Risk Sharing in the Small and in the Large

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Abstract

This paper analyzes risk sharing in economies with no aggregate uncertainty when agents have ambiguity-sensitive preferences that are not necessarily convex, i.e., uncertainty-averse in the sense of Schmeidler (1989). We consider three notions of "beliefs" for such preferences, and propose a condition under which they coincide. Our main result shows that, under this condition, betting is inefficient (i.e., every Pareto-efficient allocation provides full insurance, and conversely) if and only if agents' sets of beliefs have a non-empty intersection. Our condition implies a mild notion of aversion to ambiguity, and is consistent with experimental evidence documenting violations of convexity.

1 Introduction

Over the last two decades, increasing attention to the findings of psychologists and experimental economists has prompted a reexamination of many of the classical results on economic and financial decision making. One of these results states that, if there is no aggregate uncertainty, agents who maximize expected utility (EU) will engage in betting if and only if they have different probabilistic beliefs. There is, however, compelling evidence that agents

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may violate (subjective) EU when some events relevant for the economy are more ambiguous than others (Ellsberg, 1961). Motivated by this evidence, Billot, Chateauneuf, Gilboa, and Tallon (2000) and Rigotti, Shannon, and Strzalecki (2008, henceforth RSS) have shown that a version of the noted result on the absence of betting holds even if agents do not maximize EU, so long as their preferences over uncertain consumption are convex and sufficiently well-behaved. Yet, the interpretation of convexity in the presence of ambiguity has also been questioned. From a conceptual perspective, Epstein (1999), Ghirardato and Marinacci (2002, GM henceforth) and Baillon, L'Haridon, and Placido (2011), among others, have pointed out that convexity may not fully identify ambiguity-averse behavior. From an experimental perspective, recent work documents the significance of patterns of behavior that violate convexity, but are intuitively consistent with aversion to ambiguity: see e.g. L'Haridon and Placido (2010).[1]

This paper shows that the equivalence between the absence of betting and the consistency of agents' beliefs does not require convexity. We establish this equivalence under a significantly weaker condition that relates the "global" behavior of preferences to the "local" ranking of consumption plans that are nearly riskless. This condition implies a notion of aversion to ambiguity that generalizes a definition due to Ghirardato and Marinacci (2002), and is consistent with the experimental violations of convexity observed by Baillon et al. (2011) (see Example 5). Our analysis applies to arbitrary finite economies, unlike classical results in the literature on general equilibrium in large non-convex economies (e.g. Anderson, 1988).

We now provide a more detailed overview of the results. We endow each agent i with preferences over contingent consumption plans represented by a functional I_i and a Bernoulli utility u_i: plan f is preferred to g if and only if $I_i(u_i(f)) \geq I_i(u_i(g))$. We assume that each I_i is strongly monotonic and locally Lipschitz, and that each u_i is strictly increasing and strictly concave. Notably, the functionals I_i are not required to be quasiconcave, or to satisfy scale or location-invariance properties.

Statements about risk sharing involve agents' beliefs. In the case of standard EU preferences, betting is inefficient (i.e., only full-insurance allocations are efficient) if and only if

[1] In addition, experiments suggest that agents' attitudes toward ambiguity may depend upon the prospects being considered, thus contradicting convexity. See, e.g., Curley and Yates (1985), Heath and Tversky (1991) and, more recently, Abdellaoui, Baillon, Placido, and Wakker (2011).

agents share the same subjective probability. In the case of convex preferences, RSS identify sets of probabilities which they call "local beliefs." Under a further assumption on the structure of such beliefs, they show that betting is inefficient if and only if the agents' set of local beliefs have a non-empty intersection. Thus, a necessary first step in our analysis is to define what is meant by "beliefs" for the non-convex preferences we consider. We identify and discuss three possible notions of beliefs, i.e., sets of probabilities. The first is RSS's definition of local beliefs. The other two arise naturally when considering the two different implications in the equivalence between sharing priors and the inefficiency of betting.

To elaborate, Proposition 6 in Section 4 shows that, for the general preferences we consider, a *necessary* condition for betting to be inefficient is that, at every feasible, full-insurance allocation, there be at least *one* shared probability in the normalized Clarke (1983) differentials of each agent i's preference functional I_i (Ghirardato and Siniscalchi, 2012). Proposition 7 instead shows that a *sufficient* condition for betting to be inefficient if that the shared probability at every full-insurance allocation belong to a particular subset of each agent's normalized Clarke differential, which we call the *point core*. Examples 3 and 4 show that, in general, these conditions are not tight: betting may be efficient even if normalized Clarke differentials have common elements at every full-insurance allocation (Example 3), and it may be inefficient even if there is no shared probability in the point cores at some full-insurance allocation (Example 4). Thus, without further assumptions on preferences, there is no single notion of "shared probability" that is both necessary and sufficient for the inefficiency of betting.

Our main methodological contribution is to identify an assumption on preferences, called *differential quasiconcavity at certainty* (DQC), which implies that the local beliefs, point cores, and normalized Clarke differentials coincide at every riskless consumption. DQC ensures that there is a well-defined notion of "beliefs" for the purposes of risk sharing: betting is inefficient if and only if the agents' local beliefs / point cores / normalized Clarke differentials intersect at every feasible, full-insurance allocation (Theorem 9). Furthermore, DQC implies that the point cores at all riskless consumptions are non-empty; in the spirit of GM, we interpret this as a mild form of dislike for ambiguity, which we call *pointwise ambiguity aversion*.

Loosely speaking, condition DQC requires that, if a consumption f is weakly preferred to a riskless consumption x, then an agent whose endowment is precisely x would be willing to engage in an "infinitesimal" trade away from x and toward f. It turns out that this condition is satisfied by all convex preferences; this relates our analysis to that of RSS (see Section 4 for details). However, DQC is also consistent with substantial departures from convexity. Intuitively, an agent with non-convex preferences, who prefers f to x, but prefers x to $\frac{1}{2}f + \frac{1}{2}x$, may satisfy DQC provided she prefers $\alpha f + (1-\alpha)x$ to x when α is very small (indeed, infinitesimal). Example 2 illustrates this intuition; Example 5 corroborates it by exhibiting a behaviorally interesting class of non-convex preferences that satisfy DQC.

As mentioned above, in addition to convexity, RSS's risk-sharing result employs an assumption called "translation invariance at certainty," or TIC. This assumption implies that local belief sets are invariant across riskless consumptions. However, we illustrate in Example 8 that risk sharing can obtain even if local beliefs at certainty are *not* invariant in this sense (the preferences in that example are convex). Our risk-sharing result, as per Theorem 9, is consistent with this example: our assumptions do not include (or imply) TIC. In particular, for betting to be inefficient, local beliefs must intersect at every *feasible* full-insurance allocation, but the shared probability at each such allocation may well be different. If we also impose TIC, then (Theorem 10) each agent i's local beliefs (equivalently, point cores or normalized Clarke differentials) are invariant across riskless consumptions, and coincide with the *core* of the preference functional I_i introduced by GM.[2] In this case, the risk-sharing result takes a simpler form: betting is inefficient if and only if the cores of the agents' preference functionals intersect.

This paper is organized as follows. Section 2 introduces the formal setup. Section 3 introduces and analyzes the different notions of beliefs. Section 4 contains the main risk-sharing results. Section 5 illustrates the results via additional examples. Finally, Section 6 concludes by analyzing two extensions.

Related literature The relation with RSS and Billot et al. (2000) has already been discussed. We note that these papers allow for an arbitrary state space; for simplicity, we restrict attention to finite states.

[2]Notice that, under the assumptions of Theorem 10, each I_i has non-empty core, and therefore all agents are ambiguity-averse in the sense of GM.

Strzalecki and Werner (2011) extend and adapt the risk-sharing results in RSS to economies with aggregate uncertainty and convex preferences. In section 6 we follow their approach and provide a counterpart to our Proposition 7 for non-convex economies with aggregate uncertainty. A full investigation of risk sharing in such economies is left to future research.

Billot, Chateauneuf, Gilboa, and Tallon (2002) provide a version of Proposition 7 for Choquet-expected utility preferences (CEU; Schmeidler, 1989). They also prove a risk-sharing result for such preferences that does not assume convexity (or DQC) but requires large economies, with a continuum of agents of each "type."

Dominiak, Eichberger, and Lefort (2012) consider an economy with two CEU agents and riskless (full-insurance) endowments, extending the prior analysis of Kajii and Ui (2006) which assumed convexity. They provide a condition which is necessary and sufficient for the non-existence of Pareto-improving trades. Their analysis relies on the fact that there are only two agents in the economy, whose initial endowment is constant; on the other hand, it does not require either convexity or pointwise ambiguity aversion.

Marinacci and Pesce (2013) consider preferences that are both GM-ambiguity averse and invariant biseparable (Ghirardato, Maccheroni, and Marinacci, 2004). They study the impact of changes in GM-ambiguity aversion on efficient and equilibrium allocations. Though they do not focus on risk sharing, they independently derive a version of our Proposition 7. See however Example 6 in Section 5 of this paper on the implications of invariant biseparability for risk sharing.

The notion of point core also plays a role in the results of Cerreia-Vioglio, Maccheroni, Marinacci, and Rustichini (2015) on the structure of variational preferences.

Finally, Ghirardato and Siniscalchi (2012) provides a behavioral foundation for the analysis in the present paper. Leveraging the results therein, Appendix A.2 in this paper characterizes differential quasiconcavity at certainty in terms of the agent's preferences.

2 Setup

2.1 Decision-theoretic framework

We consider an Arrow-Debreu economy under uncertainty with finitely many states S, a single good that can be consumed in non-negative quantity, and N consumers. This section and the next focus on the preferences of an individual consumer; to simplify notation, we do not use consumer indices. These will be introduced in Section 4, which deals with efficient and equilibrium allocations in the Arrow-Debreu economy under consideration.

Behavior is described by a preference relation $\succcurlyeq$ over bundles (contingent consumption plans) $f \in \mathbb{R}_+^S$. Our results apply to any preference that admits the following representation: for every $f, g \in \mathbb{R}_+^S$,

$$f \succcurlyeq g \quad \Leftrightarrow \quad I\Big(\big(u(f(s))\big)_{s\in S}\Big) \geq I\Big(\big(u(g(s))\big)_{s\in S}\Big),$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ and $I : u(\mathbb{R}_+)^S \to \mathbb{R}$ satisfy the following assumptions:

- u is continuously differentiable, strictly concave, and strictly increasing;
- I is normalized ($I(1_S\gamma) = \gamma$ for every $\gamma \in \mathbb{U}$), locally Lipschitz and strongly monotonic (that is, $f \geq g$ and $f \neq g$ imply $f \succ g$).[3]

Henceforth, we denote the representation of the preference $\succcurlyeq$ simply by (I, u).[4]

Ghirardato and Siniscalchi (2012) argue that most parametric models of ambiguity-sensitive preferences admit a representation (I, u) where I is locally Lipschitz.[5] They also provide an axiomatization of preferences that admit a representation (I, u) where I is normalized and locally Lipschitz, in a setting a la Anscombe and Aumann (1963). In such a setting, strict concavity of u has the usual characterization. Similar results can be obtained in a Savage (1954)-style

[3] I is monotonic if $f \geq g$ implies $f \succcurlyeq g$; it is strictly monotonic if $f(s) > g(s)$ for all s implies $f \succ g$. Many, but not all results in the Appendix hold if I is strictly, but not strongly monotonic.

[4] If a preference admits multiple representations (I, u) that satisfy these assumptions, our results apply to any such representation.

[5] For most preference models, either I is normalized, or else an equivalent, normalized representation can be readily obtained.

setting with rich outcomes, a la Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2003), using subjective rather than lottery mixtures (see also Ghirardato and Pennesi, 2012).

It is convenient to let $\mathbb{U} = u(\mathbb{R}_+)$, and for every $h = (h_1, \ldots, h_S) \in \mathbb{R}^S_+$, denote by $u \circ h$ the vector $\left(u(h_1), \ldots, u(h_S)\right) \in \mathbb{U}^S$. Also, to simplify notation, if Q is any measure (not necessarily a probability measure) on S, then for every $a \in \mathbb{R}^S$, $Q(a) = \sum_s Q(s) a_s$. [Of course here a measure is characterized by a vector in $\mathbb{R}^S$, and sometimes we will treat Q as such.]

2.2 Clarke differential

Given an open subset B of $\mathbb{R}^S$, the **Clarke derivative** of a locally Lipschitz function $J : B \to \mathbb{R}$ at $b \in B$ in the direction $a \in \mathbb{R}^S$ is defined by

$$J^\circ(b; a) \equiv \limsup_{t \downarrow 0, c \to b} \frac{J(c + t a) - J(c)}{t}. \tag{1}$$

The **Clarke differential** of J at $b \in B$ is then

$$\partial J(b) = \{Q \in \mathbb{R}^S : \forall a \in \mathbb{R}^S, Q(a) \leq J^\circ(b; a)\}. \tag{2}$$

If J is monotonic, every element Q of its Clarke differential at any given point is non-negative (Rockafellar, 1980, Theorem 6, Corollary 3).

The function J is **nice** at $b \in B$ if $0_S \notin \partial J(b)$, where $0_S = (0, \ldots, 0) \in \mathbb{R}^S$: this notion is discussed and axiomatized in Ghirardato and Siniscalchi (2012). In particular, if J is monotonic and concave, or if it is translation-invariant, it is nice everywhere in the interior of its domain.

The function J is **regular** at $b \in B$ if its directional derivative

$$J'(b; a) = \lim_{t \downarrow 0} \frac{J(b + t a) - J(b)}{t} \tag{3}$$

is well-defined for all $a \in \mathbb{R}^S$, and coincides with $J^\circ(b; a)$: see Clarke (1983, Def. 2.3.4). If J is continuously differentiable at b, then it is regular there (Clarke, 1983, Corollary to Proposition 2.2.1, and Proposition 2.3.6 (a)).

For the following two definitions, recall that preferences $\succcurlyeq$ are represented by (I, u). First, the **normalized Clarke differential** of I at $h \in \mathbb{U}^S$ is

$$C(h) = \left\{ \frac{Q}{Q(S)} : Q \in \partial I(u \circ h), Q \neq 0_S \right\}. \tag{4}$$

Ghirardato and Siniscalchi (2012) provide behavioral characterizations of the (normalized) Clarke differential in an Anscombe and Aumann (1963) setting.

Second, define $I^u : \mathbb{R}^S_+ \to \mathbb{R}$ by $I^u(f) = I(u \circ f)$ for all $f \in \mathbb{R}^S_+$; thus, $f \succcurlyeq g$ iff $I^u(f) \geq I^u(g)$.[6]

Remark 1 For every $i \in N$, the Clarke differential at $f \in \mathbb{R}^S_{++}$ of I^u is

$$\partial I^u(f) = \left\{ Q^u \in \mathbb{R}^S : \forall h \in \mathbb{R}^S,\ Q^u(h) = \sum_s Q(s) u'(f(s)) h(s) \text{ for some } Q \in \partial I(u \circ f) \right\}.$$

3 The core, local beliefs, and normalized Clarke differentials

We now introduce three sets of measures that will play a role in our analysis of risk sharing. The first is due to RSS. For every bundle $f \in \mathbb{R}^S_+$, let

$$\pi(f) = \{P \in \Delta(S) : \forall g \in \mathbb{R}^S_+,\ I(u \circ g) \geq I(u \circ f) \Longrightarrow P(g) \geq P(f)\}. \tag{5}$$

That is, $\pi(f)$ is the set of prices (normalized to lie in the unit simplex) such that any bundle that is weakly preferred to f is not less expensive than f. This is the usual notion of "quasi-optimality." Alternatively, we can interpret each $P \in \pi(f)$ as representing a risk-neutral SEU preference whose better-than set at f contains the better-than set of $\succcurlyeq$ at f.

RSS interpret $\pi(\cdot)$ as a definition of *local beliefs*. They also introduce a condition, 'Translation invariance at certainty,' that ensures that $\pi(1_S x) = \pi(1_S)$ for all $x > 0$.

The second set of measures of interest is

$$\pi^c(f) = \{P \in \Delta(S) : \forall g \in \mathbb{R}^S_+,\ I(u \circ g) \geq I(u \circ f) \Longrightarrow P(u \circ g) \geq P(u \circ f)\}. \tag{6}$$

Notice that, if one assumes that u is linear, as in RSS (so that I also reflects risk attitudes), then $\pi = \pi^c$. Thus, $\pi(f)$ and $\pi^c(f)$ differ only in that the measures in $\pi(f)$ are effectively risk-adjusted probabilities, whereas the measures in $\pi^c(f)$ are not.

