \notin N$. Then a fortiori this holds for all $\omega \in F \setminus N$, so by assumption $f \succ_F g$. Thus, N is negligible given F.

(Only if): assume that N is negligible given F. Consider f, g such that $f(\omega) \succ g(\omega)$ for all $\omega \in F \setminus N$. Fix $x, y \in X$ with $x \succ y$ (these exist by Axioms 2 and 5). Then $fFx(\omega) \succ gFy(\omega)$ for all $\omega \in (F \setminus N) \cup (\Omega \setminus F)$, hence a fortiori for all $\omega \notin N$. Since N is negligible given F, $fFx \succ_F gFy$. By Observation 1, $fFx \sim_F f$ and $gFy \sim_F g$. Therefore, by Transitivity, $f \succ_F g$, so the property in the Remark holds. ■

Independence implies the following, standard dynamic-consistency property.

Remark 4 (Dynamic Consistency) *Assume Axioms 1–4. Then, for every collection $E_1, \dots, E_N \in \Sigma$ of disjont events and every $f, g \in \mathscr{A}$, if $f \succcurlyeq_{E_n} g$ for every $n = 1, \dots, N$, then $f \succcurlyeq_{\cup_{n=1}^N E_n} g$; furthermore, if $f \succ_{E_m} g$ for some m, then $f \succ_{\cup_{n=1}^N E_n} g$.*

Proof: Let $f_n = f(\cup_{\ell=1}^n E_\ell)g$, so $f_0 = g$. For every $n = 1, \dots, N$, by assumption $f \succcurlyeq_{E_n} g$; by construction, $f_n = fE_n f_{n-1}$ and $f_{n-1} = gE_n f_{n-1}$, so by the definition of conditional preferences

$f_n \succcurlyeq f_{n-1}$. Then, by Transitivity, $f(\cup_{\ell=1}^N E_\ell)g = f_N \succcurlyeq f_0 = g$, i.e., $f \succcurlyeq_{\cup_{\ell=1}^N E_\ell} g$. Furthermore, if $f \succ_{E_m} g$ for some m, then $f_m \succ f_{m-1}$, and so $f \succ_{\cup_{\ell=1}^N E_\ell} g$. ■

Next, I obtain a von Neumann-Morgenstern representation for (conditional) preferences over constant acts. Notice that the representation extends to preferences conditional upon events that do not belong to $\mathscr{F}$, but are supersets of some $F \in \mathscr{F}$.

Lemma 6 *Assume Axioms 1–6. Then there exists a cardinally unique, non-constant, affine function* $u : X \to \mathbb{R}$ *such that, for all* $x, y \in X$*, and all* $E \in \Sigma$ *such that* $E \supseteq F$ *for some* $F \in \mathscr{F}$*,* $x \succcurlyeq_E y$ *iff* $u(x) \geq u(y)$.

Proof: Under the assumed axioms (in particular, taking $F = \Omega$ in Axioms 5 and 6), the restriction of $\succcurlyeq$ to constant acts satisfies the von Neumann and Morgenstern (1947) axioms; hence, there exists a unique, affine $u : X \to \mathbb{R}$ such that $x \succcurlyeq y$ iff $u(x) \geq u(y)$.

Now consider $F \in \mathscr{F}$. By the definition of conditional preferences, $x \succcurlyeq_F y$ iff in particular $xFy \succcurlyeq y$. By Axiom 2, either $x \succcurlyeq y$ or $x \preccurlyeq y$; then, Axiom 3 implies that either $x \succcurlyeq_F y$ or $y \succcurlyeq_F x$ respectively. Thus, the restriction of $\succcurlyeq_F$ to X is complete. By standard arguments, it is transitive by Axiom 1, satisfies Independence by Axiom 6, and the Archimedean axiom by Axiom 6. Furthermore, it is non-degenerate by Axiom 5. Hence, it is represented by a unique, affine $u_F : X \to \mathbb{R}$. Furthermore, if $x \succcurlyeq y$, then $xFy \succcurlyeq y$ by Axiom 3, so $x \succcurlyeq_F y$. Thus, $u(x) \geq u(y)$ implies $u_F(x) \geq u_F(y)$. By Corollary B.3 in Ghirardato, Maccheroni, and Marinacci (2004), u and u_F agree up to a positive affine transformation, so we can take $u_F = u$.

Finally, consider $E \in \Sigma$ such that $E \supset F$ for some $F \in \mathscr{F}$. Suppose that $x \sim y$: then Monotonicity implies that $xEy \sim y$, and so $x \sim_E y$. Suppose that $x \succ y$; then also $x \succ_F y$, which, by Observation 1, implies $xEy \succ_F y$. For every $\omega \notin F$, $xEy(\omega) \succcurlyeq y = y(\omega)$; thus, by Monotonicity and the definition of conditional preferences, $xEy \succcurlyeq_{\Omega \setminus F} y$. Then, by dynamic consistency (Remark 4), $xEy \succ y$. By the definition of conditional preferences, $x \succ_E y$. Similarly, $x \prec y$ implies $x \prec_E y$. Therefore, $x \succcurlyeq_E y$ iff $x \succcurlyeq y$ iff $u(x) \geq u(y)$. ■

I now provide an equivalent condition for negligibility. The proof of sufficiency uses the notion of complementary acts from Siniscalchi (2009).

Lemma 7 *Assume Axioms 1–6. Fix events $F, N \in \Sigma$. Then N is negligible given F if and only if, for every $x, y, z \in X$ with $y \succ z$, $y \succ_F xNz$.*

Proof: Necessity is immediate. For sufficiency, consider $f, g \in \mathscr{A}$ such that $f(\omega) \succ g(\omega)$ for all $\omega \notin N$. Under the stated axioms, $\succcurlyeq$ has an EU representation on X by Lemma 6, with utility u Let $y \in X$ be such that $u(y) = \frac{1}{2}\min_\omega u(f(\omega)) + \frac{1}{2}\max_\omega u(f(\omega))$, and let $\bar{f} \in \mathscr{A}$ be such that $u(\bar{f}(\omega)) = 2u(y) - u(f(\omega))$; such an act exists because, for every ω, $2u(y) - u(f(\omega)) = \min_{\omega'} u(f(\omega')) + \max_{\omega'} u(f(\omega')) - u(f(\omega)) \in \big(\min_\omega u(f(\omega)), \max_\omega u(f(\omega))\big)$. By the definition of y and the fact that u represents $\succcurlyeq$ on X, $\frac{1}{2}f(\omega) + \frac{1}{2}\bar{f}(\omega) \sim y$ for every ω, so by Monotonicity (Axiom 3), $\frac{1}{2}f + \frac{1}{2}\bar{f} \sim y$. By Independence (Axiom 4), $f \succ_F g$ iff $y \sim \frac{1}{2}f + \frac{1}{2}\bar{f} \succ_F \frac{1}{2}g + \frac{1}{2}\bar{f}$.

Let $x, z \in X$ be such that $u(x) = \max_{\omega \in N} \frac{1}{2}u(g(\omega)) + \frac{1}{2}u(\bar{f}(\omega))$ and $u(z) = \max_{\omega \notin N} \frac{1}{2}u(g(\omega)) + \frac{1}{2}u(\bar{f}(\omega))$. Since u represents $\succcurlyeq$ on X, by Monotonicity (Axiom 3) $xNz \succcurlyeq_F \frac{1}{2}g + \frac{1}{2}\bar{f}$; furthermore, $\frac{1}{2}u(g(\omega)) + \frac{1}{2}u(\bar{f}(\omega)) = \frac{1}{2}u(g(\omega)) + u(y) - \frac{1}{2}u(f(\omega)) = u(y) - \frac{1}{2}[u(f(\omega)) - u(g(\omega))]$; since $f(\omega) \succ g(\omega)$ for all $\omega \notin N$, and u represents $\succcurlyeq$ on X, $\frac{1}{2}u(g(\omega)) + \frac{1}{2}u(\bar{f}(\omega)) < u(y)$ for every $\omega \notin N$, so in particular $y \succ z$ (recall that attention is restricted to simple acts). Then, if the sufficient condition holds, $y \succ_F xNz \succcurlyeq_F \frac{1}{2}g + \frac{1}{2}\bar{f}$, and therefore $f \succ_F g$. ■

Corollary 1 *For every $E \in \Sigma$ such that $F \subseteq E$ for some $F \in \mathscr{F}$, $\Omega \setminus E$ is negligible given E.*

Proof: Fix x, y, z with $y \succ z$. Note that $x(\Omega \setminus E)z = zFx$. Hence $\Omega \setminus F$ is negligible given E iff $y \succ_E zEx$. By Observation 1, this is equivalent to $y \succ_E z$. But by Lemma 6, this holds iff $y \succ z$. ■

Results concerning negligible events.

