

Risk Sharing in the Small and in the Large

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Abstract

This paper analyzes risk sharing in economies with no aggregate uncertainty when agents have non-convex preferences. In particular, agents need not be globally risk-averse, or uncertainty-averse in the sense of [Schmeidler \(1989\)](#). We identify a behavioral condition under which betting is inefficient (i.e., every Pareto-efficient allocation provides full insurance, and conversely) if and only if agents' subjective beliefs (defined as in [Rigotti, Shannon, and Strzalecki, 2008](#)) have a non-empty intersection. Our condition is consistent with empirical and experimental evidence documenting violations of convexity in either outcomes or utilities. Our results show that the connection between speculative betting and inconsistent beliefs is robust to substantial departures from convexity.

1 Introduction

Consider an exchange economy with a single consumption good and no aggregate uncertainty. It is well understood that in such economy risk-averse and (subjective) expected-utility-maximizing agents will choose to introduce individual uncertainty in the final allocation if and

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only if they have different beliefs. That is, betting can occur in equilibrium only if agents disagree on the probabilities of some events (Milgrom and Stokey, 1982). At the same time, it is also well understood that agents may not have unique probabilistic beliefs when some events relevant for the economy are more ambiguous than others (Ellsberg, 1961). The result on the connection between disagreement of beliefs and betting has been extended to take this into account. For instance, Billot, Chateauneuf, Gilboa, and Tallon (2000) show that risk-averse agents whose preferences satisfy the maxmin expected utility model of Gilboa and Schmeidler (1989) will bet if and only if they do not *share a priori*; i.e., if the sets of beliefs that they employ (in the maxmin representation) do not intersect. This result has been significantly extended by Rigotti et al. (2008, henceforth RSS). They showed that, with a suitable definition of “subjective beliefs,” a similar result holds for any collection of agents whose preferences are suitably well-behaved and, notably, satisfy strict *convexity in consumption*: given any two contingent consumption plans f and g , any nondegenerate convex combination¹ $\alpha f + (1 - \alpha)g$ is (strictly) preferred to the worse of the two. In other words, agents have a (strict) preference for consumption smoothing across states.

Since risk-sharing results imply that, in equilibrium, agents will attain a state-independent (i.e., maximally smooth) consumption profile, one may expect convexity to play a key role. This paper shows that this is not the case. We identify a behavioral assumption that is sufficient to deliver the equivalence between risk sharing and consistency of subjective beliefs, but permits substantial departures from convexity in consumption (though it is implied by it).

We are motivated by empirical, experimental, and theoretical concerns about the convexity assumption. With expected-utility preferences and a single consumption good, convexity in consumption characterizes risk aversion. Beyond expected utility, it implies a combination of risk and ambiguity aversion (we will be more precise below). However, there is empirical evidence that investment decisions are sometimes risk-seeking, either because of intrinsic preferences (e.g., Kumar, 2009), or due to the nature of incentive contracts for fund managers (e.g., Chevalier and Ellison, 1997). In the context of insurance decisions, Wakker, Timmermans, and Machielse (2007) documents ambiguity-seeking behavior; for similar findings in

¹Note that convex combinations (here and in RSS) are *not* mixtures in the Anscombe and Aumann (1963) sense: they are the usual vector-space notion. They represent convex combinations of *consumption*, not *utilities*.

asset markets, see [Brenner and Izhakian \(2012\)](#).

In experimental settings, the classic findings of [Curley and Yates \(1985\)](#) and [Heath and Tversky \(1991\)](#) raise questions about the pervasiveness of aversion to ambiguity (broadly defined). More recent papers cast doubts on the specific formalization of uncertainty aversion as convexity in *utilities*, due to [Schmeidler \(1989\)](#). To elaborate, most parametric representations of ambiguity-sensitive preferences associate with each contingent consumption bundle $f = (f_1, \dots, f_s)$ a utility index $I(u(f))$, where $u(f) = (u(f_1), \dots, u(f_s))$ is the state-contingent utility vector associated with f and I is a function defined over utility vectors.² [Schmeidler \(1989\)](#) defines “uncertainty aversion” as quasiconcavity of I (hence, convexity in the induced preferences over state-contingent utility vectors). However, [L’Haridon and Placido \(2010\)](#) document patterns of behavior that, while intuitively consistent with aversion to ambiguity, cannot be represented by a utility index of the form $f \mapsto I(u(f))$, if I is quasiconcave and consistent with EU for unambiguous bundles f ([Baillon, L’Haridon, and Placido, 2011](#)).

Finally, from a theoretical perspective, the connection between convexity and aversion to risk or ambiguity is not clear-cut. For preferences that are probabilistically sophisticated ([Machina and Schmeidler, 1992](#)) but not EU, intuitive notions of risk aversion do *not* imply convexity in consumption (though they are implied by it): see e.g. [Deikel \(1989\)](#). For preferences that are not probabilistically sophisticated, [Epstein \(1999\)](#) and [Ghirardato and Marinacci \(2002, henceforth GM\)](#) question the identification of ambiguity aversion with convexity in utilities, as do [Baillon et al. \(2011\)](#).

Our results aim at addressing these concerns. We illustrate this by means of three examples in Section 2. First, we exhibit an Edgeworth-box economy in which agents are “locally” ambiguity-seeking at the endowment point. Despite this, our results apply, and equilibrium entails full risk-sharing. Second, we exhibit a probabilistically-sophisticated non-EU preference (based on [Deikel, 1989](#)) that is risk-averse but not convex in consumption, yet satisfies our key behavioral condition. Third, we exhibit a preference that is ambiguity-averse in the sense of GM and can accommodate the behavior documented in [L’Haridon and Placido \(2010\)](#) (and hence violates convexity). Once again, our conditions are satisfied for this preference. These

²In particular, all parametric representations analyzed in of RSS (see their Section 2.4) have this form, with u strictly concave.

example demonstrate that risk sharing may still obtain despite significant departures from convexity in consumption—including locally ambiguity-seeking behavior.

We now describe our main results in greater detail. As noted above, we identify a behavioral condition under which betting obtains if and only if agents’ subjective beliefs are inconsistent. That is, the presence of speculative trade can be interpreted as evidence of heterogeneous beliefs. We adopt the notion of “subjective beliefs as supporting price vectors” used by RSS. In general, subjective beliefs may be different at different consumption bundles; RSS employ an axiom that ensures that they are constant across riskless consumptions. Our main risk-sharing result (Theorem 3) does not adopt this assumption; however, to facilitate comparison with RSS, we also provide one that does (Theorem 4).

The condition we propose, *strict pseudoconcavity at certainty*, (SPC), admits geometric, economic, and decision-theoretic interpretations.³ Suppose there are two states, and consider a constant consumption bundle in that yields x units of the good in each state. Suppose further that the indifference curve through x is smooth at x .⁴ Then SPC requires that the entire indifference curve through x lie strictly above the tangent line. A natural economic interpretation is that, if the tangent at x is viewed as a budget line, then any point in the budget set is strictly worse than x . From a decision-theoretic perspective, view the tangent line at x as the level curve of the expected-value function $f \mapsto P \cdot f$ going through x , where P is a probability distribution. Then any bundle f with expectation $P \cdot f = x$ is strictly worse than x . That is, the expectation of any non-constant contingent consumption bundle is strictly better than the bundle itself—a (weak) notion of risk aversion. For preferences which can be represented in the form $I(u(f))$, Corollary 6 provides further insight into condition SPC, and also simplifies the task of verifying whether it holds. Finally, in the main text, we state condition SPC as an assumption on the functional representation of preferences. However, in Appendix B we characterize it purely in terms of preferences, leveraging the results of [Ghirardato and Siniscalchi \(2012\)](#). Section 7 relates condition SPC to notions of aversion to ambiguity that have been

³See Remark 1 and the ensuing discussion for details.

⁴Without smoothness, SPC is a condition on the elements of the [Clarke \(1983\)](#) differential of the representation at certainty. If preferences are convex, the Clarke differential coincides with the convex-analysis differential, and SPC follows from strict convexity (Proposition 2).

considered in the literature. In particular, we show that it is closely related to GM-ambiguity aversion.

This paper is organized as follows. Section 2 provides examples of non-convex preferences that satisfy our assumptions. Section 3 introduces the formal setup. Section 4 contains the main risk-sharing results. Section 5 analyzes condition SPC for preferences that admit a representation of the form $I(u(f))$, with u strictly concave. Section 6 provides a sketch of the proof of Theorems 3 and 4, and contains additional results. Section 7 describes the relationship with different notions of aversion to ambiguity. Finally, Appendix A contains further examples, and Appendix B provides a behavioral characterization of condition SPC.

Related literature The relation with RSS and [Billot et al. \(2000\)](#) has already been discussed. We note that these papers allow for an arbitrary state space; for simplicity, we restrict attention to finite states.

[Strzalecki and Werner \(2011\)](#) extend and adapt the risk-sharing results in RSS to economies with aggregate uncertainty and convex preferences. An investigation of risk sharing in economies with aggregate uncertainty and non-convex preferences is left to future research.⁵

[Araujo, Chateauneuf, Gama-Torres, Novinski, et al. \(2014\)](#) study the existence of equilibrium in economies with aggregate uncertainty, where both uncertainty-averse and uncertainty-loving agents are present. They also study equilibrium risk sharing. Our analysis is complementary to theirs; we do not require any agent to be globally uncertainty-averse, and do not require the presence of uncertainty-loving agents. Indeed condition SPC is inconsistent with global uncertainty appeal.

[Billot, Chateauneuf, Gilboa, and Tallon \(2002\)](#) provide a version of Proposition 10 for Choquet-expected utility preferences (CEU; [Schmeidler, 1989](#)). They also prove a risk-sharing result for such preferences that does not assume convexity (or SPC) but requires large economies, with a continuum of agents of each “type.”

[Dominiak, Eichberger, and Lefort \(2012\)](#) consider an economy with two CEU agents and riskless (full-insurance) endowments, extending the prior analysis of [Kajii and Ui \(2006\)](#) which

⁵An earlier version of this paper (available from our web pages) provided a counterpart to our Proposition 10 for non-convex economies with aggregate uncertainty.

assumed convexity. They provide a condition which is necessary and sufficient for the non-existence of Pareto-improving trades. Their analysis relies on the fact that there are only two agents in the economy, whose initial endowment is constant; on the other hand, it does not require either convexity or pointwise ambiguity aversion.

[Marinacci and Pesce \(2013\)](#) consider preferences that are both GM-ambiguity averse and invariant biseparable ([Ghirardato, Maccheroni, and Marinacci, 2004](#)). They study the impact of changes in GM-ambiguity aversion on efficient and equilibrium allocations. Though they do not focus on risk sharing, they independently derive a version of our Proposition 10. See however Example 4 in Appendix A on the implications of invariant biseparability.

There is a large literature on equilibrium analysis with non-convexities. For a survey that focuses on non-convex production sets, see [Brown \(1991\)](#). Our proof of Proposition 8 employs a result by [Bonnisseau and Cornet \(1988\)](#) that allows for non-convexities in consumption. An alternative approach to circumvent violations of convexity is to consider large economies (see e.g. [Mas-Colell, Whinston, and Green, 1995](#), §17.I); we do not follow this approach, and instead consider a fixed, finite number of agents.

Finally, [Ghirardato and Siniscalchi \(2012\)](#) provides a behavioral foundation for the analysis in the present paper. Leveraging the results therein, Appendix B in this paper characterizes condition SPC in terms of the agent's preferences.

2 Risk sharing without convexity: examples

The first example (Section 2.1) demonstrates that risk sharing can obtain in an economy in which agents are ambiguity-seeking near the endowment point. The second example (Section 2.2) describes a probabilistically sophisticated, risk-averse non-EU preference that does not satisfy convexity in consumption, but satisfies condition SPC. The third example (Section 2.3) describes a preference that is ambiguity-averse in the sense of [Ghirardato and Marinacci \(2002\)](#), though not in the sense of [Schmeidler \(1989\)](#), and can thus account for the behavior in the reflection example of [Machina \(2009\)](#); again, condition SPC holds, whereas convexity in consumption fails.

In all three examples, as in the rest of the paper, we consider a finite state space S and

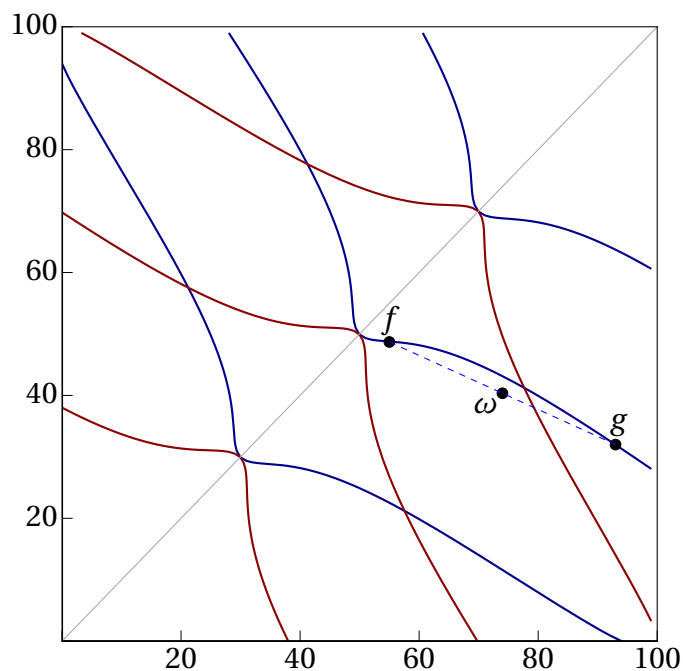


Figure 1: An economy with locally ambiguity-seeking agents

preferences over contingent consumption bundles $f \in \mathbb{R}_+^S$.

2.1 Risk- and ambiguity-seeking behavior near the endowment point

We begin with a graphical illustration. Consider the single-good, two-state Edgeworth-box economy of Fig. 1. The endowment point ω is the midpoint between the allocations f and g , and lies below Agent 1's indifference curve going through these points. In particular, starting at ω , Agent 1 would strictly prefer to carry out the trade $g - 1_S \omega$ and move to g , even though this entails increasing the volatility of her consumption across the two states.

Despite this, preferences are such that the only points of tangency between 1's and 2's indifference curves are along the certainty line. Thus, at every efficient allocation, agents fully insure one another, i.e., they share risks. By the First Welfare Theorem, this holds a fortiori for all equilibrium allocations.

We now describe the preferences in Fig. 1 analytically. Both agents have VEU preferences

(Siniscalchi, 2009), which are not necessarily convex:

$$V(f) = \sum_{s \in S} P_s u(f_s) + A \left(\sum_{s \in S} P_s \zeta_{0,s} u(f_s), \dots, \sum_{s \in S} P_s \zeta_{J-1,s} u(f_s) \right), \quad (1)$$

where u is a strictly increasing, differentiable, and strictly concave Bernoulli utility function, $P \in \Delta(S)$ is the *baseline prior*, $\zeta_0, \dots, \zeta_{J-1} \in \mathbb{R}^S$ are *adjustment factors* that satisfy $\sum_s P_s \zeta_{j,s} = 0$ for each j , and $A: \mathbb{R}^J \rightarrow \mathbb{R}$ (the *adjustment function*) satisfies $A(\phi) = A(-\phi)$ for all $\phi \in \mathbb{R}^J$.

We show in Proposition 7 that, if the adjustment function A is also smooth and non-positive, and an additional joint assumption on P, ζ , and A holds, the functional V thus defined satisfies all the assumptions of Theorem 4, our stronger risk-sharing result. Furthermore, the set of subjective beliefs consists of a single element, the baseline prior P . Thus, in an economy in which all agents have such VEU preferences, with possibly different parameters, risk-sharing obtains *if and only if they have a common baseline prior*.

Figure 1 is obtained using the following parameterization: for both agents, P is uniform, $J = 1$, $\zeta_0 \equiv \zeta = [-1, 1]$, and the adjustment function takes the form

$$A(\phi) = -\frac{1}{2} \theta \log \left(1 + \frac{\phi^2}{\theta} \right)$$

where $\theta \in (0, 4)$; note that $A \leq 0$. The two agents differ in the value of θ , and in their utility function u ; in Fig. 1, $\theta_1 = 3$ and $u_1(x) = x^{0.9}$ for agent 1, and $\theta_2 = 3.5$ and $u_2(x) = x^{0.95}$ for agent 2. This parameterization satisfies all the assumptions of Proposition 7: see Appendix E.

The example in Section 2.3 below employs the same class of VEU preferences, and discusses its properties further.

2.2 Probabilistically sophisticated, non-expected utility preferences

This example is based upon the proof of Proposition 1 in Dekel (1989). Fix a probability distribution P over the state space S , and consider the preferences represented by

$$V(f) = g \left(\sum_s P_s u(f_s) \right) + g \left(\sum_s P_s f_s \right), \quad (2)$$

where $u: \mathbb{R}_+ \rightarrow \mathbb{R}$ is strictly increasing and strictly concave, and $g: \{u(r) : r \in \mathbb{R}_+\} \rightarrow \mathbb{R}$ is strictly increasing and differentiable.

These preferences are probabilistically sophisticated and risk-averse, in the (strong) sense that they exhibit aversion to mean-preserving spreads. To see this, let \mathcal{D} be the set of all cumulative distribution functions (CDFs) on $[0, \infty)$, and define a functional $W : \mathcal{D} \rightarrow \mathbb{R}$ by letting

$$W(F) = g\left(\int u(x)dF(x)\right) + g\left(\int x dF(x)\right).$$

Then, for every bundle $f \in \mathbb{R}_+^S$, $V(f) = W(F_f)$, where F_f is the CDF induced by f and the probability P by letting $F_f(x) = P(\{s : f(s) \leq x\})$. Thus, a decision-maker with the preferences represented by Eq. (2) reduces uncertainty to risk. Appendix D verifies that the functional W satisfies monotonicity with respect to first-order stochastic dominance, a suitable form of continuity, and aversion to mean-preserving spreads.⁶

However, these preferences do not necessarily satisfy convexity in consumption. For instance, the indifference curves in Figure 2 are drawn for a two-point state space, with $u(x)$ a positive affine transformation⁷ of $-\frac{1}{1+x}$, P uniform, and $g(r) = e^{2(r-3)}$.

Yet, these preferences satisfy our condition SPC. We show in the Appendix that the Clarke differential of V at every constant bundle x is a multiple of the probability vector P . It suffices to show that, for every non-constant bundle f , if $x = P \cdot f$, then $V(f) < V(x)$ (see Remark 1). This follows from the assumptions that u is strictly concave and g is strictly increasing: by Jensen's inequality, $\sum_s u(f_s)P_s < u(x)$, and so $V(f) = g(\sum_s P_s u(f_s)) + g(P \cdot f) < g(u(x)) + g(x) = V(1_S x)$.