If we restrict attention to constant bundles $f = 1_S x$, for some $x \geq 0$, since I is normalized,

$$\pi^c(1_S x) = \{P \in \Delta(S) : \forall g \in \mathbb{R}^S_+,\ I(u \circ g) \geq u(x) \Longrightarrow P(u \circ g) \geq u(x)\}.$$

[6] In RSS, preferences are assumed to be represented by some functional $J : \mathbb{R}^S_+ \to \mathbb{R}$: $f \succcurlyeq g$ iff $J(f) \geq J(g)$. The functional I^u just defined corresponds to their J.

Fixing the Bernoulli utility u in the representation of $\succcurlyeq$, each element P^* of this set corresponds to an SEU preference $\succcurlyeq^*$ whose better-than set at $1_S x$ contains the better-than set of $\succcurlyeq$ at $1_S x$.[7] GM interpret a probability that satisfies this condition at *all* riskless consumptions as an "ambiguity-neutral" model for $\succcurlyeq$. They then show[8] that the collection of all such ambiguity-neutral models equals the **core** of the preference functional I (given the utility function u):

$$\text{Core } I = \{P \in \Delta(S) : \forall f \in \mathbb{R}_+^S,\ I(u \circ f) \leq P(u \circ f)\}. \tag{7}$$

By analogy, we call the set $\pi^c(1_S x)$, for $x \in \mathbb{R}_+$, the **point core of** $\succcurlyeq$ **at** x (given the utility function u), and interpret its elements as ambiguity-neutral models for $\succcurlyeq$ *at* x.

A preference is **GM-ambiguity-averse** if it has a non-empty core; it is **pointwise ambiguity averse at** $x \in \mathbb{R}_+$ if its point core at x is non-empty; and it is **pointwise ambiguity averse** if this is true at every $x \in \mathbb{R}_+$. Clearly, a GM-ambiguity-averse preference is pointwise ambiguity averse; the converse is not true, as we show in Example 7.

The following result illustrates the relationship between the sets of measures introduced thus far; it is central to our analysis.[9]

Proposition 1

1. *for every* $x > 0$, $\pi^c(1_S x) \subseteq \pi(1_S x)$*;*
2. *for every* $x > 0$*, if* I *is nice at* $1_S u(x)$*, then* $\pi(1_S x) \subseteq C(1_S x)$*;*
3. $\text{Core } I = \bigcap_{x>0} \pi^c(1_S x)$*;*
4. *for every* $x > 0$, $\text{Core } I \subseteq \partial I(1_S u(x))$*; hence,* $\text{Core } I \subseteq C(1_S x)$*.*

We illustrate some of these definitions and results in the following

[7] Remark 4 in the Appendix implies that $\pi^c(1_S x)$ is also the (normalized) Greenberg-Pierskalla differential of I at $1_S u(x)$ (Greenberg and Pierskalla, 1973; Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio, 2008).

[8] See also Proposition 8 in Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2011).

[9] In all parts of this result, the case $x = 0$ is excluded. This is because $\pi(1_S 0) = \pi^c(1_S 0) = \Delta(S)$, and furthermore $\partial I(1_S \gamma)$ is not defined for $\gamma = 0$.

Example 1 Let $S = \{s_1, s_2\}$ and consider the risk-neutral preferences represented by

$$I(h) = \max\left(\left[\frac{1}{2}\sqrt{h_1} + \frac{1}{2}\sqrt{h_2}\right]^2, \epsilon + \min_{p\in[0.3,0.7]}[p h_1 + (1-p)h_2]\right)$$

for some small $\epsilon > 0$. Three indifference curves are depicted in Figure 1 (thick lines). The

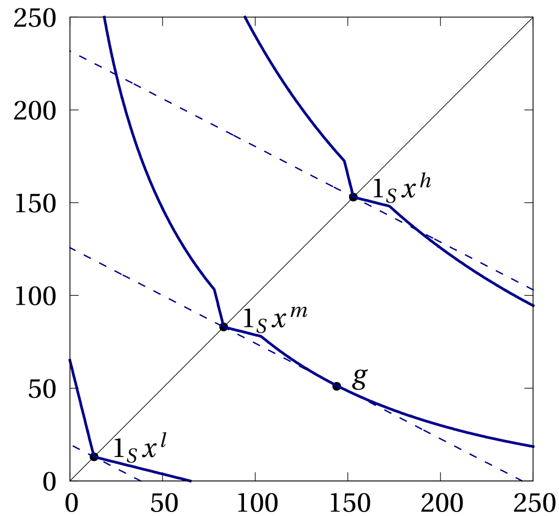

Figure 1: Relationship between the core, Clarke differential, and local beliefs

indifference curves have two features of interest. First, there is a small inward "dent" at certainty; in a neighborhood of the 45° line, this preference coincides with the risk-neutral MEU preference with priors $C = \{P \in \Delta(S) : 0.3 \leq P(s_1) \leq 0.7\}$. Second, away from the certainty line, indifference curves "flatten out" as they move farther away from the origin; thus, sufficiently far away from the origin, preferences are close to being risk-neutral EU with a uniform probability distribution on S, except in a neighborhood of the certainty line.[10]

Since u is the identity, the core of this preference is the set of probabilities that support the indifference curves of I at *every* $1_S x$. There is a single such probability, namely the uniform distribution. For any other probability P there is a sufficiently high prize $x > 0$ such that the line with slope determined by P and going through $1_S x$ intersects the indifference curve of I going through $1_S x$.

[10]For bundles g with high values of one of the coordinate and low values of the other, the preference again coincides with MEU (not shown in Figure 1). This is immaterial for the present purposes.

Next, consider the local belief sets $\pi(1_S x)$ and the point cores $\pi^c(1_S x)$. Since u is the identity, these sets coincide and are equal to the collection of probabilities that induce supporting lines at $1_S x$. Clearly, the core of I—the uniform distribution—is included in each $\pi^c(1_S x)$. However, the sets $\pi^c(1_S x)$ contain additional points. Furthermore, these sets are not constant: they shrink as x increases. For instance, the thin lines in Figure 1 are the level curves of probability distribution P that supports the indifference curve at $1_S x^m$, and hence belongs to $\pi_1^c(1_S x^m)$. Furthermore, since P is also tangent to the same indifference curve at the bundle $g \neq 1_S x^m$, any probability distribution that puts more weight on the vertical coordinate (and hence induces flatter level curves) cannot belong to $\pi^c(1_S x^m)$. However, by inspecting the level curves of P going through $1_S x^\ell$ and $1_S x^h$ respectively it is apparent that (i) there are probabilities $P' \in \pi^c(1_S x^l)$ that induce flatter level curves than P, and (ii) P itself does *not* belong to $\pi^c(1_S x^h)$.

Finally, since I behaves like a MEU preference with priors C around certainty, we have $C(1_S x) = \partial I(1_S u(x)) = C$ for all $x > 0$. In particular, this functional is nice at certainty, so part 2 of Proposition 1 applies. Note however that, for this preference, Core $I \subsetneq \pi(1_S x) = \pi^c(1_S x) \subsetneq C(1_S x)$ for every $x > 0$. $\square$

Example 1 shows that the sets of measures defined above may differ. We now introduce a sufficient condition under which these sets coincide. Consider the following definition.[11]

Definition 1 The functional I is **differentially quasiconcave at** $b \in (\mathrm{int}(\mathbb{U}))^S$ if

$$\forall a \in \mathbb{U}^S, \qquad I(a) \geq I(b) \implies \forall Q \in \partial I(b), \quad Q(a-b) \geq 0. \tag{8}$$

The functional I satisfies **differential quasiconcavity at certainty (DQC)** if it is differentially quasiconcave at $1_S \gamma$ for all $\gamma \in \mathrm{int}(\mathbb{U})$.

As noted in the Introduction, a behavioral characterization of differential quasiconcavity can be provided via Theorem 7 in Ghirardato and Siniscalchi (2012): see Appendix A.2.

The intuition for this definition is sharpest in case I is continuously differentiable at a point $b = u \circ g$, in which case the Clarke differential equals the gradient of I at b. In this case, I is

[11] $\mathrm{int}(\mathbb{U})$ denotes the interior of $\mathbb{U}$.

differentially quasiconcave at b if, whenever a bundle f (having utility profile a) is weakly preferred to g, then moving from g in the direction of f by a small (infinitesimal) amount is also beneficial. Proposition 3.1 in Penot and Quang (1997) implies that a continuous and strictly monotonic function is quasiconcave *if and only if* it satisfies Eq. (8) everywhere on its domain. The key observation is that condition DQC requires that Eq. (8) hold *only at certainty*. This allows for violations of quasiconcavity elsewhere on its domain, as the following example illustrates. We shall also demonstrate in Section 5 that such violations can accommodate interesting patterns of behavior.

Example 2 Let $S = \{s_1, s_2\}$ and consider the risk-neutral VEU preferences defined by

$$I(h) = \frac{1}{2}(h_1 + h_2) - \max\left(\log\left(1 + \frac{1}{4}(h_1 - h_2)^2\right), \left|\frac{1}{2}\theta(h_1 - h_2)\right|\right).$$

At each point $1_S x$, the upper-contour sets of this preference are contained in the upper-contour sets of the risk-neutral MEU preference characterized by the priors $C = \{P \in \Delta(S) : \frac{1}{2}(1+\theta) \geq P(\{s_1\}) \geq \frac{1}{2}(1-\theta)\}$; denote the functional representation of this MEU preference by J. For h sufficiently close to the 45° line, and for h sufficiently far from it, $I(h) = J(h)$; for bundles h at an intermediate distance from the 45° line, the indifference curves of I are bent inward, so $I(h) \leq J(h)$. See Figure 2.

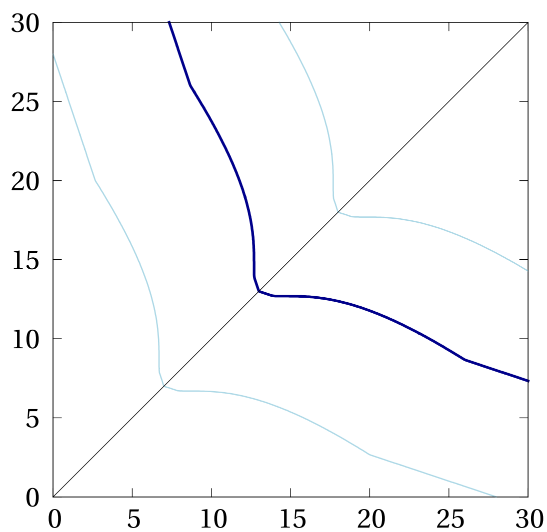

Figure 2: Indifference curves of a non-smooth, non-convex preference ($\theta = \frac{1}{2}$)

This preference is thus neither convex nor smooth. Its core is C, which is also its Clarke differential at any point on the 45° line. It then follows from Proposition 1 parts 1, 2 and 3 that the local belief sets and point cores at any constant bundle also coincide with C. Condition DQC holds; to see this, note that, if $I(h) \geq I(x)$, then also $J(h) \geq J(x)$; since J is concave, Proposition 4 below implies that it satisfies condition DQC, so $Q(1_S x; h-x) \geq 0$ for every $Q \in \partial J(1_S x) = \partial I(1_S x)$. □

On the other hand, the preferences in Example 1 do not satisfy condition DQC. For instance, consider the point g in Figure 1: since it lies on the indifference curve going through $1_S x^m$, it is indifferent to it, but if $Q \in \partial I(1_S x^m)$ is the probability that assigns weight 1 to the vertical coordinate, clearly $Q(g - 1_S x^m) < 0$.

The following result shows that condition DQC provides a tight connection between the normalized differentials $C(1_S x)$ and the sets $\pi(1_S x)$ and $\pi^{\mathrm{c}}(1_S x)$:

Proposition 2 *If DQC holds, then for every $x > 0$, $C(1_S x) \subseteq \pi^{\mathrm{c}}(1_S x)$, and therefore $C(1_S x) \subseteq \pi(1_S x)$. If in addition I is nice at $1_S u(x)$, then $C(1_S x) = \pi^{\mathrm{c}}(1_S x) = \pi(1_S x)$.*

Recall that, in Example 1, the set $C(1_S x)$ is strictly larger than $\pi(1_S x)$; as noted above, the preferences therein violate condition DQC. On the other hand, the preference in Example 2 is not quasiconcave, but it does satisfy DQC and niceness (because in a neighborhood of certainty it coincides with a MEU preference). Consistently with Proposition 2, the normalized Clarke differential and local belief sets coincide (and also happen to be constant across all $x > 0$).

Condition DQC also provides a tight connection between the core of each I and its normalized Clarke differentials at certainty. Specifically, if DQC holds, then the core of I coincides with the intersection of these Clarke differentials. Conversely, if the Clarke differential at some $x > 0$ is contained in the core of I, then I is differentially quasiconcave at x.

Proposition 3

1. *If DQC holds, then $\bigcap_{x>0} C(1_S x) \subseteq \mathrm{Core}\ I$ (so, by Proposition 1, $\bigcap_{x>0} C(1_S x) = \mathrm{Core}\ I$);*

2. *For every $x > 0$, if $C(1_S x) \subseteq \mathrm{Core}\ I$, then I is differentially quasiconcave at $1_S u(x)$.*

3. *If $\bigcup_{x>0} C(1_S x) \subseteq \mathrm{Core}\ I$, then DQC holds.*

Condition DQC always holds in two important cases (the first was already noted above):

Proposition 4 *DQC holds if one of the conditions below is satisfied:*

1. *I is quasiconcave, or*

2. Core $I \neq \emptyset$ *and I is regular at every* $1_S\gamma$, $\gamma > 0$.

Thus, condition DQC holds for all preference models considered in RSS, because they all satisfy convexity. Furthermore, recall that if a function is continuously differentiable at a point, it is regular there; hence, condition DQC also applies to all smooth representations of GM-ambiguity-averse preferences. However, as Example 2 suggests, this condition allows for non-differentiabilities, in addition to non-convexities.

Finally, we observe that pointwise ambiguity aversion is (essentially) *necessary* for condition DQC to hold. By Proposition 2, the normalized Clarke differential at $1_S x$ is contained in the point core at x. So long as the non-normalized Clarke differential at $1_S x$ does not consist solely of the zero measure 0_S (in particular, if I is nice at $1_S x$), the set $C(1_S x)$ is non-empty, and therefore so is the point core at x. We record this fact for future reference:

Remark 2 If DQC holds then, for every $x \in \mathbb{R}_+$, $C(1_S x) \neq \emptyset$ implies that $\succcurlyeq$ is pointwise ambiguity averse at x.

4 Risk Sharing

4.1 Notation and preliminaries

An economy is a tuple $(N, (\succcurlyeq_i, \omega_i)_{i \in N})$, where N is the collection of agents, and for every i, agent i is characterized by preferences $\succcurlyeq_i$ over $\mathbb{R}^S_+$ and has an endowment $\omega_i \in \mathbb{R}^S_+$. As in RSS, we assume that there is no aggregate uncertainty: formally, $\sum_i \omega_i = 1_S \bar{\omega}$ for some $\bar{\omega} > 0$.

An *allocation* is tuple $(f_1, \dots, f_N)$ such that $f_i \in \mathbb{R}^S_+$ for every $i \in N$; as usual f_i is the contingent-consumption bundle assigned to agent i. The allocation $(f_1, \dots, f_N)$ is *feasible* if $\sum_i f = \sum_i \omega_i$; it is a *full-insurance allocation* if, for every consumer i, $f = 1_S x$ for some $x \in \mathbb{R}_+$; it is *Pareto-efficient* if it is feasible, and there is no other feasible allocation $(g_1, \dots, g_N)$ such that $g \succcurlyeq f$ for all i, and $g_j \succ_j f_j$ for some j.

It is useful to state the main result of Rigotti et al. (2008) for convex preferences.[12]

Theorem 5 (cf. Rigotti et al. (2008), Proposition 9) *In addition to the maintained assumptions in Sec. 2, suppose that every $\succcurlyeq_i$ is strictly convex,[13] and that $\pi_i(1_S x) = \pi_i(1_S)$ for every $x > 0$. Then the following are equivalent:*

(i) There exists an interior, full-insurance Pareto-efficient allocation;

(ii) Every Pareto-efficient allocation is a full-insurance allocation;

(iii) Every feasible, full-insurance allocation is Pareto-efficient;

(iv) $\bigcap_i \pi_i(1_S) \neq \emptyset$.

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) hold for general preferences.[14] However, the implication (i) $\Rightarrow$ (iv) uses the standard Second Welfare Theorem, which requires convexity. The argument for (iv) $\Rightarrow$ (ii) also invokes convexity.

4.2 Necessary and sufficient conditions for efficiency

We now state two results that are reminiscent of the implications (i) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (ii) of Theorem 5, but do not require convexity. These results are economically interesting in their own right.

The first result generalizes the standard result that smooth indifference curves must be tangent at any interior Pareto-efficient allocation. With convex preferences, the common slope at the point of tangency determines a supporting price vector; as we discuss momentarily, a "local price vector" is also identified in the non-convex, non-smooth case, though the sense in which it "supports" the allocation is more delicate (see below). Thus, the following result can also be viewed as a local version of the Second Welfare Theorem.

