

# A SUBJECTIVE SPIN ON ROULETTE WHEELS

BY PAOLO GHIRARDATO, FABIO MACCHERONI, MASSIMO MARINACCI AND  
MARCIANO SINISCALCHI<sup>1</sup>

## ABSTRACT

We provide a simple behavioral definition of ‘subjective mixture’ of acts for a large class of (not necessarily expected-utility) preferences. Subjective mixtures enjoy the same algebraic properties as the ‘objective mixtures’ used to great advantage in the decision setting introduced by Anscombe and Aumann (1963).

This makes it possible to formulate mixture-space axioms in a fully subjective setting. For illustration, we present simple subjective axiomatizations of some models of choice under uncertainty, including Bewley’s model of choice with incomplete preferences (2002).

## Introduction

The axiomatizations of subjective expected utility (SEU) of Savage (1954) and Anscombe and Aumann (1963, AA for short) are often contrasted in terms of their analytical complexity and behavioral content. On one hand, Savage’s theory relies solely upon behavioral data, namely preferences among *acts* (i.e., maps assigning consequences to states); in contrast, the AA decision setting features pre-assigned, ‘objective’ probabilities embedded in the consequence space. On the other hand, the latter is much more amenable to mathematical treatment than Savage’s. This is especially apparent in Fishburn’s (1970) well-known reformulation and extension of AA’s analysis, which employs familiar vector-space arguments.<sup>2</sup>

The main contribution of this note is to show that it is possible to exploit all the advantages of the approach pioneered by AA and Fishburn (‘AA approach’ for short) relying solely on behavioral data, and hence retaining the conceptual appeal of Savage’s approach.

In the AA setting, payoffs are *lotteries* contingent on the output of a randomizing device, or ‘roulette wheel.’ Postulating the existence of such a device, characterized by objective probabilities, is generally considered unappealing and philosophically debatable (cf. the references cited in Sec. 4). On the other hand, the existence of roulette wheels enables one to

---

<sup>1</sup>An extended version of this note is available as an ICER Working Paper (2003). We thank Faruk Gul, Mark Machina, an editor and three referees and the audiences at the V SAET Conference (Ischia, July 2001) and the 2001 RUD Conference (Venice, July 2001) for helpful comments. We are also grateful for the hospitality of: the Department of Economics at Boston University and the Division of HSS at Caltech (Maccheroni), the MEDS Department at the Kellogg Graduate School of Management and the Department of Economics of Northwestern University (Siniscalchi). This work was begun while Ghirardato was with the Division of HSS at Caltech.

<sup>2</sup>AA assume that the state space is finite, and their original arguments employ the ‘linearity’ property of von Neumann-Morgenstern utility directly. Most textbook presentations of ‘the AA axiomatization of SEU’ are actually based on Fishburn’s. See also Schmeidler (1989, p. 578).

define ‘objective mixtures’ of acts; this considerably simplifies the axiomatic derivation of the SEU model, as well as extensions thereof, as we discuss below.

This note shows that it is possible to define a mixture operation with convenient algebraic properties in a fully behavioral setting like Savage’s, without the help of a randomizing device. Our construction requires that preferences satisfy some mild conditions, and that the set of possible outcomes be sufficiently rich (e.g., an interval in the real line). We show that if these conditions are met, we can use purely behavioral data to identify the prize  $z$  that sits at the mid-point of the preference interval between two prizes  $x$  and  $y$ ; that is, which satisfies

$$u(z) = \frac{1}{2}u(x) + \frac{1}{2}u(y). \tag{1}$$

Building on Eq. (1), one can define ‘subjective mixtures’ of acts with arbitrary weights.

Subjective mixtures share two important features with AA-style objective mixtures. First, the set of acts, equipped with the mixture operation, can be viewed as a mixture set; this makes it possible to formulate axioms in the style of von Neumann and Morgenstern (1947). Second, the utility profile of the mixture of two acts (with weights  $\alpha$  and  $1 - \alpha$ ) is the convex combination of the utility profiles of the latter (with the same weights).<sup>3</sup> This simplifies the construction of mathematical representations.

Thus, subjective mixtures enable us to readily extend AA-style axiomatics and techniques to a fully subjective environment. For instance, we employ subjective mixtures to offer simple axiomatizations of Schmeidler’s ‘Choquet expected utility’ (CEU) model (1989), and Gilboa and Schmeidler’s ‘maxmin expected utility’ (MEU) model (1989). Both models generalize SEU and were first axiomatized in the AA setting. We provide a similar treatment of Bewley’s model of choice with incomplete preferences (2002). While previous axiomatizations of CEU and MEU in fully subjective settings exist, this is to the best of our knowledge the first such axiomatization of Bewley’s model. Moreover, our explicit adoption of the AA approach results in more transparent axiomatics and analysis than earlier work (see Section 4 for discussion). Other fruitful applications are possible; as an example, we briefly discuss ‘menu choice’ models in the style of Kreps (1979).