Lemma 8 *Assume Axioms 1–6. Fix* $F, G, N, M \in \Sigma$.

1. *If* N *is negligible given* F *and* $M \subset N$, *then* M *is negligible given* F. *Thus, if* M *is not negligible given* F, *neither is* N.

2. *If* $N \subseteq F$ *is negligible given* F, *and* $F \subseteq G$, *then* N *is negligible given* G. *Thus, if* N *is not negligible given* G, *neither is it negligible given* F.

3. *If* N *and* M *are both negligible given* F, *then so is* $N \cup M$.

Proof: 1: if $f(\omega) \succ g(\omega)$ for all $\omega \notin M$, then a fortiori this holds for all $\omega \notin N \supset M$; since N is negligible given F, $f \succ_F g$. Since this is the case for all $f, g \in \mathscr{A}$, M is negligible given F.

2: if $f(\omega) \succ g(\omega)$ for all $\omega \notin N$, then $f \succ_F g$ because N is negligible given F. By the definition of conditional preferences, in particular $fFg \succ g$. By Monotonicity (Axiom 3), since in particular $f(\omega) \succ g(\omega)$ for all $\omega \in G \setminus F$ and $G \supseteq F \supseteq N$, $fGg \succcurlyeq fFg$, so by Transitivity $fGg \succ g$. But then, by the definition of conditional preferences, $f \succ_G g$. Since f, g were arbitrary, N is negligible for G.

3: Fix F, N, M as in the statement. It is enough to consider the case $N \cap M = \emptyset$: this is because (i) for arbitrary negligible events N, M, $N \cup M = (N \setminus M) \cup (N \cap M) \cup (M \setminus N)$, and the sets on the rhs are all negligible by part 1; and (ii) if the union of two disjoint negligible events is negligible, by induction so is the union of finitely many (in particular, three) disjoint negligible events.

Since $\succcurlyeq$ satisfies Axioms 1–6, one can invoke the alternative characterization of negligibility in Lemma 7; also, the restriction of $\succcurlyeq$ to X is represented by the non-constant, affine function $u : X \to \mathbb{R}$. It is without loss of generality to assume that $[-1, 1] \subseteq u(X)$; also, let $z_0 \in X$ be such that $u(z_0) = 0$.

For arbitrary $x, y, z \in X$, there exists $\alpha \in (0, 1)$ such that $u \circ [\alpha x + (1-\alpha) z_0] = \alpha u(x) \in [-\frac{1}{2}, \frac{1}{2}]$, and similarly $u \circ [\alpha y + (1-\alpha) z_0], u \circ [\alpha z + (1-\alpha) z_0] \in [-\frac{1}{2}, \frac{1}{2}]$. Then, by Independence (Axiom 4), $y \succ z$ iff $\alpha y + (1-\alpha) z_0 \succ \alpha z + (1-\alpha) z_0$, and $y \succ_F xNy$ iff $\alpha y + (1-\alpha) z_0 \succ_F [\alpha x + (1-$

$\alpha)z_0]N[\alpha z+(1-\alpha)z_0]$. Thus, it is enough to show that $M\cup N$ satisfies the condition of Lemma 7 for $x,y,z\in X$ such that $u(x),u(y),u(z)\in[-\frac{1}{2},\frac{1}{2}]$.

Furthermore, for any such tuple $x,y,z\in X$, there is $\bar{z}\in X$ such that $u(\bar{z})=-u(z)$, so that $u\circ[\frac{1}{2}z+\frac{1}{2}\bar{z}]=0=u(z_0)$, i.e., $\frac{1}{2}z+\frac{1}{2}\bar{z}\sim z_0$. Again, by Independence, $y\succ z$ iff $\frac{1}{2}y+\frac{1}{2}\bar{z}\succ\frac{1}{2}z+\frac{1}{2}\bar{z}\sim z_0$, and $y\succ_F xNz$ iff $\frac{1}{2}y+\frac{1}{2}\bar{z}\succ_F(\frac{1}{2}x+\frac{1}{2}\bar{z})N(\frac{1}{2}z+\frac{1}{2}\bar{z})\sim(\frac{1}{2}x+\frac{1}{2}\bar{z})Nz_0$, where the indifferences follow from Monotonicity. Clearly, $u\circ[\frac{1}{2}y+\frac{1}{2}\bar{z}],u\circ[\frac{1}{2}x+\frac{1}{2}\bar{z}]\in[-\frac{1}{2},\frac{1}{2}]$. Thus, it is enough to show that, for all $x,y\in X$ with $u(x),u(y)\in[-\frac{1}{2},\frac{1}{2}]$, $y\succ z_0$ implies $y\succ_F x[M\cup N]z_0$.

Fix such $x,y\in X$. Suppose that $y\succ z_0$. Since $u(x)\in[-\frac{1}{2},\frac{1}{2}]\subset[-1,1]\subseteq u(X)$, there is $x'\in X$ such that $u(x')=2u(x)$. Since both N and M are negligible given F, $y\succ z_0$ implies both $y\succ_F x'Nz_0$ and $y\succ_F x'Mz_0$. Therefore, by Independence (Axiom 4)[4], $y\succ_F(\frac{1}{2}x'+\frac{1}{2}z_0)(N\cup M)z_0$. But $u\circ[\frac{1}{2}x'+\frac{1}{2}z_0]=\frac{1}{2}u(x')+\frac{1}{2}u(z_0)=\frac{1}{2}\cdot 2u(x)=u(x)$. Therefore, by Monotonicity, $(\frac{1}{2}x'+\frac{1}{2}z_0)(N\cup M)z_0\sim x(N\cup M)z_0$. By Transitivity, $y\succ_F x(N\cup M)z_0$. This proves the claim. ■

Corollary 2 *For any $F,N\in\Sigma$, if N is negligible given F, then $F\setminus N$ is not negligible given F.*

Proof: If both N and $F\setminus N$ are negligible, then $F\cap N$ is negligible given F by Part 1, and so $F=(F\cap N)\cup(F\setminus N)$ is negligible given F by Part 3. Using the characterization in Lemma 7, choose $x,y,z\in X$ with $x\succ y\succ z$; then $y\succ_F xFz$. By the definition of conditional preferences, in particular $yFz\succ(xFz)Fz=xFz$. Since $x\succ y$, this contradicts Monotonicity (Axiom 3). ■

Recall that, for every $s_{-i}\in S_{-i}$, $[s_{-i}]=\{s_{-i}\}\times W$.