[12] Strictly speaking, the assumptions in Theorem 5 are slightly stronger than those in RSS's Proposition 9. Specifically, we maintain the assumption that each I_i is locally Lipschitz; RSS only assume continuity. We retain all our assumptions to streamline the exposition. Also note that all the parametric representations considered in RSS are concave, and hence locally Lipschitz.

[13] That is, for $f, g \in \mathbb{R}^S_+$ with $f \neq g$, $f \succcurlyeq_i g$ implies $\alpha f + (1-\alpha)g \succ_i g$ for all $\alpha \in (0,1)$.

[14] For (ii) $\Rightarrow$ (iii), the key step is in Remark 6, which follows from standard results. See also the proof of Proposition 7.

Proposition 6 *Let* $(f_i)_{i\in N}$ *be an allocation such that each functional* I_i *is nice at* $u_i \circ f_i$*. If* $(f_i)_{i\in N}$ *is Pareto-efficient, then there exists a price vector* $p \in \mathbb{R}^S_+ \setminus \{0\}$ *and, for each* $i \in N$*, scalars* $\lambda_i > 0$ *and vectors* $Q^u_i \in \partial I^u_i(f_i)$ *such that* $p = \lambda_i Q^u_i$ *for every* i*. If in addition* $(f_i)_{i\in N}$ *is a full-insurance allocation, then for each* $i \in N$ *there are scalars* $\mu_i > 0$ *and vectors* $Q_i \in \partial I_i(u_i \circ f_i)$ *such that* $p = \mu_i Q_i$*; therefore,* $\bigcap_{i\in N} C_i(f_i) \neq \emptyset$*.*

The key step in the proof of the first claim is provided by Bonnisseau and Cornet (1988), who show that, under the stated assumptions, there is a vector p such that $-p$ lies in the intersection of the Clarke normal cones of the upper contour set of I^u_i at the bundle f_i (see the Appendix for a precise statement and a definition of the required terms). If preferences are convex, this set coincides with the normal cone of the upper contour set of I^u_i at f_i in the sense of convex analysis. (Indeed Clarke's notion of normal cone is meant as a generalization of the normal cone of convex analysis.) This suggests interpreting p as a "local price vector."

The second claim states that, if the Pareto-efficient allocation $(f_i)_{i\in N}$ is a *full-insurance* allocation, then the normalized Clarke differentials of the functionals I_i themselves have non-empty intersection. For arbitrary Pareto-efficient allocations, this conclusion only applies to the (normalized) Clarke differentials of the composite functional I^u_i.

Thus, if a full-insurance allocation is Pareto-efficient, then (up to rescaling) the Clarke differentials of the agents' functionals I_i at that allocation have non-empty intersection. One may then wonder if the converse is also true: is it the case that, if the normalized Clarke differentials at some full-insurance allocation intersect, that allocation is Pareto-efficient? The next example shows that the answer is negative.

Example 3 Interpret Figure 1 as an Edgeworth box: agent 1 has preferences inducing the thick indifference curves, whereas agent 2 has risk-neutral preferences inducing the thin lines as indifference curves; of course, for agent 2, utility increases in the south-western direction. (Both consumers could be made strictly risk-averse without changing the analysis.)

Notice that the allocation $\left(1_S x^h, 1_S(\bar{\omega} - x^h)\right)$ provides full insurance. The normalized Clarke differential of I_1 at $1_S x^h$ contains that of I_2, which coincides with the probability P representing 2's preferences. However, this allocation is not Pareto-efficient.

Note also that the allocation $(g, 1_S\bar{\omega} - g)$ is Pareto-efficient, but does not provide full insurance. $\square$

The intuition behind this example is as follows. Clarke differentials provide information about the local behavior of preferences (again, see Ghirardato and Siniscalchi, 2012, for a precise characterization). If the normalized Clarke differentials have non-empty intersection at an allocation, then *locally* there are no mutually beneficial trades. However, the notion of Pareto efficiency involves more than just local comparisons: there may be Pareto-superior allocations sufficiently far from the given one. This is indeed the case for the allocation $\left(1_S x^h, 1_S(\bar{\omega} - x^h)\right)$. Thus, the example suggests that, in order to obtain a converse to Proposition 6, one needs to refer to a set of priors that also conveys *global* information about preferences.

The second result we present shows that the point core sets $\pi^c(\cdot)$ do provide the required information.

Proposition 7 *Assume that, for every feasible, full-insurance allocation* $(1_S x_1, \ldots, 1_S x_N)$*, it is the case that* $\bigcap_i \pi_i^c(1_S x_i) \neq \emptyset$*. Then, a feasible allocation is Pareto-efficient if and only if it provides full insurance. Moreover, such an allocation is a competitive equilibrium allocation (with transfers).*

In words, if, at every feasible, full-insurance allocation, agents share at least one ambiguity-neutral model, then the set of Pareto-efficient allocations coincides with the set of feasible full-insurance allocations. Recall that, in Example 3 the condition $\bigcap_i \pi_i^c(1_S x_i) \neq \emptyset$ fails at $(1_S x^h, 1_S(\bar{\omega} - x^h))$, and in that economy there is both a full-insurance allocation that is not Pareto-efficient and a non-full-insurance allocation that is Pareto-efficient.

A necessary condition for $\bigcap_i \pi_i^c(1_S x_i) \neq \emptyset$ is, of course, that every agent i's point core at x_i be non-empty. Thus, the assumption in Proposition 7 implies that each agent is pointwise ambiguity averse at every feasible, full-insurance allocation.

In light of part 3 of Proposition 1, the condition $\bigcap_i \pi_i^c(1_S x_i) \neq \emptyset$ will be satisfied *a fortiori* at every feasible, full-insurance allocation if the cores of the representing functionals I_i have non-empty intersection. We then have

Corollary 8 *If* $\bigcap_i \text{Core } I_i \neq \emptyset$*, then a feasible allocation is Pareto-efficient if and only if it* *provides full insurance. Moreover, such an allocation is a competitive equilibrium allocation (with transfers).*

Thus, under the conditions of Proposition 7 or Corollary 8, not only are full-insurance allocations efficient: they are the *only* efficient allocations. This generalizes the standard result that, if agents share a common prior, then an interior allocation is Pareto-efficient if and only if it provides full insurance.

The key intuition behind Proposition 7 is that, under the stated assumptions, if $(f_1, \ldots, f_N)$ is a feasible allocation that does *not* provide full insurance, and c_i is the certainty equivalent of f_i for each agent i, then $\sum_i c_i < \bar{\omega}$. It is then immediate to construct a full-insurance allocation that Pareto-dominates the initial one.

In the case of SEU preferences with a common prior P, the inequality $\sum_i c_i < \bar{\omega}$ follows from the basic fact that, for a strictly risk-averse individual, the expected value of a non-constant act is strictly greater than its certainty equivalent. Formally, in our setting, $P(f_i) > c_i$; then, since $(f_1, \ldots, f_N)$ is feasible, we get $\sum_i c_i < \sum_i P(f_i) = P(\sum_i f_i) = \bar{\omega}$.

Now consider the case in Corollary 8 and let $P \in \bigcap_i \text{Core } I_i$. Whenever f_i is non-constant, strict concavity of u_i implies that $u_i(P(f_i)) > P(u_i \circ f_i)$; furthermore, by the definition of the core, $P(u_i \circ f_i) \geq I_i(u_i \circ f_i) = u_i(c_i)$. Thus, we again conclude that $u_i(P(f_i)) > u_i(c_i)$, i.e., $P(f_i) > c_i$. In other words, ambiguity aversion reinforces the effects of risk aversion. The general case in Proposition 7 uses a different argument but relies on a similar intuition.

Proposition 7 also strengthens the conclusion of Proposition 6. If the point cores intersect, one can find a price vector that supports any feasible, full-insurance allocation as a competitive equilibrium in the usual sense. By way of contrast, the vector p identified in Proposition 6 is a supporting price only in the local sense discussed above.

In Example 3, there is a feasible, full-insurance allocation that is not Pareto-efficient, and a Pareto-efficient allocation that does not provide full insurance. The preferences considered there do not satisfy the condition in Proposition 7.[15] The following example instead shows

[15]Consider the allocation $(1_S x^h, 1_S(\bar{\omega} - x^h))$ in Figure 1. At this allocation, $\pi_2^c(1_S(\bar{\omega} - x^h))$ consists of the probability P that represents 2's preferences, but P clearly is not an element of $\pi_1^c(1_S x^h)$: there are bundles that agent 1 strictly prefers to $1_S x^h$, and that an agent with risk-neutral EU preferences with prior P would consider worse that $1_S x^h$. (This continues to be the case if preferences are perturbed slightly so that both consumers have a

that this condition, while sufficient, is *not* necessary for Pareto-efficient and full-insurance allocations to coincide.

Example 4 Let $S = \{s_1, s_2\}$. Assume that consumer 2 has EU preferences, with a prior P that assigns probability 0.4 to state s_1 (on the horizontal axis) and power utility $u(x) = x^{0.2}$. Consumer 1 has preferences represented by

$$I_1(h) = \max\left(\frac{1}{2}h_1 + \frac{1}{2}h_2, \delta + \min_{p\in[0,1]}[p h_1 + (1-p)h_2]\right).$$

Thus, consumer 1's preferences are risk-neutral EU, with a uniform prior, except within δ of the certainty line. The value of δ is chosen so that, given the curvature of 2's utility function, there is no tangency anywhere except at certainty; see Figure 3. Then, in this economy, a fea-

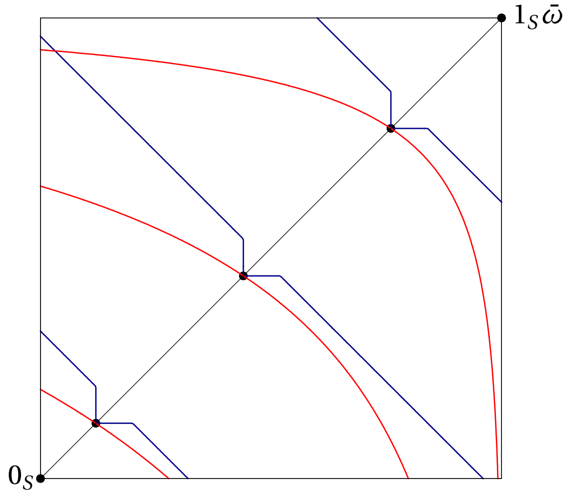

Figure 3: Relationship between the core, Clarke differential, and local beliefs

sible allocation is Pareto-efficient if and only if it provides full insurance. Both preferences are GM-ambiguity-averse. In particular, as in Example 1, the core of I_1 consists solely of the uniform measure, whereas the core of 2's EU preference functional is $\{P\}$. Thus, the intersection of the cores is empty, even though the sets of Pareto-efficient and full-insurance allocations

strictly concave utility function.) Thus, the intersection of the point cores $\pi_i^c(\cdot)$ at the allocation $(1_S x^h, 1_S(\bar{\omega} - x^h))$ is empty.

coincide. Note also that the normalized Clarke differentials are constant at certainty: they equal $\Delta(S)$ for consumer 1, and $\{P\}$ for consumer 2.

The sufficient condition in Proposition 7 is violated. For consumer 2, the point core is equal to $\pi_2^c(1_S x_2) = \{P\}$ at every $x_2 \geq 0$; however, $P \notin \pi_1^c(1_S x_1)$ for x_1 sufficiently large. For instance, consider the rightmost point of tangency on the certainty line; denote the corresponding feasible allocation by (x_1, x_2). Since P is not uniform, if the dent in 1's preferences at certainty (which depends upon the parameter δ) is sufficiently small, the tangent to 2's indifference curve at (x_1, x_2) will eventually cross 1's indifference curve going through that point. Therefore, P does not belong to the point core $\pi_1(1_S x_1)$ at x_1. □

Finally, Example 8 in Section 5 describes an economy in which the sufficient condition of Proposition 7 is satisfied, but that of Corollary 8 is not. Notably, both preferences in that economy are convex.

4.3 Closing the gap

The preceding results show that the condition that $\bigcap_i C_i(1_S x_i) \neq \emptyset$ is necessary for the full-insurance allocation $(1_S x_1, \ldots, 1_S x_N)$ to be Pareto-efficient (Proposition 6), but it is not sufficient (Example 3). On the other hand, the condition that $\bigcap_i \pi_i^c(1_S x_i) \neq \emptyset$ on the certainty line (a fortiori, the condition $\bigcap_i \text{Core } I_i \neq \emptyset$) is sufficient for full-insurance allocations to be the *only* Pareto-efficient ones (Proposition 7), but it is not necessary (Example 4). This points to a gap between Propositions 6 and 7.

It turns out that differential quasiconcavity at certainty provides a simple way to close this gap. The key insight is that, under Condition DQC (and niceness), the point core $\pi_i^c(1_S x_i)$ and the normalized Clarke differential $C_i(1_S x_i)$ coincide for every $x_i > 0$. [Items (ii)–(iv) in the statement below correspond to items (ii)–(iv) in Theorem 5; there is no condition corresponding to (i). See Example 7 for further discussion.]

Theorem 9 *Assume that, for every $i \in N$, DQC holds and I_i is nice at $1_S u_i(x_i)$ for every $x_i > 0$. Then the following are equivalent:*

(ii) Every Pareto-efficient allocation is a full-insurance allocation;

(iii) Every feasible, full-insurance allocation is Pareto-efficient;

(iv) For every feasible, full insurance allocation $(1_S x_1, \ldots, 1_S x_N)$*,*

$$\bigcap_i \pi_i^{c}(1_S x_i) = \bigcap_i \pi_i(1_S x_i) = \bigcap_i C_i(1_S x_i) \neq \emptyset.$$

Furthermore, under the above equivalent conditions, every interior, feasible full-insurance allocation is a competitive equilibrium with transfers.

Proof: As just noted, for every i and $x_i > 0$, since DQC holds and I_i is nice at $1_S u_i(x_i)$, Proposition 2 implies that $\pi_i^{c}(1_S x_i) = \pi_i(1_S x_i) = C_i(1_S x_i)$.

The implication (ii) ⇒ (iii) is standard. Now assume (iii) and fix a feasible, full-insurance allocation $(1_S x_1, \ldots, 1_S x_N)$. Then this allocation is Pareto-efficient. By Proposition 6, $\bigcap_i C_i(1_S x_i) \neq \emptyset$; since $\pi_i^{c}(1_S x_i) = \pi_i(1_S x_i) = C_i(x_i)$ for all i, (iv) holds. Finally, assume (iv): then, by Proposition 7, every Pareto-efficient allocation is a full-insurance allocation, i.e., (ii) holds. ■

Theorem 9 is a counterpart to RSS's result (Theorem 5 in this paper) for preferences that are not necessarily convex, i.e., uncertainty-averse. The assumptions only imply that every agent is pointwise ambiguity averse (see Remark 2). Condition DQC plays the same role in Theorem 9 as convexity does in RSS's result: it allows one to derive a "global" conclusion about Pareto efficiency from a "local" assumption about the intersection of the point cores at certainty. Yet, as discussed in Section 3, while convexity implies DQC, the latter allows for significant departures from convexity (see also Example 5 in Section 5).

There are two additional differences between RSS's result and Theorem 9. On one hand, RSS assume that the local belief sets $\pi_i(1_S x_i)$ are constant at certainty; there is no corresponding assumption in Theorem 9 (cf. Example 8 in Section 5). On the other hand, the condition in item (iv) of Theorem 5 (RSS's result) involves agents' preferences alone, whereas condition (iv) in Theorem 9 involves both preferences and endowments—agents' point core sets must have a non-empty intersection at all *feasible* allocations.

It turns out that, if we adopt a counterpart to RSS's assumption that local beliefs at certainty are constant, then we can similarly state condition (iv) purely in terms of preferences. Consider the following definition:

Definition 2 Let $\succcurlyeq$ be represented by (I, u). Then $\succcurlyeq$ satisfies **condition IDC** (Invariant normalized Differentials at Certainty) if $C(1_S x) = C(1_S)$ for all $x > 0$.

Appendix A.2 characterizes this condition in terms of preferences. Under DQC and niceness, the normalized Clarke differential $C_i(1_S x)$ at every $x > 0$ coincides with the local belief set $\pi_i(1_S x)$. Thus, under these assumptions, Condition IDC has the same implication as RSS's axiom TIC, namely that the sets $\pi_i(1_S x)$ do not depend upon x.