## 1 Preliminaries

The Savage-style setting we use consists of a non-empty set  $S$  of *states of the world*, an algebra  $\Sigma$  of subsets of  $S$  called *events*, and a non-empty set  $X$  of *consequences*. We denote by  $\mathcal{F}$  the set of all the *simple acts*: finite-valued and  $\Sigma$ -measurable functions  $f : S \rightarrow X$ . For  $x \in X$ , with the usual slight abuse of notation we define  $x \in \mathcal{F}$  to be the constant act such that  $x(s) = x$  for all  $s \in S$ . Given  $x, y \in X$  and  $A \in \Sigma$ ,  $xAy$  denotes the binary act which yields  $x$  if  $s \in A$  and  $y$  otherwise;  $\mathcal{F}_A$  denotes the set of all such acts.

---

<sup>3</sup>In the AA setting, objective mixtures satisfy this property if, as is typically assumed (but see Machina and Schmeidler (1995)), the decision maker’s preferences over lotteries conform to expected utility. On the other hand, our subjective mixtures satisfy this property by construction. See Section 2.2 for further details.

The decision maker's preferences are given by a binary relation  $\succsim$  on  $\mathcal{F}$ , whose symmetric and asymmetric components are denoted by  $\sim$  and  $\succ$ . A functional  $V : \mathcal{F} \rightarrow \mathbb{R}$  is called: a *representation of  $\succsim$*  if  $V(f) \geq V(g)$  if and only if  $f \succsim g$ ; *monotonic* if  $V(f) \geq V(g)$  whenever  $f(s) \succsim g(s)$  for all  $s \in S$ ; *nontrivial* if  $V(f) \neq V(g)$  for some  $f, g \in \mathcal{F}$ .

A set-function  $\rho : \Sigma \rightarrow \mathbb{R}_+$  is a *capacity* if it is monotone and normalized; that is,  $\rho(A) \leq \rho(B)$  if  $A \subseteq B$ ,  $\rho(S) = 1$  and  $\rho(\emptyset) = 0$ . A capacity is called a *probability* if it is additive; that is  $\rho(A \cup B) = \rho(A) + \rho(B)$  if  $A \cap B = \emptyset$ .

## 1.1 Biseparable Preferences

We first define subjective mixtures for the class of biseparable preferences introduced and axiomatized by Ghirardato and Marinacci (2001, henceforth GM).<sup>4</sup> We do not need all the structure entailed by that model (see Remark 2), but this simplifies the exposition. Given a binary relation  $\succsim$ , we say that event  $E \in \Sigma$  is *essential* if  $x \succ x E y \succ y$  for some  $x, y \in X$ .

**Definition 1** *A binary relation  $\succsim$  on  $\mathcal{F}$  is a biseparable preference if it has some essential event  $E \in \Sigma$  and a nontrivial and monotonic representation  $V : \mathcal{F} \rightarrow \mathbb{R}$  for which: 1) there exists  $\rho : \Sigma \rightarrow [0, 1]$  such that, for all consequences  $x \succsim y$  and all  $A \in \Sigma$ ,*

$$V(x A y) = u(x) \rho(A) + u(y) (1 - \rho(A)), \quad (2)$$

where  $u(x) \equiv V(x)$  for all  $x \in X$ ; 2)  $V(X)$  is convex.

GM argue that biseparable preferences are the weakest model achieving a separation of cardinal state-independent utility and a unique representation of beliefs.<sup>5</sup> It encompasses CEU, MEU (hence SEU) and other well-known decision models, like Gul's (1991) 'disappointment aversion' model.<sup>6</sup> It can be seen that if  $\succsim$  is a biseparable preference, then: 1)  $u$  is cardinal; 2)  $\rho$  is a capacity and it is unique; 3) every act  $f \in \mathcal{F}$  has a *certainty equivalent*  $c_f$ , i.e. an arbitrarily chosen element of the set  $\{x \in X : x \sim f\}$ . Further discussion of the properties of biseparable preferences is found in GM.

To motivate the requirement that  $u$  be cardinal, recall that our chief objective is to identify for every pair of consequences  $x$  and  $y$  a third consequence  $z$  satisfying Eq. (1). The equality should hold independently of the normalization of utility: if  $u$  and  $v$  both represent  $\succsim$  on  $X$  and  $z$  satisfies Eq. (1), we should also have  $v(z) = (1/2)v(x) + (1/2)v(y)$ . If the set  $X$  is rich (i.e.,  $u(X)$  is convex), this is easily seen to imply that  $v$  is an affine transformation of  $u$ .

<sup>4</sup>A similar representation appears as an intermediate result in Nakamura (1990) and Casadesus-Masanell, Klibanoff and Ozdenoren (2000a, 2000b).

<sup>5</sup>GM's definition does not require an essential event, but this does not affect the following remarks.