Hence, Theorem 3 implies that agents with such preferences (but possibly different functions u and g) will engage in mutually beneficial bets if and only if their beliefs P differ. \square

2.3 Machina's reflection paradox

Consider the family of VEU preferences in Eq. (1). As illustrated in Section 2.1, these preferences are not necessarily convex in either consumption or utility; however, under suitable

⁶Dekel (1989, Proposition 1) shows this for CDFs on a compact interval $[0, M]$. Since the set of consumption levels is unbounded above in the present paper, for completeness we extend the results in the Appendix.

⁷Similarly to the construction in Dekel (1989), we fix two bundles f_1, f_2 such that $P \cdot f_1 > P \cdot f_2$ and f_2 is strictly preferred to f_1 by an EU decision maker with beliefs P and utility $v(x) = -\frac{1}{1+x}$. To obtain u , we normalize v so that $P \cdot u \circ f_1 = P \cdot f_2$ and $P \cdot u \circ f_2 = P \cdot f_1$.

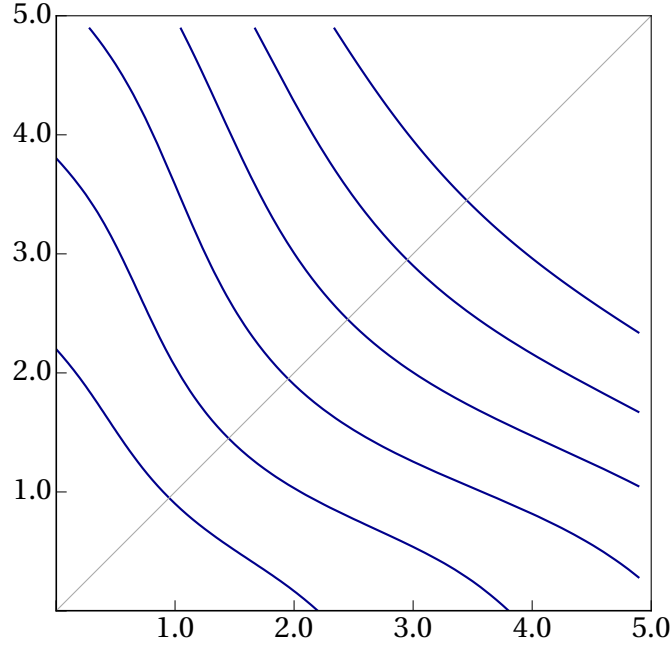


Figure 2: Risk-averse, non-convex preferences (see §2.2)

assumptions on the parameters P, ζ, A , they satisfy the assumptions of our risk-sharing results. We now demonstrate that these preferences can accommodate the modal preferences in the “reflection example” of Machina (2009) (see also L’Haridon and Placido, 2010; Baillon et al., 2011). Let $S = \{s_1, s_2, s_3, s_4\}$ and assume that the events $\{s_1, s_2\}$ and $\{s_3, s_4\}$ are unambiguous and equally likely, but no further information is provided as to the relative likelihood of s_1 vs. s_2 and s_3 vs. s_4 . Furthermore, the draw of s_1 vs. s_2 and s_3 vs. s_4 are perceived as being independent. Consider the bets in Table 1.

	s_1	s_2	s_3	s_4
f^1	\$4,000	\$8,000	\$4,000	\$0
f^2	\$4,000	\$4,000	\$8,000	\$0
f^3	\$0	\$8,000	\$4,000	\$4,000
f^4	\$0	\$4,000	\$8,000	\$4,000

Table 1: Machina’s reflection example. Reasonable preferences: $f^1 < f^2$ and $f^3 > f^4$

Machina (2009) argues on the basis of symmetry considerations that the preference ranking $f^1 < f^2$ and $f^3 > f^4$ is plausible and intuitively consistent with aversion to ambiguity; L’Haridon and Placido (2010) verify that these rankings do occur in an experimental setting. However, Baillon et al. (2011) show that preference models that satisfy uncertainty aversion a la Schmeidler (1989), and are consistent with EU in the absence of ambiguity, cannot accommodate this behavior. We now provide a parameterization of the VEU preferences in Eq. (1) that can, and is consistent with GM’s “comparative” notion of aversion to ambiguity.⁸

Assume a uniform baseline prior P and two adjustment factors $\zeta_0, \zeta_1 \in \mathbb{R}^S$:

$$\zeta_0 = [1, -1, 0, 0] \quad \text{and} \quad \zeta_1 = [0, 0, 1, -1].$$

The adjustment function is a two-factor analog of the one considered in Section 2.1:

$$A(\phi) = A(\phi_0, \phi_1) = -\frac{1}{2}\theta \sum_{j=0,1} \log\left(1 + \frac{\phi_j^2}{\theta}\right)$$

where $\theta \in (0, 4)$. Finally, let $u(0) = 0$, $u(8,000) = 4$, and $u(4,000) = 4\alpha$, for some $\alpha \in (\frac{1}{2}, 1)$. Appendix E shows that this specification of the parameters P, A, ζ_0, ζ_1 yields a strictly monotonic preference. Furthermore, while this parameterization does not satisfy the Uncertainty Aversion axiom of Schmeidler (1989), it is ambiguity-averse in the sense of GM: see Siniscalchi (2009), Proposition 2.⁹ Finally, Appendix E shows that the rankings $f^1 < f^2$ and $f^3 > f^4$ obtain iff $0 < \theta < \frac{\alpha(1-\alpha)}{2}$. □

3 Setup

We consider an Arrow-Debreu economy under uncertainty with finitely many states S , a single good that can be consumed in non-negative quantity, and N consumers.

⁸A similar example is provided in Siniscalchi (2009), but the VEU preferences described therein are not smooth and violate SPC, our main preference assumption.

⁹For VEU preferences, $A \leq 0$ characterizes preferences that are ambiguity-averse in the sense of GM; on the other hand, uncertainty aversion (convexity) requires that A be non-positive and concave.

3.1 Decision-theoretic assumptions

We begin by describing consumers' preferences. To simplify notation, in this section we do not use consumer indices. We complete the description of the economy in section 3.2.

Behavior is described by a preference relation \succsim over bundles (contingent consumption plans) $f \in \mathbb{R}_+^S$. We assume that \succsim is represented by a function $V : \mathbb{R}_+^S \rightarrow \mathbb{R}$: that is, for every pair $f, g \in \mathbb{R}_+^S$, $f \succsim g$ if and only if $V(f) \geq V(g)$.

Given $n \geq 1$, an open subset B of \mathbb{R}^n , and a function $F : B \rightarrow \mathbb{R}$, the **Clarke differential** of F at $b \in B$ (Clarke, 1983) is

$$\partial F(b) = \text{cl conv} \left\{ \lim_{k \rightarrow \infty} \nabla F(b^k) : (b^k) \rightarrow b, \nabla F(b^k) \text{ exists } \forall k \right\}; \quad (3)$$

The Clarke differential is non-empty at any $b \in B$ where F is locally Lipschitz. The function F is **nice** at $b \in B$ if $0_S \notin \partial F(b)$, where $0_S = (0, \dots, 0) \in \mathbb{R}^n$. Appendix C.1 provides additional results on the characterization of Clarke differentials.

We summarize our basic decision-theoretic assumptions in the following:

Assumption 1 The relation \succsim admits a representation V satisfying the following properties:

1. V is strongly monotonic: that is, $f \geq g$ and $f \neq g$ imply $V(f) > V(g)$;¹⁰
2. V is locally Lipschitz at every $f \in \mathbb{R}_{++}^S$;
3. for every $x > 0$, V is nice at $1_S x$.

Ghirardato and Siniscalchi (2012) (henceforth GS) argue that most parametric models of ambiguity-sensitive preferences admit a representation where V satisfies properties 2 and 3; for instance, this is the case if V is monotonic and concave, or if it is translation-invariant. Loosely speaking, V is nice at a point $f \in \mathbb{R}_+^S$ if preferences remain non-trivial (i.e., not flat) in arbitrarily small neighborhoods of f . GS also provide axioms that ensure the existence of a locally Lipschitz, nice representation; see Appendix B for details. Since strong monotonicity can clearly be expressed in terms of preferences, it follows that all of the requirements in Assumption 1 admit a behavioral characterization.

¹⁰ V is monotonic if $f \geq g$ implies $V(f) \geq V(g)$; it is strictly monotonic if $f(s) > g(s)$ for all s implies $V(f) > V(g)$. Many, but not all results in the Appendix hold if V is strictly, but not strongly monotonic.

RSS introduce a set of measures, which they call “subjective beliefs,” that plays a key role in the analysis of risk sharing. Denote by $\Delta(S)$ the unit simplex in \mathbb{R}^S . For every $f \in \mathbb{R}_+^S$, let

$$\pi(f) = \{P \in \Delta(S) : \forall g \in \mathbb{R}_+^S, V(g) \geq V(f) \implies P \cdot g \geq P \cdot f\}. \quad (4)$$

That is, $\pi(f)$ is the set of (normalized) prices such that any bundle that is weakly preferred to f is not less expensive than f . This is the usual notion of “quasi-optimality” in equilibrium theory. Alternatively, we can interpret each $P \in \pi(f)$ as representing a risk-neutral SEU preference whose better-than set at f contains the better-than set of \succsim at f .

3.2 The economy

An economy is a tuple $(N, (\succsim_i, \omega_i)_{i \in N})$, where N is the collection of agents, and for every i , agent i is characterized by preferences \succsim_i over \mathbb{R}_+^S and has an endowment $\omega_i \in \mathbb{R}_+^S$. As in RSS, we assume that there is no aggregate uncertainty: formally, $\sum_i \omega_i = 1_S \bar{x}$ for some $\bar{x} > 0$.

An *allocation* is a tuple (f_1, \dots, f_N) such that $f_i \in \mathbb{R}_+^S$ for each $i \in N$; as usual f_i is the contingent-consumption bundle assigned to agent i . The allocation (f_1, \dots, f_N) is *feasible* if $\sum_i f_i = \sum_i \omega_i$; it is a *full-insurance allocation* if, for every consumer i , $f_i = 1_S x$ for some $x \in \mathbb{R}_+$; it is *Pareto-efficient* if it is feasible, and there is no other feasible allocation (g_1, \dots, g_N) such that $g \succsim f$ for all i , and $g_j \succ_j f_j$ for some j .

For each $i \in N$, we denote by V_i and, respectively, $\pi_i(\cdot)$ the representation of i 's preferences and her sets of subjective beliefs.

4 Risk Sharing

To begin, it is useful to restate the main result of RSS for convex preferences.¹¹

¹¹Strictly speaking, the assumptions in Theorem 1 are slightly stronger than those in RSS's Proposition 9. Specifically, we maintain the assumption that each V_i is locally Lipschitz; RSS only assume continuity. We retain all our assumptions to streamline the exposition. Also note that all the parametric representations analyzed in RSS are concave, hence locally Lipschitz.

Theorem 1 (cf. RSS, Proposition 9) *Suppose that, for each $i \in N$, Assumption 1 holds, and that furthermore \succsim_i is strictly convex¹² and $\pi_i(1_S x) = \pi_i(1_S)$ for every $x > 0$. Then the following are equivalent:*

- (i) *There exists an interior, full-insurance Pareto-efficient allocation;*
- (ii) *Every Pareto-efficient allocation is a full-insurance allocation;*
- (iii) *Every feasible, full-insurance allocation is Pareto-efficient;*
- (iv) $\bigcap_i \pi_i(1_S) \neq \emptyset$.

The implications (ii) \Rightarrow (iii) \Rightarrow (i) hold for all strongly monotonic and continuous preferences.¹³ However, the implication (i) \Rightarrow (iv) is proved by invoking the Second Welfare Theorem, which requires convexity. RSS's argument for (iv) \Rightarrow (ii) also invokes strict convexity.

We now introduce our main assumption on preferences.

Definition 1 The function V is **strictly pseudoconcave at $f \in \mathbb{R}_{++}^S$** if

$$\forall g \in \mathbb{R}_+^S \setminus \{f\}, \quad V(g) \geq V(f) \implies \forall Q \in \partial V(f), \quad Q \cdot (g - f) > 0. \quad (5)$$

The functional V satisfies **strict pseudoconcavity at certainty (SPC)** if it is strictly pseudoconcave at $1_S x$ for all $x > 0$.

The intuition for this condition is sharpest in case V is continuously differentiable at a point f , in which case the Clarke differential equals the gradient of V at f . Then, V is strictly pseudoconcave at f if, whenever a bundle g is weakly preferred to f , moving from f in the direction of g by a small (infinitesimal) amount is strictly beneficial. Appendix B provides a behavioral characterization of strict pseudoconcavity that formalizes this intuition leveraging Theorem 7 in GS.¹⁴

In the Introduction we described alternative interpretations of condition SPC. These build upon the following characterization:

¹²That is, for $f, g \in \mathbb{R}_+^S$ with $f \neq g$, $f \succsim_i g$ implies $\alpha f + (1 - \alpha)g \succ_i g$ for all $\alpha \in (0, 1)$. Equivalently, V_i is strictly quasiconcave: that is, $f \neq g$ and $V_i(f) \geq V_i(g)$ implies $V_i(\lambda f + (1 - \lambda)g) > V_i(g)$ for all $\lambda \in (0, 1)$.

¹³For (ii) \Rightarrow (iii), the key step is in Remark 6, which follows from standard results.

¹⁴The notion of (non-strict) pseudoconcavity was introduced by [Mangasarian \(1965\)](#) for differentiable functions; for a definition of (strict) pseudoconvexity for non-smooth functions and related results, see e.g. [Penot and Quang \(1997\)](#).

Remark 1 If V is nice at $f \in \mathbb{R}_{++}^S$, then it is strictly pseudoconcave at f if and only if

$$\forall g \in \mathbb{R}_+^S \setminus \{f\}, Q \in \partial V(f): \quad Q \cdot g = Q \cdot f \implies V(g) < V(f). \quad (6)$$

Thus, first, the hyperplane associated with the vector Q and going through f is tangent to the indifference curve of V going through f , and strictly supports it. Second, interpreting Q as a price vector, any point on the associated budget line (hence, by monotonicity, any point in the budget set) is strictly worse than f . Third, in the special case of a constant bundle $f = 1_S x$, if one interprets the normalized vector $P = Q/(Q \cdot 1_S)$ as a “local belief,” then any bundle g with P -expected value x is worse than x .

The following result shows that condition SPC holds in particular when preferences are strictly convex, as is assumed in RSS’s risk-sharing result.

Proposition 2 *If V satisfies Assumption 1 and is strictly quasiconcave, then it is strictly pseudoconcave at every $f \in \mathbb{R}_{++}^S$ where it is nice; in particular, it satisfies condition SPC.¹⁵*

Thus, assuming *global* strict quasiconcavity ensures that condition SPC holds. However, condition SPC restricts the behavior of the functional V and of its differential *only at certainty*. This allows for violations of (strict or weak) quasiconcavity elsewhere on its domain. As the example in Section 2.3 illustrates, such violations are consistent with interesting patterns of behavior. We provide additional examples in Appendix A.

With this, our main result is:

Theorem 3 *Suppose that, for each $i \in N$, Assumption 1 holds, and furthermore V_i satisfies SPC. Then the following are equivalent:*

- (ii) *Every Pareto-efficient allocation is a full-insurance allocation;*
- (iii) *Every feasible, full-insurance allocation is Pareto-efficient;*
- (iv) *For every feasible, full insurance allocation $(1_S x_1, \dots, 1_S x_N)$,*

$$\bigcap_i \pi_i(1_S x_i) \neq \emptyset.$$

¹⁵For the converse implication, [Penot and Quang \(1997\)](#) show that a locally Lipschitz function on a Banach space that satisfies strict pseudoconcavity *everywhere* is strictly quasiconcave.

Furthermore, under the above equivalent conditions, every interior, feasible full-insurance allocation is a competitive equilibrium with transfers.

Items (ii)–(iv) in the statement above correspond to items (ii)–(iv) in Theorem 1; there is no condition corresponding to (i). See Example 5 in Appendix A for further discussion.

In addition to replacing strict convexity with condition SPC, Theorem 3 differs from RSS’s risk-sharing result in two related aspects. On one hand, RSS assume that the subjective belief sets $\pi_i(1_S x_i)$ are constant at certainty; there is no corresponding assumption in Theorem 3 (cf. Example 6 in Appendix A). On the other hand, the condition in item (iv) of Theorem 1 (RSS’s result) involves agents’ preferences alone, whereas condition (iv) in Theorem 3 involves both preferences and endowments—agents’ point core sets must have a non-empty intersection at all *feasible* allocations.

It turns out that, if we additionally adopt RSS’s assumption that subjective beliefs at certainty are constant, then we can similarly state condition (iv) purely in terms of preferences. Consider the following definition:

Definition 2 V_i satisfies **condition BIC** (Belief Invariance at Certainty) if $\pi_i(1_S x) = \pi_i(1_S)$ for all $x > 0$.

RSS introduce an axiom that characterizes condition BIC for convex preferences. In Appendix B, we show that their axiom characterizes BIC for non-convex preferences as well.

Theorem 4 Suppose that, for each $i \in N$, Assumption 1 holds, and furthermore V_i satisfies SPC and BIC. Then the following are equivalent:

- (i) There exists an interior, full-insurance Pareto-efficient allocation;
- (ii) Every Pareto-efficient allocation is a full-insurance allocation;
- (iii) Every feasible, full-insurance allocation is Pareto-efficient;
- (iv) $\bigcap_i \pi(1_S) \neq \emptyset$.

Furthermore, under the above equivalent conditions, every interior, feasible full-insurance allocation is a competitive equilibrium with transfers.

Thus, under assumptions SPC and BIC, we obtain a close counterpart to RSS’s risk-sharing result (Theorem 1). The preferences in the example in Section 2.3 satisfy both BIC and SPC,

though they are not convex.

5 Preference representations with strictly concave utility

Most representations of ambiguity-sensitive preferences used in applications decompose the functional V into a Bernoulli utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$, and an aggregator $I : \mathbb{R}^S \rightarrow \mathbb{R}$: that is, for every $f \in \mathbb{R}_+^S$,

$$V(f) = I(u \circ f),$$

where $u \circ f = (u(f(s)))_{s \in S} \in \mathbb{R}^S$ is the utility vector associated with the bundle f . For instance, MEU preferences admit such a representation, with $I : \mathbb{R}^S \rightarrow \mathbb{R}$ given by

$$I(a) = \min_{P \in D} \int a dP \quad \forall a \in \mathbb{R}^S$$

where $D \subseteq \Delta(S)$. Analogously, for smooth ambiguity-averse preferences (Klibanoff, Marinacci, and Mukerji, 2005),

$$I(a) = \phi^{-1} \left(\int_{\Delta(S)} \phi \left(\int_S a dP \right) d\mu \right) \quad \forall a \in u(X)^S,$$

where μ is a (second-order) probability on $\Delta(S)$ and ϕ is a concave (second-order) utility defined on the range of u .