We also note that, under DQC and niceness, condition IDC also implies that preferences are GM-ambiguity-averse (rather than just pointwise ambiguity averse).[16]

Theorem 10 *Assume that, for every $i \in N$, I_i is nice at $1_S u_i(x)$ for every $x > 0$, and that conditions DQC and IDC hold. Then the following are equivalent:*

(i) There exists an interior, full-insurance Pareto-efficient allocation;

(ii) Every Pareto-efficient allocation is a full-insurance allocation;

(iii) Every feasible, full-insurance allocation is Pareto-efficient;

(iv) $\bigcap_i \mathrm{Core} I_i = \bigcap_i \pi^c(1_S) = \bigcap_i \pi_i(1_S) = \bigcap_i C_i(1_S) \neq \emptyset$.

Furthermore, under the above equivalent conditions, every interior, feasible full-insurance allocation is a competitive equilibrium with transfers.

Proof: It follows by Proposition 2 that, under DQC, IDC, and niceness, $\pi_i(1_S x) = \pi_i^c(1_S x) = C_i(1_S x) = C_i(1_S) = \pi_i^c(1_S) = \pi_i(1_S)$. By Proposition 1 part 3 and IDC, Core $I_i = \bigcap_{x>0} \pi_i^c(1_S x) = \pi_i^c(1_S)$. Therefore, $\bigcap_i \mathrm{Core} I_i = \bigcap_i \pi_i^c(1_S) = \bigcap_i \pi_i(1_S) = \bigcap_i C_i(1_S)$, and furthermore the condition in (iv) of Theorem 10 is equivalent to the condition in (iv) of Theorem 9. Hence, the equivalence of (ii), (iii) and (iv) follows from Theorem 9.

For (i) $\Rightarrow$ (iv), if $(1_S x_1, \dots, 1_S x_n)$ is an interior, full-insurance Pareto-efficient allocation, since each I_i is nice at $1_S u_i(x_i)$, Proposition 6 implies that $\bigcap_i C_i(1_S x_i) \neq \emptyset$, so (iv) holds by the equalities established above. Finally, (iii) $\Rightarrow$ (i) is immediate. ■

To sum up, Theorem 10 provides a more direct counterpart to RSS's result (Theorem 5 in this paper).[17] On the other hand, Theorem 9 puts the emphasis on the role of differential

[16]By Proposition 1 part 3, Core $I_i = \bigcap_{x>0} \pi_i^c(1_S x) = \pi_i^c(1_S)$; by Proposition 2, $\pi_i^c(1_S) = C_i(1_S)$; under niceness, $C_i(1_S) \neq \emptyset$, so Core $I_i \neq \emptyset$.

[17]Another difference between Theorems 10 and 5 is the assumption of niceness at certainty. If preferences have a concave representation, niceness is automatically satisfied, so for such preferences Theorem 5 follows

quasiconcavity at certainty in risk sharing. In particular, risk sharing does not require that preferences satisfy condition IDC (or RSS's TIC). We discuss an alternative to conditions DQC and IDC in Section 6 below.

5 More Examples

In this section we provide four additional examples that illustrate our results. Example 5 shows that our Theorem 10 can accommodate behaviorally interesting preferences that are not covered by prior results on risk sharing. Example 6 considers the special case of invariant biseparable preferences (Ghirardato et al., 2004). Example 7 illustrates a novel implication of our results for convex preferences that are nice at certainty: a non-empty intersection of the point cores is *necessary* for risk sharing. Finally, Example 8 illustrates how risk sharing may obtain when condition IDC (or RSS's Translation Invariance at Certainty) is violated; in the convex, non-pathological economy described therein, Theorem 9 (and Proposition 7) apply, but Theorem 10 (and Corollary 8) do not.

Example 5 (Smooth VEU preferences) A convenient class of preferences that satisfies all conditions of Theorem 10, but is not necessarily covered by RSS's result, is the family of VEU preferences that are smooth (hence, regular) and GM-ambiguity-averse, but not necessarily convex. These preferences admit a representation (I, u) with[18]

$$I(a) = P(a) + A\big(P(\zeta_0 a), \dots, P(\zeta_{J-1} a)\big),$$

where $P \in \Delta(S)$, $0 \leq J \leq |S|$, each $\zeta_j \in \mathbb{R}^S$ (an *adjustment factor*) satisfies $P(\zeta_j) = 0$, and $A : \mathbb{R}^J \to \mathbb{R}$ (the *adjustment function*) is continuously differentiable and satisfies $A(\phi) = A(-\phi)$ for all $\phi \in \mathbb{R}^J$, and $A \leq 0$. To ensure strict monotonicity, additionally assume that $P(\{s\}) > 0$ for all s and, for all $a \in \mathbb{U}^S$ and $s \in S$, $1 + \sum_{0 \leq j < J} \frac{\partial A}{\partial \phi_j}(P(\zeta_0 a), \dots, P(\zeta_{J-1} a))\zeta_j(s) > 0$. Note that I is translation-invariant, so IDC holds; it is GM-ambiguity-averse, so Core $I \neq \emptyset$; and it is regular at certainty, so by Proposition 4 it satisfies DQC. Furthermore, by Propositions 1 and 2, and

directly from Theorem 10.

[18] If $a, b : \mathbb{U}^S \to \mathbb{R}$, "$ab$" denotes the function that assigns the value $a(s)b(s)$ to each state s.

straightforward calculations (in the Appendix),

$$\text{Core } I = \pi(1_S) = \pi^c(1_S) = C(1_S) = \{P\}.$$

Therefore, in an economy where all agents have preferences satisfying these conditions, risk-sharing obtains if and only if agents have the same baseline prior P (but possibly different adjustment factors and functions).

Smooth, GM-AA VEU preferences can provide a tractable model of behavior that can be deemed averse to ambiguity, even though it is inconsistent with convexity of preferences. To illustrate, we show that they can accommodate the modal preferences in the "reflection example" of Machina (2009) (see also Baillon et al., 2011). Let $S = \{s_1, s_2, s_3, s_4\}$ and assume that the events $\{s_1, s_2\}$ and $\{s_3, s_4\}$ are unambiguous and equally likely, but no further information is provided as to the relative likelihood of s_1 vs. s_2 and s_3 vs. s_4. Furthermore, the draw of s_1 vs. s_2 and s_3 vs. s_4 are perceived as being independent. Consider the bets in Table 1.

	s_1	s_2	s_3	s_4
f^1	\$4,000	\$8,000	\$4,000	\$0
f^2	\$4,000	\$4,000	\$8,000	\$0
f^3	\$0	\$8,000	\$4,000	\$4,000
f^4	\$0	\$4,000	\$8,000	\$4,000

Table 1: Machina's reflection example. Reasonable preferences: $f^1 \prec f^2$ and $f^3 \succ f^4$

Machina (2009) argues on the basis of symmetry considerations that the preference ranking $f^1 \prec f^2$ and $f^3 \succ f^4$ is plausible and consistent with aversion to ambiguity; L'Haridon and Placido (2010) verify that these rankings do occur in an experimental setting. However, Baillon et al. (2011) show that preference models that satisfy convexity cannot accommodate this behavior while respecting natural probabilistic formulations of the noted symmetry and independence assumptions. We now demonstrate that, by way of contrast, smooth, GM-AA VEU preferences can do so. A similar example is provided in Siniscalchi (2009), but the VEU preferences described there are not smooth and violate DQC.

Assume a uniform baseline prior P and two adjustment factors $\zeta_0, \zeta_1 \in \mathbb{R}^S$:

$$\zeta_0 = [1, -1, 0, 0] \qquad \text{and} \qquad \zeta_1 = [0, 0, 1, -1].$$

Act	$P(\zeta_0 u \circ f^k)$	$P(\zeta_1 u \circ f^k)$	Adjustment (omitting $\frac{1}{2}\theta$)
f^1	$\alpha - 1$	α	$-\log(1+\theta^{-1}(\alpha-1)^2) - \log(1+\theta^{-1}\alpha^2)$
f^2	0	1	$-\log(1+\theta^{-1})$
f^3	-1	0	$-\log(1+\theta^{-1})$
f^4	$-\alpha$	$1-\alpha$	$-\log(1+\theta^{-1}\alpha^2) - \log(1+\theta^{-1}(1-\alpha)^2)$

Table 2: Adjustments

The adjustment function takes the form

$$A(\phi) = A(\phi_0, \phi_1) = -\frac{1}{2}\theta \sum_{j=0,1} \log\left(1 + \frac{\phi_j^2}{\theta}\right)$$

where $\theta \in (0,4)$; note that $\lim_{\theta\to 0} A(\phi) = 0$, so the limiting case $\theta = 0$ corresponds to EU. We verify in Appendix A.6 that this specification of the parameters P, A, ζ_0, ζ_1 yields a strictly monotonic preference, and that higher values of θ correspond to greater ambiguity aversion in the sense of GM. Finally, let $u(0) = 0$, $u(8,000) = 4$, and $u(4,000) = 4\alpha$, for some $\alpha \in (0,1)$.

All four acts $f^1, \dots, f^4$ have the same expected baseline utility: $P(u \circ f^k) = 2\alpha + 1$ for $k = 1, \dots, 4$. Hence, their ranking is entirely determined by the adjustment terms $A(P(\zeta_0 u \circ f^k), P(\zeta_i u \circ f^k))$. These are displayed in Table 2.

In order to generate the preferences $f^1 \prec f^2$, we need to ensure that $A(P(\zeta_0 u \circ f^1), P(\zeta_1 u \circ f^1)) < A(P(\zeta_0 u \circ f^2), P(\zeta_1 u \circ f^2))$. Notice that, since $(\alpha-1)^2 = (1-\alpha)^2$, this will also ensure that $A(P(\zeta_0 u \circ f^3), P(\zeta_1 u \circ f^3)) > A(P(\zeta_0 u \circ f^4), P(\zeta_1 u \circ f^4))$ and therefore $f^3 \succ f^4$, as the adjustments for f^1 and f^2 are the same as the adjustments for f^4 and f^3 respectively. Thus, we require

$$-\log(1+\theta^{-1}(\alpha-1)^2) - \log(1+\theta^{-1}\alpha^2) < -\log(1+\theta^{-1})$$

which, as shown in Appendix A.6, holds iff $0 < \theta < \frac{\alpha(1-\alpha)}{2}$. $\square$

Example 6 (Invariant Biseparable preferences) A preference is *invariant biseparable* (Ghirardato et al., 2004) if its representation (I, u) is such that I is positively homogeneous and translation-invariant on its domain. We now show that MEU preferences are the only invariant biseparable preferences for which the conditions of Theorems 9 or 10 hold. Recall that, similarly, the only invariant biseparable preferences to which the results in RSS apply —i.e., the convex invariant biseparable preferences— are MEU preferences. Thus, both RSS's main risk-sharing result and our results generalize the one in Billot et al. (2000) only insofar as they apply to preferences that do not satisfy either positive homogeneity or translation invariance.

Recall from Ghirardato et al. (2004) that, for an invariant biseparable preference represented by (I, u), the functional I admits a unique extension to all of $\mathbb{R}^S$, and the Clarke differential at zero, i.e., $\partial I(0_S)$, consists of probability measures and coincides with $\partial I(1_S u(x))$ for all $x > 0$. Hence, I is nice at $1_S u(x)$ for every $x > 0$, and condition IDC holds.

Let $C = \partial I(0_S) = \partial I(1_S u(x)) \subseteq \Delta(S)$ for any $x \geq 0$. By Proposition 3, condition DQC holds *if and only if* $C = \bigcap_{x>0} C_i(1_S x) = \bigcup_{x>0} C_i(1_S x) = \text{Core } I$. But by Proposition 16 in Ghirardato et al. (2004), $C = \text{Core } I$ if and only if I is concave, in which case $I(\cdot) = \min_{P \in C} P(\cdot)$.

Hence, an invariant biseparable preference satisfies condition DQC if and only if it is MEU. Equivalently, for an invariant biseparable preference, there is no gap between Propositions 6 and 7 if and only if preferences are in fact MEU. □

Example 7 Let $S = \{s_1, s_2\}$. Agent 1's preferences are represented by the utility function $u(x) = x^{0.6}$ and the differentiable, quasiconcave, but not concave functional

$$I(a) = \frac{1}{2}a_2 + \sqrt{4 + \frac{1}{4}a_2^2 + 2a_1} - 2.$$

Agent 2 has EU preferences, with probability P and utility $u(x) = x^{0.8}$. Figure 4 shows indifference curves for these preferences, drawn as solid blue and red lines respectively. Agent 1's and 2's indifference curves are tangent at the allocation $(1_S x^l, 1_S(\bar{\omega} - x^l))$; their common slope there equals the slope of the two parallel, straight purple lines. (Thus, this slope identifies P.)

The figure shows that the slope of 1's indifference curves at $1_S x^l$ and $1_S x^h$ is different; indeed, it may be verified that the slope of the indifference curve of I^u at $1_S x$ is $-\frac{2}{u(x)+2}$, which is non-zero and strictly decreasing in x. Hence (cf. Remark 1), I is nice at certainty. Furthermore, since I is quasiconcave, it satisfies DQC by Proposition 4, and therefore by Proposition 2, $\pi_1^c(1_S x) = \pi_1(1_S x) = C_1(1_S x)$ for all $x > 0$. On the other hand, since agent 2's preferences are consistent with EU, $\pi_2(1_S x) = \pi_2^c(1_S x) = C_2(1_S x) = \text{Core } I_2 = \{P\}$.

From a decision-theoretic perspective, we observe that agent 1's preference is convex (hence uncertainty-averse in the sense of Schmeidler, 1989), as well as pointwise ambiguity-averse,

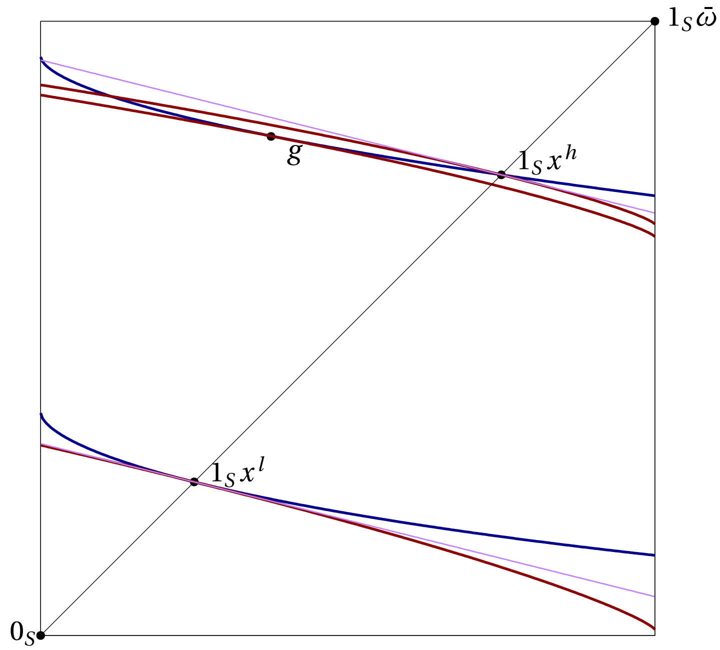

Figure 4: A convex preference with empty core.

but not GM-ambiguity-averse.[19] To see this, note that, by Proposition 1 the core must be contained in the sets $\pi_1^c(1_S x)$ for all $x > 0$, but as noted above these sets are all singleton (hence, non-empty) and different for different x, so Core $I_1 = \emptyset$. For the same reason, IDC fails.

Turn now to risk sharing. The assumptions of Theorem 9 hold. The purple line going through x^h is tangent to agent 2's indifference curve, but does *not* support agent 1's indifference curve: therefore, $\pi_1^c(1_S x^h)$ does not intersect $\pi_2^c(1_S(\bar{\omega}-x^h))$. Thus, condition (iv) in Theorem 9 is violated. Correspondingly, conditions (ii) and (iii) also fail: the allocation $(g, 1_S\bar{\omega}-g)$ is Pareto-efficient, but does not provide full insurance, whereas the interior, full-insurance allocation $(1_S x^h, 1_S(\bar{\omega}-x^h))$ is not Pareto-efficient.

Finally, note that the interior, full-insurance allocation $(1_S x^l, 1_S(\bar{\omega}-x^\ell))$ is Pareto-efficient; thus, in this economy, condition (i) in Theorem 5 holds. However, as just noted, conditions (ii)-(iv) in Theorem 9 do not hold. Thus, this example demonstrates that condition (i) cannot

[19]Another example of a preference which is convex but not GM-ambiguity-averse can be found in Cerreia-Vioglio et al. (2011).

be included in the statement of Theorem 9. □

Example 8 Modify Example 7 by assuming that agent 2's preferences are MEU, with priors $\Delta(S)$ and utility $u(x) = \sqrt{x}$. Refer to Figure 5.