Corollary 3 *For any $E\in\Sigma$, $\sigma(E)$ is not negligible given E, and $E\setminus\sigma(E)$ is negligible given E.*

Proof: Let $N\equiv\bigcup\{[s_{-i}]:[s_{-i}]$ is negligible given $E\}$. By part 3 of the Lemma, N is negligible

[4] Suppose that $f\succcurlyeq g$ and $f'\succcurlyeq g'$. Then Independence implies $\frac{1}{2}f+\frac{1}{2}f'\succcurlyeq\frac{1}{2}g+\frac{1}{2}f'$ and $\frac{1}{2}f'+\frac{1}{2}g\succcurlyeq\frac{1}{2}g'+\frac{1}{2}g$. Thus, by Transitivity, $\frac{1}{2}f+\frac{1}{2}f'\succcurlyeq\frac{1}{2}g+\frac{1}{2}g'$.

given E. By definition, $\sigma(E) = E \setminus N$. Since $E \setminus \sigma(E) \subseteq N$,[5] by part 1 of the Lemma it is also negligible given E. Finally, since N is negligible given E, by Corollary 2, $\sigma(E) = E \setminus N$ is not negligible given E. ■

D Sufficiency: main argument

Basic EU representation:

Lemma 9 *Assume Axioms 1–8. Fix an n-sequence $F_1, \ldots, F_L$. Then:*

1. $\succcurlyeq_{\sigma(\cup_{\ell=1}^{L} F_\ell)}$ *admits an EU representation (u, p), where $p \in pr(\Sigma)$ satisfies $p(\sigma(\cup_{\ell=1}^{L} F_\ell)) = 1$ and $u : X \in \mathbb{R}$ is the function that represents $\succcurlyeq$ on X per Lemma 6.*

2. *for every $f, g \in \mathscr{A}$, $\mathrm{E}_p u \circ f > \mathrm{E}_p u \circ g$ implies $f \succ_{\cup_{\ell=1}^{L} F_\ell} g$ (thus, $f \succcurlyeq_{\cup_{\ell=1}^{L} F_\ell} g$ implies $\mathrm{E}_p u \circ f \geq \mathrm{E}_p u \circ g$).*

3. *an event $N \in \Sigma$ is negligible given $\cup_{\ell=1}^{L} F_\ell$ if and only if $p(N) = 0$.*

Proof: To streamline the notation, let $F = \cup_{\ell=1}^{L} F_\ell$.

1: The relation $\succcurlyeq_{\sigma(F)}$ satisfies Monotonicity and Independence by Axioms 3 and 4; it satisfies Transitivity by Axiom 1, and Completeness by Axiom 7; and it satisfies the Archimedean property by Axiom 8.

Now fix $x, y \in X$ such that $x \succ y$ (these exist by Axiom 5). By Corollary 3, $F \setminus \sigma(F)$ is negligible for F. Therefore, $y[F \setminus \sigma(F)]x \succ_F y$. By Observation 1, $y[F \setminus \sigma(F)]x \sim_F x\sigma(F)y$, because both acts yield x on $\sigma(F)$ and y on $F \setminus \sigma(F)$. By Transitivity, $x\sigma(F)y \succ_F y$. By the definition

[5] In general, N need not be a subset of E. Hence we allow for inclusion rather than equality. However, if $E = (\mathrm{proj}_{S_{-i}} E) \times W$, as is the case for unions of elements from $\mathscr{F}$, one obtains an equality.

of conditional preferences, in particular $x\sigma(F)y \succ y$; but for the same reason $x \succ_{\sigma(F)} y$. Thus, $\succcurlyeq_{\sigma(F)}$ is non-degenerate.

Therefore, $\succcurlyeq_{\sigma(F)}$ has an EU representation (v, p), with v non-constant. We showed that $x \succ y$ implies $x \succ_{\sigma(F)} y$, and thus symmetrically $x \prec y$ implies $x \prec_{\sigma(F)} y$. Finally, if $x \sim y$, then Monotonicity and the definition of conditional preferences immediately implies that $x \sim_{\sigma(F)} y$. It follows that v and u can only differ by a positive affine transformation, so one can take $v = u$.

Finally, let $x, y \in X$ be such that $x \succ y$ (these exist by Axiom 5). By Observation 1, $y\sigma(F)x \sim_{\sigma(F)} y$. By the EU representation, $p(\sigma(F))u(y) + [1 - p(\sigma(F))]u(x) = u(y)$, i.e., $[1 - p(\sigma(F))][u(x) - u(y)] = 0$. Since $u(x) > u(y)$, this implies $p(\sigma(F)) = 1$, as claimed.

2: let $x, y \in X$ be such that $x \sim_{\sigma(F)} f$ and $y \sim_{\sigma(F)} g$; these exist because $\succcurlyeq_{\sigma(F)}$ has an EU representation. Since $f \sim_{F\setminus\sigma(F)} f$ and $g \sim_{F\setminus\sigma(F)} g$ (by the definition of conditional probabilities), dynamic consistency (Remark 4) implies $f \sim_F x\sigma(F)f$ and $g \sim_F y\sigma(F)g$. Now by assumption $\mathrm{E}_p u \circ f > \mathrm{E}_p u \circ g$; thus, $f \succ_{\sigma(F)} g$, and so by Transitivity $x \succ_{\sigma(F)} y$, hence $x \succ y$. By Corollary 3, $F \setminus \sigma(F)$ is negligible given F; thus, by Remark 3, $x\sigma(F)f \succ_F y\sigma(F)g$. But then, Transitivity implies $f \succ_F g$, as claimed.

3: fix $N \in \Sigma$. Suppose that $p(N) = 0$. Consider $f, g \in \mathscr{A}$. Then $f(\omega) \succ g(\omega)$ for $\omega \notin N$ implies $\mathrm{E}_p u \circ f > \mathrm{E}_p u \circ g$, so $f \succ_F g$ by part 2 of this Lemma. Hence N is negligible for F.

Conversely, suppose $p(N) > 0$. Pick $x, y \in X$ so $x \succ y$, and consider $f_\epsilon = \epsilon x + (1-\epsilon)y$ and $g = xNy$, where $\epsilon \in (0,1)$. For $\omega \notin N$, $f_\epsilon(\omega) = \epsilon x + (1-\epsilon)y \succ y = g(\omega)$. However, $\mathrm{E}_p u \circ f_\epsilon = \epsilon u(x) + (1-\epsilon)u(y)$ and $\mathrm{E}_p u \circ g = p(N)u(x) + [1 - p(N)]u(y)$; for $\epsilon < p(N)$, $g \succ_F f_\epsilon$ by part 2 of this Lemma. Thus, N is not negligible given F. ■

Next, we establish the chain rule of conditioning.

Lemma 10 *Assume Axioms 1–8. Let $F_1, \ldots, F_L$ and $G_1, \ldots, G_M$ be n-sequences; denote by p and, respectively, q the probabilities whose existence is asserted by Lemma 9. For every $E \in \Sigma$,* *if* $E \subseteq \cup_{m=1}^{M} G_m \subseteq \cup_{\ell=1}^{L} F_\ell$*, then* $p(E) = q(E) \cdot p(\cup_{m=1}^{M} G_m)$*.*

Proof: Let $F = \cup_{\ell=1}^{L} F_\ell$ and $G = \cup_{m=1}^{M} G_m$. If $p(G) = 0$, then also $p(E) = 0$ and there is nothing to show, so assume $p(G) > 0$.

I claim that $q(E) = 0$ implies $p(E) = 0$. Proof: assume $q(E) = 0$. Then, by Lemma 9 part 3, E is negligible given G, and $E \subseteq G \subseteq F$; thus, by Lemma 8 part 2, E is also negligible given F. Then, Lemma 9 part 3 implies that $p(E) = 0$.

Since $G \setminus \sigma(G)$ is negligible given G by Corollary 3, $p(G \setminus \sigma(G)) = 0$, and so $p(G) = p(\sigma(G))$.

Next, I claim that it is enough to consider $E \subseteq \sigma(G)$. Consider an arbitrary $E \in \Sigma$. Since, by Lemma 9 part 1, $q(\sigma(G)) = 1$, it must be the case that $q(E) = q(E \cap \sigma(G))$. Therefore, if $q(E \cap \sigma(G))p(G) = p(E \cap \sigma(G))$, then also $q(E)p(G) = q(E \cap \sigma(G))p(G) = p(E \cap \sigma(G)) = p(E \setminus \sigma(G)) + p(E \cap \sigma(G)) = p(E)$, because $q(E \setminus \sigma(G)) = 0$ implies $p(E \setminus \sigma(G)) = 0$. Thus, henceforth, assume $E \subseteq \sigma(G)$.