Furthermore, the Bernoulli utility function u is often assumed to be strictly concave and otherwise well-behaved in applications. When this is the case, it is possible to provide easy-to-check conditions on the functional I that imply that $V = I \circ u$ satisfies SPC. We first summarize the relevant assumptions.

Assumption 2

1. $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is strictly increasing, strictly concave, and differentiable;
2. $I : \mathbb{R}^S \rightarrow \mathbb{R}$ is normalized, strongly monotonic, locally Lipschitz and nice at each $1_S u(x)$, $x > 0$.

The following key assumption is the counterpart of Condition SPC. Unlike the latter, it refers to the function I , which is defined over utility vectors, rather than acts.

Definition 3 Fix an act $f \in \mathbb{R}_{++}^S$. The function I is ∂ -**quasiconcave** at $u \circ f$ if

$$\forall g \in \mathbb{R}_+^S, \quad I(u \circ g) \geq I(u \circ f) \implies \forall Q \in \partial I(u \circ f), \quad Q \cdot (u \circ g - u \circ f) \geq 0. \quad (7)$$

The function I satisfies **differential quasiconcavity at certainty (DQC)** if it is ∂ -quasiconcave at $1_S u(x)$ for all $x > 0$.

To compare with SPC, consider an act g such that $V(g) = I(u \circ g) \geq I(u \circ f) = V(f)$. Strict pseudoconcavity at f requires that $Q \cdot (g - f) > 0$ for all $Q \in \partial V(f)$; on the other hand, ∂ -quasiconcavity at $u \circ f$ requires that $Q \cdot (u \circ g - u \circ f) \geq 0$ for all $Q \in \partial I(u \circ f)$. As noted above, these assumptions pertain to different objects: in general, $\partial V(f) \neq \partial I(u \circ f)$. Yet the interpretation is similar: if g is at least as good as f , then moving from the utility vector $u \circ f$ towards the vector $u \circ g$ should increase the value of I , at least weakly. Note also that SPC requires a strict inequality, whereas DQC allows for a weak one. As Proposition 5 shows, when u is strictly concave, this is sufficient to imply that V satisfies SPC.

The main result of this section is

Proposition 5 *If (I, u) satisfy Assumption 2, then V satisfies Assumption 1. If in addition I satisfies DQC, then SPC holds.*

(The term “ ∂ -quasiconcave” is due to [Penot and Quang, 1997](#).)

Condition DQC holds in two interesting special cases. We need two additional definitions. The first is due to GM, to which we refer the reader for interpretation.

$$\text{Core } I = \{P \in \Delta(S) : \forall f \in \mathbb{R}_+^S, I(u \circ f) \leq P \cdot u \circ f\}. \quad (8)$$

The second is due to [Clarke \(1983\)](#). A locally Lipschitz function $J : \mathbb{R}^S \rightarrow \mathbb{R}$ is **regular** at $b \in \mathbb{R}^S$ if its directional derivative

$$J'(b; a) = \lim_{t \downarrow 0} \frac{J(b + t a) - J(b)}{t} \quad (9)$$

is well-defined for all $a \in \mathbb{R}^S$, and coincides with $\max_{Q \in \partial J(b)} Q \cdot a$: see [Clarke \(1983, Def. 2.3.4\)](#). If J is continuously differentiable at b , then it is regular there ([Clarke, 1983, Corollary to Proposition 2.2.1](#), and [Proposition 2.3.6 \(a\)](#)).

Corollary 6 *Suppose that (I, u) satisfy Assumption 2. Then I satisfies DQC, and hence SPC holds, if one of the conditions below is satisfied:*

1. I is quasiconcave, or
2. $\text{Core } I \neq \emptyset$ and I is regular at every $1_S u(x)$, $x > 0$.

For instance, the functional I in the MEU representation is normalized, strongly monotonic, globally Lipschitz, and nice everywhere on its domain; furthermore, it is concave. The functional I in the smooth ambiguity-averse representation is also normalized, and if ϕ is continuously differentiable, satisfies $\phi'(r) > 0$ for all $r \in u(\mathbb{R}_{++})$, and is strictly concave, then I is also strongly monotonic, locally Lipschitz, nice at certainty, and quasiconcave.¹⁶ Thus, condition 1 of Corollary 6 holds, and SPC is satisfied. Analogous conclusions hold for other parametric uncertainty-averse models, such as variational preferences (Maccheroni, Marinacci, and Rustichini, 2006) and confidence-function preferences (Chateauneuf and Faro, 2009). Of course, these preference models also satisfy the assumptions in RSS.

A convenient class of preferences that satisfies Condition 2 of Corollary 6, but is not necessarily covered by RSS's result, is the family of VEU preferences (Siniscalchi, 2009) that are continuously differentiable (hence, regular) and GM-ambiguity-averse, but not necessarily convex. A VEU representation is a pair (I, u) with¹⁷

$$I(a) = P \cdot a + A(P \cdot (\zeta_0 a), \dots, P \cdot (\zeta_{J-1} a)), \quad (10)$$

where $P \in \Delta(S)$ (the *baseline prior*), $0 \leq J \leq |S|$, each $\zeta_j \in \mathbb{R}^S$ (an *adjustment factor*) satisfies $P(\zeta_j) = 0$, and $A : \mathbb{R}^J \rightarrow \mathbb{R}$ (the *adjustment function*) satisfies $A(\phi) = A(-\phi)$ for all $\phi \in \mathbb{R}^J$. The preferences in the example of Section 2.3 are a special case in which $J = 1$.

Proposition 7 *Assume that (I, u) is a VEU representation such that u is strictly increasing, differentiable and strictly concave, A is continuously differentiable with $A(\phi) \leq 0$ for all $\phi \in \mathbb{R}^J$,*

¹⁶A direct calculation shows that, under the stated assumptions on ϕ , I is continuously differentiable, and thus locally Lipschitz; furthermore, its gradient at any $1_S \gamma$, $\gamma \in u(\mathbb{R}_{++})$, is $\int_{\Delta(S)} P d\mu \neq 0_S$, so I is nice at certainty. Finally, since $a \mapsto \int_{\Delta(S)} \phi(\int a dP) d\mu$ is concave and ϕ^{-1} is strictly increasing, I is quasiconcave.

¹⁷If $a, b : u(X)^S \rightarrow \mathbb{R}$, “ ab ” denotes the function that assigns the value $a(s)b(s)$ to each state s .

$P(\{s\}) > 0$ for all s and, for all $a \in u(X)^S$ and $s \in S$, $1 + \sum_{0 \leq j < J} \frac{\partial A}{\partial \phi_j}(P(\zeta_0 a), \dots, P(\zeta_{J-1} a)) \zeta_j(s) > 0$.¹⁸ Then (I, u) satisfies Assumption 2; furthermore, SPC holds. Finally,

$$\text{Core } I = \pi(1_S) = \{P\}.$$

Thus, under the assumptions of Proposition 7, our risk-sharing results apply. In particular, since VEU preferences also satisfy constant-additivity, BIC holds, so we can invoke Theorem 4. One specific implication is that, in an economy where all agents have VEU preferences satisfying these conditions, risk-sharing obtains *if and only if agents have the same baseline prior P* (but possibly different adjustment factors and functions).

While Corollary 6 provides easy-to-check sufficient conditions for SPC, these conditions are not necessary. Thus, Theorems 3 and 4 cover a broader set of preferences than the ones that admit a decomposable representation satisfying the assumptions in Corollary 6. Example 3 in Appendix A illustrates this.

Furthermore, the conditions in Corollary 6 are restrictive in conjunction with certain structural properties of the functional I . For instance, this is the case for Choquet-expected utility preferences (Schmeidler, 1989). Example 4 in Appendix A provides the details.

6 Analysis of the main results, and the role of SPC

This section provides a discussion of the key steps in the proof of Theorem 3. Omitted proofs and remaining details can be found in Appendix C.5. The analysis highlights the role of condition SPC, and also provides results on risk sharing that may be of independent interest. For this reason, each individual result specifies the assumptions on preferences that are needed.

As noted above, the key implications in Theorem 3 are (iii) \Rightarrow (iv)—if full-insurance allocations are efficient, then agents share some subjective belief—and (iv) \Rightarrow (ii)—if agents share some subjective belief, then only full-insurance allocations are efficient (i.e., betting is inefficient).

We begin with a result that is reminiscent of the implication (iii) \Rightarrow (iv). It generalizes the standard result that smooth indifference curves must be tangent at any interior Pareto-

¹⁸This last condition ensures that I is strictly monotonic.

efficient allocation. With convex preferences, the common slope at the point of tangency determines a supporting price vector; as we discuss momentarily, a “local price vector” is also identified in the non-convex, non-smooth case, though the sense in which it “supports” the allocation is more delicate (see below). With this caveat, the following result can also be viewed as a local version of the Second Welfare Theorem.

Proposition 8 *For each $i \in N$, assume that V_i is locally Lipschitz and monotonic. Let $(f_i)_{i \in N}$ be an interior allocation such that each functional V_i is nice at f_i . If $(f_i)_{i \in N}$ is Pareto-efficient, then there exists a price vector $p \in \mathbb{R}_+^S \setminus \{0\}$ and, for each $i \in N$, scalars $\lambda_i > 0$ and vectors $Q_i \in \partial V_i(f_i)$ such that $p = \lambda_i Q_i$ for every i .*

The key step in the proof of the first claim is provided by [Bonnisseau and Cornet \(1988\)](#), who show that, under the stated assumptions, there is a vector p such that $-p$ lies in the intersection of the Clarke normal cones of the upper contour set of V_i at the bundle f_i (see [Appendix C.2](#) for a precise statement and definitions of the required terms). If preferences are convex, the Clarke normal cone coincides with the normal cone in the sense of convex analysis. This suggests interpreting p as a “local price vector.”

For our purposes, it is convenient to restate the above result slightly. Recall that the elements of the local-belief sets $\pi_i(\cdot)$ that appear in [Theorem 3](#) are probabilities—that is, they are non-negative vectors normalized to lie in the unit simplex. On the other hand, the elements of the Clarke differential of the functional V_i are arbitrary non-negative vectors—they are not normalized. Following GS, the **normalized Clarke differential** of V_i at $h \in \mathbb{R}_{++}^S$ is

$$C_i(f) = \left\{ \frac{Q}{Q(S)} : Q \in \partial V_i(f), Q \neq 0_S \right\}. \quad (11)$$

GS provide a behavioral characterization of the normalized Clarke differential (see [Appendix B](#)). This notion allows us to restate [Proposition 8](#) as follows:

Corollary 9 *Let $(f_i)_{i \in N}$ be an interior allocation such that each functional V_i is nice at f_i . If $(f_i)_{i \in N}$ is Pareto-efficient, then $\bigcap_{i \in N} C_i(f_i) \neq \emptyset$.*

The difference between this result and the implication (iii) \Rightarrow (iv) in [Theorem 3](#) is that [Corollary 9](#) involves the normalized Clarke differentials $C_i(\cdot)$ instead of the local-belief sets $\pi_i(\cdot)$. Specif-

ically, Proposition 11 below shows that $\pi_i(f) \subseteq C_i(f)$ for every bundle f ; thus, the conclusion in Corollary 9 is weaker than implication (iii) \Rightarrow (iv) in Theorem 3.

In light of this Corollary, one may conjecture that, if the intersection of normalized Clarke differentials is non-empty at every full-insurance allocation, then betting is inefficient. (Such a conclusion would be analogous to the implication (iv) \Rightarrow (ii).)

Example 1 in Appendix A shows that this is not the case. Intuitively, if the normalized Clarke differentials have non-empty intersection at an allocation, then *locally* there are no mutually beneficial trades. However, the notion of Pareto efficiency involves more than just local comparisons: there may be Pareto-superior allocations sufficiently far from the given one.¹⁹ Thus, a converse to Corollary 9 requires a condition involving sets of priors that also convey *global* information about preferences.

A natural candidate for this role is the set $\pi_i(\cdot)$ of subjective beliefs. However, for general preferences, this set is not suitable. For instance, consider an Edgeworth-box economy where both agents have risk-neutral expected-utility preferences with the *same* subjective probability P . In this economy, every feasible allocation is Pareto-efficient, including ones that do not provide full insurance; yet, both agents' subjective belief sets (at any bundle) only consist of the probability P .²⁰

To rule out such pathologies, we introduce a strict version of RSS's notion of subjective beliefs:

$$\pi_i^s(f) = \{P \in \Delta(S) : \forall g \in \mathbb{R}_+^S \setminus \{f\}, V_i(g) \geq V_i(f) \implies P(g) > P(f)\}. \quad (12)$$

Recall that an element $P \in \pi_i(f)$ is required to assign *weakly* higher price to a bundle g that i weakly prefers to f ; by way of contrast, an element $P \in \pi_i^s(f)$ is required to assign *strictly* higher price to such a bundle g . It follows that $\pi_i^s(f) \subseteq \pi_i(f)$. We show in Proposition 11 below that, if preferences are strictly convex (as in RSS), then $\pi_i^s(f) = \pi_i(f)$; however, the inclusion may be strict for more general preferences.

The notion of “strict subjective beliefs” allows us to state a result analogous to the implication (iv) \Rightarrow (ii) in Theorem 3:

¹⁹This is indeed the case in Example 1: consider the allocation $(1_S x^h, 1_S(\bar{x} - x^h))$.

²⁰We owe this example to a referee of an earlier version of this paper.

Proposition 10 *Assume that, for every $i \in N$, V_i is continuous and strongly monotonic. Assume further that, for every feasible, full-insurance allocation $(1_S x_1, \dots, 1_S x_N)$, it is the case that $\bigcap_i \pi_i^s(1_S x_i) \neq \emptyset$. Then every Pareto-efficient allocation provides full insurance. Moreover, such an allocation is a competitive equilibrium allocation (with transfers).*

In words, if, at every feasible, full-insurance allocation, agents share at least one strict subjective belief, then *betting is Pareto-inefficient*. Indeed, as noted above, by standard arguments, if every Pareto-efficient allocation provides full insurance, then it is also the case that every feasible, full-insurance allocation is Pareto-efficient. Therefore, under the assumptions of Proposition 10, the set of Pareto-efficient allocations coincides with the set of feasible, full-insurance allocations.

However, it turns out that the condition on strict subjective beliefs in Proposition 10, while sufficient, is *not* necessary for betting to be Pareto-inefficient. Example 2 in Appendix A demonstrates this. Thus, to sum up, the condition that $\bigcap_i C_i(1_S x_i) \neq \emptyset$ is necessary for the full-insurance allocation $(1_S x_1, \dots, 1_S x_N)$ to be Pareto-efficient (Proposition 8), but it is not sufficient (Example 1). On the other hand, the condition that $\bigcap_i \pi_i^s(1_S x_i) \neq \emptyset$ on the certainty line is sufficient for full-insurance allocations to be the *only* Pareto-efficient ones (Proposition 10), but it is not necessary (Example 2). This points to a gap between Propositions 8 and 10.

Strict pseudoconcavity at certainty closes this gap. The key insight is that, under Condition SPC (and niceness), the strict subjective beliefs $\pi_i^s(1_S x_i)$, the subjective beliefs $\pi_i(1_S x_i)$, and the normalized Clarke differential $C_i(1_S x_i)$ coincide for every $x_i > 0$. In fact, under niceness, these sets coincide *if and only if* SPC holds. The following result provides the details.

Proposition 11 *Fix $i \in N$, and assume that V_i is locally Lipschitz. Consider $f \in \mathbb{R}_{++}^S$.*

1. *for every $f \in \mathbb{R}_{++}^S$, if V_i is nice at f , then $\pi_i(f) \subseteq C_i(f)$;*
2. *for every $f \in \mathbb{R}_{++}^S$, $\pi_i^s(f) \subseteq \pi_i(f)$; if V_i is strictly quasiconcave, then $\pi_i^s(f) = \pi_i(f)$;*
3. *if V_i is strictly pseudoconcave at f , then $C_i(f) \subseteq \pi_i^s(f)$;*
4. *conversely, if V_i is monotonic and nice at f , and $C_i(f) \subseteq \pi_i^s(f)$, then V_i is strictly pseudoconcave at f .*

Thus, if V_i is monotonic and nice at f , it is strictly pseudoconcave at f if and only if $C_i(f) = \pi_i^s(f) = \pi_i(f)$.

By items 1 and 2 of this Proposition, assuming niceness at certainty, $\bigcap_i \pi_i^s(1_S x_i) \neq \emptyset$ implies that $\bigcap_i C_i(1_S x_i) \neq \emptyset$. This provides further insight into the relationship between the necessary and sufficient conditions in Propositions 8 and 10, as also illustrated in Examples 1 and 2 in Appendix A.

Since it ensures that $\pi_i^s(1_S x_i) = \pi_i(1_S x_i) = C_i(1_S x_i)$, Condition SPC implies that a non-empty intersection of these sets is both necessary and sufficient for the set of full-insurance allocations to coincide with the Pareto set.

To conclude, we leverage Propositions 8, 10, and 11 to prove Theorems 3 and 4.

Begin with Theorem 3. As just noted, for every i and $x_i > 0$, since SPC holds and V_i is nice at $1_S x_i$, Proposition 11 implies that $\pi_i^s(1_S x_i) = \pi_i(1_S x_i) = C_i(1_S x_i)$. The implication (ii) \Rightarrow (iii) is standard. Now assume (iii) and fix a feasible, full-insurance allocation $(1_S x_1, \dots, 1_S x_N)$. Then this allocation is Pareto-efficient. By Proposition 8, $\bigcap_i C_i(1_S x_i) \neq \emptyset$; since $\pi_i(1_S x_i) = C_i(x_i)$ for all i , (iv) holds. Finally, assume (iv): then, since $\pi_i^s(1_S x_i) = \pi_i(1_S x_i)$, by Proposition 10, every Pareto-efficient allocation provides full-insurance, i.e., (ii) holds.

Turn now to Theorem 4. Under SPC, BIC, and niceness, $\pi_i(1_S x) = \pi_i(1_S) = C_i(1_S x) = C_i(1_S)$. Therefore, the condition in (iv) of Theorem 4 is equivalent to the condition in (iv) of Theorem 3. Hence, the equivalence of (ii), (iii) and (iv) follows from Theorem 3. For (i) \Rightarrow (iv), if $(1_S x_1, \dots, 1_S x_N)$ is an interior, full-insurance Pareto-efficient allocation, since each V_i is nice at $1_S x_i$, Proposition 8 implies that $\bigcap_i C_i(1_S x_i) \neq \emptyset$, so (iv) holds by the equalities established above. Finally, (iii) \Rightarrow (i) is immediate.

7 Discussion: SPC, DQC, and aversion to ambiguity

We have identified condition SPC as a central assumption that allows us to generalize the risk-sharing result of RSS to non-convex preferences. If preferences admit a representation of the form $V(f) = I(u \circ f)$ with u strictly concave, then the key condition is DQC. We conclude by relating these two conditions to notions of aversion to ambiguity that have been discussed in

the literature.