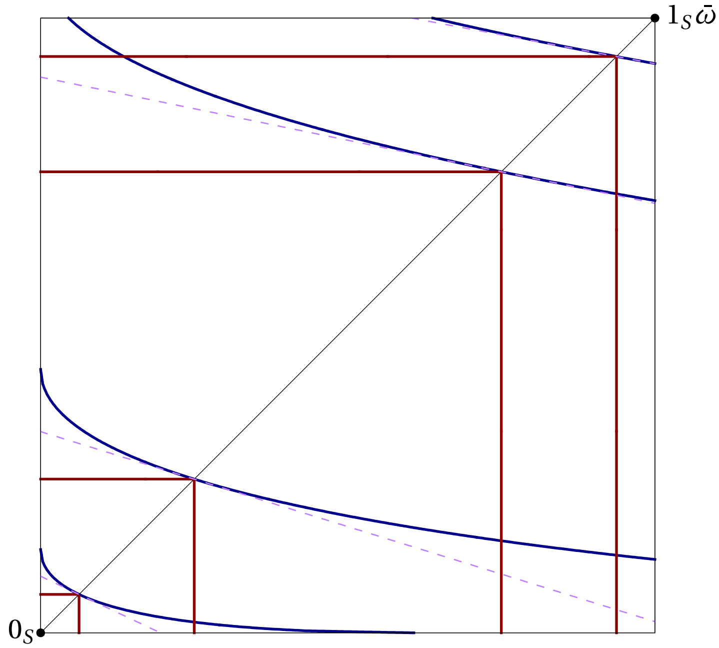

Figure 5: Risk sharing with non-constant local beliefs at certainty.

Both preferences are convex, and hence satisfy DQC. Bernoulli utility is strictly concave. The representing functionals I_i are nice at certainty, and in addition agent 2's preferences satisfy IDC: this follows because $\partial I_2(1_S x) = C_2(1_S x) = \Delta(S)$ for every $x > 0$. From Proposition 2, $\pi_2(1_S x) = \pi_2^c(1_S x) = C_2(1_S x) = \text{Core } I_2 = \Delta(S)$ for every $x > 0$. Therefore, for every $x > 0$, $\pi_1^c(1_S x) \cap \pi_2^c(1_S x) \neq \emptyset$. Proposition 7 and Theorem 9 apply, and indeed the set of Pareto-efficient and full-insurance allocations coincide.

Since agent 1's preferences do not satisfy IDC, Theorem 10 does not apply. Indeed, neither does RSS's original risk-sharing result (Proposition 9 in their paper): while both agents' preferences are convex (and indeed 2's preferences have a concave representation), agent 1's local beliefs do not coincide at certainty. Finally, since Core $I_1 = \emptyset$ (as shown in Example 7), Corollary 8 does not apply. □

6 Extensions

6.1 Alternative conditions for risk sharing

The assumption of differential quasiconcavity at certainty in Theorem 9 is sufficient to obtain the equivalence of conditions (ii)–(iv); however, it is not necessary. We discuss an alternative that relaxes DQC significantly, at the cost of imposing more structure on the point cores at certainty. Specifically, we can assume that differential quasiconcavity and niceness hold at only *some* full-insurance allocation, where preferences are "least ambiguity-averse."

Theorem 11 *Assume that there exists a feasible, full-insurance allocation* $(1_S x_1^*, \ldots, 1_S x_N^*)$ *such that, for every* $i \in N$*, (1)* $\pi_i^c(1_S x_i^*) \subseteq \pi_i^c(1_S x)$ *for all* $x > 0$*, and (2)* I_i *is differentially quasiconcave and nice at* $1_S u_i(x_i^*)$*.Then the following are equivalent:*

(i) The allocation $(1_S x_1^*, \ldots, 1_S x_N^*)$ *is Pareto-efficient;*

(ii) Every Pareto-efficient allocation is a full-insurance allocation;

(iii) Every feasible, full-insurance allocation is Pareto-efficient;

(iv) $\bigcap_i \mathrm{Core} I_i \neq \emptyset$*.*

Furthermore, under the above equivalent conditions, every interior, feasible full-insurance allocation is a competitive equilibrium with transfers.

Proof: (ii) $\Rightarrow$ (iii) is standard, and (iii) $\Rightarrow$ (i) is immediate. Finally, if $\bigcap_i \mathrm{Core}\ I_i \neq \emptyset$, then Corollary 8 implies that every Pareto-efficient allocation must be a full-insurance allocation; thus, (iv) $\Rightarrow$ (ii). To complete the argument, we show that (i) $\Rightarrow$ (iv). Since under (i) the allocation $(1_S x_1^*, \ldots, 1_S x_N^*)$ is Pareto-efficient and provides full insurance, by Proposition 6 $\bigcap_i C_i(1_S x_i^*) \neq \emptyset$. Consider an agent i. Since by assumption $\pi_i^c(1_S x_i^*) \subseteq \pi_i^c(1_S x)$ for all $x > 0$, $\pi_i^c(1_S x_i^*) = \bigcap_{x>0} \pi_i^c(1_S x)$; then, by Proposition 1 part 3, Core $I_i = \pi_i^c(1_S x_i^*)$; finally, since I_i is quasiconcave and nice at $1_S u_i(x_i^*)$, by Proposition 2 $\pi_i^c(1_S x_i^*) = C_i(1_S x_i^*) \neq \emptyset$. Thus, Core $I_i = C_i(1_S x_i^*)$. Since this is true for all agents $i \in N$, $\bigcap_i \mathrm{Core}\ I_i = \bigcap_i C_i(1_S x_i^*) \neq \emptyset$. ■

Assumption (1) requires that the point cores at each x_i^* are minimal with respect to set inclusion. Invoking the interpretation of the point core discussed in Section 3, this means that every ambiguity-neutral model at x_i^* is also an ambiguity-neutral model at every other point on the certainty line. In this interpretation, x_i^* is a point at certainty where the agent is *least* pointwise ambiguity-averse. As the above proof shows, it then turns out that the point core at x_i^* coincides with the (global) core of I_i.

Assumption (2) is notable in that niceness and differential quasiconcavity are only imposed at a single point on the certainty line. Thus, as noted above, the full strength of DQC (and niceness) is not required to obtain risk sharing.

However, Theorem 11 combines assumptions about preferences and endowments to a greater extent than Theorem 9: in Theorem 9, feasibility only appears in Condition (iv), whereas in Theorem 11 it is part of assumptions (1) and (2). Furthermore, while there are examples of preferences that satisfy the assumptions of Theorem 11, we do not know of an economically meaningful such example that does not also satisfy the assumptions of Theorem 9.

In this spirit, notice that Theorem 11 is neither more nor less general than Theorem 9: in Example 7, the assumptions of Theorem 9 are satisfied, but those of Theorem 11 are not.

Finally, Assumptions (1) and (2) still imply that all point cores are non-empty, and so agents are pointwise ambiguity averse—as they must be in Theorems 9 and 10.

6.2 Aggregate uncertainty

Strzalecki and Werner (2011) consider risk sharing in economies with aggregate uncertainty and convex preferences. While we leave a full investigation of such environments to future work, we can state a partial generalization of Corollary 8 to the case in which the aggregate endowment is non-constant, but "unambiguous" in a suitable sense. (We conjecture that Proposition 7 can be similarly extended.)

We need two definitions. The first is a weakening of the notion of "unambiguous act" studied in Cerreia-Vioglio et al. (2011); the second adapts Strzalecki and Werner (2011)'s notion of "conditional beliefs" to the class of preferences we consider.

Definition 3 Consider a preference $\succcurlyeq$ on $\mathbb{R}_+^S$ represented by (I, u). An act $f : S \to \mathbb{R}_+$ is **core-unambiguous** for (I, u) if $P(u \circ f) = I(u \circ f)$ for all $P \in \text{Core } I$. A partition $\mathscr{G}$ of S is core-unambiguous for (I, u) if every $\mathscr{G}$-measurable[20] bundle f is core-unambiguous for (I, u).

[20] A bundle f is $\mathscr{G}$-measurable if $f(s) = f(s')$ for all $s, s' \in G$ and $G \in \mathscr{G}$.

In Cerreia-Vioglio et al. (2011) an act f is *unambiguous* if $P(u \circ f) = I(u \circ f)$ for all priors P in the set C that represents the largest independent subrelation of $\succcurlyeq$ in the sense of Bewley (2002).[21] It turns out that Core $I \subseteq C$, so Definition 3 is less demanding.[22]

Definition 4 (cf. Strzalecki and Werner, 2011, Definition 3) Consider a preference $\succcurlyeq$ on $\mathbb{R}^S_+$ represented by (I, u), and a partition $\mathscr{G}$ of S. The $\mathscr{G}$**-conditional core** of I, written $\mathrm{Core}_{\mathscr{G}}\, I$, is the collection of all probabilities $Q \in \Delta(S)$ such that

(i) $Q(G) > 0$ for all $G \in \mathscr{G}$; and

(ii) there exists $P \in \mathrm{Core}\, I$ with $P(G) > 0$ and $P(\cdot|G) = Q(\cdot|G)$ for all $G \in \mathscr{G}$.[23]

Loosely speaking, $\mathrm{Core}_{\mathscr{G}}\, I$ is the set of all probabilities that "match" the probabilities conditional upon each event $G \in \mathscr{G}$ induced by some $P \in \mathrm{Core}\, I$. If every P in the core assigns positive probability to the elements of $\mathscr{G}$, then $\mathrm{Core}_{\mathscr{G}}\, I$ is a larger set than Core I. Note also that, if $\mathscr{G} = \{S\}$, then $\mathrm{Core}_{\mathscr{G}}\, I = \mathrm{Core}\, I$. For further interpretation, see Strzalecki and Werner (2011).

We can now state the promised partial generalization of Corollary 8. Let $\mathscr{E}$ be the partition induced by the aggregate endowment $\omega \equiv \sum_i \omega_i \in \mathbb{R}^S_+$: that is, the coarsest partition $\mathscr{G}$ such that ω is $\mathscr{G}$-measurable. If the aggregate endowment is constant, then $\mathscr{E} = \{S\}$ and a bundle is $\mathscr{E}$-measurable if and only if it provides full insurance.

Proposition 12 *If, for every $i \in N$, $\mathscr{E}$ is core-unambiguous for (I_i, u_i), and $\bigcap_i \mathrm{Core}_{\mathscr{E}}\, I_i \neq \emptyset$, then every Pareto-efficient allocation is $\mathscr{E}$-measurable.*

The other assertions in Corollary 8 do not generalize under the assumption that $\bigcap_i \mathrm{Core}_{\mathscr{G}}\, I_i \neq \emptyset$. Consider a two-state, two-agent economy with aggregate uncertainty: then $\mathscr{E}$ is the discrete

[21] Ghirardato and Siniscalchi (2012) show that C is the union of all sets $C(a)$, for all $a \in \left(\mathrm{int}(\mathbb{U})\right)^S$.

[22] Consider a risk-neutral CEU preference with $S = \{s_1, s_2, s_3\}$ and capacity ν given by $\nu(\{s_1\}) = \frac{1}{3}$, $\nu(\{s_2\}) = \nu(\{s_3\}) = 0$, and $\nu(E) = \frac{2}{3}$ for every 2-element set E. The core of this preference consists solely of the uniform probability P_u, so e.g. for $a = (3, 2, 1)$ (obvious notation), $P_u(a) = 2 = I(a) = 3 \cdot \nu(\{s_1\}) + 2 \cdot [\nu(\{s_1, s_2\}) - \nu(\{s_1\})] + 3 \cdot [1 - \nu(\{s_1, s_2\})]$. Thus, a is core-unambiguous. However, it is not unambiguous in the sense of Cerreia-Vioglio et al. (2011): by results in Ghirardato et al. (2004), the set C contains, for example, the measure $P = (\frac{1}{3}, 0, \frac{2}{3})$ (obvious notation) for which $P(a) = \frac{5}{3}$.

[23] $P(\cdot|G)$ and $Q(\cdot|G)$ denote conditional probabilities given G.

partition, so every bundle is $\mathscr{E}$-measurable, and furthermore the $\mathscr{E}$-conditional cores are degenerate and always intersect no matter what the preferences. However (except for degenerate cases) not every feasible allocation is Pareto-efficient; moreover, one can easily construct examples of Pareto-efficient allocations that are not competitive equilibria. Whether one can obtain positive results under stronger assumptions is left to future research.

A Appendix: Proofs

A.1 Preliminaries

Proof of Remark 1: The map $F : \mathbb{R}^S_+ \to \mathbb{U}^S$ defined by $F(f) = (u(f_1), \ldots, u(f_S))$ is strictly differentiable (pp. 30-31 Clarke, 1983) and, furthermore, it maps every neighborhood of f to a neighborhood of $F(f)$.[24] Hence, since $I^u = I \circ F$, by Theorem 2.3.10 in Clarke $\partial I^u(f) = \partial I(u \circ f) \circ D_s F(f)$; that is, more explicitly, every $Q^u \in \partial I^u(f)$ is defined by

$$\forall h \in \mathbb{R}^S, \qquad Q^u(h) = \sum_s Q(s) u'(f_s) h_s$$

for some $Q \in \partial I(u \circ f)$. ■

The following geometric notions will be useful. For every bundle $f \in \mathbb{R}^S_+$, let

$$U(f) = \{g \in \mathbb{R}^S_+ : g \succcurlyeq f\},$$

the upper countour set of the preference $\succcurlyeq$ at f. For every set $C \subset \mathbb{R}^S_+$ and bundle $f \in \mathbb{R}^S_+$, let

$$d_C(f) = \inf\{\|f - g\| : g \in C\}$$

The *Clarke tangent cone* to C at some $f \in C$ is

$$T_C(f) = \{v \in \mathbb{R}^S : (d_C)^0(f; v) = 0\},$$

[24]To see this, fix a strictly positive bundle f and consider the set $\{g \in \mathbb{R}^S_+ : f_s - \epsilon < g_s < f_s + \epsilon \forall s \in S\}$, which is open. The image of this set via F is $\{v \in \mathbb{U}^S : u(f_s - \epsilon) < v_s < u(f_s + \epsilon) \forall s \in S\}$, because u is continuous and strictly increasing. This set is also open.

i.e. the set of directions v for which the Clarke derivative of the distance function (which is Lipschitz and convex) is zero. The following characterization (Clarke, 1983, Theorem 2.4.5) is useful:

$$T_C(f) = \{v \in \mathbb{R}^S : \forall (f^k, t^k) \subset C \times \mathbb{R}_{++} \text{ s.t. } f^k \to f,\ t^k \downarrow 0,\ \exists (v^k) \subset \mathbb{R}^S \text{ s.t. } v^k \to v,\ f^k + t^k v^k \in C\ \forall k\}.$$

Finally, define the *Clarke normal cone* to C at f by polarity:

$$N_C(f) = \{Q \in ba(S) = \mathbb{R}^S : Q(v) \leq 0\ \forall v \in T_C(f)\}.$$

Specializing to our environment, we have

$$T(f) \equiv T_{U(f)}(f) = \big\{v \in \mathbb{R}^S : \forall (f^k, t^k) \subset \mathbb{R}^S_+ \times \mathbb{R}_{++} \text{ s.t. } f^k \succcurlyeq f\ \forall k,\ f^k \to f,\ t^k \downarrow 0,$$
$$\exists (v^k) \subset \mathbb{R}^S \text{ s.t. } v^k \to v,\ f^k + t^k v^k \succcurlyeq f\ \forall k\big\}.$$

and it is convenient to define

$$N(f) \equiv N_{U(f)}(f) = \{Q \in \mathbb{R}^S : Q(v) \leq 0\ \forall v \in T(f)\}.$$

Loosely speaking, $T(f)$ is the set of directions v with the property that any sequence of bundles preferred to f and converging to it can be perturbed in the direction v without leaving the upper contour set of f. More informally, moving from bundles near f in the direction v by a small amount leads to an act that is at least as good as f. Then, if Q is in the normal cone, $-Q$ is a price vector that assigns non-negative value to such changes.

The following two results pertain to the Clarke normal cone. Note that the first does not require any particular assumption on the functional I.

Remark 3 For every bundle $f \in \mathbb{R}^S_{++}$, $-\pi(f) \subseteq N(f)$.