Fix x, y with $x \succ y$. Since $\succcurlyeq_{\sigma(G)}$ is an EU preference, there is $\alpha \in [0,1]$ such that $xEy \sim_{\sigma(G)} \alpha x + (1-\alpha)y$. Since $y \sim_{F \setminus \sigma(G)} y$, Observation 1 and dynamic consistency (Remark 4) implies $(xEy)\sigma(G)y \sim_F [\alpha x + (1-\alpha)y]\sigma(G)y$. By Lemma 9 part 2, since both $(xEy)\sigma(G)y \succcurlyeq_F [\alpha x + (1-\alpha)y]\sigma(G)y$ and $(xEy)\sigma(G)y \preccurlyeq_F [\alpha x + (1-\alpha)y]\sigma(G)y$, and $E \subseteq \sigma(G)$,

$$p(E)u(x) + [1-p(E)]u(y) = p(\sigma(G))[\alpha u(x) + (1-\alpha)u(y)] + [1-p(\sigma(G))]u(y).$$

Rewrite:

$$p(E)[u(x) - u(y)] + u(y) = \alpha p(\sigma(G))[u(x) - u(y)] + u(y).$$

Since $u(x) > u(y)$, this holds if and only if $p(E) = \alpha p(\sigma(G))$. But by the EU representation of $\succcurlyeq_{\sigma(G)}$ in Lemma 1, $\alpha = q(E)$; furthermore, $p(\sigma(G)) = p(G)$. This proves the claim. ■

The following immediate Corollary is key in the construction of a CPS.

Corollary 4 *If* $\cup_{\ell=1}^{L} F_\ell = \cup_{m=1}^{M} G_m$*, then* $p = q$*.*

Construction of the CPS μ, plausibility relation $\triangleright$, and basis p: let $\mathscr{F}^* = \{\cup_{\ell=1}^L F_\ell : F_1, \dots, F_L$ is an n-sequence$\}$. Note that $\mathscr{F} \subseteq \mathscr{F}^*$. Define a function $\mu^* : \Sigma \times \mathscr{F}^* \to [0,1]$ by letting $\mu^*(E|\cup_{\ell=1}^L F_\ell) = p(E)$ for each n-sequence $F_1, \dots, F_L$, where p is the probability associated with $F_1, \dots, F_L$ as per Lemma 9. By Corollary 4, this definition is well-posed (conditional probability depends only on the conditioning event, not on the specific n-sequence used to define it). Let μ denote the restriction of μ^* to $\Sigma \times \mathscr{F}$; also let $\triangleright \in \mathscr{F} \times \mathscr{F}$ be the plausibility relation induced by μ according to Def. 2.

Lemma 11 *Assume Axioms 1–8.*

1. $\mu^* \in cpr(\Sigma, \mathscr{F}^*)$ *and* $\mu \in cpr(\Sigma, \mathscr{F})$;
2. $N \in \Sigma$ *is negligible given* $F \in \mathscr{F}$ *iff* $\mu(N|F) = 0$; *hence* $F_1, \dots, F_L \in \mathscr{F}$ *is an n-sequence iff it is a* μ*-sequence, and* $\mathscr{F}^* = \mathscr{F}_\mu$;
3. $\mathbf{p} = (p_F)$, *where* $p_F = \mu^*(\cdot | \cup \{G \in \mathscr{F} : F \triangleright G, G \triangleright F\})$ *for all* $F \in \mathscr{F}$, *is a basis for* μ.

Furthermore, consider a μ*-sequence* $F_1, \dots, F_L \in \mathscr{F}$. *Let* $u : X \in \mathbb{R}$ *be the function that represents* $\succcurlyeq$ *on* X *per Lemma 6, and define* $p : \Sigma \to [0,1]$ *by*

$$\forall E \in \Sigma, \qquad p(E) = \frac{p_{F_L}(E \cap [\cup_\ell F_\ell])}{p_{F_L}(\cup_\ell F_\ell))}. \tag{4}$$

Then

4. $p \in pr(\Sigma)$, *and if* $\{F_1, \dots, F_L\}$ *is an equivalence class of* $\triangleright$, *then* $p = p_{F_L}$;
5. $p(\sigma(\cup_{\ell=1}^L F_\ell)) = 1$, *and* $\succcurlyeq_{\sigma(\cup_{\ell=1}^L F_\ell)}$ *admits the EU representation* (u, p);
6. $\forall f, g \in \mathscr{A}$, $\mathrm{E}_p u \circ f > \mathrm{E}_p u \circ g$ *implies* $f \succ_{\cup_{\ell=1}^L F_\ell} g$, *so* $f \succcurlyeq_{\cup_{\ell=1}^L F_\ell} g$ *implies* $\mathrm{E}_p u \circ f \geq \mathrm{E}_p u \circ g$.
7. *an event* $N \in \Sigma$ *is negligible given* $\cup_{\ell=1}^L F_\ell$ *iff* $p(N) = 0$, *hence iff* $p_{F_L}(N \cap [\cup_\ell F_\ell]) = 0$;
8. $\sigma(\cup_\ell F_\ell) = \sigma_\mu(\cup_\ell F_\ell)$.

Note that The ratio in Eq. (4) is well defined because, by part (3) of Def. 3, $p_{F_L}(F_L) > 0$.

Proof: (1): by the definition of μ^* and Lemma 9 part 1, for every $F \in \mathscr{F}^*$, $\mu^*(\sigma(F)|F) = 1$, so a fortiori (cf. footnote 5) $\mu^*(F|F) = 1$. By Lemma 10, μ^* satisfies the chain rule: if $E \in \Sigma$ and $F, G \in \mathscr{F}^*$ are such that $E \subseteq F \subseteq G$, then $\mu^*(E|G) = \mu^*(E|F)\mu^*(F|G)$. Thus, $\mu^* \in cpr(\Sigma, \mathscr{F}^*)$.

(2): consider the trivial n-sequence $F_1 \equiv F$, where $F \in \mathscr{F}$. From the definition of $\mu(\cdot|F) = \mu^*(\cdot|F)$ and Lemma 9 part 3, $N \in \Sigma$ is negligible given F iff $\mu(N|F) = 0$. The other statements follow immediately.

(3): by Theorem 3, since $\mathscr{F}^* = \mathscr{F}_\mu$ and so μ is the restriction of a CPS $\mu^* \in cpr(\Sigma, \mathscr{F}_\mu)$ to $\Sigma \times \mathscr{F}$, the collection **p** in the statement is the unique basis for μ.

(4) by Lemma 2 part 2, the set function p defined in Eq. (4) equals $\mu^*(\cdot|\cup_\ell F_\ell) \in pr(\Sigma)$. If in addition $\{F_1, \ldots, F_L\}$ is an equivalence class of $\triangleright$, then in particular $\{F_1, \ldots, F_L\} = \{G \in \mathscr{F} : G \triangleright F_L, F_L \triangleright G\}$, and so by part 3, $\mu^*(\cdot|\cup_\ell F_\ell) = p_{F_L}$.

(5, 6, 7): since $p = \mu^*(\cdot|\cup_\ell F_\ell)$ and the latter is defined as the probability measure delivered by Lemma 9, the claims follow, respectively, from parts 1, 2 and 3 of that Lemma.

(8): by part (7), $\sigma(\cup_\ell F_\ell) = \cup\{[s_{-i}] : p_{F_L}([s_{-i}] \cap (\cup_\ell F_\ell)) > 0\} = \sigma_\mu(\cup_\ell F_\ell)$. ∎

For the remainder of this Section, *I will not distinguish between n-sequences and μ-sequences, or between σ and σ_μ.* I will also invoke the definitions and results in Section A, as they only rely upon the properties of CPSs and bases.

I now **construct the representation**. Consider the following algorithm.

Definition 8 (Covering Algorithm) *Consider acts $f, g \in \mathscr{A}$. Set $r = 1$ and $\mathscr{F}_1 = \mathscr{F}$.*

1. *Find $F_r \in \mathscr{F}_r$ such that $\int u \circ f \, dp_{F_r} > \int u \circ g \, dp_{F_r}$. If there is none, then STOP.*

2. *Let $\mathscr{C}_r = \{G \in \mathscr{F}_r : F_r \triangleright G\}$.*

3. *Let* $\mathscr{F}_{r+1} = \mathscr{F}_r \setminus \mathscr{C}_r$.

4. *Let* $r := r+1$ *and GOTO 1.*

Let R *be the value of* r *when the algorithm stops. A tuple* $(R, (F_r, \mathscr{C}_r)_{r=1,\dots,R-1})$ *is a* ***run of the Covering Algorithm (for*** f, g***)*** *if it can be generated by following the above steps.*

Since there may be several different possible choices in Step 1, the algorithm may produce different runs. This is immaterial to the analysis.