First, we reiterate that SPC is strictly weaker than convexity in consumption, i.e., preference for diversification. Similarly, for representations of the form $V = I \circ u$, DQC is strictly weaker than convexity in utilities, i.e. uncertainty aversion à la [Schmeidler \(1989\)](#). The examples in Section 2 illustrate these points.

Second, when $V = I \circ u$ with I regular, ambiguity aversion in the sense of GM implies DQC. In general, the latter is strictly weaker: see Example 3 in Appendix A. However, DQC implies GM-ambiguity aversion under an additional condition that, for instance, is implied by BIC (Definition 2): see Appendix C.6 for a formal statement. Similar results hold for condition SPC and a general V : in the interest of conciseness, we relegate details to Appendix C.6.

Third, [Chateauneuf and Tallon \(2002\)](#) introduce weakenings of convexity in consumption and utility. A representation of the form $V = I \circ u$ satisfies *preference for sure EU diversification* if, for all bundles f_1, \dots, f_N that are mutually indifferent and such that, for suitable weights, their *utility* mixture is a constant $u(x)$, it is the case that $x \succcurlyeq f_n$ for all n . We show in Appendix C.6 that condition DQC implies a preference for sure EU diversification. The converse holds in certain special cases: for example, it is true for Choquet and VEU preferences, by results in [Chateauneuf and Tallon \(2002\)](#) and [Siniscalchi \(2009\)](#). Whether it holds more generally is an open question.

[Chateauneuf and Tallon \(2002\)](#) also define *preference for sure diversification*: for all bundles f_1, \dots, f_N that are mutually indifferent and such that, for suitable weights, their *outcome* mixture is a constant x , it is the case that $x \succcurlyeq f_n$ for all n . We show in Appendix C.6 that condition SPC implies a preference for sure diversification. As above, to what extent the converse implication holds is an open question.

To conclude, we remark that Proposition 11 provides an alternative characterization of condition SPC: under our maintained assumptions, it is equivalent to the statement that subjective beliefs and normalized Clarke differentials coincide at certainty. (Also recall that we provide a behavioral characterization in Appendix B.)

A Examples

A.1 Necessary and sufficient conditions for risk sharing

We begin with two examples that emphasize the different roles of the conditions in Proposition 8 (Corollary 9) and Proposition 10. Example 1 shows that the condition $\bigcap_i C_i(1_S x_i) \neq \emptyset$ is not sufficient for risk sharing. On the other hand, Example 2 shows that the condition $\bigcap_i \pi_i^s(1_S x_i) \neq \emptyset$ is not necessary.

Example 1 Consider the following two-agent economy. Let $S = \{s_1, s_2\}$; agent 1's preferences are represented by

$$V_1(h) = \max \left(\left[\frac{1}{2} \sqrt{h_1} + \frac{1}{2} \sqrt{h_2} \right]^2, \epsilon + \min_{p \in [0.3, 0.7]} [p h_1 + (1-p) h_2] \right)$$

for some $\epsilon > 0$. Agent 2 has risk-neutral expected-utility preferences, with probability $P_2 = (\frac{1}{3}, \frac{2}{3})$. Figure 3 represents this economy in an Edgeworth box. The solid indifference curves refer to agent 1's preferences and the dashed ones represent agent 2's preferences; as usual, for agent 2, utility increases in the south-western direction. The indifference curves of agent

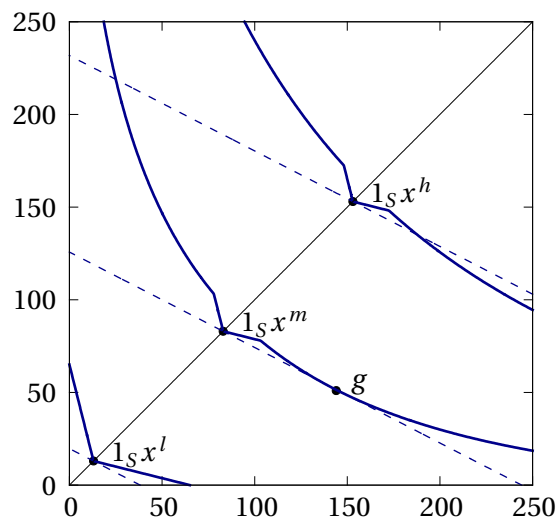


Figure 3: $\bigcap_i C_i(1_S x_i) \neq \emptyset$ is not sufficient for risk sharing.

1 have a small inward “dent” at certainty; in a neighborhood of the 45° line, this preference coincides with the risk-neutral MEU preference with priors $D_1 = \{P \in \Delta(S) : 0.3 \leq P(s_1) \leq 0.7\}$.

Notice that the allocation $(1_S x^h, 1_S(\bar{x} - x^h))$ provides full insurance. The normalized Clarke differential of V_1 at $1_S x^h$ (indeed, everywhere on the 45° line) is D_1 ; hence, it contains that of V_2 , which coincides with the probability P_2 . Moreover, $\pi_2(1_S(\bar{x} - x^h)) = \{P_2\}$ (see RSS), but $P_2 \notin \pi_1(1_S x^h)$: this follows because, as is apparent from Figure 3, P_2 does not support 1's indifference curve at x^h . Thus, $\pi_1(1_S x^h) \cap \pi_2(1_S(\bar{x} - x^h)) = \emptyset$.

However, $(1_S x^h, 1_S(\bar{x} - x^h))$ is not Pareto-efficient. Furthermore, the allocation $(g, 1_S \bar{x} - g)$ is Pareto-efficient, but does not provide full insurance. \square

Example 2 Let $S = \{s_1, s_2\}$. Assume that agent 2 has EU preferences, with a prior P_2 that assigns probability 0.4 to state s_1 (on the horizontal axis) and power utility $u(x) = x^{0.2}$. Consumer 1 has preferences represented by

$$V_1(h) = \max\left(\frac{1}{2}h_1 + \frac{1}{2}h_2, \delta + \min_{p \in [0,1]} [ph_1 + (1-p)h_2]\right).$$

Thus, agent 1's preferences are risk-neutral EU, with a uniform prior, except within δ of the certainty line. See Figure 4.

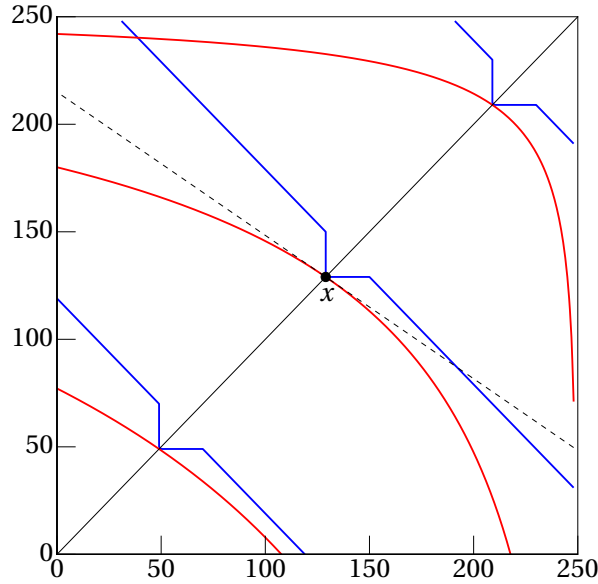


Figure 4: $\bigcap_i \pi_i^s(1_S x_i) \neq \emptyset$ is not necessary for risk sharing.

For agent 2, the strict local-belief set is equal to $\pi_2^s(1_S x_2) = \{P_2\}$ at every $x_2 > 0$; this is because his preferences are strictly convex, so $\pi_2^s(\cdot) = \pi_2(\cdot)$, and the claim follows from RSS's

characterization of subjective beliefs for EU preferences. For agent 1, $\pi_1^s(1_S x) \neq \emptyset$ for all $x > 0$. Note that $P_2 \notin \pi_1^s(1_S x_1)$ for x_1 sufficiently large. For instance, consider the point on the certainty line labelled x in Figure 4. Observe that the tangent to 2's indifference curve at x (the dashed black line) crosses 1's indifference curve going through x . This occurs because P_2 is not uniform, and the dent in 1's preferences at certainty (which depends upon the parameter δ) is sufficiently small.

Thus, $\pi_1^s(1_S x) \cap \pi_2^s(1_S(\bar{x} - x)) = \emptyset$. However, the value of δ in Figure 4 is chosen so that, given the curvature of 2's utility function, the agents' indifference curves are not tangent anywhere except at certainty. That is, betting is inefficient in this economy: a feasible allocation is Pareto-efficient if and only if it provides full insurance. \square

A.2 Preference representations with strictly concave utility

The first example we provide in this section illustrates that the conditions in Theorem 3 are strictly more general than those discussed in Section 5. The second example demonstrates that the conditions in Section 5 may be restrictive when combined with specific assumptions about the functional I .

Example 3 Let $S = \{s_1, s_2\}$. We define the function $V : \mathbb{R}_+^2 \rightarrow \mathbb{R}$, depicted in Figure 5, in three steps.

First, we define $W_1 : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ by $W(f) = \frac{1}{2}\sqrt{f_2} + \sqrt{4 + \frac{1}{4}f_2 + 2\sqrt{f_1}} - 2$. Note that the slope of the indifference curve of W_1 going through the point $1_S x$ (drawn as a dashed black line in Figure 5) equals $-\frac{2}{2+\sqrt{x}}$.

Second, we define $W_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ by specifying the features of its indifference curves. Fix a constant α (in the picture, $\alpha = 1.05$). For any $x > 0$, the indifference curve of W_2 going through $1_S(\alpha x)$ is linear, and parallel to the tangent to the indifference curve of W_1 at $1_S x$. Furthermore, $W_2(1_S \alpha x) = \sqrt{x}$ for all $x > 0$.²¹

²¹The details are as follows. Since the indifference curve of W_2 is linear, it consists of points $f = (f_1, f_2)$ such that $f_2 = m f_1 + q$; for $f_1 = f_2 = \alpha x$, by assumption the slope is $-\frac{2}{2+\sqrt{x}}$, so $q = \frac{4+\sqrt{x}}{2+\sqrt{x}}\alpha x$. Hence the indifference curve of W_2 going through $1_S \alpha x$ has equation $f_2 = -\frac{2}{2+\sqrt{x}}f_1 + \frac{4+\sqrt{x}}{2+\sqrt{x}}\alpha x$. Since any $f \in \mathbb{R}_+^2$ lies on a *unique* indifference curve, and each indifference curve is parameterized by x , the value of $W_2(f)$ for an arbitrary $f \in \mathbb{R}_+^2$ can be

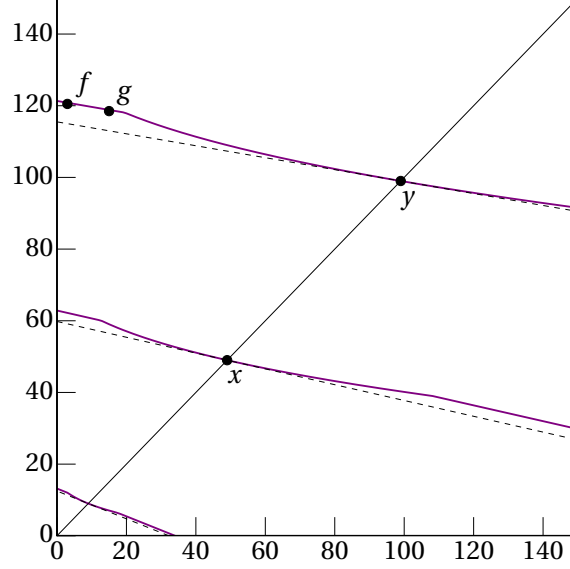


Figure 5: A non-convex preference with an empty core that nevertheless satisfies SPC

Finally, we let $V(f) = \max(W_1(f), W_2(f))$. By construction, at the point $1_S x$, $W_2(1_S x) < W_1(1_S x)$. Thus, for bundles f near the certainty line, $V(f) = W_1(f)$. However, since the indifference curves of W_2 are flat, whereas those of W_1 bend inward, for bundles f sufficiently far from the 45° line, $W_2(f) > W_1(f)$, so $V(f) = W_2(f)$.

Fix an arbitrary, strictly concave function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$, and assume that $I : u(X)^2 \rightarrow \mathbb{R}$ is strictly monotonic and such that $V = I \circ u$. We argue that I is not quasiconcave, and its core is empty; thus, neither condition 1 nor condition 2 in Corollary 6 applies.

Consider two bundles f, g such that $V(f) = V(g) = W_2(f) = W_2(g)$: that is, f and g lie on the same indifference curve for V , in a region where V coincides with W_2 (see Figure 5). Since in that region the indifference curve is linear, $V(\frac{1}{2}f + \frac{1}{2}g) = V(f)$. Therefore, $I(u \circ (\frac{1}{2}f + \frac{1}{2}g)) = I(u \circ f)$. However, for every state s , since u is strictly concave and $f(s) \neq g(s)$ because indifference curves are not parallel to the axes, $u(\frac{1}{2}f(s) + \frac{1}{2}g(s)) > \frac{1}{2}u(f(s)) + \frac{1}{2}u(g(s))$. Since I is strictly monotonic, $I(u \circ (\frac{1}{2}f + \frac{1}{2}g)) > I(\frac{1}{2}u \circ f + \frac{1}{2}u \circ g)$. Conclude that $I(u \circ f) > I(\frac{1}{2}u \circ f + \frac{1}{2}u \circ g)$:

computed as follows. First, find the unique x such that f satisfied the linear equation parameterized by x ; this can be done numerically, using any one-dimensional search algorithm. Second, note that then f lies on the same indifference curve as $1_S \alpha x$, so by assumption $W_2(f) = W_2(1_S \alpha x) = \sqrt{x}$.

but then, I is not quasiconcave.

Next, Proposition 11 in Section 6 and Proposition 16 in the Appendix imply that $\text{Core } I \subseteq \bigcap_{x>0} C(1_S x)$.²² But, since $V = W_1$ near every constant bundle $1_S x$, and W_1 is smooth, $C(1_S x) = \{P_x\}$, where P_x identifies the line supporting the upper contour set of W_1 at $1_S x$, which therefore has slope $-\frac{2}{2+\sqrt{x}}$. Since $x \neq y$ implies $P_x \neq P_y$, $\text{Core } I = \emptyset$.

Finally, we show that Condition SPC is satisfied. Suppose that $V(f) \geq V(1_S x) = W_1(1_S x)$. If $V(f) = W_1(f)$, then $W_1(f) \geq W_1(1_S x)$ implies that $\nabla V(1_S x) \cdot (f - 1_S x) = \nabla W_1(1_S x) \cdot (f - 1_S x) > 0$ because W_1 is strictly quasiconcave. If instead $V(f) = W_2(f)$, let $y \geq 0$ be such that $V(f) = W_2(f) = V(1_S y) = W_1(1_S y)$; this is the case for the points labelled f, x, y in Figure 5. Then $y \geq x$, and f lies on an indifference curve for W_2 that is parallel to, but *higher* than the indifference curve for W_1 through $1_S y$, and hence also *higher* than the indifference curve for W_1 through $1_S x$. But this means that, again, $\partial V(1_S x) \cdot (f - 1_S x) = \partial W_1(1_S x) \cdot (f - 1_S x) > 0$.

Example 4 (Invariant Biseparable preferences) A preference is *invariant biseparable* (Ghirardato et al., 2004) if its representation (I, u) is such that I is positively homogeneous and translation-invariant on its domain. We now show that MEU preferences are the only invariant biseparable preferences for which condition DQC in Section 5 holds.

Recall from Ghirardato et al. (2004) that, for an invariant biseparable preference represented by (I, u) , the functional I admits a unique extension to all of \mathbb{R}^S , and the Clarke differential at zero, i.e., $\partial I(0_S)$, consists of probability measures and coincides with $\partial I(1_S u(x))$ for all $x > 0$. Hence, I is nice at $1_S u(x)$ for every $x > 0$.

If preferences are MEU, then I is concave, so DQC holds by Corollary 6. Conversely, assume that DQC holds. Let $D = \partial I(0_S) = \partial I(1_S) \subseteq \Delta(S)$. Then, Proposition 16 and Corollary 17 in the Appendix imply that $D = \text{Core } I$. But by Proposition 16 in Ghirardato et al. (2004), $D = \text{Core } I$ if and only if I is concave, in which case the preference is MEU.

Thus, for invariant biseparable preferences, the sufficient condition for SPC provided by Proposition 16 only holds for the special case of MEU. It is an open question whether condition SPC may hold for invariant biseparable preferences that are not MEU. \square

²²For every $x > 0$, Proposition 16 implies that $\text{Core } I \subseteq \pi^s(1_S x)$; by parts 1 and 2 of Proposition 11, $\pi^s(1_S x) \subseteq \pi(1_S x) \subseteq C(1_S x)$ [note that $\partial W_1(1_S x) \neq 0_S$, so V is nice at $1_S x$]; the claim follows.

A.3 Convex preferences

We conclude with two examples with convex preferences. Example 5 shows that, even when all preferences are strictly convex, a non-empty intersection of the subjective belief sets $\pi_i(\cdot)$ at *every* full-insurance allocation is necessary for risk sharing. Example 6 instead illustrates how risk sharing may obtain when BIC fails—that is, when Theorem 3 applies but 4 does not.

Example 5 Let $S = \{s_1, s_2\}$. Agent 1's preferences are represented by the utility function $u_1(x) = x^{0.6}$ and the differentiable, quasiconcave, but not concave functional

$$I(a) = \frac{1}{2}a_2 + \sqrt{4 + \frac{1}{4}a_2^2 + 2a_1} - 2.$$

Agent 2 has EU preferences, with probability P and utility $u_2(x) = x^{0.8}$. Figure 6 shows indifference curves for these preferences, drawn as solid blue and red lines respectively. Agent 1's and 2's indifference curves are tangent at the allocation $(1_S x^l, 1_S(\bar{x} - x^l))$; their common slope there equals the slope of the two parallel, straight purple lines. (Thus, this slope identifies P .)

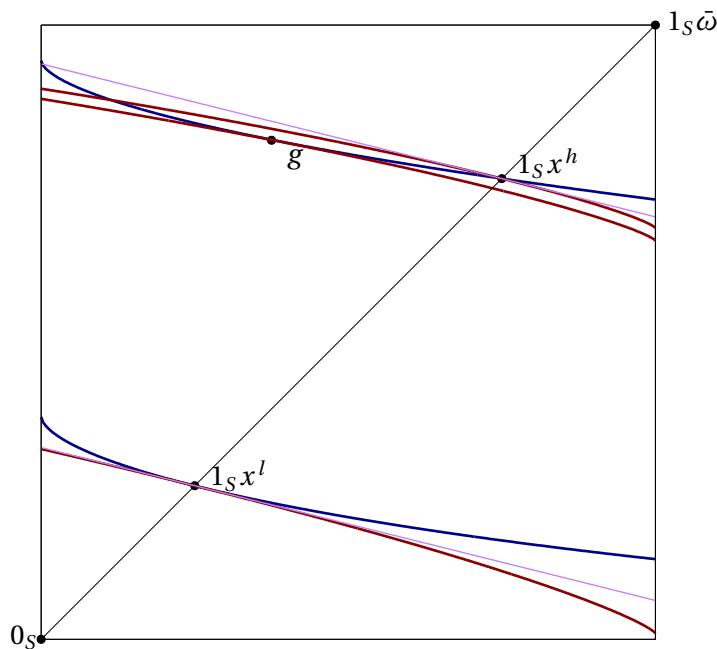


Figure 6: A convex preference with empty core.