Proof: Fix $P \in \pi(f)$. Consider $v \in T(f)$, the constant sequence $f^k \equiv f$, and an arbitrary sequence $(t^k) \downarrow 0$. Since $v \in T(f)$, there exists a sequence $(v^k) \to v$ such that, for every k, $f^k + t^k v^k \succcurlyeq f$, i.e., $I(u \circ [f + t^k v^k]) \geq I(u \circ f)$. Since $P \in \pi(f)$, $P(f + t^k v^k) \geq P(f)$, and therefore $P(v^k) \geq 0$ for every k. By continuity, $P(v) \geq 0$. Therefore, $-P \in N(f)$. ■

Lemma 13 *For any agent i and bundle $f \in \mathbb{R}^S_+$, if I is nice at $u \circ f$, then $N(f) \subseteq \bigcup_{\lambda \geq 0} \lambda\big(-\partial I^u(f)\big)$. In particular, for any $x > 0$, if $R \in N(1_S x) \setminus \{0_S\}$, then there is $\mu > 0$ and $Q \in \partial I(1_S u(x))$ such that $R = -\mu Q$.*

Proof: Let $J^u = -I^u$, and note that $U(f) = \{g \in \mathbb{R}^S_+ : J^u(g) \leq J^u(f)\}$. By Proposition 2.3.1 in Clarke (1983), $\partial J^u(f) = -\partial I^u(f)$. Recall that every $Q^u \in \partial I^u(1_S x)$ maps $a \in \mathbb{R}^S$ to $\sum_s Q(s) u'(f_s) a_s = u'(x) Q(a)$ for some $Q \in \partial I(1_S u(x))$; since u is strictly increasing, and Q is non-negative because I is monotonic, it follows that $Q^u = 0_S$ only if $Q = 0$; but I is nice at $u \circ f$ by assumption, so $0_S \notin \partial I^u(f)$, i.e., I^u and hence J^u are nice at f. Then, by Corollary 1 to Theorem 2.4.7 in Clarke (1983), $N(f) \subset \bigcup_{\lambda \geq 0} \lambda \partial J^u(f) = \bigcup_{\lambda \geq 0} \lambda\big(-\partial I^u(f)\big)$, as claimed.

If $f = 1_S x$ for some $x > 0$, then every $Q^u \in \partial I^u(1_S x)$ maps $a \in \mathbb{R}^S$ to $\sum_s Q(s) u'(x) a_s = u'(x) Q(a_s)$ for some $Q \in \partial I(1_S u(x))$, where $u'(x) > 0$ by assumption. By the preceding claim, every $R \in N(1_S x)$ not equal to 0_S can be written as $R = -\lambda Q^u$ for some $\lambda > 0$ and $Q^u \in \partial I^u(1_S x)$, and hence also as $R = -\lambda u'(x) Q$ for some $Q \in \partial I(1_S u(x))$. The second claim follows by taking $\mu = \lambda u'(x)$. ■

Conclude this section with a remark that restates the definition of $\pi^c(f)$ for $f = 1_S x$.

Remark 4 For every $x > 0$, $\pi^c(1_S x) = \{P \in \Delta(S) : \forall g \in \mathbb{R}^S_+,\ u(x) \geq P(u \circ g) \Longrightarrow u(x) \geq I(u \circ g)\}$.

Proof: Denote the set on the rhs of the Remark by $\hat{\pi}(1_S x)$. Suppose that $P \in \pi^c(1_S x)$. We show that, for every $g \in \mathbb{R}^S_+$, $I(u \circ g) > u(x)$ implies $P(u \circ g) > u(x)$, so $P \in \hat{\pi}(1_S x)$. Fix g and suppose $I(u \circ g) > u(x)$. Since $P \in \pi^c(1_S x)$, $P(u \circ g) \geq u(x)$. By contradiction, suppose $P(u \circ g) = u(x)$. By monotonicity, there must be a state s such that $g(s) > x$. There are two cases. First, if there is some $s \in S$ such that $g(s) > x$ and $P(\{s\}) > 0$, by continuity of u and I there is $\epsilon > 0$ such that the bundle g' defined by $g'(s) = g(s) - \epsilon \geq 0$ and $g'(s') = g(s')$ for $s' \neq s$ satisfies $I(u \circ g') > u(x)$ and $P(u \circ g') = P(u \circ g) + P(\{s\})[u(g(s) - \epsilon) - u(g(s))] < P(u \circ g) = u(x)$; this contradicts the assumption that $P \in \pi^c(1_S x)$. Second, if $g(s) > x$ implies $P(\{s\}) = 0$, then, since $P(u \circ g) = u(x)$, it must be the case that $P(\{s\}) > 0$ implies $g(s) = x$. Fix one such state s; since $x > 0$, by continuity of u and I we can find $\epsilon > 0$ such that the bundle g' defined by $g'(s) = g(s) - \epsilon = x - \epsilon \geq 0$ and $g'(s') = g(s')$ for $s' \neq s$ satisfies $I(u \circ g') > u(x)$ and $P(u \circ g') =$

$P(u \circ g) + P(\{s\})[u(x-\epsilon) - u(x)] < P(u \circ g) = u(x)$; again, we obtain a contradiction. Hence, $P(u \circ g) > u(x)$, as claimed.

Conversely, suppose that $P \in \hat{\pi}(1_S x)$. We show that, for every $g \in \mathbb{R}^S_+$, $u(x) > P(u \circ g)$ implies $u(x) > I(u \circ g)$, so $P \in \pi^c(1_S x)$. Fix g and suppose that $u(x) > P(u \circ g)$. Since $P \in \hat{\pi}(1_S x)$, $u(x) \geq I(u \circ g)$. By contradiction, suppose $u(x) = I(u \circ g)$. By continuity of u and P, there is $\epsilon > 0$ such that $u(x) > P(u \circ (g + 1_S \epsilon))$; however, by strong monotonicity $I(u \circ (g + 1_S \epsilon)) > I(u \circ g) = u(x)$, which contradicts the assumption that $P \in \hat{\pi}(1_S x)$. Hence, $u(x) > I(u \circ g)$. ■

Corollary 14 *For all $x > 0$ and $P \in \pi^c(1_S x)$, $P(\{s\}) > 0$ for all $s \in S$.*

Proof: Fix $x > 0$, $s \in S$ and $P \in \pi^c(1_S x)$. Define g by $g(s) = x + 1$ and $g(s') = x$ for all $s' \neq s$. If $P(\{s\}) = 0$, then $u(x) = P(u \circ g)$, and therefore, by Remark 4 $u(x) \geq I(u \circ g)$: this contradicts the assumption that I is strongly monotonic. ■

A.2 Behavioral characterization of conditions DQC and IDC

Fix a preference $\succcurlyeq$ represented by a pair (I, u) that satisfies the assumptions of Section 2; in particular, recall that u is strictly increasing and continuous. In order to introduce the key definition from Ghirardato and Siniscalchi (2012), two ancillary notions are required.

First, define a mixture operation on $\mathbb{R}_+$ as follows: for any $x, y \in \mathbb{R}_+$ and $\alpha \in [0,1]$, $\alpha x \oplus (1-\alpha) y$ denotes the (unique) $z \in \mathbb{R}_+$ such that $u(z) = \alpha u(x) + (1-\alpha) u(y)$. If, following Anscombe and Aumann (1963), one extends preferences to acts mapping states to lotteries over $\mathbb{R}_+$, one can alternatively take z to be the gamble that yields x and y with probabilities α and $1-\alpha$ respectively. Ghirardato et al. (2003) and Ghirardato and Pennesi (2012) provide alternative characterizations of the mixture operation $\oplus$ that do not require objective lotteries.

Convergence of acts is in the usual Euclidean topology. Since u is strictly increasing and continuous, and the state space S is finite, this is equivalent to the convergence notion considered by Ghirardato and Siniscalchi (2012).

Definition 5 For any pair of acts f, g and prize $x \in \mathbb{R}_+$, say that f is a *(weakly) better deviation than g near x*, written $f \succcurlyeq^*_x g$, if, for every $(\lambda^n)_{n\geq 0} \subset [0,1]$ and $(h^n)_{n\geq 0}$ such that $\lambda^n \downarrow 0$ and $h^n \to 1_S x$,

$$\lambda^n f \oplus (1-\lambda^n) h^n \succcurlyeq \lambda^n g \oplus (1-\lambda^n) h^n \quad \text{eventually.}$$

The basic intuition is that f is a better deviation than g at x if, starting from an initial riskless consumption bundle $1_S x$, the DM prefers to move by a vanishingly small amount in the direction of the bundle f rather than in the direction of the bundle g. Furthermore, this remains true if the initial bundle is not exactly $1_S x$, but is close to it. We then have:

Proposition 15 *Assume that I is nice at every $1_S u(x)$, $x > 0$.*

(i) $\succcurlyeq$ satisfies IDC if and only if, for every $x, y > 0$ and $f, g \in \mathbb{R}^S_+$, $f \succcurlyeq_x g$ implies $f \succcurlyeq_y g$.

(ii) I satisfies differential quasiconcavity at $1_S\gamma$, where $\gamma = u(x)$ and $x > 0$, if and only if, for every $f, g \in \mathbb{R}^S_+$ such that $f(s) \succ g(s)$ for all s, and all $y \in (0, x)$, $g \succcurlyeq x$ implies $f \succcurlyeq_x y$.

Note that, by (ii), a simple sufficient condition for differential quasiconcavity at $x > 0$ is

$$\forall g \in \mathbb{R}^S_+,\ x > 0, \qquad g \succcurlyeq x \quad \Longrightarrow \quad g \succcurlyeq_x x.$$

This condition is not also necessary because the relation $\succcurlyeq_x$ is in general not continuous: this is discussed in Ghirardato and Siniscalchi (2012) (Example 4). However, it confirms the basic insight provided in Section 3: if g is at least as good as x, and hence a *global* improvement over x, it is also a better *local* deviation from x.

Proof: (i) is immediate from Theorems 6 or 7 in Ghirardato and Siniscalchi (2012).

For (ii), suppose that $\succcurlyeq$ satisfies the stated condition. Consider $\gamma > u(0)$ and $a \in \mathbb{U}^S$, so there are $x \in \mathbb{R}_{++}$ and $g \in \mathbb{R}^S_+$ such that $\gamma = u(x)$ and $a = u \circ g$. Suppose that $I(a) \geq \gamma$, so $g \succcurlyeq x$. Then $f \succcurlyeq_x y$ for all $f \in \mathbb{R}^S_+$ with $f(s) \succ g(s)$ and $y \in (0, x)$. Since S is finite and $\succcurlyeq_x$ is monotonic, this also implies that, for every $f' \in \mathbb{R}^S_+$ with $y \succ f'(s)$ for every s, $f \succcurlyeq_x f'$. That is, for every spread (f, f') of $(g, 1_S x)$, $f \succcurlyeq_x f'$. Since I is nice at $1_S u(x)$, by Theorem 7 in Ghirardato and Siniscalchi (2012), $P(u \circ g) \geq u(x)$ for all $P \in C(1_S x)$. But this implies that $Q(u \circ g) \geq Q(1_S u(x))$, i.e., $Q(a - 1_S\gamma) \geq 0$, for all $Q \in \partial I(1_S\gamma)$; thus, I is differentially quasiconcave at γ.

Conversely, suppose that I is differentially quasiconcave at $1_S\gamma$ for some $\gamma \in \operatorname{int} u(X)$. Let $x = u^{-1}(\gamma) > 0$ and consider $f, g \in \mathbb{R}^S_+$ with $f(s) \succ g(s)$ for all s, and $y \in (0, x)$. Suppose that

$g \succcurlyeq x$, so $I(u \circ g) \geq u(x) = \gamma$. Then, by differential quasiconcavity, $Q(u \circ g - 1_S \gamma) \geq 0$ for all $Q \in \partial I(1_S \gamma)$; since I is nice at $1_S \gamma$, this implies that $P(u \circ g) \geq u(x)$ for all $P \in C(1_S x)$. By Theorem 7 in Ghirardato and Siniscalchi (2012), $g' \succcurlyeq_x g''$ for all spreads (g', g'') of $(g, 1_S x)$; in particular, this is true for $(f, 1_S y)$, so the stated condition holds. ■

A.3 Proof of the results in Section 3

Proof of Proposition 1: (1): fix $x > 0$. Suppose $P \in \text{Core}\, I$ and consider a bundle $g \in \mathbb{R}^S_+$. Assume that $P(1_S u(x)) \geq P(u \circ g)$. Since I is normalized and P is in the core of I, $I(1_S u(x)) = u(x) = P(1_S u(x)) \geq P(u \circ g) \geq I(u \circ g)$. Since this holds for all $g \in \mathbb{R}^S_+$, $P \in \pi^{\mathrm{c}}(1_S x)$. Hence, $\text{Core}\, I \subseteq \pi^{\mathrm{c}}(1_S x)$.

For the second inclusion, note that $P \in \pi^{\mathrm{c}}(1_S x)$ iff, for all $g \in \mathbb{R}^S$, $I(u \circ g) > u(x)$ implies $P(u \circ g) > u(x)$. Now fix $P \in \pi^{\mathrm{c}}(1_S x)$ and consider a bundle $g \in \mathbb{R}^S_+$. Assume that $I(u \circ g) \geq u(x)$. Since $\mathbb{U}$ does not include its upper bound, there is $\bar{\epsilon} > 0$ such that, for all $\epsilon \in (0, \bar{\epsilon})$, $u \circ g + 1_S \epsilon \in \mathbb{U}^S$ (i.e., there is $g_\epsilon \in \mathbb{R}^S_+$ with $u \circ g_\epsilon = u \circ g + 1_S \epsilon$). Then, for any such ϵ, by strong monotonicity, $I(u \circ g + 1_S \epsilon) > I(u \circ g) \geq u(x)$, and so $P(u \circ g) + \epsilon = P(u \circ g + 1_S \epsilon) > u(x)$ because $P \in \pi^{\mathrm{c}}(1_S x)$. Since this holds for all $\epsilon \in (0, \bar{\epsilon})$, $P(u \circ g) \geq u(x)$; since u is (strictly) concave, $u(P(g)) \geq P(u \circ g) \geq u(x)$; since u is strictly increasing, $P(g) \geq x$; and since $g \in \mathbb{R}^S_+$ was arbitrary, $P \in \pi(1_S x)$.

(2): fix $x > 0$ and consider $P \in \pi(1_S x)$. By Remark 3, $-P \in N(1_S x)$. By Lemma 13, if I is nice at $1_S u(x)$, then $N(1_S x) \setminus \{0_S\} \subseteq \bigcup_{\mu > 0} (-\partial I(1_S u(x)))$. Therefore, there are $\mu > 0$ and $Q \in \partial I(1_S u(x))$ such that $-P = \mu(-Q)$, i.e., $P = \mu Q$. Furthermore, $1 = P(S) = \mu Q(S)$, so $\mu = Q(S)^{-1}$ and $P = \frac{Q}{Q(S)} \in C(1_S x)$, as required.

(3): By the first inclusion in part (1), $\text{Core}\, I \subseteq \bigcap_{x>0} \pi^{\mathrm{c}}(1_S x)$. Conversely, suppose $P \in \bigcap_{x>0} \pi^{\mathrm{c}}(1_S x)$. We claim that $P(\{s\}) > 0$ for all $s \in S$. By contradiction, suppose that $P(\{s\}) = 0$ for some $s \in S$. Then, for every $x > 0$, $P(1_S u(x)) = u(x) > u(0) = P(u \circ 1_{\{s\}})$, and therefore, since $P \in \pi^{\mathrm{c}}(1_S x)$, $I(1_S u(x)) \geq I(u \circ 1_{\{s\}})$. Since this holds for all $x > 0$, by continuity of u and I, $I(1_S u(0)) \geq I(u \circ 1_{\{s\}})$, i.e., $0 \succcurlyeq 1_{\{s\}}$, which contradicts strong monotonicity.

Now fix $g \in \mathbb{R}^S_+$ and let $x \in \mathbb{R}_+$ be such that $u(x) = P(u \circ g)$. If $x = 0$, then the preceding claim implies that $g=0$, and so $P(1_S u(0)) = u(0) = I(u\circ g)$. Otherwise, since by assumption $P\in\pi^{c}(1_S x)$, $P(1_S u(x)) = u(x) = P(u\circ g)$ implies $I(1_S u(x)) \geq I(u\circ g)$; but $I(1_S u(x)) = u(x) = P(1_S u(x)) = P(u\circ g)$, so indeed $P(u\circ g)\geq I(u\circ g)$.

(4): Fix $P\in\text{Core } I$. $x>0$, and $a\in\mathbb{R}^S$. Let $\gamma = u(x)$. We calculate:

$$\begin{aligned} I^{\circ}(1_S\gamma;a) &= \limsup_{c\to 1_S\gamma, t\downarrow 0}\frac{I(c+ta)-I(c)}{t} = \limsup_{d\to 1_S\gamma, t\downarrow 0}\frac{I(d)-I(d-ta)}{t}\geq \\ &\geq \limsup_{t\downarrow 0}\frac{I(1_S\gamma)-I(1_S\gamma - ta)}{t}\geq \limsup_{t\downarrow 0}\frac{\gamma - P(1_S\gamma - ta)}{t} = \\ &= \limsup_{t\downarrow 0}\frac{\gamma-\gamma+tP(a)}{t} = P(a). \end{aligned}$$

The second equality follows because, if $c\to 1_S\gamma$ and $t\downarrow 0$, then $d\equiv c+ta\to 1_S\gamma$; conversely, if $d\to 1_S\gamma$ and $t\downarrow 0$, then $c\equiv d-ta\to 1_S\gamma$. The first inequality follows by considering the constant sequence $d\equiv 1_S\gamma$. The second inequality follows from normalization and the fact that $P\in\text{Core } I$: $I(1_S\gamma - ta)\leq P(1_S\gamma - ta)$, so $-I(1_S\gamma - ta)\geq -P(1_S\gamma - ta)$.