Lemma 12 *Let* $(R, (F_r, \mathscr{C}_r)_{r=1,\dots,R-1})$ *be a run of the Covering Algorithm for acts* $f, g \in \mathscr{A}$. *Then:*

1. *for every* $q, r = 1, \dots, R-1$ *with* $q \neq r$, $\mathscr{C}_r \cap \mathscr{C}_q = \emptyset$;

2. *for every* $r = 1, \dots, R-1$, $\mathscr{C}_r = \cup_{q=0}^{Q(r)} \mathscr{C}_q^r$, *where each* $\mathscr{C}_q^r$ *is an equivalence class of* $\triangleright$, *and* $F_r \in \mathscr{C}_0^r$;

3. *for every* $r = 1, \dots, R-1$ *and* $q = 1, \dots, Q(r)$, $\cup \mathscr{C}_q^r$ *is negligible given* $\cup[\mathscr{C}_0^r \cup \mathscr{C}_q^r]$;

4. *for every* $r = 1, \dots, R-1$, $\cup\left[\mathscr{C}_r \setminus \mathscr{C}_0^r\right]$ *is negligible given* $\cup \mathscr{C}_r$.

5. *for every* $r = 1, \dots, R-1$, $f \succ_{\bigcup_{q=0}^{Q(r)} \sigma(\cup \mathscr{C}_q^r)} g$.

Proof: (1): immediate from Def. 8.

(2): consider first the following *Claim*: for every $r = 1, \dots, R-1$ and $F, G \in \mathscr{F}$, if $F \in \mathscr{C}_r$ and $F \triangleright G, G \triangleright F$ then $G \in \mathscr{C}_r$. To see this, let $r = 1$. Recall that the algorithm selects $F_1 \in \mathscr{F}_1 = \mathscr{F}$, and sets $\mathscr{C}_1 = \{G \in \mathscr{F}_1 : F_1 \triangleright G\}$. If $F \in \mathscr{C}_1$, then $F_1 \triangleright F$; thus, if $G \triangleright F, F \triangleright G$, by transitivity $F_1 \triangleright G$ and so $G \in \mathscr{C}_1$. Inductively, suppose the statement is true for $q = 1, \dots, r-1$, and consider $q = r$. Again, the algorithm picks $F_r \in \mathscr{F}_r = \mathscr{F} \setminus \cup_{q=1}^{r-1} \mathscr{C}_q$ and sets $\mathscr{C}_r = \{G \in \mathscr{F}_r : F_r \triangleright G\}$. Again, if $F \in \mathscr{C}_r$ and $G \triangleright F, F \triangleright G$, then $F_r \triangleright F$, and by transitivity $F_r \triangleright G$. To conclude that $G \in \mathscr{C}_r$, it is enough to show that $G \in \mathscr{F}_r$. If this is not the case, then $G \in \mathscr{C}_q$ for some $q = 1, \dots, r-1$. But the inductive hypothesis implies that, in this case, $F \in \mathscr{C}_q$ as well; thus, $F \notin \mathscr{F}_r$ and so $F \notin \mathscr{C}_r$, contradiction. Hence, $G \in \mathscr{F}_r$ and so $G \in \mathscr{C}_r$. This proves the claim.

Therefore, $\mathscr{C}_r$ is a union of equivalence classes of $\triangleright$; and since $F_r \in \mathscr{C}_r$ (because $\triangleright$ is reflexive), one such class contains $\mathscr{F}_r$.

(3): by Remark 2 part 5, there are μ-sequences $G_1,\ldots,G_L$ and $G''_1,\ldots,G''_M$ such that $\{G_1,\ldots,G_L\} = \mathscr{C}^r_0$, $G_1 = G_L = F_r$, $\{G'_1,\ldots,G''_{L''}\} = \mathscr{C}^r_q$, and $G''_1 = G''_{L''}$. Since $G''_1 \in \mathscr{C}^r_q \subseteq \mathscr{C}_r$, $G_1 = F_r \triangleright G''_{L''}$ by Def. 8. Hence, by Remark 2 part 3, there is a μ-sequence $G'_1,\ldots,G'_{L'}$ such that $G'_{L'} = G_1$ and $G'_1 = G''_{L''}$. Therefore,

$$G''_1,\ldots,G''_{L''},G'_1,\ldots,G'_{L'},G_1,\ldots,G_L$$

is a μ-sequence. For all $\ell = 1,\ldots,L''$, $G''_\ell \notin \mathscr{C}^r_0$ and $G_L = F_r \triangleright G''_\ell$, so not $G''_\ell \triangleright F_r = G_L$. Moreover, for all $\ell = 1,\ldots,L$, $G_\ell \in \mathscr{C}^r_0$ and so in particular $G_\ell \triangleright F_r = G_L$; for these events, $p_{F_r}(G_\ell) > 0$ by Def. 3 part (3). Hence, by Lemma 2 part 1, $p_{F_r}(G''_\ell) = 0$ for all $\ell = 1,\ldots,L''$, and there is $m \in \{1,\ldots,L'\}$ such that $p_{F_r}(G'_\ell) > 0$ and $G'_\ell \triangleright G_L = F_r$ iff $\ell = m,\ldots,L$. Note that, therefore, $G'_\ell \in \mathscr{C}^r_0$ for $\ell = m,\ldots,L'$; furthermore, $p_{F_r}(\cup\mathscr{C}^r_q) = p_{F_r}(\cup_{\ell=1}^{m-1} G'_\ell) = 0$.

Next, fix $x,y,z \in X$ with $y \succ z$. Let $F = \cup\mathscr{C}^r_0$ and $N = \cup\mathscr{C}^r_q$. Also let $G = F \cup N \cup \bigcup_{\ell=1}^{L'} G'_\ell = G''_1 \cup\ldots\cup G''_{L''} \cup G'_1 \cup\ldots\cup G'_{L'} \cup G_1 \cup\ldots\cup G_L$. Note that $N \subset F \cup N \subseteq G$. I claim that $y(F\cup N)z \succ_G xNz$. Notice that, since $p_{F_r}(N) = p_{F_r}(\cup_{\ell=1}^{m-1} G'_\ell) = 0$,

$$p_{F_r}(F \cup N \cup [\cup_{\ell=1}^{L'} G'_\ell]) \geq p_{F_r}(F \cup [\cup_{\ell=m}^{L'} G'_\ell]) = p_{F_r}(F) = 1,$$

because $F = \cup\mathscr{C}^r_0$ and $\mathscr{C}^r_0$ is the equivalence class containing F_r, and furthermore $G'_\ell \in \mathscr{C}^r_0$ for $\ell = m,\ldots,L'$. Therefore, the measure p defined in Eq. (4) of Lemma 11 coincides with p_{F_r}. Hence, by part 6 of that Lemma, it is enough to show that $\mathrm{E}_{p_{F_r}} u \circ y(F\cup N)z > \mathrm{E}_{p_{F_r}} u \circ xNz$ (recall that $F_r = G_L$). Again because $p_{F_r}(N) = 0$ and $p_{F_r}(F\cup N) \geq p_{F_r}(F) = 1$,

$$\mathrm{E}_{p_{F_r}} u \circ y(F\cup N)z = u(z) + p_{F_r}(F\cup N)[u(y) - u(z)] = u(y)$$

and

$$\mathrm{E}_{p_{F_r}} u \circ xNz = u(z) + p_{F_r}(N)[u(x) - u(z)] = u(z).$$

Thus, $\mathrm{E}_{p_{F_r}} u \circ y(F\cup N)z = u(y) > u(z) = \mathrm{E}_{p_{F_r}} u \circ xNz$, and so $y(F\cup N)z \succ_G xNz$.

By the definition of conditional preferences, since $N \subset (F \cup N) \subseteq G$, conclude that, for all x, y, z with $y \succ z$, $y(F \cup N)z \succ xNz$; again by the definition of conditional preferences, since $N \subset F \cup N$, $y \succ_{F \cup N} xNz$. Then, by Lemma 7, $N = \cup \mathscr{C}_q^r$ is negligible given $F \cup N = \cup[\mathscr{C}_0^r \cup \mathscr{C}_q^r]$.