The figure shows that the slope of 1's indifference curves at $1_S x^l$ and $1_S x^h$ is different; indeed, it may be verified that the slope of the indifference curve of I at $1_S \gamma$ is $-\frac{2}{\gamma+2}$ for every $\gamma > u_1(0)$; this is non-zero and strictly decreasing in γ . Hence, I is nice at certainty. Furthermore, since I is quasiconcave and u_1 is strictly concave, $V_1 = I \circ u_1$ satisfies SPC by Corollary 6, and therefore by Proposition 11, $\pi_1^s(1_S x) = \pi_1(1_S x) = C_1(1_S x)$ for all $x > 0$. In particular, $\pi_1(1_S x)$ is a singleton set. On the other hand, since agent 2's preferences are consistent with EU, $\pi_2^s(1_S x) = \pi_2(1_S x) = C_2(1_S x) = \{P\}$.

From a decision-theoretic perspective, we observe that agent 1's preference is uncertainty-averse in the sense of Schmeidler, 1989, because I is quasiconcave; however, it is not GM-ambiguity-averse.²³ To see this, note that, by Corollary 6 part 1, together with Corollary 17 in Appendix C.4, the core of I must be contained in the sets $\pi_1(1_S x)$ for all $x > 0$, but as noted above these sets are all singleton (hence, non-empty) and different for different x , so $\text{Core } I = \emptyset$. For the same reason, BIC fails.

Turn now to risk sharing. The assumptions of Theorem 3 hold. The purple line going through x^h is tangent to agent 2's indifference curve, but does *not* support agent 1's indifference curve: therefore, $\pi_1(1_S x^h)$ does not intersect $\pi_2(1_S(\bar{x} - x^h))$. Thus, condition (iv) in Theorem 3 is violated. Correspondingly, conditions (ii) and (iii) also fail: the allocation $(g, 1_S \bar{x} - g)$ is Pareto-efficient, but does not provide full insurance, whereas the interior, full-insurance allocation $(1_S x^h, 1_S(\bar{x} - x^h))$ is not Pareto-efficient.

Finally, note that the interior, full-insurance allocation $(1_S x^l, 1_S(\bar{x} - x^l))$ is Pareto-efficient; thus, in this economy, condition (i) in Theorem 1 holds. However, as just noted, conditions (ii)-(iv) in Theorem 3 do not hold. Thus, this example demonstrates that condition (i) cannot be included in the statement of Theorem 3. \square

Example 6 Modify Example 5 by assuming that agent 2's preferences are MEU, with priors $D \equiv \{P \in \Delta(S) : P(\{s\}) \in [0.4, 0.6]\}$; furthermore, assume that both agents have utility $u_i(x) = \sqrt{x}$. Refer to Figure 7.

Both preferences are strictly convex and admit a decomposable representation $V_i = I_i \circ u_i$,

²³Another example of a preference which is convex but not GM-ambiguity-averse can be found in Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2011).

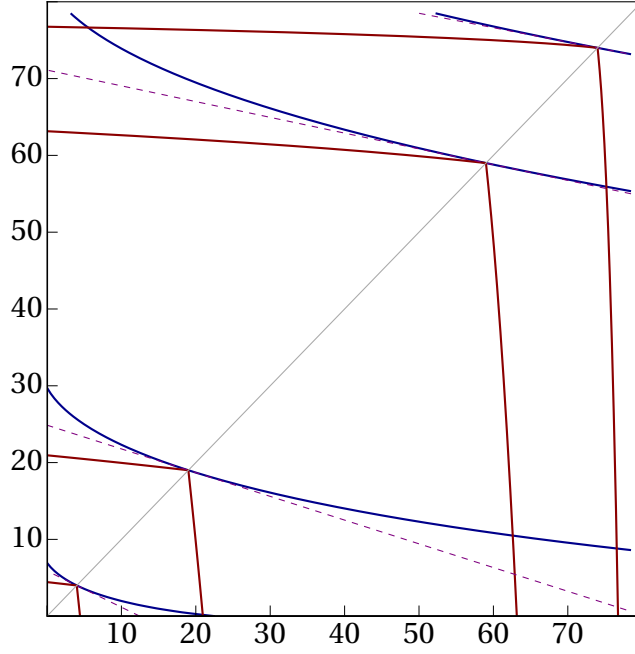


Figure 7: Risk sharing with non-constant subjective beliefs at certainty.

$i = 1, 2$. Observe that Assumption 2 holds; furthermore, the functionals I_i are both quasiconcave, so both preferences satisfy SPC by Corollary 6. In addition, agent 2's preferences satisfy BIC: this follows because $\partial I_2(1_S x) = C_2(1_S x) = D$ for every $x > 0$.²⁴

From Corollary 6 and Corollary 17 in Appendix C.4, $\pi_2(1_S x) = \pi_2^c(1_S x) = C_2(1_S x) = D$ for every $x > 0$. From Proposition 11, since each V_i satisfies SPC, $\pi_i(1_S x) = \pi_i^s(1_S x)$ for $i = 1, 2$ and $x > 0$. Therefore, for every $x > 0$, $\pi_1(1_S x) \cap \pi_2(1_S x) \neq \emptyset$. Proposition 10 and Theorem 3 apply, and indeed the set of Pareto-efficient and full-insurance allocations coincide.

Since agent 1's preferences do not satisfy BIC, Theorem 4 does not apply. Indeed, neither does RSS's original risk-sharing result (Proposition 9 in their paper): while both agents' preferences are convex (and indeed 2's preferences have a concave representation), agent 1's subjective beliefs are not constant at certainty. \square

²⁴By Proposition 16, $C_2(1_S x) = C_2^u(1_S x)$, and $C_2^u(1_S x) = D$ because I_2 is the MEU preference functional, with priors D (Ghirardato et al., 2004).

B Behavioral characterization of conditions SPC and BIC

Throughout this section of the Appendix, and the next, we identify a probability distribution $P \in \Delta(S)$ with the linear function it induces on \mathbb{R}^S . Thus, we write $P(f)$ instead of $P \cdot f$.

Fix a preference \succsim represented by a functional V that satisfies Assumption 1.

We first recall an axiom from RSS, and show that it implies condition BIC.

Axiom 1 (Translation Invariance at Certainty) *For all $x, x' \in \mathbb{R}_+$ and $g \in \mathbb{R}^S$: if there is $\lambda > 0$ such that $1_S x + \lambda g \in \mathbb{R}_+^S$ and $1_S x + \lambda g \succsim x$, then there is $\lambda' > 0$ such that $1_S x' + \lambda' g \in \mathbb{R}_+^S$ and $1_S x' + \lambda' g \succsim x$.*

Proposition 12 *Assume that \succsim is represented by V . If Axiom 1 holds, then V satisfies condition BIC.*

Proof: Fix $x, x' \in \mathbb{R}_{++}$ and $P \in \pi(1_S x)$. Consider $g \in \mathbb{R}_+^S$ such that $V(g) \geq V(1_S x')$, i.e., $g \succsim 1_S x'$. Equivalently, $1_S x' + 1 \cdot (g - 1_S x') \succsim 1_S x'$. Therefore, by Axiom 1, there exists $\lambda > 0$ such that $1_S x + \lambda(g - 1_S x') \in \mathbb{R}_+^S$ and $1_S x + \lambda(g - 1_S x') \succsim 1_S x$. Since $P \in \pi(1_S x)$, $P(1_S x + \lambda(g - 1_S x')) \geq P(1_S x) = x$, i.e., $x + \lambda[P(g) - x'] \geq x$, i.e., $P(g) \geq x'$. Since g was arbitrary, $P \in \pi(1_S x')$.

Therefore, $\pi(1_S x) \subseteq \pi(1_S x')$. Repeating the argument switching x and x' yields the required conclusion. ■

Observe that Axiom 1 is only sufficient for condition BIC.

Next, we characterize BIC and SPC using a key notion from GS. That paper considers a preference defined over acts mapping states to a convex subset X of a vector space, endowed with a mixture operation, that admits a Bernoulli separable representation (I, u) . The utility u is affine with respect to the assumed mixture operation on X . We take $X = \mathbb{R}_+$ and convex combination as the mixture operation. As a result, the utility function is affine on \mathbb{R}_+ , so it can be taken to be the identity; consequently, in the notation of this paper, $I = V$. This implies that $I = V$ subsumes both risk and ambiguity attitudes, whereas in GS risk attitudes are captured by u . Moreover, since u is taken to be the identity, convergence of acts as defined in GS is convergence in the usual Euclidean topology.

Definition 4 For any pair of acts $f, g \in \mathbb{R}_+^S$ and prize $x \in \mathbb{R}_+$, say that f is a (weakly) better deviation than g near x , written $f \succcurlyeq_x^* g$, if, for every $(\lambda^n)_{n \geq 0} \subset [0, 1]$ and $(h^n)_{n \geq 0}$ such that $\lambda^n \downarrow 0$ and $h^n \rightarrow 1_S x$,

$$\lambda^n f \oplus (1 - \lambda^n) h^n \succcurlyeq \lambda^n g \oplus (1 - \lambda^n) h^n \quad \text{eventually.}$$

The basic intuition is that f is a better deviation than g at x if, starting from an initial riskless consumption bundle $1_S x$, the DM prefers to move by a vanishingly small amount in the direction of the bundle f rather than in the direction of the bundle g . Furthermore, this remains true if the initial bundle is not exactly $1_S x$, but is close to it. We then have:

Proposition 13 Assume that V is nice at every $1_S x$, $x > 0$.

- (i) \succcurlyeq satisfies BIC if and only if, for every $x, y > 0$ and $f, g \in \mathbb{R}_+^S$, $f \succcurlyeq_x g$ implies $f \succcurlyeq_y g$.
- (ii) for every $x > 0$, V is strictly pseudoconcave at $1_S x$ if and only if

$$\forall g \in \mathbb{R}_+^S, \quad g \succcurlyeq 1_S x \quad \implies \quad \exists \delta > 0: \quad g \succcurlyeq_x 1_S(x + \delta) \quad (13)$$

Proof: (i) is immediate from GS.

(ii) We first show that the preference \succcurlyeq_x is “translation-invariant:” for every $f, g \in \mathbb{R}_+^S$ and $\Delta > 0$, $f \succcurlyeq_x g$ iff $f + 1_S \Delta \succcurlyeq_x g + 1_S \Delta$. Supplementary Appendix S.E of GS shows that \succcurlyeq_x satisfies Independence: that is, for all $f', g', h' \in \mathbb{R}_+^S$ and $\lambda \in (0, 1)$, $f' \succcurlyeq_x g'$ iff $\lambda f' + (1 - \lambda) h' \succcurlyeq_x \lambda g' + (1 - \lambda) h'$. Let $f' = 2f$ and $g' = 2g$; then $f', g' \in \mathbb{R}_+^S$. Take $h' = 21_S \Delta$. Then $f + 1_S \Delta \succcurlyeq_x g + 1_S \Delta$ is equivalent to $\frac{1}{2} f' + \frac{1}{2} h' \succcurlyeq_x \frac{1}{2} g' + \frac{1}{2} h'$; by independence, this holds iff $f' \succcurlyeq_x g'$. Now apply independence again with $h'' = 0_S$ to conclude that $f' \succcurlyeq_x g'$ iff $\frac{1}{2} f' + \frac{1}{2} h'' \succcurlyeq_x \frac{1}{2} g' + \frac{1}{2} h''$. But the latter preference statement is equivalent to $f \succcurlyeq_x g$. This proves the claim.

Now fix $\Delta > 0$ throughout. Assume first that Eq. (13) holds, and consider $g \in \mathbb{R}_+^S$ such that $g \succcurlyeq x$. Then there is $\delta > 0$ such that $g \succcurlyeq_x 1_S(x + \delta)$. By the above claim, $g' \equiv g + 1_S \Delta \succcurlyeq_x 1_S(x + \delta + \Delta)$. Furthermore, in the language of GS, g' is an interior act: that is, there are $y, y' \geq 0$ such that $y > g'(s) > y'$ for all s . Since $x, \delta, \Delta > 0$, so is $1_S(x + \delta + \Delta)$. Let (f, f') be a spread of $g', 1_S(x + \delta + \Delta)$: that is, $f(s) > g'(s)$ and $x + \delta + \Delta > f'(s)$ for all s . Since \succcurlyeq_x is monotonic, $f \succcurlyeq_x f'$. Then, by Theorem 7 in GS, $P(g) + \Delta = P(g') \geq P(1_S[x + \delta + \Delta]) = x + \delta + \Delta$ for every $P \in C(1_S x)$. Equivalently, $P(g) \geq x + \delta$ for all $P \in C(1_S x)$. Since V is nice at $1_S x$, this implies

that, for every $Q \in \partial V(1_S x)$, $Q(g)/Q(S) \geq x + \delta$, or $Q(g - 1_S x) \geq Q(S)\delta$. Since $\partial V(1_S x)$ is compact, $\min_{Q' \in \partial V(1_S x)} Q'(S)$ is attained by some $Q^* \in \partial V(1_S x)$; since V is monotonic at nice at $1_S x$, $Q^*(S) > 0$. Therefore, $Q(g - 1_S x) \geq Q(S)\delta \geq Q^*(S)\delta > 0$ for all $Q \in \partial V(1_S x)$, which shows that V is strictly pseudoconcave at $1_S x$.

Conversely, assume that V is strictly pseudoconcave at $1_S x$ and consider $g \in \mathbb{R}_+^S$ such that $g \succcurlyeq 1_S x$. Then $Q(g - 1_S x) > 0$ for all $Q \in \partial V(1_S x)$. Equivalently, $Q([g + 1_S \Delta] - 1_S[x + \Delta]) > 0$ for all $Q \in \partial V(1_S x)$. Since $\partial V(1_S x)$ is compact, there are $Q^+, Q^- \in \partial V(1_S x)$ such that $Q^-([g + 1_S \Delta] - 1_S[x + \Delta]) = \min_{Q' \in \partial V(1_S x)} Q'([g + 1_S \Delta] - 1_S[x + \Delta])$ and $Q^+(S) = \max_{Q' \in \partial V(1_S x)} Q'(S)$. By strict pseudoconcavity at $1_S x$, $Q^-([g + 1_S \Delta] - 1_S[x + \Delta]) > 0$; and by niceness at $1_S x$, $Q^+(S) > 0$.

Let

$$\eta = \frac{Q^-([g + 1_S \Delta] - 1_S[x + \Delta])}{Q^+(S)} > 0.$$

Observe that, for every $Q \in \partial V(1_S x)$,

$$Q([g + 1_S \Delta] - 1_S[x + \Delta]) \geq Q^-([g + 1_S \Delta] - 1_S[x + \Delta]) = \eta Q^+(S) \geq \eta Q(S).$$

Now let $\epsilon > 0$ be such that $\epsilon < \Delta$ and $\epsilon < \frac{1}{2}\eta$. Then

$$Q([g + 1_S(\Delta - \epsilon)] - 1_S[x + \Delta + \epsilon]) = Q([g + 1_S \Delta] - 1_S[x + \Delta]) - Q(S)(2\epsilon) \geq (\eta - 2\epsilon)Q(S)$$

for every $Q \in \partial V(1_S x)$. Furthermore, by the choice of ϵ , the act $g + 1_S(\Delta - \epsilon)$ is interior, and $\delta \equiv \eta - 2\epsilon > 0$. Further rewrite this as

$$Q([g + 1_S(\Delta - \epsilon)] - 1_S[x + \Delta + \epsilon + \delta]) \geq 0$$

for all $Q \in \partial V(1_S x)$. Hence,

$$P(g + 1_S(\Delta - \epsilon)) \geq P(1_S[x + \Delta + \epsilon + \delta])$$

for all $P \in C(1_S x)$. The act $1_S[x + \Delta + \epsilon + \delta]$ is of course also interior. Then, by Theorem 7 in GS, for all spreads (f, f') of $(g + 1_S(\Delta - \epsilon), 1_S[x + \Delta + \epsilon + \delta])$, $f \succcurlyeq_x f'$. One particular such spread is $(g + 1_S \Delta, 1_S[x + \Delta + \delta])$. Thus, $g + 1_S \Delta \succcurlyeq_x 1_S[x + \Delta + \delta]$. By the above claim, this holds iff $g \succcurlyeq_x 1_S[x + \delta]$. Since g was arbitrary, Eq. (13) holds. ■

C Proofs

As in the previous Appendix, for $P \in \Delta(S)$ and $f \in \mathbb{R}^S$, we write $P(f)$ instead of $P \cdot f$. For expositional reasons, Proposition 11 is proved “out of order,” in Appendix C.3.

C.1 Preliminaries: Clarke derivatives and differentials

We introduce additional notation and definitions related to differentials and their properties.

Fix a function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ that is locally Lipschitz on an open set $B \subset \mathbb{R}^n$. The **Clarke upper derivative** of F at $b \in B$ in the direction $a \in \mathbb{R}^n$ is

$$F^\circ(b; a) = \limsup_{t \downarrow 0, c \rightarrow b} \frac{F(c + ta) - F(c)}{t}; \quad (14)$$

Clarke (1983) shows that the set $\partial F(b) \equiv \{Q \in \mathbb{R}^n : Q \cdot a \leq F^\circ(b; a)\}$ is such that $F^\circ(b; a) = \max_{Q \in \partial F(b)} Q(a)$. Furthermore, it admits the characterization given in Eq. (3) in Section 3.

The **Clarke lower derivative** (cf. Ghirardato et al., 2004, pp. 150 and 157) is instead

$$F^\ell(b; a) = \liminf_{t \downarrow 0, c \rightarrow b} \frac{F(c + ta) - F(c)}{t}; \quad (15)$$

It is readily verified that $F^\ell(b; a) = -F^\circ(b; -a)$ and, therefore, $F^\ell(b; a) = \min_{Q \in \partial F(b)} Q(a)$ for all $b \in B$ and all $a \in \mathbb{R}^n$.