Hence, for every $a\in\mathbb{R}^S$, $\max_{Q\in\partial I(1_S\gamma)} Q(a) = I^{\circ}(1_S\gamma;a)\geq \max_{P\in\text{Core } I} P(a)$, so by standard results (e.g., Clarke, 1983, Prop. 2.1.4 (b)), Core $I\subseteq\partial I(1_S\gamma)$. Furthermore, by definition, $Q\in\text{Core } I$ implies $Q(S)=1$, so $Q=\frac{Q}{Q(S)}\in C(1_S x)$. ■

Proof of Proposition 2: Fix $x>0$ and $P\in C(1_S x)$. We first claim that $P(\{s\})>0$ for all $s\in S$. By assumption, there is $Q\in\partial I(1_S u(x))$ such that $Q(S)>0$ and $P=\frac{Q}{Q(S)}$. By strong monotonicity, $I(1_S u(x)+1_{\{s\}})>u(x)$; by continuity, there exists $\epsilon\in(0,u(x))$ such that $1_S u(x)+1_{\{s\}}-\epsilon 1_{S\setminus\{s\}}\in\mathbb{U}^S$ and

$$I(1_S u(x)+1_{\{s\}}-\epsilon 1_{S\setminus\{s\}})>u(x).$$

Therefore, since I is differentially quasiconcave at $1_S u(x)$ by DQC,

$$Q(1_S u(x)+1_{\{s\}}-\epsilon 1_{S\setminus\{s\}}-1_S u(x))\geq 0 \quad\Leftrightarrow\quad Q(\{s\})\geq\epsilon Q(S\setminus\{s\}) \quad\Leftrightarrow\quad P(\{s\})\geq\epsilon P(S\setminus\{s\}).$$

If $P(\{s\})=0$, the last inequality reduces to $0\geq\epsilon$, a contradiction. Thus, $P(\{s\})>0$.

We now show that, for any $g\in\mathbb{R}^S_+$, $u(x)\geq P(u\circ g)$ implies $u(x)\geq I(u\circ g)$; thus, $P\in\pi^{c}(1_S x)$. We show that the contrapositive holds. Suppose that $I(u\circ g)>u(x)$; notice that we cannot have $g(s)=0$ for all s, because by assumption $x>0$ and so $u(x)>u(0)=I(1_S u(0))$ by strong monotonicity of u and normalization. Hence, for every $\alpha \in (0,1)$, $g(s) \geq \alpha g(s)$ in every state s, and there is at least one state s^* such that $g(s^*) > \alpha g(s^*)$. Furthermore, by continuity there is $\alpha^* \in (0,1)$ such that $I(u \circ (\alpha^* g)) > u(x)$. By DQC, I is differentially quasiconcave at $1_S u(x)$, so $Q(u \circ (\alpha^* g) - 1_S u(x)) \geq 0$, and so $P(u \circ (\alpha^* g)) \geq u(x)$. Finally, since there is at least one state s^* with $g(s^*) > \alpha^* g(s^*)$, and we showed above that $P(\{s^*\}) > 0$, $P(u \circ g) > P(u \circ (\alpha^* g)) \geq u(x)$.

Hence, $C(1_S x) \subseteq \pi^c(1_S x)$, and indeed by Proposition 1 part 1, $C(1_S x) \subseteq \pi^c(1_S x) \subseteq \pi(1_S x)$. If in addition I is nice at $1_S u(x)$, part 2 of Proposition 1 implies that $\pi(1_S x) \subseteq C(1_S x)$, so $C(1_S x) = \pi^c(1_S x) = \pi(1_S x)$. ■

Note: the above argument shows that it is enough to assume quasiconcavity at $1_S u(x)$ in order to obtain the noted inclusions.

Proof of Proposition 3 (1): Assume that DQC holds, and let $P \in \bigcap_{x>0} C(1_S x)$. Fix $a \in \mathbb{U}^S$ and let $\gamma = I(a)$. By monotonicity of I, $\min_s a_s \leq \gamma \leq \max_s a_s$. Since $a \in \mathbb{U}^S$, by continuity of u there is $x \geq 0$ such that $\gamma = u(x)$.

If $x = 0$, then, since $u(x) = u(0) = \min \mathbb{U}$, $a \geq 1_S u(0)$, and so $P(a) \geq P(1_S u(0)) = u(0) = u(x) = \gamma = I(a)$. Now consider $x > 0$, so $\gamma > \min \mathbb{U}$. By definition, since $P \in C(1_S x)$, there is $Q \in \partial I(1_S \gamma)$ such that $Q(S) > 0$ and $P = \frac{Q}{Q(S)}$. Since $\mathbb{U} = u(\mathbb{R}_+)$ does not contain its upper bound and is connected because u is continuous, $\gamma \in \mathrm{int}(\mathbb{U})$. Hence, by DQC, $I(a) = \gamma$ implies $Q(a - 1_S \gamma) \geq 0$, i.e., $Q(a) \geq Q(1_S \gamma)$; hence also $P(a) \geq P(1_S \gamma)$. Therefore, P satisfies $P(a) \geq P(1_S \gamma) = \gamma = I(a)$. Since this holds for all $a \in \mathbb{U}^S$, $P \in \mathrm{Core}\, I$. Thus, $\bigcap_{x>0} C(1_S x) \subseteq \mathrm{Core}\, I$.

(2): Assume that $C(1_S x) \subseteq \mathrm{Core}\, I$ for some $x > 0$, let $\gamma = u(x)$, and fix $a \in \mathbb{U}^S$. Suppose that $I(a) \geq \gamma$: then, for every $P \in \mathrm{Core}\, I$, $P(a) \geq I(a) \geq \gamma = P(1_S \gamma)$, i.e, $P(a - 1_S \gamma) \geq 0$. Since, by assumption, $C(1_S x) \subseteq \mathrm{Core}\, I$, if $Q \in \partial I(1_S \gamma)$, so that $\frac{Q}{Q(S)} \in C(1_S x)$, one has $Q(a - 1_S \gamma) \geq 0$. Hence, $I^\ell(1_S \gamma; a - 1_S \gamma) = \min_{Q \in \partial I(1_S \gamma)} Q(a - 1_S \gamma) \geq 0$, i.e., I is differentially quasiconcave at $1_S \gamma$.

(3): since $\gamma \in \mathrm{int}(\mathbb{U})$ iff $u^{-1}(\gamma) > 0$, the result is immediate from (2). ■

To prove results involving the condition in DQC, it is convenient to define the **Clarke lower derivative** of I (cf. Ghirardato et al., 2004, pp. 150 and 157) as

$$I^\ell(b; a) = \liminf_{t \downarrow 0, c \to b} \frac{I(c + ta) - I(c)}{t};$$

It is readily verified that $I^\ell(b;a) = -I^\circ(b;-a)$ and, therefore, $I^\ell(b;a) = \min_{Q\in\partial I(b)} Q(a)$ for all interior $b \in \mathbb{U}^S$ and all $a \in \mathbb{R}^S$. Then, the condition in DQC can equivalently be restated as

$$\forall\gamma \in \mathrm{int}(\mathbb{U}),\ a \in \mathbb{U}^S, \qquad I(a) \geq \gamma \quad \Longrightarrow \quad I^\ell(1_S\gamma; a - 1_S\gamma) \geq 0. \tag{9}$$

Proof of Proposition 4: For both results, we use the equivalent characterization in Eq. (9). As noted in the text, part (1) follows from a result in Penot and Quang (1997); however, since their assumptions are formulated somewhat differently from ours, invoking their result requires some work. We provide a direct proof.

(1) Fix $\gamma \in \mathrm{int}(\mathbb{U})$ and $a \in \mathbb{U}^S$ such that $I(a) \geq \gamma$. Also fix $\epsilon > 0$ such that $a + 1_S\epsilon \in \mathbb{U}^S$ (this must exist, because $\mathbb{U} = u(\mathbb{R}_+)$ does not contain its supremum). By strong monotonicity, $I(a + 1_S\epsilon) > \gamma$. Consider sequences $(c^k) \subset \mathbb{U}^S$ and $(t^k) \subset \mathbb{R}_{++}$ such that $c^k \to 1_S\gamma$ and $t^k \downarrow 0$. Note that

$$t^k[(a+1_S\epsilon) - 1_S\gamma] + c^k = t^k[(a+1_S\epsilon) - 1_S\gamma + c^k] + (1-t^k)c^k$$

and, since $c^k \to 1_S\gamma$, eventually $(a+1_S\epsilon) - 1_S\gamma + c^k \in \mathbb{U}^S$; furthermore, by continuity $I(a+1_S\epsilon - 1_S\gamma + c^k) \to I(a+1_S\epsilon)$ and $I(c^k) \to I(1_S\gamma) = \gamma$. Therefore, for k sufficiently large, $I(a+1_S\epsilon - 1_S\gamma + c^k) > I(c^k)$. Then, by quasiconcavity, for all such k,

$$I(t^k[(a+1_S\epsilon) - 1_S\gamma] + c^k) = I(t^k[(a+1_S\epsilon) - 1_S\gamma + c^k] + (1-t^k)c^k) \geq I(c^k).$$

It follows that

$$I^\ell(1_S\gamma; (a+1_S\epsilon) - 1_S\gamma) = \liminf_{c \to 1_S\gamma, t\downarrow 0} \frac{I(t[(a+1_S\epsilon) - 1_S\gamma] + c) - I(c)}{t} \geq 0.$$

Finally, by continuity of $I^\ell(1_S\gamma;\cdot)$, $I^\ell(1_S\gamma; a - 1_S\gamma) \geq 0$ as well.

(2): if I is regular, $I^\ell(1_S\gamma; a - 1_S x) = -I^\circ(1_S\gamma; 1_S\gamma - a) = -I'(1_S\gamma; 1_S\gamma - a)$; furthermore, if $I(a) \geq I(1_S\gamma) = \gamma$, by GM-ambiguity aversion and normalization, for any $P \in \mathrm{Core}\, I$,

$$\begin{aligned}
-I^\ell(1_S\gamma; a - 1_S\gamma) = I'(1_S\gamma; 1_S\gamma - a) = \lim_{t\downarrow 0} \frac{I(1_S x + t[1_S x - a]) - I(1_S x)}{t} = \\
= \lim_{t\downarrow 0} \frac{I(1_S x + t[1_S x - a]) - x}{t} \leq \lim_{t\downarrow 0} \frac{P(1_S x + t[1_S x - a]) - x}{t} = \\
= \lim_{t\downarrow 0} \frac{x + tx - tP(a) - x}{t} = x - P(a) \leq I(a) - P(a) \leq 0,
\end{aligned}$$

so DQC holds. ■

A.4 Proof of the results in Section 4

The key step in the proof of Proposition 6 is contained in the following result.

Lemma 16 *If $(f_i)_{i\in N}$ is a Pareto-efficient allocation, then there exists a price vector $p \in \mathbb{R}^S_+ \setminus \{0\}$ such that $-p \in N_i(f_i)$ for all $i \in N$.*

Proof: Apply Prop. 2.1 (a) and (e) and Theorem 2.1 in Bonnisseau and Cornet (1988) to get $-p \in \bigcap_{i\in N} N_i(f_i)$. We only need to show that p is non-negative. By monotonicity, $\mathbb{R}^S_+ \subset T_i(f_i)$: to see this, note that, if $v \in \mathbb{R}^S_+$, then for any sequence (f^k, t^k) such that $f^k \succcurlyeq_i f_i$, $f^k \to f_i$, and $t \downarrow 0$, the constant sequence $v^k = v$ satisfies $f^k + t^k v^k \geq f^k \succcurlyeq_i f_i$ for all k.

Now consider $v \in \mathbb{R}^S_+$ s.t. $v_s = 0$ iff $p_s \geq 0$, and $v_s = 1$ otherwise. If $p_s < 0$ for some s, then $p \cdot v < 0$, i.e. $-p \cdot v > 0$, which contradicts the fact that $v \in T_i(f_i)$ and $-p \in N_i(f_i)$ for all i. Thus, $p \geq 0$. ■

Proof of Proposition 6: For the first implication, Lemma 16 yields $p \in \mathbb{R}^S_+ \setminus \{0_S\}$ such that $-p \in N_i(f_i)$ for all i; by Lemma 13, $-p \in \bigcup_{\lambda>0} \lambda\big(-\partial I_i^u(f)\big)$ for all $i \in N$, and the claim follows. The second claim follows from the second part of Lemma 13. Finally, at a full-insurance allocation $(1_S x_1, \dots, 1_S x_N)$, $p = \mu_i Q_i$ for every i, where $\mu_i > 0$ and $Q_i \in \partial I_i(1_S u_i(x_i))$; then $Q_i(S) = \frac{\sum_s p_s}{\mu_i}$, and therefore $\frac{Q_i}{Q_i(S)} = \frac{\mu_i^{-1} p}{\mu_i^{-1} \sum_s p_s} = \frac{p}{\sum_s p_s} \equiv P$; hence, $P \in \bigcap_i C_i(1_S x)$. ■

Remark 5 If $-I_i$ is regular, then by Theorem 2.3.10 and Corollary 1 to Theorem 2.4.7 in Clarke (1983) $-I_i^u$ is also regular, and $N_i(f_i) = \bigcup_{\lambda\geq 0} \lambda\big(-\partial I_i^u(f_i)\big)$.

The next Remark follows from standard arguments; we include the proof for completeness. Observe that the argument relies on continuity and strong monotonicity.

Remark 6 If a feasible allocation $(f_1, \dots, f_N)$ is not Pareto-efficient, then it is Pareto-dominated by a Pareto-efficient allocation.

Proof: By assumption, there exists a feasible allocation $(g_1, \dots, g_N)$ that Pareto-dominates $(f_1, \dots, f_N)$. Assume wlog that $g_1 \succ_1 f_1$. Consider the following problem: maximize $I_1(u_1 \circ h_1)$

subject to $(h_1,\dots,h_N)$ being feasible and $h_i \succcurlyeq_i g_i$ for all $i = 2,\dots,N$. Notice that the allocation $(g_1,\dots,g_N)$ satisfies these constraints. By standard arguments (e.g. Mas-Colell, Whinston, and Green, 1995, §16.F), since preferences are continuous and strongly monotonic, a solution $(h_1^*,\dots,h_N^*)$ to this problem exists and is Pareto-efficient. Furthermore, for every $i > 1$, $h_i^* \succcurlyeq_i g_i \succcurlyeq_i f_i$, and $h_1^* \succcurlyeq_i g_1 \succ_1 f_1$; that is, $(h_1^*,\dots,h_N^*)$ is a Pareto-efficient allocation that Pareto-dominates $(f_1,\dots,f_N)$. ■

Proof of Proposition 7: Assume that $\bigcap_i \pi_i^c(1_S x_i) \neq \emptyset$ for every feasible, full-insurance allocation $(1_S x_1,\dots,1_S x_N)$, with $x_i \geq 0$ for all $i \in N$.

We first show that every Pareto-efficient allocation must provide full insurance. To do so, consider a feasible allocation $(f_1,\dots,f_N)$. We show that, if this allocation does not provide full insurance, there is a full-insurance allocation that Pareto-dominates it.

For every $i \in N$, let c_i be the certainty equivalent of f_i: that is, $u_i(c_i) = I_i(u_i \circ f_i)$. There are two cases to consider.

Case 1: $\sum_i c_i \geq \bar{\omega} > 0$. Define a new allocation $(1_S x_1,\dots,1_S x_N)$ as follows: for every $i \in N$, let $x_i = \frac{\bar{\omega}}{\sum_j c_j} c_i$. Then $\sum_i x_i = \frac{\bar{\omega}}{\sum_j c_j}\sum_i c_i = \bar{\omega}$, i.e., $(1_S x_1,\dots,1_S x_N)$ is feasible. Since $(f_1,\dots,f_N)$ is not a full-insurance allocation, there is at least one agent i for whom f_i is non-constant; wlog let that be agent 1. By strong monotonicity, $u_1(c_1) = I_1(u_1 \circ f_1) > u_1(0)$; since u_1 is strictly increasing, $c_1 > 0$, and therefore, $x_1 = \frac{\bar{\omega}}{\sum_j c_j} c_1 > 0$. By assumption, there is $P \in \bigcap_i \pi_i^c(1_S x_i)$; by Corollary 14, since in particular $P \in \pi_1^c(1_S x_1)$ and $x_1 > 0$, $P \gg 0$.