(4): Let $F = \cup \mathscr{C}_0^r$ and, for every $q = 1, \ldots, Q(r)$, $N_q = \cup \mathscr{C}_q^r$. By part 3, for every $q = 1, \ldots, Q(r)$, N_q is negligible given $F \cup N_q$. It must be shown that $\cup_q N_q$ is negligible given $F \cup (\cup_q N_q)$. I show by induction that, for every $\bar{q} = 1, \ldots, Q(r)$, $\cup_{q=1}^{\bar{q}} N_q$ is negligible given $F \cup (\cup_{q=1}^{\bar{q}} N_q)$.

The claim is trivially true for $\bar{q} = 1$. Suppose it is true for $\bar{q} \geq 1$, and consider $N_{\bar{q}+1}$, which by assumption is negligible given $F \cup N_{\bar{q}+1}$. Since $N_{\bar{q}+1} \subsetneq F \cup N_{\bar{q}+1} \subseteq F \cup (\cup_{q=1}^{\bar{q}+1} N_q)$, $N_{\bar{q}+1}$ is also negligible given $F \cup (\cup_{q=1}^{\bar{q}+1} N_q)$ by Lemma 8 part 2. By the induction hypothesis, $\cup_{q=1}^{\bar{q}} N_q$ is negligible given $F \cup (\cup_{q=1}^{\bar{q}} N_q) \subseteq F \cup (\cup_{q=1}^{\bar{q}+1} N_q)$, and hence by Lemma 8 part 2 also for $F \cup (\cup_{q=1}^{\bar{q}+1} N_q)$. By Lemma 8 part 3, $\cup_{q=1}^{\bar{q}+1} N_q = N_{\bar{q}+1} \cup \bigcup_{q=1}^{\bar{q}+1} N_q$ is then negligible given $F \cup (\cup_{q=1}^{\bar{q}+1} N_q)$.

(5): as in part 4, let $F = \cup \mathscr{C}_0^r$ and, for every $q = 1, \ldots, Q(r)$, $N_q = \cup \mathscr{C}_q^r$. Also let $E = F \cup \bigcup_{q=1}^{Q(r)} N_q$. Part 4 of this Lemma states that $\cup_{q=1}^{Q(r)} N_q$ is negligible given E. Furthermore, by Corollary 3, $F \setminus \sigma(F) \subset F$ is negligible given F, and hence also given $E \supseteq F$ by part 2 of Lemma 8. Therefore, by part 3 of Lemma 8, $[F \setminus \sigma(F)] \cup \bigcup_{q=1}^{Q(r)} N_q$ is negligible given E.

As in the proof of part 3, there is a μ-sequence $G_1, \ldots, G_L$ such that $\mathscr{C}_0^r = \{G_1, \ldots, G_L\}$ and $G_1 = G_L = F_r$. Since by construction $\mathrm{E}_{p_{F_r}} u \circ f > \mathrm{E}_{p_{F_r}} u \circ g$, Lemma 11 parts 4 and 5 impliy that $f \succ_{\sigma(F)} g$.

Fix $z, z', z'' \in X$ with $z' \succ z''$, and define acts f', g' such that $f'(\omega) = f(\omega)$ and $g'(\omega) = g(\omega)$ for all $\omega \in \sigma(F) \cup [\cup_{q=1}^{Q(r)} \sigma(N_q)]$, $f'(\omega) = g'(\omega) = z$ for all $\omega \in E \setminus \{\sigma(F) \cup [\cup_{q=1}^{Q(r)} \sigma(N_q)]\}$, and $f'(\omega) = z'$ and $g'(\omega) = z''$ for $\omega \notin E$.

By Lemma 11 part 5, $\succcurlyeq_{\sigma(F)}$ is an EU relation, so there are $x, y \in X$ such that $x \sim_{\sigma(F)} f$ and $y \sim_{\sigma(F)} g$. Then $x \succ y$. Furthermore, since $f(\omega) = f'(\omega)$ and $g(\omega) = g'(\omega)$ for $\omega \in \sigma(F)$, Observation 1 and Transitivity imply $x \sim_{\sigma(F)} f'$ and $y \sim_{\sigma(F)} g'$. By the definition of conditional preferences, in particular $x\sigma(F)f' \sim f'$ and $y\sigma(F)g' \sim g'$, which, since $\sigma(F) \subseteq E$, implies $x\sigma(F)f' \sim_E f'$ and $y\sigma(F)g' \sim_E g'$.

Finally, consider the acts $x\sigma(F)f'$ and $y\sigma(F)g'$. Observe that $x\sigma(F)f'(\omega) \succ \sigma(F)g'(\omega)$ for all $\omega \notin [F \setminus \sigma(F)] \cup [\cup_{q=1}^{Q(r)} N_q)]$. To see this, note that any such ω cannot belong to $F \setminus \sigma(F)$ or to any of the sets N_q. One possibility is $\omega \notin E = F \cup [\cup_q] N_q]$, and for such ω, $x\sigma(F)f'(\omega) = f'(\omega) = z' \succ z'' = g'(\omega) = y\sigma(F)g'$. If instead $\omega \in E$, then it cannot belong to any of the sets N_q, so it must belong to F; but it cannot belong to $F \setminus \sigma(F)$, so it *must* belong to $\sigma(F)$, and for such ω, $x\sigma(F)f'(\omega) = x \succ y = y\sigma(F)g'$. Then, since $[F \setminus \sigma(F)] \cup \bigcup_{q=1}^{Q(r)} N_q$ is negligible given E, $x\sigma(F)f' \succ_E y\sigma(F)g'$.

Now Transitivity implies that $f' \succ_E g'$. By the definition of conditional preferences, in particular $f'Ez \succ g'Ez$. Due to the way f' and g' were defined above, $f'Ez(\omega) = f(\omega)$ and $g'Ez(\omega) = g(\omega)$ for $\omega \in \sigma(F) \cup [\cup_{q=1}^{Q(r)} \sigma(N_q)]$, and $f'Ez(\omega) = g'Ez(\omega) = z$ for $\omega \notin \sigma(F) \cup [\cup_{q=1}^{Q(r)} \sigma(N_q)]$. Then, the definition of conditional preferences implies that

$$f \succ_{\sigma(F) \cup [\cup_{q=1}^{Q(r)} \sigma(N_q)]} g,$$

as required. ■

It is now possible to establish the representation. Let $\mathbf{C}$ be the collection of equivalence classes of $\triangleright$. By Lemma 3 part 1, for all $\mathscr{C}, \mathscr{D} \in \mathbf{C}$, $\sigma(\cup\mathscr{C}) \cap \sigma(\cup\mathscr{D}) = \emptyset$.

Let $N = \Omega \setminus \cup_{\mathscr{C} \in \mathbf{C}} \sigma(\cup\mathscr{C})$. Consider $F \in \mathscr{F}$ and let $\mathscr{C} \in \mathbf{C}$ be the equivalence class such that $F \in \mathscr{C}$. Since $N \cap \sigma(\cup\mathscr{C}) = \emptyset$ and $p_F(\sigma(\cup\mathscr{C})) = 1$ by Lemma 11 parts 4 and 5, $p_F(N) = 0$, so a fortiori $p_F(N \cap F) = 0$. By Lemma 11 part 7, considering the degenerate sequence $F_1 = F$, N is negligible given F. Finally, since this is the case for all $F \in \mathscr{F}$, Axiom 9 implies that N is Savage-null for $\succcurlyeq$.

Thus, fix $z \in X$ and let $f' = zNf$ and $g' = zNg$. Since N is Savage-null for $\succcurlyeq$, $f' \sim f$ and $g' \sim g$. Thus, by Transitivity, $f \succcurlyeq g$ iff $f' \succcurlyeq g'$. Furthermore, for every $M \geq 1$ and $\mathscr{C}_1, \ldots, \mathscr{C}_M \in \mathbf{C}$, $f'(\omega) = f(\omega)$ and $g'(\omega) = g(\omega)$ for all $\omega \in \cup[\cup_{m=1}^M \mathscr{C}_m]$; by Axiom 3 (Monotonicity), $f'[\cup_{m=1}^N \sigma(\cup\mathscr{C})]z \sim f[\cup_{m=1}^N \sigma(\cup\mathscr{C})]z$ and $g'[\cup_{m=1}^N \sigma(\cup\mathscr{C})]z \sim g[\cup_{m=1}^N \sigma(\cup\mathscr{C})]z$, so by the definition of conditional preferences and Transitivity $f' \succcurlyeq_{[\cup_{m=1}^N \sigma(\cup\mathscr{C})]} g'$ iff $f \succcurlyeq_{[\cup_{m=1}^N \sigma(\cup\mathscr{C})]} g$.