C.2 Preliminaries: Clarke tangent and normal cones; subjective beliefs

The following geometric notions will be useful. For every bundle $f \in \mathbb{R}_+^S$, let

$$U(f) = \{g \in \mathbb{R}_+^S : g \succcurlyeq f\},$$

the upper countour set of the preference \succcurlyeq at f . For every set $C \subset \mathbb{R}_+^S$ and bundle $f \in \mathbb{R}_+^S$, let

$$d_C(f) = \inf\{\|f - g\| : g \in C\}$$

The *Clarke tangent cone* to C at some $f \in C$ is

$$T_C(f) = \{v \in \mathbb{R}^S : (d_C)^0(f; v) = 0\},$$

i.e. the set of directions v for which the Clarke derivative of the distance function (which is Lipschitz and convex) is zero. The following characterization (Clarke, 1983, Theorem 2.4.5) is useful:

$$T_C(f) = \{v \in \mathbb{R}^S : \forall (f^k, t^k) \in C \times \mathbb{R}_{++} \text{ s.t. } f^k \rightarrow f, t^k \downarrow 0, \exists (v^k) \subset \mathbb{R}^S \text{ s.t. } v^k \rightarrow v, f^k + t^k v^k \in C \forall k\}.$$

Finally, define the *Clarke normal cone* to C at f by polarity:

$$N_C(f) = \{Q \in \text{ba}(S) = \mathbb{R}^S : Q(v) \leq 0 \forall v \in T_C(f)\}.$$

Specializing to our environment, we have

$$\begin{aligned} T(f) \equiv T_{U(f)}(f) = & \{v \in \mathbb{R}^S : \forall (f^k, t^k) \in \mathbb{R}_+^S \times \mathbb{R}_{++} \text{ s.t. } f^k \succcurlyeq f \forall k, f^k \rightarrow f, t^k \downarrow 0, \\ & \exists (v^k) \subset \mathbb{R}^S \text{ s.t. } v^k \rightarrow v, f^k + t^k v^k \succcurlyeq f \forall k\}. \end{aligned}$$

and it is convenient to define

$$N(f) \equiv N_{U(f)}(f) = \{Q \in \mathbb{R}^S : Q(v) \leq 0 \forall v \in T(f)\}.$$

Loosely speaking, $T(f)$ is the set of directions v with the property that any sequence of bundles preferred to f and converging to it can be perturbed in the direction v without leaving the upper contour set of f . More informally, moving from bundles near f in the direction v by a small amount leads to an act that is at least as good as f . Then, if Q is in the normal cone, $-Q$ is a price vector that assigns non-negative value to such changes.

The following two results pertain to the Clarke normal cone. Note that the first does not require any specific assumption on the functional V .

Remark 2 For every bundle $f \in \mathbb{R}_{++}^S$, $-\pi(f) \subseteq N(f)$.

Proof: Fix $P \in \pi(f)$. Consider $v \in T(f)$, the constant sequence $f^k \equiv f$, and an arbitrary sequence $(t^k) \downarrow 0$. Since $v \in T(f)$, there exists a sequence $(v^k) \rightarrow v$ such that, for every k , $f^k + t^k v^k \succcurlyeq f$, i.e., $V(f + t^k v^k) \geq V(f)$. Since $P \in \pi(f)$, $P(f + t^k v^k) \geq P(f)$, and therefore $P(v^k) \geq 0$ for every k . By continuity, $P(v) \geq 0$. Therefore, $-P \in N(f)$. ■

Remark 3 For every bundle $f \in \mathbb{R}_{++}^S$, if V is locally Lipschitz and nice at f , then $N(f) \subseteq \bigcup_{\lambda \geq 0} \lambda(-\partial V(f))$. Thus, for any such f , if $R \in N(f) \setminus \{0_S\}$, there is $\lambda > 0$ and $Q \in \partial V(f)$ such that $R = -\lambda Q$.

Proof: Let $W = -V$, and note that $U(f) = \{g \in \mathbb{R}_+^S : W(g) \leq W(f)\}$. By Proposition 2.3.1 in Clarke (1983), $\partial W(f) = -\partial V(f)$. Thus, if V is nice at f , so is W ; moreover, by Corollary 1 to Theorem 2.4.7 in Clarke (1983), $N(f) \subseteq \bigcup_{\lambda \geq 0} \lambda \partial W(f) = \bigcup_{\lambda \geq 0} \lambda(-\partial V(f))$, as claimed.

Hence, if $R \in N(f)$, there is $Q \in \partial V(f)$ and $\lambda > 0$ such that $R = -\lambda Q$. ■

The following result restates the definition of $\pi(f)$ for $f \in \mathbb{R}_{++}^S$.

Lemma 14 Assume that V is strongly monotonic and continuous. For every $f \in \mathbb{R}_{++}^S$, $\pi(f) = \{P \in \Delta(S) : \forall g \in \mathbb{R}_+^S, P(f) \geq P(g) \implies V(f) \geq V(g)\}$.

Proof: Denote the set on the rhs of the Remark by $\hat{\pi}(f)$. Suppose that $P \in \pi(f)$. We show that, for every $g \in \mathbb{R}_+^S$, $V(g) > V(f)$ implies $P(g) > P(f)$, so $P \in \hat{\pi}(f)$. Fix g and suppose $V(g) > V(f)$. Since $P \in \pi(f)$, $P(g) \geq P(f)$. By contradiction, suppose $P(g) = P(f)$. Then, there must be a state s such that $g(s) \geq f(s)$, thus $g(s) > 0$, and $P(\{s\}) > 0$. By continuity of V , there is $\epsilon > 0$ such that $g(s) - \epsilon > 0$ and the bundle g' defined by $g'(s) = g(s) - \epsilon$ and $g'(s') = g(s')$ for $s' \neq s$ satisfies $V(g') > V(f)$. But $P(g') = P(g) - P(\{s\})\epsilon < P(g) = P(f)$, which contradicts the assumption that $P \in \pi(f)$. Thus $P(g) > P(f)$.

Conversely, suppose that $P \in \hat{\pi}(f)$. We show that, for every $g \in \mathbb{R}_+^S$, $P(f) > P(g)$ implies $V(f) > V(g)$, so $P \in \pi(f)$. Fix g and suppose that $P(f) > P(g)$. Since $P \in \hat{\pi}(f)$, $V(f) \geq V(g)$. By contradiction, suppose $V(f) = V(g)$. Then there is $\epsilon > 0$ such that $P(f) > P(g + 1_S \epsilon)$; however, by strong monotonicity $V(g + 1_S \epsilon) > V(g) = V(f)$, which contradicts the assumption that $P \in \hat{\pi}(f)$. Hence, $V(f) > V(g)$. ■

Corollary 15 For every $f \in \mathbb{R}_{++}^S$ and $P \in \pi(f)$, $P(\{s\}) > 0$ for all $s \in S$.

Proof: Fix $f \in \mathbb{R}_{++}^S$, $s \in S$ and $P \in \pi(f)$. Define g by $g(s) = f(s) + 1$ and $g(s') = f(s')$ for all $s' \neq s$. If $P(\{s\}) = 0$, then $P(f) = P(g)$, and therefore, by Lemma 14, $V(f) \geq V(g)$: this contradicts the

assumption that V is strongly monotonic. ■

C.3 Proof of Remark 1 and Propositions 11 and 2

Proof of Remark 1: suppose that V is strictly pseudoconcave at f , and consider $g \in \mathbb{R}_{++}^S$, $Q \in \partial V(f)$ with $g \neq f$ and $Q(g) = Q(f)$. If $V(g) \geq V(f)$, strict pseudoconcavity at f implies $Q(g - f) > 0$, contradiction: thus, $V(g) < V(f)$. Conversely, assume that Eq. (6) holds at f , and consider $g \in \mathbb{R}_{++}^S \setminus \{f\}$, $Q \in \partial V(f)$ such that $V(g) \geq V(f)$. If $Q(g - f) \leq 0$, let $g' = g + 1_S Q(f - g)/Q(1_S) \geq g$; this is well-defined because $Q(1_S) > 0$ by niceness. Then $Q(g') = Q(g) + Q(f - g)/Q(1_S) \cdot Q(1_S) = Q(f)$, so by Eq. (6) $V(g') < V(f)$. But $g' \geq g$ and monotonicity imply $V(g) < V(f)$, contradiction. Thus, $Q(g - f) > 0$. ■

We prove Proposition 11 first, because it is used in the proof of Proposition 2.

Proof of Proposition 11: (1): fix $f \in \mathbb{R}_{++}^S$ and consider $P \in \pi(f)$. By Remark 2, $-P \in N(f)$. By Remark 3, if V is nice at f , then there are $\lambda > 0$ and $Q \in \partial V(f)$ such that $-P = \lambda(-Q)$, i.e., $P = \lambda Q$. Furthermore, $1 = P(S) = \lambda Q(S)$, so $\lambda = Q(S)^{-1}$ and $P = \frac{Q}{Q(S)} \in C(f)$, as required.

(2) fix $f \in \mathbb{R}_{++}^S$. The statement that $\pi^s(f) \subseteq \pi(f)$ is immediate from the definitions. Now suppose that V is strictly quasiconcave. Fix $P \in \pi(f)$ and $g \in \mathbb{R}_{++}^S \setminus \{f\}$ such that $V(g) \geq V(f)$. Since $P \in \pi(f)$, $P(g) \geq P(f)$. By contradiction, suppose that $P(g) = P(f)$. We consider two cases.

First, if $V(g) > V(f)$, then by continuity of V there is $\lambda \in (0, 1)$ such that $V(\lambda g) > V(f)$. However, since $f \in \mathbb{R}_{++}^S$, $P(g) = P(f) > 0$ and so $P(\lambda g) = \lambda P(g) < P(g) = P(f)$. This contradicts the fact that $P \in \pi(f)$.

Second, if $V(g) = V(f)$, then by strict quasiconcavity of V , if $g' = \frac{1}{2}f + \frac{1}{2}g$, then $V(g') > V(f)$. Moreover, $P(g') = \frac{1}{2}P(f) + \frac{1}{2}P(f) = P(f)$. The argument just given shows that $V(g') > V(f)$ and $P(g') = P(f)$ contradicts $P \in \pi(f)$, so the proof is complete.

(3): let $P \in C(f)$ and consider $g \in \mathbb{R}_+^S \setminus \{f\}$ such that $V(g) \geq V(f)$. By strict pseudoconcavity, this implies that, for every $Q \in \partial V(f)$, $Q(g - f) > 0$, i.e., $Q(g) > Q(f)$. In particular, since $P = Q/Q(S)$ for some $Q \in \partial V(f)$ with $Q(S) > 0$, $P(g) > P(f)$. Thus, $P \in \pi^s(f)$.

(4): consider $g \in \mathbb{R}_+^S$ such that $V(g) \geq V(f)$ and $Q \in \partial V(f)$. Since V is nice at f and V is monotonic, $Q(S) > 0$ and so $Q/Q(S) \in C(f)$. By assumption, also $Q/Q(S) \in \pi^s(f)$, and therefore $V(g) \geq V(f)$ implies $Q(g)/Q(S) > Q(f)/Q(S)$, i.e., $Q(g) > Q(f)$, or $Q(g-f) > 0$. Since $Q \in \partial V(f)$ was arbitrary, V is strictly pseudoconcave at f .

By part 1, if V is nice at f then $\pi(f) \subseteq C(f)$; by part 2, $\pi^s(f) \subseteq \pi(f)$, so $\pi^s(f) \subseteq \pi(f) \subseteq C(f)$. Hence, part 4 can be restated as follows: if V is monotonic and nice at f , then it is strictly pseudoconcave at f if and only if $\pi^s(f) = \pi(f) = C(f)$. The last statement follows. ■

Proof of Proposition 2: We first show that V is ∂ -quasiconcave.

Fix $f \in \mathbb{R}_{++}^S$ and $g \in \mathbb{R}_+^S$ such that $V(g) \geq V(f)$. Also fix $\epsilon > 0$ and let $g_\epsilon = g + 1_S \epsilon$. By strong monotonicity, $V(g_\epsilon) > V(g) \geq V(f)$. Consider sequences $(c^k) \subset \mathbb{R}_+^S$ and $(t^k) \subset \mathbb{R}_{++}$ such that $c^k \rightarrow f$ and $t^k \downarrow 0$. Note that

$$t^k[g_\epsilon - f] + c^k = t^k[g_\epsilon - f + c^k] + (1 - t^k)c^k$$

and, since $c^k \rightarrow f \in \mathbb{R}_{++}^S$, eventually $g_\epsilon - f + c^k \in \mathbb{R}_{++}^S$; furthermore, by continuity $V(g_\epsilon - f + c^k) \rightarrow V(g_\epsilon)$ and $V(c^k) \rightarrow V(f)$. Therefore, for k sufficiently large, $V(g_\epsilon - f + c^k) > V(c^k)$. Then, by (strict) quasiconcavity, for all such k ,

$$V(t^k[g_\epsilon - f] + c^k) = V(t^k[g_\epsilon - f + c^k] + (1 - t^k)c^k) \geq V(c^k).$$

It follows that

$$V^\ell(f; g_\epsilon - f) = \liminf_{c \rightarrow f, t \downarrow 0} \frac{V(t[g_\epsilon - f] + c) - V(c)}{t} \geq 0.$$

Finally, since this holds for all $\epsilon > 0$, by continuity of $V^\ell(f; \cdot)$, $V^\ell(f; g - f) \geq 0$ as well. Since $V^\ell(f; g - f) = \min_{Q \in \partial V(f)} Q(g - f)$, it follows that $Q(g) \geq Q(f)$ for all $Q \in \partial V(f)$.

We now show that this implies that $C(f) \subseteq \pi(f)$. Since, for all $g \in \mathbb{R}_+^S$, $V(g) \geq V(f)$ implies $Q(g) \geq Q(f)$ for all $Q \in \partial V(f)$, this is true in particular for $Q \in \partial V(f)$ such that $Q(S) > 0$. Therefore, if $P \in C(f)$, then $P(g) \geq P(f)$. Hence, $P \in \pi(f)$, as claimed.

To complete the proof, Proposition 11 part 2 shows that, if V is strictly quasiconcave, then $\pi^s(f) = \pi(f)$ for all $f \in \mathbb{R}_{++}^S$. Conclude that $C(f) \subseteq \pi^s(f)$ for all such f . By Proposition 11 part 4, if in addition V is nice at f , then it is strictly pseudoconcave at f . ■

C.4 Results in Section 5

Throughout this section, we assume that I is normalized, strongly monotonic, and locally Lipschitz, and that u is strictly increasing, strictly concave and twice differentiable.

The normalized Clarke differential of I at $h \in \mathbb{R}_+^S$ is

$$C^u(h) = \left\{ \frac{Q^u}{Q^u(S)} : Q^u \in \partial I(u \circ h), Q^u \neq 0_S \right\}. \quad (16)$$

Remark 4 For every $i \in N$, the Clarke differential at $f \in \mathbb{R}_{++}^S$ of $V = I \circ u$ is

$$\partial V(f) = \left\{ Q \in \mathbb{R}^S : \forall h \in \mathbb{R}^S, Q(h) = \sum_s Q^u(s) u'(f(s)) h(s) \text{ for some } Q^u \in \partial I(u \circ f) \right\}.$$

Proof: Let $\mathbb{U} = u(\mathbb{R}_+)$. The map $F : \mathbb{R}_+^S \rightarrow \mathbb{U}^S$ defined by $F(f) = (u(f_1), \dots, u(f_S))$ is strictly differentiable (pp. 30-31 [Clarke, 1983](#)) and, furthermore, it maps every neighborhood of f to a neighborhood of $F(f)$.²⁵ Hence, since $V = I \circ F$, by Theorem 2.3.10 in Clarke $\partial V(f) = \partial I(u \circ f) \circ D_s F(f)$; that is, more explicitly, every $Q \in \partial V(f)$ is defined by

$$\forall h \in \mathbb{R}^S, \quad Q(h) = \sum_s Q^u(s) u'(f_s) h_s$$

for some $Q^u \in \partial I(u \circ f)$. ■

We define a set of probabilities that is related to $\pi(\cdot)$, but employs the decomposition of V in terms of I and u .

$$\pi^c(f) = \{P \in \Delta(S) : \forall g \in \mathbb{R}_+^S, I(u \circ g) \geq I(u \circ f) \implies P(u \circ g) \geq P(u \circ f)\}. \quad (17)$$

Recall that one can interpret $P \in \pi(f)$ as a risk-neutral SEU preference whose better-than set at f contains the better-than set of \succsim at f . Similarly, $P \in \pi^c(f)$ identifies an SEU preference, with risk attitudes described by u , whose better-than set at f contains that of \succsim at f .

The following result is a consequence of the concavity of u .

Remark 5 For all $x \in \mathbb{R}_{++}$, $\pi^c(1_S x) \subseteq \pi(1_S x)$.

²⁵To see this, fix a strictly positive bundle f and consider the set $\{g \in \mathbb{R}_+^S : f_s - \epsilon < g_s < f_s + \epsilon \forall s \in S\}$, which is open. The image of this set via F is $\{v \in \mathbb{U}^S : u(f_s - \epsilon) < v_s < u(f_s + \epsilon) \forall s \in S\}$, because u is continuous and strictly increasing. This set is also open.

Proof: Fix $P \in \pi^c(1_S x)$ and suppose that $g \in \mathbb{R}_+^S$ satisfies $V(g) \geq V(1_S x)$. Then $I(u \circ g) \geq I(u(x)) = u(x)$, and since $P \in \pi^c(1_S x)$, $P(u \circ g) \geq P(1_S u(x)) = u(x)$. Since u is (strictly) concave, $u(P(g)) \geq P(u \circ g)$, so $u(P(g)) \geq u(x)$. Since u is strictly increasing, $P(g) \geq x$. Thus, $P \in \pi(1_S x)$. ■

This is our “portmanteau” theorem.

Proposition 16 For every $x > 0$:

1. I is nice at $1_S u(x)$ if and only if V is nice at $1_S x$.
2. $C^u(1_S x) = C(1_S x)$
3. $\pi^c(1_S x) \subseteq \pi^s(1_S x)$
4. if I is ∂ -quasiconcave at $1_S u(x)$, then $C(1_S x) \subseteq \pi^c(1_S x)$

Furthermore, $\text{Core } I = \bigcap_{x>0} \pi^c(1_S x)$.

Corollary 17 Assume that I is nice and ∂ -quasiconcave at $1_S u(x)$ for every $x > 0$. Then, for every such $x > 0$, V is nice at $1_S x$, and $\pi(1_S x) = \pi^c(1_S x) = \pi^s(1_S x) = C(1_S x)$. Furthermore, V satisfies SPC.

Proof: (1): by Remark 4,

$$\partial V(1_S x) = \{Q \in \mathbb{R}^S : \forall h \in \mathbb{R}^S, Q(h) = u'(x)Q^u(h) \text{ for some } Q^u \in \partial I(u \circ f)\}.$$

Since $u'(x) > 0$ by assumption, $0_S \in \partial V(1_S x)$ iff $0_S \in \partial I(1_S u(x))$.