For every $i \in N$, by construction $I_i(u_i \circ f_i) = u_i(c_i) \geq u_i(x_i)$. Since $P \in \pi_i^c(1_S x_i)$, this implies that $P(u_i \circ f_i) \geq u_i(x_i)$. Finally, since u_i is strictly concave, $u_i(P(f_i)) \geq P(u_i \circ f_i)$, and this inequality is strict unless f_i is constant; in particular, it is strict for $i = 1$. Therefore,

$$u_i(P(f_i)) \geq P(u_i \circ f_i) \geq u_i(x_i)$$

and the first inequality is strict for agent $i = 1$ and every other agent i for whom f_i is non-constant. Since u_i is strictly increasing, $P(f_i) \geq x_i$ for all i, with strict inequality for at least one agent.

Conclude that $\sum_i P(f_i) > \sum_i x_i = \bar{\omega}$. However, $\sum_i P(f_i) = P(\sum_i f_i) = P(1_S \bar{\omega}) = \bar{\omega}$, because $(f_1,\dots,f_N)$ is feasible: contradiction. Thus, this case cannot occur.

Case 2: $\sum_i c_i < \bar{\omega}$. Let $\epsilon = \frac{\bar{\omega}-\sum_i c_i}{N}$: then, the full-insurance allocation $\big(1_S(c_1+\epsilon),\dots,1_S(c_N+\epsilon)\big)$ is feasible and Pareto-dominates $(f_1,\dots,f_N)$ by strong monotonicity, as claimed.

Conversely, consider a feasible, full-insurance allocation $(1_S y_1,\dots,1_S y_N)$, and suppose that it is not Pareto-efficient. Then, by Remark 6, it is Pareto-dominated by a Pareto-efficient allocation; by the result just proved, under the maintained assumptions, this allocation must be a full-insurance allocation, say $(1_S x_1,\dots,1_S x_N)$. Since preferences are strongly monotonic, this implies that $x_i \geq y_i$ for all i, and the inequality is strict for at least one i. But then $\sum_i x_i > \sum_i y_i = \bar{\omega}$, i.e., $(1_S x_1,\dots,1_S x_N)$ is not feasible: contradiction. Thus, every full-insurance allocation is Pareto-efficient.

Finally, let $(1_S x_1,\dots,1_S x_N)$ be a full-insurance, hence Pareto-efficient allocation. Fix $P \in \bigcap_i \pi_i^{\mathrm{c}}(1_S x_i)$. Since $\sum_i x_i = \bar{\omega} > 0$, there must be some $i \in N$ for whom $x_i > 0$; since $P \in \pi_i^{\mathrm{c}}(1_S x_i)$, by Corollary 14, $P \gg 0$.

Now suppose that, for some $i \in N$ and $g \in \mathbb{R}^S_+$, $g \succ_i 1_S x_i$. Since $P \in \pi_i^{\mathrm{c}}(1_S x_i)$, $P(u_i \circ g) \geq u_i(x_i)$. If g is constant, i.e., $g = 1_S y$ for some $y \in \mathbb{R}^S_+$, then $u_i(y) = I_i(u_i \circ g) > u_i(x_i)$ implies that $y > x_i$ because I_i is normalized and u_i is strictly increasing: thus, $P(g) = y > x_i = P(1_S x_i)$. If instead g is non-constant, then, by strict concavity of u_i and strict positivity of P, $u_i(P(g)) > P(u_i \circ g) \geq u_i(x_i)$, i.e., again $P(g) > x_i = P(1_S x_i)$.

Hence, for all g, $g \succ_i 1_S x_i$ implies $P(g) > P(1_S x_i) = x_i$; equivalently, $P(g) \leq x_i$ implies $1_S x_i \succcurlyeq_i g$. We can then let $t = P(1_S x_i) - P(\omega_i) = x_i - P(\omega_i)$: we get $\sum_i t = \sum_i x_i - \sum_i P(\omega_i) = \bar{\omega} - P(\sum_i \omega_i) = \bar{\omega} - P(1_S \bar{\omega}) = 0$. Hence $t_1,\dots,t_N$ define feasible transfers. Since preferences are strongly monotonic (hence local non-satiated), consumers will exhaust their budget $P(\omega_i) + t_i = x_i$, and the argument just given shows that they will demand $1_S x_1,\dots,1_S x_N$. ■

A.5 Proof of Proposition 12

We need a generalization of the first inclusion in part 3 of Proposition 1:

Lemma 17 *If $\mathscr{G}$ is core-unambiguous for (I,u), then* Core $I \subseteq \pi^{\mathrm{c}}(g)$ *for every $\mathscr{G}$-measurable bundle $g \in \mathbb{R}^S_+ \setminus \{0_S\}$.*

Proof: Fix a $\mathscr{G}$-measurable $g \in \mathbb{R}^S_+ \setminus \{0_S\}$. Suppose $P \in \text{Core}\ I$ and consider a bundle $f \in \mathbb{R}^S_+$. Assume that $P(u \circ g) \geq P(u \circ f)$. Since P is in the core of I and g is $\mathscr{G}$-measurable, hence core-unambiguous, $I(u \circ g) = P(u \circ g) \geq P(u \circ f) \geq I(u \circ f)$. Since this holds for all $f \in \mathbb{R}^S_+$, $P \in \pi^c(g)$. Hence, Core $I \subseteq \pi^c(g)$. ■

Proof of Proposition 12: Let $Q \in \bigcap_i \text{Core}_{\mathscr{E}}\ I_i$ and, for every $i \in N$, let $P_i \in \text{Core}\ I_i$ be the probability that satisfies Condition (ii) in Definition 4. Since each I_i is strongly monotonic, $P_i(\{s\}) > 0$ for every $s \in S$: to see this, note that, if $f \in \mathbb{R}^S_+$ is such that $f(s) = 1$ and $f(s') = 0$ for all $s' \neq s$, then $P_i(\{s\})u_i(1) + [1 - P_i(\{s\})]u_i(0) = P_i(u_i \circ f) \geq I_i(u_i \circ f) > u_i(0)$, which implies $P_i(\{s\}) > 0$. Therefore, by Condition (ii) in Definition 4, $Q(\{s\}) > 0$ for every $s \in S$ as well.

By contradiction, suppose $(f_1, \ldots, f_N)$ is a Pareto-efficient allocation but some bundle f_i, say wlog f_1, is not $\mathscr{E}$-measurable. We construct a new allocation $(g_1, \ldots, g_N)$ that is $\mathscr{E}$-measurable and Pareto-dominates it. For every $i \in N$, every $G \in \mathscr{E}$, and every $s \in G$, let

$$g_i(s) \equiv \sum_{s' \in G} Q(\{s'\}|G) f_i(s') = \sum_{s' \in G} P_i(\{s'\}|G) f_i(s'), \tag{10}$$

where the equality follows from the choice of P and Condition (ii) in Definition 4. That is, $g_i(s)$ is the conditional expectation of f_i given G, where $s \in G$.

First, verify feasibility: for every $G \in \mathscr{E}$ and $s \in G$,

$$\sum_i g_i(s) = \sum_i \sum_{s' \in G} Q(\{s'\}|G) f_i(s') = \sum_{s' \in G} Q(\{s'\}|G) \sum_i f_i(s') = \sum_{s' \in G} \omega(s') = \omega(s).$$

The next-to-last equality follows from the assumption that $(f_1, \ldots, f_N)$ is feasible. The last equality follows from the assumption that $\mathscr{E}$ is the partition induced by ω, so that, if $s \in G$, then $\omega(s') = \omega(s)$ for all $s' \in G$.

Turn to Pareto-dominance. For every $G \in \mathscr{E}$, fix $s_G \in G$. For every $i \in N$, since u_i is strictly concave,

$$P_i(u_i \circ f_i) = \sum_{G \in \mathscr{E}} P_i(G) \sum_{s \in G} P_i(\{s\}|G) u_i(f_i(s)) \leq \sum_{G \in \mathscr{E}} P_i(G) u_i\left(\sum_{s \in G} P_i(\{s\}|G) f_i(s)\right) = \sum_{G \in \mathscr{E}} P_i(G) u_i(g_i(s_G)) = P_i(u_i \circ g_i).$$

The inequality follows from Jensen's inequality, and it is strict for agent 1 and any other agent for whom f_i is not $\mathscr{E}$-measurable (i.e., for which f_i is not constant on every $G \in \mathscr{E}$). The penultimate equality follows from the fact that $\sum_{s \in G} P_i(\{s\}|G) f_i(s) = \sum_{s \in G} Q(\{s\}|G) f_i(s) = g_i(s_G)$.

If $g_i = 0_S$, then, since P_i is strictly positive, $f_i = 0_S$ as well, and so trivially $g_i \succcurlyeq_i f_i$. Furthermore, recall that by assumption there is $G \in \mathscr{E}$ such that f_1 is not constant on G; then, $g_1(s_G) > 0$.

Now consider $i \in N$ such that $g_i \in \mathbb{R}^+_S \setminus \{0_S\}$ (including $i = 1$). Since $P_i \in \text{Core } I_i$, by Lemma 17, also $P_i \in \pi^c_i(g_i)$, because by construction g_i is $\mathscr{E}$-measurable and $\mathscr{E}$ is core-unambiguous for (I_i, u_i). Thus, $P_i(u_i \circ g_i) \geq P_i(u_i \circ f_i)$ implies $g_i \succcurlyeq_i f_i$.

Conclude that $g_i \succcurlyeq_i f_i$ for all $i \in N$. Furthermore, for $i = 1$, since $g_1(s_G) > 0$ for some $G \in \mathscr{E}$, and $P_1(u_1 \circ g_1) > P_1(u_1 \circ f_1)$, by continuity of u_1 and the fact that $P_i(G) > 0$ there is $\epsilon \in (0, g_1(s_G))$ such that $P_1(u_1 \circ (g_1 - 1_G \epsilon)) > P_1(u_1 \circ f_1)$ as well. Then, again by Lemma 17, $g_1 - 1_G \epsilon \succcurlyeq_1 f_1$. Since preferences are strongly monotonic, $g_1 \succ_1 f_1$. This contradicts the assumption that $(f_1, \ldots, f_N)$ was Pareto-efficient. ■

A.6 Calculations for Example 5

Observe first of all that, for all $\phi \in \mathbb{R}^n$,

$$\nabla I(a) \equiv \left(\frac{\partial I(a)}{\partial a(s)}\right)_{s \in S} = \left(P(\{s\}) \left[1 + \sum_{0 \leq j < J} \frac{\partial A(P(\zeta_0 a), \ldots, P(\zeta_{n-1} a))}{\partial \phi_j} \zeta_j(s)\right]\right)_{s \in S}. \tag{11}$$

Thus, the condition in the text ensuring that preferences are strongly monotonic is simply the requirement that all partial derivatives be strictly positive almost everywhere on $\mathbb{U}^S$.

Next, we show that $\nabla A(0_J) = 0_J$. Fix $0 \leq j < J$. Since A is continuously differentiable at 0_J, satisfies $A(0_J) = 0$ and is symmetric about 0_J,

$$\nabla A(0_J) \cdot 1_j = \lim_{t \downarrow 0} \frac{A(0_J + t 1_j) - A(0_j)}{t} = \lim_{t \downarrow 0} \frac{A(t 1_j)}{t} = \lim_{t \downarrow 0} \frac{A(t(-1_j))}{t} = \lim_{t \downarrow 0} \frac{A(0_J + t(-1_j)) - A(0_J)}{t} = \nabla A(0_J) \cdot (-1_j),$$

which clearly requires that $\nabla A \cdot 1_j = \frac{\partial A(0_J)}{\partial \phi_j} = 0$, as claimed. Since $P(\zeta_j 1_S x) = x P(\zeta_j) = 0$, it follows that $\nabla I(1_S x) = P$ for all $x > 0$.

Next, we verify that the specification of adjustment factors and function in Example 5, together with a uniform baseline prior, ensures strong monotonicity. We use Eq. (11): first, note that

$$\frac{\partial A}{\partial \phi_j} = -\frac{1}{2} \theta \cdot \frac{2\theta^{-1} \phi_j}{1 + \theta^{-1} \phi_j^2} = -\frac{\phi_j}{1 + \theta^{-1} \phi_j^2}.$$

Hence,

$$\left|\frac{\partial A}{\partial \phi_j}\right| = \frac{|\phi_j|}{1+\theta^{-1}\phi_j^2} = \frac{|\phi_j|}{1+\theta^{-1}|\phi_j|^2}.$$

Letting $t = |\phi_j|$, this is less then one iff $t < 1+\theta^{-1}t^2$, i.e. iff $t^2-\theta t+\theta > 0$. We study the function $t \mapsto t^2-\theta t+\theta$ for $t \geq 0$. If $t=0$, the function takes the value θ, so we need $\theta > 0$. The derivative of this function at any $t > 0$ (which is also the right derivative at 0) is $2t-\theta$, which shows that this function is strictly convex and has a minimum at $t = \frac{1}{2}\theta$, where it is equal to $\frac{1}{4}\theta^2 - \frac{1}{2}\theta^2+\theta$. This is strictly positive iff $-\frac{1}{4}\theta+1 > 0$, i.e. iff $\theta < 4$, as claimed.

Now consider states $s = s_1, s_2$. Only ζ_0 has non-zero values, and $\zeta_0(s) \in \{1,-1\}$. Therefore, if $\theta \in (0,4)$,

$$1-\frac{\phi_0}{1+\theta^{-1}\phi_0^2}\zeta_0(s)-\frac{\phi_1}{1+\theta^{-1}\phi_1^2}\zeta_1(s) \geq 1-\left|\frac{\phi_0}{1+\theta^{-1}\phi_0^2}\right| > 0.$$

Similarly, in states $s = s_3, s_4$, $\zeta_0(s) = 0$ and $\zeta_1(s) \in \{1,-1\}$, so

$$1-\frac{\phi_0}{1+\theta^{-1}\phi_0^2}\zeta_0(s)-\frac{\phi_1}{1+\theta^{-1}\phi_1^2}\zeta_1(s) \geq 1-\left|\frac{\phi_1}{1+\theta^{-1}\phi_1^2}\right| > 0.$$

so I is strictly increasing.

We now show that, if θ increases, the resulting preference is more GM-ambiguity-averse. By the characterization result in Siniscalchi (2009), it suffices to show that $A(\phi)$ is decreasing in θ for every ϕ. Differentiating $A(\phi)$ with respect to θ,

$$\frac{\partial A(\phi)}{\partial \theta} = -\frac{1}{2}\sum_j \log(1+\theta^{-1}\phi_j^2) - \frac{1}{2}\theta\sum_j \frac{1}{1+\theta^{-1}\phi_j^2}(-\theta^{-2}\phi_j^2);$$

it suffices to show that, for every j and ϕ_j, $\log(1+\theta^{-1}\phi_j^2) > \frac{\theta^{-1}\phi_j^2}{1+\theta^{-1}\phi_j^2}$. Let $t \equiv \theta^{-1}\phi_j^2$, so we need to show that $\log(1+t) > \frac{t}{1+t}$. Both functions equal zero at $t=0$. For $t > 0$, the derivatives of the lhs and rhs are $\frac{1}{1+t}$ and $\frac{1\cdot(1+t)-t(1)}{(1+t)^2} = \frac{1}{(1+t)^2}$ respectively. Since $(1+t)^2 > 1+t$ for $t > 0$, $\frac{1}{1+t} < \frac{1}{(1+t)^2}$, and therefore, for all $t > 0$, $\log(1+t) = \int_0^t \frac{1}{1+s}ds > \int_0^t \frac{1}{(1+s)^2}ds = \frac{t}{1+t}$, as claimed.

Finally, we derive the condition on θ for the desired rankings to hold:

$$\begin{aligned}
& -\log(1+\theta^{-1}(\alpha-1)^2)-\log(1+\theta^{-1}\alpha^2) < -\log(1+\theta^{-1}) \\
\Leftrightarrow \quad & (1+\theta^{-1}(1-\alpha)^2)(1+\theta^{-1}\alpha^2) > 1+\theta^{-1} \Leftrightarrow \quad 1+\theta^{-1}(1-\alpha)^2+\theta^{-1}\alpha^2+\theta^{-2}(1-\alpha)^2\alpha^2 > 1+\theta^{-1} \\
\Leftrightarrow \quad & (1-\alpha)^2+\alpha^2+\theta^{-1}(1-\alpha)^2\alpha^2 > 1 \\
\Leftrightarrow \quad & \theta^{-1} > \frac{1-\alpha^2-(1-\alpha^2)}{\alpha^2(1-\alpha)^2} = \frac{1-\alpha^2-1-\alpha^2+2\alpha}{\alpha^2(1-\alpha)^2} = \frac{2\alpha(1-\alpha)}{\alpha^2(1-\alpha)^2} = \frac{2}{\alpha(1-\alpha)} \Leftrightarrow \quad \theta < \frac{\alpha(1-\alpha)}{2}.
\end{aligned}$$

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