Let $(R,(F_r,\mathscr{C}_r)_{r=1}^{R-1})$ be a run of the Covering Algorithm.

Suppose first that $f \sim^{u,\mu} g$. Then $R=0$, and for every equivalence class $\mathscr{C} \in \mathbf{C}$ and $F \in \mathscr{C}$, $\mathrm{E}_{p_F} u\circ f = \mathrm{E}_{p_F} u\circ g$. By Lemma 11 parts 4 and 5, $f \sim_{\sigma(\cup\mathscr{C})} g$, and so, as argued above, $f' \sim_{\sigma(\cup\mathscr{C})} g'$. By Lemma 3, the supports $\sigma(\cup\mathscr{C})$ for distinct equivalence classes $\mathscr{C}$ are disjoint. Therefore, by dynamic consistency (Remark 4), and $\bigcup_{\mathscr{C}\in\mathbf{C}} \sigma(\cup\mathscr{C}) = \Omega\setminus N$, $f' \sim_{\Omega\setminus N} g'$. Since $f'(\omega)=g'(\omega)=z$ for $\omega\in N$, the definition of conditional probability implies in particular $f' \sim g'$. Hence $f \sim g$.

Next, suppose that $f \succ^{u,\mu} g$, so $R>0$. By Lemma 12, there are distinct equivalence classes $\mathscr{C}^r_q$, $r=1,\dots,R-1$, $q=0,\dots,Q(r)$, such that $f \succ_{\cup_{q=0}^{Q(r)}\sigma(\cup\mathscr{C}^r_q)} g$ for each r; hence, also $f' \succ_{\cup_{q=0}^{Q(r)}\sigma(\cup\mathscr{C}^r_q)} g'$. Furthermore, consider an equivalence class $\mathscr{C}\in\mathbf{C}$ such that $\mathscr{C} \neq \mathscr{C}^r_q$ for all q,r. Fix $F\in\mathscr{C}$. I claim that $\mathrm{E}_{p_F} u\circ f = \mathrm{E}_{p_F} u\circ g$, so that $f \sim_{\sigma(\cup\mathscr{C})} g$ and $f' \sim_{\sigma(\cup\mathscr{C})} g'$.

To see this, suppose by contradiction that $\mathrm{E}_{p_F} u\circ f > \mathrm{E}_{p_F} u\circ g$. Since by assumption $\mathscr{C}\neq\mathscr{C}^r_q$ for all q,r, setting $F_R = F$ is a possible choice in step R; however, by assumption the algorithm terminates after $R-1$ steps: contradiction. Suppose instead that $\mathrm{E}_{p_F} u\circ f < \mathrm{E}_{p_F} u\circ g$. Then, since by assumption $f \succ^{u,\mu} g$, so in particular $f \not\succcurlyeq^{u,\mu} g$, there is $G\in\mathscr{F}$ such that $\mathrm{E}_{p_G} u\circ f > \mathrm{E}_{p_G} u\circ g$ and $G \triangleright F$. Let $\mathscr{D}$ be the equivalence class of $\triangleright$ containing G. Suppose by contradiction that $\mathscr{D}\neq\mathscr{C}^r_q$ for all q,r: then, as in the previous case, $F_R = G$ would have been a possible choice in step R of the algorithm, which however terminated at step $R-1$. Therefore, there must be q,r such that $\mathscr{D}=\mathscr{C}^r_q$. Then in particular $F_r \triangleright G$, so by transitivity $F_r \triangleright F$. This implies that $F_r \triangleright F'$ for all $F'\in\mathscr{C}$, and therefore $\mathscr{C}=\mathscr{C}^r_{q'}$ for some q': contradiction.

Therefore, applying dynamic consistency (Remark 4) to the sets $\bigcup_{q=0}^{Q(r)}\sigma(\cup\mathscr{C}^r_q)$ for $r=1,\dots,R-1$, and $\sigma(\mathscr{C})$ for $\mathscr{C}\in\mathbf{C}\setminus\{\mathscr{C}^r_q : r=1,\dots,R-1, q=1,\dots,Q(r)\}$, since in particular preferences conditional upon the first $R-1$ such sets is strict, yields $f' \succ_{\Omega\setminus N} g'$; as above, this implies $f' \succ g'$, and so $f \succ g$.

Equivalently, it was just shown that $f \succcurlyeq^{u,\mu} g$ implies $f \succcurlyeq g$, and $f \succ^{u,\mu} g$ implies $f \succ g$. **This establishes Theorem 2.**

For **Theorem 1**, since necessity of the axioms was shown in Appendix B, and uniqueness of u and μ in Lemmata 6 and 11, it remains to be shown that, if not $f \succcurlyeq^{u,\mu} g$, then not $f \succcurlyeq g$. Thus, $f \succcurlyeq^{u,\mu} g$ iff $f \succcurlyeq g$, which proves the equivalence statement in Theorem 1.

Thus, assume that not $f \succcurlyeq^{u,\mu} g$, and define two auxiliary acts f', g' as follows. Fix $z \in X$ and, for every $\mathscr{E} \in \mathbf{C}$, let $x_{\mathscr{C}}, y_{\mathscr{C}} \in X$ be such that $x_{\mathscr{C}} \sim_{\sigma(\cup\mathscr{C})} f$ and $y_{\mathscr{C}} \sim_{\sigma(\cup\mathscr{C})} g$; these exist because, by Remark 2 part 5 and Lemma 11 parts 4 and 5, $\succcurlyeq_{\sigma(\cup\mathscr{C})}$ has an EU representation for every $\mathscr{C} \in \mathbf{C}$. Then,

$$f' = \begin{cases} x_{\mathscr{C}} & \omega \in \sigma(\cup\mathscr{C}), \mathscr{C} \in \mathbf{C} \\ z & \omega \in N \end{cases} \quad \text{and} \quad g' = \begin{cases} y_{\mathscr{C}} & \omega \in \sigma(\cup\mathscr{C}), \mathscr{C} \in \mathbf{C} \\ z & \omega \in N. \end{cases}$$

The acts f', g are well-defined because, by Lemma 3, the sets $\sigma(\cup\mathscr{C})$, $\mathscr{C} \in \mathbf{C}$, are disjoint, and by definition $N = \Omega \setminus \cup\{\sigma(\cup\mathscr{C}) : \mathscr{C} \in \mathbf{C}\}$. Furthermore, by dynamic consistency (Remark 4), $f \sim fNf'$ and $g \sim gNg'$; since N is Savage-null for $\succcurlyeq$, $fNf' \sim f'$ and $gNg' \sim g'$; and by Transitivity, $f \sim f'$ and $g \sim g'$.

As above, fix a run $(R, (F_r, \mathscr{C}_r)_{r=1,\dots,R-1})$ of the Covering Algorithm. I first show that if there is an equivalence class $\mathscr{D}$ of $\triangleright$ such that $\mathscr{D} \neq \mathscr{C}^r_q$ for all q, r, and $\mathrm{E}_{p_F} u \circ f < \mathrm{E}_{p_F} u \circ g$, where $F \in \mathscr{D}$, then not $f \succcurlyeq g$.

Suppose there is one such equivalence class $\mathscr{D}$. By Remark 2 part 5, $\mathscr{D} = \{G_1, \dots, G_M\}$ for some n-sequence (hence μ-sequence) $G_1, \dots, G_M$: I will rely on this fact without further reference, when invoking Lemma 9 or Lemma 11. Fix $F \in \mathscr{D}$.