(2): Again from Remark 4,

$$C(1_S x) = \left\{ \frac{Q}{Q(S)} : Q \in \partial V(1_S x), Q \neq 0_S \right\} = \left\{ \frac{u'(x)Q^u}{u'(x)Q^u(S)} : Q^u \in \partial I(1_S u(x)), Q^u \neq 0_S \right\} = C^u(1_S x),$$

where by part 1, $Q^u = 0_S$ iff $0_S \in \partial I(1_S u(x))$.

(3): by Remark 5, $\pi^c(1_S x) \subseteq \pi(1_S x)$. Also recall that, since I is strongly monotonic and u is strictly increasing, $V = I \circ u$ is strongly monotonic. Finally, since I and u are both continuous, so is V .

Fix $P \in \pi^c(1_S x)$. Consider $g \in \mathbb{R}_+^S$ such that $g \neq 1_S x$ and $V(g) \geq V(1_S x)$.

Suppose first that g is constant, i.e., $g = 1_S y$ for some $y \geq 0$. Since $V(1_S y) = V(g) \geq V(1_S x)$, $y \geq x > 0$ by strong monotonicity of $V = I \circ u$. Since $1_S y = g \neq 1_S x$, $y > x$. Therefore, $P(g) = y > x$.

Now suppose that g is non-constant. As noted above, $P \in \pi^c(1_S x)$ implies $P \in \pi(1_S x)$. Since $V = I \circ u$ is strongly monotonic and continuous, by Corollary 15 $P \gg 0$. Then, since g is non-constant and u is strictly concave, $u(P(g)) > P(u \circ g)$. To see this, suppose that $s, s' \in S$ are such that $g(s) \neq g(s')$. Then, since $P(s) > 0$ and $P(s') > 0$,

$$\sum_{t \in \{s, s'\}} \frac{P(t)}{P(\{s, s'\})} u(g(t)) < u \left(\sum_{t \in \{s, s'\}} \frac{P(t)}{P(\{s, s'\})} g(t) \right),$$

and therefore

$$\begin{aligned} P(u \circ g) &= \sum_{t \in S} P(t) u(g(t)) = \\ &= [1 - P(\{s, s'\})] \sum_{t \in S \setminus \{s, s'\}} \frac{P(t)}{1 - P(\{s, s'\})} u(g(t)) + P(\{s, s'\}) \sum_{t \in \{s, s'\}} \frac{P(t)}{P(\{s, s'\})} u(g(t)) < \\ &< [1 - P(\{s, s'\})] u \left(\sum_{t \in S \setminus \{s, s'\}} \frac{P(t)}{1 - P(\{s, s'\})} g(t) \right) + P(\{s, s'\}) u \left(\sum_{t \in \{s, s'\}} \frac{P(t)}{P(\{s, s'\})} g(t) \right) \leq \\ &\leq u \left(\sum_{t \in S} P(t) g(t) \right) = u(P(g)). \end{aligned}$$

To conclude the argument, $P \in \pi^c(1_S x)$ and $V(g) \geq V(1_S x)$ imply $P(u \circ g) \geq u(x)$; but since $u(P(g)) > P(u \circ g)$ and u is strictly increasing, $P(g) > x$.

Since g was chosen arbitrarily, $P \in \pi^s(1_S x)$.

(4): fix $P \in C(1_S x)$. By part 2, $P \in C^u(1_S x)$, so there exists $Q^u \in \partial I(1_S u(x))$ such that $P = Q^u / Q^u(S)$. Consider $f \in \mathbb{R}_+^S$ such that $I(u \circ f) \geq u(x)$. Since I satisfies ∂ -quasiconcavity at $1_S u(x)$, in particular $Q^u(u \circ f - 1_S u(x)) \geq 0$. This implies that $P(u \circ f) \geq u(x)$. Since f was arbitrary, $P \in \pi^c(1_S x)$.

For the last statement, fix $P \in \text{Core } I$ and $x > 0$. Consider $f \in \mathbb{R}_+^S$ such that $V(f) \geq V(1_S x)$, i.e., since $V = I \circ u$ and I is normalized, $I(u \circ f) \geq u(x)$. Since $P \in \text{Core } I$, $P(u \circ f) \geq I(u \circ f)$. Therefore, $P(u \circ f) \geq u(x)$. Since f was arbitrary, $P \in \pi^c(1_S x)$.

Conversely, suppose that $P \in \bigcap_{x > 0} \pi^c(1_S x)$. Fix $f \in \mathbb{R}_+^S$. If $f = 0_S$, then $P(u \circ f) = u(0) = I(u \circ f)$, where the second equality follows because I is normalized. If instead $f \neq 0_S$, let c be

the certainty equivalent of f : that is, $I(u \circ f) = u(c)$. By strong monotonicity, $c > 0$. Therefore, $P \in \pi^c(1_S c)$. Then, $I(u \circ f) = u(c)$ implies that $P(u \circ f) \geq u(c)$. But then $P(u \circ f) \geq u(c) = I(u \circ f)$, so $P \in \text{Core } I$.

Turn now to the Corollary. By assumption, I is nice and ∂ -quasiconcave at every $1_S u(x)$, $x > 0$. By part 1, V is nice at $1_S x$, $x > 0$. Moreover, for every such $x > 0$, by parts 3 and 4, $C(1_S x) \subseteq \pi^c(1_S x) \subseteq \pi^s(1_S x)$; by Proposition 11 part 4, V is strictly pseudoconcave at $1_S x$. Therefore, V satisfies SPC. Furthermore, by Proposition 11 parts 2 and 1, $\pi^s(1_S x) \subseteq \pi(1_S x) \subseteq C(1_S x)$. Therefore, since $C(1_S x) \subseteq \pi^c(1_S x)$, all these sets are equal. ■

Proof of Proposition 5: since u is strictly increasing and I is strongly monotonic, V is strongly monotonic. Since u is concave and strictly increasing, it is locally Lipschitz (cf. Clarke, 1983, Prop. 2.2.6). Since I is locally Lipschitz, $V = I \circ u$ is also locally Lipschitz (cf. Clarke, 1983, p. 42). Finally, since I is nice at every $1_S u(x)$, $x > 0$, Proposition 16 part 1 implies that V is nice at every $x > 0$. Therefore, Assumption 1 holds. By Corollary 17, since I satisfies DQC, it is ∂ -quasiconcave at every $1_S u(x)$, $x > 0$; thus, V is strictly pseudoconcave at every $1_S x$, $x > 0$, i.e., SPC holds. ■

Proof of Corollary 6. Note: Part (1) follows from a result in Penot and Quang (1997); however, since their assumptions are formulated somewhat differently from ours, invoking their result requires some work. We provide a direct proof.

It is convenient to let $\mathbb{U} = u(\mathbb{R}_+) = \{r : \exists x \geq 0, r = u(x)\}$. Fix $\gamma \in \text{int}(\mathbb{U})$ and $a \in \mathbb{U}^S$ such that $I(a) \geq \gamma$. For both conditions, we use the properties of the Clarke lower derivative in Eq. (15); in particular, it is enough to show that $I^\ell(1_S \gamma; a - 1_S \gamma) \geq 0$.

(1): fix $\epsilon > 0$ such that $a + 1_S \epsilon \in \mathbb{U}^S$ (this must exist, because $\mathbb{U} = u(\mathbb{R}_+)$ does not contain its supremum). By strong monotonicity, $I(a + 1_S \epsilon) > \gamma$. Consider sequences $(c^k) \subset \mathbb{U}^S$ and $(t^k) \subset \mathbb{R}_{++}$ such that $c^k \rightarrow 1_S \gamma$ and $t^k \downarrow 0$. Note that

$$t^k[(a + 1_S \epsilon) - 1_S \gamma] + c^k = t^k[(a + 1_S \epsilon) - 1_S \gamma + c^k] + (1 - t^k)c^k$$

and, since $c^k \rightarrow 1_S \gamma$, eventually $(a + 1_S \epsilon) - 1_S \gamma + c^k \in \mathbb{U}^S$; furthermore, by continuity $I(a + 1_S \epsilon - 1_S \gamma + c^k) \rightarrow I(a + 1_S \epsilon)$ and $I(c^k) \rightarrow I(1_S \gamma) = \gamma$. Therefore, for k sufficiently large, $I(a + 1_S \epsilon -$

$1_S\gamma + c^k > I(c^k)$. Then, by quasiconcavity, for all such k ,

$$I(t^k[(a + 1_S\epsilon) - 1_S\gamma] + c^k) = I(t^k[(a + 1_S\epsilon) - 1_S\gamma + c^k] + (1 - t^k)c^k) \geq I(c^k).$$

It follows that

$$I^\ell(1_S\gamma; (a + 1_S\epsilon) - 1_S\gamma) = \liminf_{c \rightarrow 1_S\gamma, t \downarrow 0} \frac{I(t[(a + 1_S\epsilon) - 1_S\gamma] + c) - I(c)}{t} \geq 0.$$

Finally, by continuity of $I^\ell(1_S\gamma; \cdot)$, $I^\ell(1_S\gamma; a - 1_S\gamma) \geq 0$ as well.

(2): if I is regular, $I^\ell(1_S\gamma; a - 1_Sx) = -I^\circ(1_S\gamma; 1_S\gamma - a) = -I'(1_S\gamma; 1_S\gamma - a)$; furthermore, if $I(a) \geq I(1_S\gamma) = \gamma$, by normalization, for any $P \in \text{Core } I \neq \emptyset$,

$$\begin{aligned} -I^\ell(1_S\gamma; a - 1_S\gamma) &= I'(1_S\gamma; 1_S\gamma - a) = \lim_{t \downarrow 0} \frac{I(1_S\gamma + t[1_S\gamma - a]) - I(1_S\gamma)}{t} = \\ &= \lim_{t \downarrow 0} \frac{I(1_S\gamma + t[1_S\gamma - a]) - \gamma}{t} \leq \lim_{t \downarrow 0} \frac{P(1_S\gamma + t[1_S\gamma - a]) - \gamma}{t} = \\ &= \lim_{t \downarrow 0} \frac{\gamma + t\gamma - tP(a) - \gamma}{t} = \gamma - P(a) \leq I(a) - P(a) \leq 0, \end{aligned}$$

as required. ■

Proof of Proposition 7: observe first that, for all $\phi \in \mathbb{R}^J$,

$$\nabla I(a) \equiv \left(\frac{\partial I(a)}{\partial a(s)} \right)_{s \in S} = \left(P(\{s\}) \left[1 + \sum_{0 \leq j < J} \frac{\partial A(P(\zeta_0 a), \dots, P(\zeta_{J-1} a))}{\partial \phi_j} \zeta_j(s) \right] \right)_{s \in S}. \quad (18)$$

Thus, the last condition in the Proposition is simply the requirement that all partial derivatives be strictly positive almost everywhere on $u(\mathbb{R}_+)^S$. Thus, I is strongly monotonic.

Next, we show that $\nabla A(0_J) = 0_J$. Fix $0 \leq j < J$. Since A is continuously differentiable at 0_J , satisfies $A(0_J) = 0$ and is symmetric about 0_J ,

$$\nabla A(0_J) \cdot 1_j = \lim_{t \downarrow 0} \frac{A(0_J + t1_j) - A(0_J)}{t} = \lim_{t \downarrow 0} \frac{A(t1_j)}{t} = \lim_{t \downarrow 0} \frac{A(t(-1_j))}{t} = \lim_{t \downarrow 0} \frac{A(0_J + t(-1_j)) - A(0_J)}{t} = \nabla A(0_J) \cdot (-1_j),$$

which clearly requires that $\nabla A \cdot 1_j = \frac{\partial A(0_J)}{\partial \phi_j} = 0$, as claimed. Since $P(\zeta_j 1_S x) = xP(\zeta_j) = 0$, it follows that $\nabla I(1_S x) = P$ for all $x > 0$. Hence I is nice at certainty, and $C^u(1_S x) = \{P\}$; by Proposition 16 part 2, also $C(1_S x) = \{P\}$.

Since $A \leq 0$, it is immediate that $P \in \text{Core } I$; furthermore, I is smooth, hence regular. Therefore, I and u satisfy Assumption 2; furthermore, it satisfies condition 2 in Corollary 6, so SPC holds.

Finally, by Corollary 6 part 2, I satisfies DQC, so by Corollary 17, $\pi(1_S x) = \pi^c(1_S x) = C(1_S x) = \{P\}$, and so, by Proposition 16, $\text{Core } I = \{P\}$ as well. ■

C.5 Results in Section 6

The key step in the proof of Proposition 8 is contained in the following result.

Lemma 18 *Assume that each V_i is monotonic. If $(f_i)_{i \in N}$ is a Pareto-efficient allocation, then there exists a price vector $p \in \mathbb{R}_+^S \setminus \{0\}$ such that $-p \in N_i(f_i)$ for all $i \in N$.*

Proof: Apply Prop. 2.1 (a) and (e) and Theorem 2.1 in [Bonnisseau and Cornet \(1988\)](#) to get $-p \in \bigcap_{i \in N} N_i(f_i)$. We only need to show that p is non-negative. By monotonicity, $\mathbb{R}_+^S \subset T_i(f_i)$: to see this, note that, if $v \in \mathbb{R}_+^S$, then for any sequence (f^k, t^k) such that $f^k \succsim_i f_i$, $f^k \rightarrow f_i$, and $t \downarrow 0$, the constant sequence $v^k = v$ satisfies $f^k + t^k v^k \geq f^k \succsim_i f_i$ for all k .

Now consider $v \in \mathbb{R}_+^S$ s.t. $v_s = 0$ iff $p_s \geq 0$, and $v_s = 1$ otherwise. If $p_s < 0$ for some s , then $p \cdot v < 0$, i.e. $-p \cdot v > 0$, which contradicts the fact that $v \in T_i(f_i)$ and $-p \in N_i(f_i)$ for all i . Thus, $p \geq 0$. ■

Proof of Proposition 8 and Corollary 9: Lemma 18 yields $p \in \mathbb{R}_+^S \setminus \{0_S\}$ such that $-p \in N_i(f_i)$ for all i ; by Remark 3, $-p \in \bigcup_{\lambda > 0} \lambda(-\partial V_i(f))$ for all $i \in N$. Thus, $p = \lambda_i Q_i$ for every i , where $\lambda_i > 0$ and $Q_i \in \partial V_i(f)$; then $Q_i(S) = \frac{\sum_s p_s}{\lambda_i}$, and therefore $\frac{Q_i}{Q_i(S)} = \frac{\lambda_i^{-1} p}{\lambda_i^{-1} \sum_s p_s} = \frac{p}{\sum_s p_s} \equiv P$; hence, $P \in \bigcap_i C_i(f)$. ■

The next Remark follows from standard arguments; we include the proof for completeness.

Remark 6 Assume that each V_i is continuous and strongly monotonic. If a feasible allocation (f_1, \dots, f_N) is not Pareto-efficient, then it is Pareto-dominated by a Pareto-efficient allocation.

Proof: By assumption, there exists a feasible allocation (g_1, \dots, g_N) that Pareto-dominates (f_1, \dots, f_N) . Assume wlog that $g_1 \succ_1 f_1$. Consider the following problem: maximize $V_1(h_1)$ subject to (h_1, \dots, h_N) being feasible and $h_i \succ_i g_i$ for all $i = 2, \dots, N$. Notice that the allocation (g_1, \dots, g_N) satisfies these constraints. By standard arguments (e.g. Mas-Colell et al., 1995, §16.F), since preferences are continuous and strongly monotonic, a solution (h_1^*, \dots, h_N^*) to this problem exists and is Pareto-efficient. Furthermore, for every $i > 1$, $h_i^* \succ_i g_i \succ_i f_i$, and $h_1^* \succ_1 g_1 \succ_1 f_1$; that is, (h_1^*, \dots, h_N^*) is a Pareto-efficient allocation that Pareto-dominates (f_1, \dots, f_N) . ■

Proof of Proposition 10: Assume that $\bigcap_i \pi_i^s(1_S x_i) \neq \emptyset$ for every feasible, full-insurance allocation $(1_S x_1, \dots, 1_S x_N)$, with $x_i \geq 0$ for all $i \in N$.

We first show that every Pareto-efficient allocation must provide full insurance. To do so, consider a feasible allocation (f_1, \dots, f_N) . We show that, if this allocation does not provide full insurance, there is a full-insurance allocation that Pareto-dominates it.

For every $i \in N$, let c_i be the certainty equivalent of f_i : that is, $V_i(1_S c_i) = V_i(f_i)$. There are two cases to consider.

Case 1: $\sum_i c_i \geq \bar{x} > 0$. Define a new allocation $(1_S x_1, \dots, 1_S x_N)$ as follows: for every $i \in N$, let $x_i = \frac{\bar{x}}{\sum_j c_j} c_i$. Then $\sum_i x_i = \frac{\bar{x}}{\sum_j c_j} \sum_i c_i = \bar{x}$, i.e., $(1_S x_1, \dots, 1_S x_N)$ is feasible. Since (f_1, \dots, f_N) is not a full-insurance allocation, there is at least one agent i for whom f_i is non-constant; wlog let that be agent 1. By strong monotonicity, $V_1(1_S c_1) = V_1(f_1) > V_1(0_S)$; since V_1 is strongly monotonic, $c_1 > 0$, and therefore, $x_1 = \frac{\bar{x}}{\sum_j c_j} c_1 > 0$. By assumption, there is $P \in \bigcap_i \pi_i^s(1_S x_i)$; it is immediate from the definitions that $\pi_i^s(1_S x_i) \subseteq \pi_i(1_S x_i)$, so $P \in \pi_i(1_S x_i)$. By Corollary 15, since in particular $P \in \pi_1(1_S x_1)$ and $x_1 > 0$, P is strictly positive.

For every $i \in N$, by construction $V_i(f_i) = V_i(1_S c_i) \geq V_i(1_S x_i)$. Since $P \in \pi_i^s(1_S x_i) \subseteq \pi_i(1_S x_i)$, for all i , $P(f_i) \geq x_i$. For $i = 1$, f_1 is non-constant and hence distinct from $1_S x_1$, $P(f_1) > x_1$.

Conclude that $\sum_i P(f_i) > \sum_i x_i = \bar{x}$. However, $\sum_i P(f_i) = P(\sum_i f_i) = P(1_S \bar{x}) = \bar{x}$, because (f_1, \dots, f_N) is feasible: contradiction. Thus, this case cannot occur.

Case 2: $\sum_i c_i < \bar{x}$. Let $\epsilon = \frac{\bar{x} - \sum_i c_i}{N}$: then, the full-insurance allocation $(1_S(c_1 + \epsilon), \dots, 1_S(c_N + \epsilon))$ is feasible and Pareto-dominates (f_1, \dots, f_N) by strong monotonicity, as claimed.