By Lemma 11 parts 4 and 5, $\mathrm{E}_{p_F} u \circ f < \mathrm{E}_{p_F} u \circ g$ implies $f \prec_{\sigma(\cup\mathscr{D})} g$. By the definition of $x_{\mathscr{D}} y_{\mathscr{D}}$ and Transitivity, $x_{\mathscr{D}} \sim_{\sigma(\cup\mathscr{D})} f \prec_{\sigma(\cup\mathscr{D})} g \sim_{\sigma(\cup\mathscr{D})} y_{\mathscr{D}}$. By Lemma 9 part 1, $x_{\mathscr{D}} \prec_{\sigma(\cup\mathscr{D})} y_{\mathscr{D}}$ implies $x_{\mathscr{D}} \prec y_{\mathscr{D}}$; applying the same result to the one-element μ-sequence (F) then yields $x_{\mathscr{D}} \prec_{\sigma(F)} y_{\mathscr{D}}$. Furthermore, applying Part 2 of Lemma 3 to the one-element μ-sequence (F), $\sigma(F) \subseteq \sigma(\cup\mathscr{D})$. Then, by the definition of f', g', and Observation 1 (Null Complements), $x_{\mathscr{D}} \sim_{\sigma(F)} x_{\mathscr{D}} \sigma(F) f' = f'$ and similarly $y_{\mathscr{D}} \sim_{\sigma(F)} g'$. Therefore, by Transitivity $f' \prec_{\sigma(F)} g'$. Since, by Def. 3 part (3), for every $E \in \Sigma$, $p_F(E \cap F)/p(F) = \mu(E \cap F|F) = \mu(E|F)$, Lemma 11 part 5 applied to the μ-

sequence (F) implies that $\mathrm{E}_{\mu(\cdot|F)} u \circ f' < \mathrm{E}_{\mu(\cdot|F)} u \circ g'$, and part 6 in the same Lemma finally implies that $f' \prec_F g'$.

By contradiction, assume that $f \succcurlyeq g$. Then $f' \succcurlyeq g'$, and since it was just shown that, for $F \in \mathscr{D} \subseteq \mathscr{F}$, $f' \prec_F g'$, Axiom 10 implies that there exists a n-sequence $F_1, \dots, F_L$ such that either (i) $F = F_L$ and $f' \succcurlyeq_{\sigma(\cup_\ell F_\ell)} g'$, or (ii) $F = F_1$ and $f' \succ_{\sigma(\cup_\ell F_\ell)} g'$. I argue that neither can possibly hold, which implies that assuming $f \succcurlyeq g$ leads to a contradiction.

Consider case (i). Since $F_L = F \in \mathscr{D}$, by Lemma 3 part 2 $\sigma(\cup_\ell F_\ell) \subseteq \sigma(\cup \mathscr{D})$. Therefore, by construction, $f'(\omega) = x_{\mathscr{D}} \prec y_{\mathscr{D}} = g'(\omega)$ for all $\omega \in \sigma(\cup_\ell F_\ell)$. By Lemma 9 part 1, $x_{\mathscr{D}} \prec y_{\mathscr{D}}$ implies $x_{\mathscr{D}} \prec_{\sigma(\cup_\ell F_\ell)} y_{\mathscr{D}}$, so by Transitivity and Observation 1, $f' \sim_{\sigma(\cup_\ell F_\ell)} x_{\mathscr{D}} \prec_{\sigma(\cup_\ell F_\ell)} y_{\mathscr{D}} \sim_{\sigma(\cup_\ell F_\ell)} g'$. But (i) requires $f' \succcurlyeq_{\sigma(\cup_\ell F_\ell)} g'$, and hence cannot hold.

Now consider case (ii), so $F_1 = F \in \mathscr{D}$. Let $\mathscr{C}$ be the equivalence class for $\triangleright$ that contains F_L. If $\mathscr{C} = \mathscr{D}$, then $F_L \in \mathscr{D}$, and the argument just given shows that then $f' \prec_{\sigma(\cup_\ell F_\ell)} g'$, so (ii) does not hold. If instead $\mathscr{C} \neq \mathscr{D}$, by Lemma 3 part 2, $\sigma(\cup_\ell F_\ell) \subseteq \sigma(\cup \mathscr{C})$. By construction, for all $\omega \in \sigma(\cup_\ell F_\ell) \subseteq \sigma(\cup \mathscr{C})$, $f'(\omega) = x_{\mathscr{C}}$ and $g'(\omega) = x_{\mathscr{C}}$. If, as required by (ii), $f' \succ_{\sigma(\cup_\ell F_\ell)} g'$, then by Transitivity and Observation 1 $x_{\mathscr{C}} \sim_{\sigma(\cup_\ell F_\ell)} f' \succ_{\sigma(\cup_\ell F_\ell)} g' \sim_{\sigma(\cup_\ell F_\ell)} y_{\mathscr{C}}$. By Lemma 9 part 1, this implies $x_{\mathscr{C}} \succ y_{\mathscr{C}}$, hence $x_{\mathscr{C}} \succ_{\sigma(\cup \mathscr{C})} y_{\mathscr{C}}$. By the definition of $x_{\mathscr{C}}, y_{\mathscr{C}}$ and Transitivity, $f \sim_{\sigma(\cup \mathscr{C})} x_{\mathscr{C}} \succ_{\sigma(\cup \mathscr{C})} y_{\mathscr{C}} \sim_{\sigma(\cup \mathscr{C})} g$, and finally, by Lemma 11 parts 4 and 5, $\mathrm{E}_{p_{F_L}} u \circ f > \mathrm{E}_{p_{F_L}} u \circ g$. But then, since the algorithm terminated at step $R > 1$, $\mathscr{C} = \mathscr{C}_{\bar{q}}^{\bar{r}}$ for some $\bar{q} = 0, \dots, Q(\bar{r})$ and $\bar{r} = 1, \dots, R-1$ (otherwise, a further iteration would have been possible). This implies that $F_{\bar{r}} \triangleright F_L$, where $F_{\bar{r}}$ denotes the conditioning event selected in step 1 of round $\bar{r}$ of the Covering Algorithm. But since also $F_L \triangleright F_1$, by transitivity of the plausibility relation $F_{\bar{r}} \triangleright F_1$. By the definition of the Covering algorithm, either $F_1 \in \mathscr{F}_{\bar{r}}$ (i.e., F_1 was not "covered" in any previous round of the algorithm), in which case $F_1 = F \in \mathscr{C}_{\bar{r}}$ and so $\mathscr{D} = \mathscr{C}_{\bar{q}'}^{\bar{r}}$ for some $\bar{q}' = 1, \dots, Q(\bar{r})$; or $F_1 = F \notin \mathscr{F}_{\bar{r}}$ (F_1 was already "covered"), in which case $\mathscr{D} = \mathscr{C}_{\bar{q}'}^{\bar{r}'}$ for some $\bar{r}' = 1, \dots, \bar{r}-1$ and $\bar{q}' = 1, \dots, Q(\bar{r}')$. [In both cases, $q > 0$, because by assumption $\mathrm{E}_{p_F} u \circ f < \mathrm{E}_{p_F} u \circ g$, so F could not be selected in step 1 of any round of the algorithm.] This contradicts the assumption that $\mathscr{D} \neq \mathscr{C}_q^r$ for all q, r. Thus, (ii) also cannot hold.

Thus, Axiom 10 is violated if $f \succcurlyeq g$, so as asserted it is not the case that $f \succcurlyeq g$.

To complete the proof, suppose that not $f \succcurlyeq^{u,\mu} g$. Then there is $F \in \mathscr{F}$ with $\mathrm{E}_{p_F} u \circ f < \mathrm{E}_{p_F} u \circ g$ and such that $\mathrm{E}_{p_G} u \circ f \leq \mathrm{E}_{p_G} u \circ g$ for all $G \triangleright F$. Let $\mathscr{D}$ be the equivalence class of $\triangleright$ that contains F. I claim that $\mathscr{D} \neq \mathscr{C}_q^r$ for all q, r: as was just shown, this implies that not $f \succcurlyeq g$. Clearly, $\mathscr{D} \neq \mathscr{C}_0^r$ for all r, and furthermore, if $\mathscr{D} = \mathscr{C}_q^r$ for some $q = 1, \ldots, Q(r)$ and $r = 1, \ldots, R-1$, then by construction $F_r \triangleright F$ and $\mathrm{E}_{p_{F_r}} u \circ f > \mathrm{E}_{p_{F_r}} u \circ g$, contradiction. This completes the proof.

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