Conversely, consider a feasible, full-insurance allocation $(1_S y_1, \dots, 1_S y_N)$, and suppose that it is not Pareto-efficient. Then, by Remark 6, it is Pareto-dominated by a Pareto-efficient allocation; by the result just proved, under the maintained assumptions, this allocation must be a full-insurance allocation, say $(1_S x_1, \dots, 1_S x_N)$. Since preferences are strongly monotonic, this implies that $x_i \geq y_i$ for all i , and the inequality is strict for at least one i . But then $\sum_i x_i > \sum_i y_i = \bar{x}$, i.e., $(1_S x_1, \dots, 1_S x_N)$ is not feasible: contradiction. Thus, every full-insurance allocation is Pareto-efficient.

Finally, let $(1_S x_1, \dots, 1_S x_N)$ be a full-insurance, hence Pareto-efficient allocation. Fix $P \in \bigcap_i \pi_i^s(1_S x_i)$. Since $\sum_i x_i = \bar{x} > 0$, there must be some $i \in N$ for whom $x_i > 0$; since $P \in \pi_i^s(1_S x_i)$, by Corollary 15 and the fact that $\pi_i^s(1_S x_i) \subseteq \pi_i(1_S x_i)$, P is strictly positive.

Now suppose that, for some $i \in N$ and $g \in \mathbb{R}_+^S$, $g \succ_i 1_S x_i$. Since $P \in \pi_i^s(1_S x_i)$, $P(g) > x_i$. Equivalently, $P(g) \leq x_i$ implies $1_S x_i \succsim_i g$. We can then let $t = P(1_S x_i) - P(\omega_i) = x_i - P(\omega_i)$: we get $\sum_i t = \sum_i x_i - \sum_i P(\omega_i) = \bar{x} - P(\sum_i \omega_i) = \bar{x} - P(1_S \bar{x}) = 0$. Hence t_1, \dots, t_N define feasible transfers. Since preferences are strongly monotonic (hence local non-satiated), consumers will exhaust their budget $P(\omega_i) + t_i = x_i$, and the argument just given shows that they will demand $1_S x_1, \dots, 1_S x_N$. ■

C.6 Results in Section 7

First, consider a preference \succsim that admits a representation of the form $V = I \circ u$, such that (I, u) satisfy Assumption 2.

Remark 7 If $\bigcap_{x>0} C(1_S x) \neq \emptyset$ and DQC holds, then $\text{Core } I \neq \emptyset$.

Proof: Let $P \in \bigcap_{x>0} C(1_S x)$. Consider $a \in u(X)^S$. If $a = 1_S u(0)$, then $I(a) = u(0) = P(a)$ by normalization. If $a \neq 1_S u(0)$, let x be such that $I(a) = u(x)$; this exists by standard arguments. By DQC, for all $Q \in \partial I(1_S u(x))$, $Q(a - 1_S u(x)) \geq 0$. Since $a \neq 1_S u(0)$, by strong monotonicity $I(a) > u(0)$, so $x > 0$. Hence, there is $Q \in \partial I(1_S u(x))$ such that $Q(S) > 0$ and $Q/Q(S) = P$. Therefore, $P(a - 1_S u(x)) \geq 0$, or $P(a) \geq P(1_S u(x)) = u(x) = I(a)$. Since a was arbitrary, $P \in \text{Core } I$. ■

Remark 8 (Preference for sure EU diversification) Suppose that DQC holds. Then, for all bundles f_1, \dots, f_N and weights $\alpha_1, \dots, \alpha_N \geq 0$ such that $\sum_i \alpha_i = 1$, if $f_i \sim f_j$ for all i, j , and there exists $x \geq 0$ such that $\sum_i \alpha_i u(f_i(s)) = u(x)$ for all s , then $x \succcurlyeq f_i$.

Proof: Fix $f_i, \alpha_i, i = 1, \dots, N$, and x as in the statement. Let $y \geq 0$ be such that $f_i \sim y$ for all i ; this exists by standard arguments. If $y = 0$, then $f_i = 0$ for all i , so $x = 0$ because u is strictly increasing; then, trivially, $x \sim f_i$ for all i . If instead $y > 0$, DQC implies that $Q(u \circ f_i - 1_S u(y)) \geq 0$ for all $Q \in \partial I(1_S u(y))$. By linearity,

$$Q(1_S u(x) - 1_S u(y)) = Q\left(\sum_i \alpha_i u \circ f_i - 1_S u(y)\right) = Q\left(\sum_i \alpha_i [u \circ f_i - 1_S u(y)]\right) = \sum_i \alpha_i Q(u \circ f_i - 1_S u(y)) \geq 0.$$

Since $Q(S) > 0$ because I is nice at $1_S u(y)$ by assumption, $u(x) \geq u(y)$. hence, $x \succcurlyeq y \sim f_i$ for each i . ■

Now consider a preference that admits a representation V that is not (necessarily) of the form $I \circ u$. We maintain Assumption 1; in addition, for the first two results, we will assume that V is *normalized*: that is, $V(1_S x) = x$ for all $x \geq 0$. Notice that V can then be interpreted as a certainly-equivalent functional: $V(f) = x$ if and only if $f \sim x$.

The *strict core* of V is the set $\text{score } V = \{P \in \Delta(S) : \forall f \in \mathbb{R}_+^S \text{ non-constant, } P(f) > V(f)\}$. For instance, if V is the certainty-equivalent function of an EU preference with strictly positive beliefs P and a strictly concave utility u , then for all non-constant f , $V(f) = u^{-1}P(u \circ f) < u^{-1}(u(P(f))) = P(f)$, so $P \in \text{score } V$. If instead the preference is risk-neutral, $\text{score } V = \emptyset$. Therefore, a non-empty strict core captures a notion of strict risk/ambiguity aversion.

Remark 9 If V is normalized and regular at certainty, and $\text{score } V \neq \emptyset$, then V satisfies SPC.

The proof mimics that of Corollary 6 part 2, with some additional subtleties.

Proof: Fix $x > 0$ and $f \neq 1_S x$ such that $V(f) \geq V(1_S x)$. If $f = 1_S y$, then by strong monotonicity and the assumption that $f \neq 1_S x$, $y > x$; since $Q(S) > 0$ for all $Q \in \partial V(1_S x)$ by niceness, $Q(1_S y - 1_S x) = Q(S)(y - x) > 0$ for all such Q .

Now suppose that f is non-constant. As in the proof of Corollary 6 part 2, it is enough to show that $V^\ell(1_S x; f - 1_S x) > 0$. Since V is regular, $V^\ell(1_S x; f - 1_S x) = -V^\circ(1_S x; 1_S x - f) =$

$-V'(1_S x; 1_S x - f)$; furthermore, if $V(f) \geq V(1_S x) = x$, by normalization, for any $P \in \text{score } V$,

$$\begin{aligned} -V^\ell(1_S x; f - 1_S x) &= V'(1_S x; 1_S x - f) = \lim_{t \downarrow 0} \frac{V(1_S x + t[1_S x - f]) - V(1_S x)}{t} = \\ &= \lim_{t \downarrow 0} \frac{V(1_S x + t[1_S x - f]) - x}{t} \leq \lim_{t \downarrow 0} \frac{P(1_S x + t[1_S x - f]) - x}{t} = \\ &= \lim_{t \downarrow 0} \frac{x + t x - t P(f) - x}{t} = x - P(f) \leq V(f) - P(f) < 0, \end{aligned}$$

as required. Notice that, while $P(1_S x + t[1_S x - f]) > V(1_S x + t[1_S x - f])$ for positive $t > 0$ because the argument of P and V is non-constant when f is, this may not be true in the limit as $t \downarrow 0$. Thus, the first inequality is weak. However, the last inequality is strict, because f is non-constant and $P \in \text{score } V$. ■

Remark 10 If V is normalized, $\cap_{x>0} C(1_S x) \neq \emptyset$, and SPC holds, then $\text{score } V \neq \emptyset$.

Proof: Let $P \in \cap_{x>0} C(1_S x)$ and consider f non-constant. Let $x \sim f$, which exists by standard arguments. Since $x = V(1_S x) = V(f)$, SPC implies that $Q(f - 1_S x) > 0$ for all $Q \in \partial V(1_S x)$. Hence, in particular, $P(f - 1_S x) > 0$, i.e., $P(f) > x = V(f)$. Therefore $P \in \text{score } V$. ■

Remark 11 (Preference for sure diversification) Suppose that SPC holds. Then, for all bundles f_1, \dots, f_N and weights $\alpha_1, \dots, \alpha_N \geq 0$ such that $\sum_i \alpha_i = 1$, if $f_i \sim f_j$ for all i, j , and there exists $x \geq 0$ such that $\sum_i \alpha_i f_i(s) = x$ for all s , then $x \succcurlyeq f_i$ for all i ; indeed $x \succ f_i$ for all i unless all f_i 's are constant.

Proof: Fix $f_i, \alpha_i, i = 1, \dots, N$, and x as in the statement. If all f_i 's are constant, then the assumptions imply that they must all equal $1_S x$, so the statement holds trivially. By strong monotonicity, this must be the case if $f_i \sim 0$ for all i . Thus, assume that they are not all constant, and that $f_i \succ 0$ for all i . Let $y > 0$ be such that $f_i \sim y$ for all i . SPC implies that $Q(f_i - 1_S y) > 0$ for all $Q \in \partial V(1_S y)$ and all $f_i \neq 1_S y$. By linearity,

$$Q(1_S x - 1_S y) = Q\left(\sum_i \alpha_i f_i - 1_S y\right) = Q\left(\sum_i \alpha_i [f_i - 1_S y]\right) = \sum_i \alpha_i Q(f_i - 1_S y) > 0.$$

The inequality is strict because $Q(f_i - 1_S y) > 0$ for at least one i . Since $Q(S) > 0$ because V is nice at $1_S y$ by assumption, $x > y$. Hence, $x \succ y \sim f_i$ for each i . ■

D Calculations for the example in Section 2.2

We first briefly discuss continuity and monotonicity with respect to first-order stochastic dominance. If the set \mathcal{D} of CDFs is endowed with the topology of weak convergence of measures, then continuity follows immediately from the assumption that g is continuous. Next, note that, for every CDF H , since u is a positive affine transformation of $-\frac{1}{1+x}$, it takes values in a bounded interval; therefore, $\int u^- dH > -\infty$ and $\int u^+ dH < \infty$, where as usual $u^- = \min(u, 0)$ and $u^+ = \max(u, 0)$. Now consider $F, G \in \mathcal{D}$ such that $F(x) \leq G(x)$ for all $x \geq 0$, i.e., F first-order stochastically dominates G . Then, Theorem 2.1 in [Brumelle and Vickson \(1975\)](#) implies that $\int u dF \geq \int u dG$ and $\int x dF \geq \int x dG$. Since g is strictly increasing, $W(F) \geq W(G)$, as required.

Now turn to risk aversion. Assume that G is a mean-preserving spread of F , in the sense that $\int_0^x [F(t) - G(t)] dt \leq 0$ for all $x \geq 0$, and $\int x dF = \int x dG$. Theorem 2.3 in [Brumelle and Vickson \(1975\)](#) then implies that $\int u dF > \int u dG$. In this case, again because g is strictly increasing, $W(F) \geq W(g)$.

Finally, note that, if g and u are both differentiable, for any interior bundle f ,

$$\frac{\partial V}{\partial f_s}(f) = g' \left(\sum_t u(f_t) P_t \right) u'(f_s) P_s + g'(P \cdot f) P_s.$$

If $f = 1_S x$ for some $x > 0$, then

$$\frac{\partial V}{\partial f_s}(f) = g'(u(x)) u'(x) P_s + g'(x) P_s = [g'(u(x)) + g'(x)] P_s,$$

i.e., $\partial V(1_S x) = \{\nabla V(1_S x)\} = [g'(u(x)) + g'(x)] \cdot P$, as asserted.

E Calculations for the examples in Sections 2.1 and 2.3

We first verify that the specification of adjustment factors and function in Section 2.3, together with a uniform baseline prior, ensures strong monotonicity. The same argument applies to the simpler specification in Section 2.1; we only indicate the minor, required modifications.

We use Eq. (18): first, note that

$$\frac{\partial A}{\partial \phi_j} = -\frac{1}{2}\theta \cdot \frac{2\theta^{-1}\phi_j}{1+\theta^{-1}\phi_j^2} = -\frac{\phi_j}{1+\theta^{-1}\phi_j^2}. \quad (19)$$

Hence,

$$\left| \frac{\partial A}{\partial \phi_j} \right| = \frac{|\phi_j|}{1+\theta^{-1}\phi_j^2} = \frac{|\phi_j|}{1+\theta^{-1}|\phi_j|^2}.$$

Letting $t = |\phi_j|$, this is less than one iff $t < 1 + \theta^{-1}t^2$, i.e. iff $t^2 - \theta t + \theta > 0$. We study the function $t \mapsto t^2 - \theta t + \theta$ for $t \geq 0$. If $t = 0$, the function takes the value θ , so we need $\theta > 0$. The derivative of this function at any $t > 0$ (which is also the right derivative at 0) is $2t - \theta$, which shows that this function is strictly convex and has a minimum at $t = \frac{1}{2}\theta$, where it is equal to $\frac{1}{4}\theta^2 - \frac{1}{2}\theta^2 + \theta$. This is strictly positive iff $-\frac{1}{4}\theta + 1 > 0$, i.e. iff $\theta < 4$, as claimed.

Now consider states $s = s_1, s_2$. Only ζ_0 has non-zero values, and $\zeta_0(s) \in \{1, -1\}$. Therefore, if $\theta \in (0, 4)$,

$$1 - \frac{\phi_0}{1+\theta^{-1}\phi_0^2}\zeta_0(s) - \frac{\phi_1}{1+\theta^{-1}\phi_1^2}\zeta_1(s) \geq 1 - \left| \frac{\phi_0}{1+\theta^{-1}\phi_0^2} \right| > 0.$$

Similarly, in states $s = s_3, s_4$, $\zeta_0(s) = 0$ and $\zeta_1(s) \in \{1, -1\}$, so

$$1 - \frac{\phi_0}{1+\theta^{-1}\phi_0^2}\zeta_0(s) - \frac{\phi_1}{1+\theta^{-1}\phi_1^2}\zeta_1(s) \geq 1 - \left| \frac{\phi_1}{1+\theta^{-1}\phi_1^2} \right| > 0.$$

so I is strictly increasing.

To adapt the argument to the preferences in Section 2.1, consider only states s_1 and s_2 .

We now show that, if θ increases, the resulting preference is more GM-ambiguity-averse. By the characterization result in [Siniscalchi \(2009\)](#), it suffices to show that $A(\phi)$ is decreasing in θ for every ϕ . Differentiating $A(\phi)$ with respect to θ ,

$$\frac{\partial A(\phi)}{\partial \theta} = -\frac{1}{2} \sum_j \log(1 + \theta^{-1}\phi_j^2) - \frac{1}{2}\theta \sum_j \frac{1}{1+\theta^{-1}\phi_j^2} (-\theta^{-2}\phi_j^2);$$

it suffices to show that, for every j and ϕ_j , $\log(1 + \theta^{-1}\phi_j^2) > \frac{\theta^{-1}\phi_j^2}{1+\theta^{-1}\phi_j^2}$. Let $t \equiv \theta^{-1}\phi_j^2$, so we need to show that $\log(1+t) > \frac{t}{1+t}$. Both functions equal zero at $t = 0$. For $t > 0$, the derivatives of the lhs and rhs are $\frac{1}{1+t}$ and $\frac{1 \cdot (1+t) - t(1)}{(1+t)^2} = \frac{1}{(1+t)^2}$ respectively. Since $(1+t)^2 > 1+t$ for $t > 0$, $\frac{1}{1+t} < \frac{1}{(1+t)^2}$, and therefore, for all $t > 0$, $\log(1+t) = \int_0^t \frac{1}{1+s} ds > \int_0^t \frac{1}{(1+s)^2} ds = \frac{t}{1+t}$, as claimed.

We finally turn to the analysis of the specific parameterization in Section 2.3. The four acts f^1, \dots, f^4 have the same expected baseline utility: $P(u \circ f^k) = 2\alpha + 1$ for $k = 1, \dots, 4$. Hence, their ranking is entirely determined by the adjustment terms $A(P(\zeta_0 u \circ f^k), P(\zeta_i u \circ f^k))$.

Act	$P(\zeta_0 u \circ f^k)$	$P(\zeta_1 u \circ f^k)$	Adjustment (omitting $\frac{1}{2}\theta$)
f^1	$\alpha - 1$	α	$-\log(1 + \theta^{-1}(\alpha - 1)^2) - \log(1 + \theta^{-1}\alpha^2)$
f^2	0	1	$-\log(1 + \theta^{-1})$
f^3	-1	0	$-\log(1 + \theta^{-1})$
f^4	$-\alpha$	$1 - \alpha$	$-\log(1 + \theta^{-1}\alpha^2) - \log(1 + \theta^{-1}(1 - \alpha)^2)$

Table 2: Adjustments

These are displayed in Table 2.

In order to generate the preferences $f^1 < f^2$, we need to ensure that $A(P(\zeta_0 u \circ f^1), P(\zeta_1 u \circ f^1)) < A(P(\zeta_0 u \circ f^2), P(\zeta_1 u \circ f^2))$. Notice that, since $(\alpha - 1)^2 = (1 - \alpha)^2$, this will also ensure that $A(P(\zeta_0 u \circ f^3), P(\zeta_1 u \circ f^3)) > A(P(\zeta_0 u \circ f^4), P(\zeta_1 u \circ f^4))$ and therefore $f^3 > f^4$, as the adjustments for f^1 and f^2 are the same as the adjustments for f^4 and f^3 respectively. Thus, we require

$$-\log(1 + \theta^{-1}(\alpha - 1)^2) - \log(1 + \theta^{-1}\alpha^2) < -\log(1 + \theta^{-1}).$$

We now derive a condition on θ that ensures that the above inequality holds.

$$\begin{aligned} & -\log(1 + \theta^{-1}(\alpha - 1)^2) - \log(1 + \theta^{-1}\alpha^2) < -\log(1 + \theta^{-1}) \\ \Leftrightarrow & (1 + \theta^{-1}(1 - \alpha)^2)(1 + \theta^{-1}\alpha^2) > 1 + \theta^{-1} \Leftrightarrow 1 + \theta^{-1}(1 - \alpha)^2 + \theta^{-1}\alpha^2 + \theta^{-2}(1 - \alpha)^2\alpha^2 > 1 + \theta^{-1} \\ \Leftrightarrow & (1 - \alpha)^2 + \alpha^2 + \theta^{-1}(1 - \alpha)^2\alpha^2 > 1 \\ \Leftrightarrow & \theta^{-1} > \frac{1 - \alpha^2 - (1 - \alpha^2)}{\alpha^2(1 - \alpha)^2} = \frac{1 - \alpha^2 - 1 - \alpha^2 + 2\alpha}{\alpha^2(1 - \alpha)^2} = \frac{2\alpha(1 - \alpha)}{\alpha^2(1 - \alpha)^2} = \frac{2}{\alpha(1 - \alpha)} \Leftrightarrow \theta < \frac{\alpha(1 - \alpha)}{2}. \end{aligned}$$

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