<sup>6</sup>On the other hand, the preferences of Section 3 and the 'probabilistically sophisticated' preferences of Machina and Schmeidler (1992) may not be biseparable (see GM, Remark 5.1).

## 2 Subjective Mixtures

### 2.1 Definition and Properties

In this section we introduce the key notion of ‘subjective mixture’ of two acts for a biseparable preference relation  $\succsim$  on  $\mathcal{F}$  that admits an essential event. We first define subjective mixtures of consequences in terms of the utility  $u$  of Definition 1. A behavioral characterization is provided below.

By way of motivation, temporarily assume that  $X$  is the set  $\Delta(Z)$  of lotteries over a collection of prizes  $Z$ , as in the AA ‘roulette-wheel’ setting. The axiomatizations *à la* AA (except for Machina and Schmeidler (1995)) guarantee that the function  $u$  is *affine* on  $X$ ; that is, given  $x, y \in X$  and  $\alpha \in [0, 1]$ ,  $u$  satisfies  $u(\alpha x + (1 - \alpha)y) = \alpha u(x) + (1 - \alpha)u(y)$ .<sup>7</sup> The latter equality is the result of specific structural and behavioral assumptions in the AA approach.

Consider now an arbitrary set  $X$ , and a biseparable preference with utility  $u$ . It is natural to use affinity as a way of *identifying* ‘subjective mixtures,’ as follows.

**Definition 2** *Given  $x, y \in X$  and  $\alpha \in [0, 1]$ , say that a consequence  $z \in X$  is an  $\alpha : 1 - \alpha$  utility mixture of  $x$  and  $y$  if  $u(z) = \alpha u(x) + (1 - \alpha)u(y)$ .*

The above definition, which is well-posed as  $u(X) = V(X)$  is convex, induces a mixture-space structure on the set of prizes  $X$ , corresponding to the linear structure of  $u(X)$ . To elaborate, consider the following abstract definition of mixture set, due to Fishburn (1982):

**Definition 3** *A triple  $(M, \hat{=}, \hat{+})$  is a generalized mixture set if  $\hat{=}$  is an equivalence relation on  $M$  and  $\hat{+}$  satisfies: for all  $x, y \in M$  and all  $\alpha, \beta \in [0, 1]$ ,  $\alpha x \hat{+} (1 - \alpha)y \in M$  and*

$$\mathbf{M1}(\hat{=}) \quad 1x \hat{+} 0y \hat{=} x,$$

$$\mathbf{M2}(\hat{=}) \quad \alpha x \hat{+} (1 - \alpha)y \hat{=} (1 - \alpha)y \hat{+} \alpha x,$$

$$\mathbf{M3}(\hat{=}) \quad \alpha[\beta x \hat{+} (1 - \beta)y] \hat{+} (1 - \alpha)y \hat{=} \alpha\beta x \hat{+} (1 - \alpha\beta)y.$$

If we denote by  $\alpha x \oplus (1 - \alpha)y$  an arbitrarily chosen element of the indifference class of  $\alpha : 1 - \alpha$  utility mixtures of  $x$  and  $y$ , it is clear that  $(X, \sim, \oplus)$  is a generalized mixture set. Moreover, by definition  $u(\alpha x \oplus (1 - \alpha)y) = \alpha u(x) + (1 - \alpha)u(y)$ . Consequently, it is possible to define (pointwise) utility mixtures of acts, and hence formulate AA-style axioms in the present fully subjective environment. See Section 2.2 for illustration.

Turn now to the behavioral characterization of utility mixtures.

**Definition 4** *Let  $E$  be an essential event. Given  $x, y \in X$ , if  $x \succsim y$  we say that a consequence  $z \in X$  is a preference average of  $x$  and  $y$  (given  $E$ ) if  $x \succsim z \succsim y$  and*

$$x E y \sim c_{xEz} E c_{zEy}. \quad (3)$$

*If  $y \succsim x$ ,  $z$  is said a preference average of  $x$  and  $y$  if it is a preference average of  $y$  and  $x$ .*

<sup>7</sup>Where  $\alpha x + (1 - \alpha)y$  is the lottery that yields prize  $\zeta \in Z$  with probability  $\alpha x(\zeta) + (1 - \alpha)y(\zeta)$ .

In order to interpret the above definition, consider the following analogy. Fix a function  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that  $\varphi(z', z'') = \varphi(z'', z')$  for all  $z', z'' \in \mathbb{R}$ . Then, given  $x, y, z \in \mathbb{R}$ , it may be said that  $z$  is a ‘ $\varphi$ -average’ of  $x$  and  $y$  if  $\varphi(x, y) = \varphi(z, z)$ . As is well-known, arithmetic, geometric and other types of average correspond to specific choices of the function  $\varphi$ .

With this analogy in mind, consider consequences  $z', z'' \in X$  such that  $x \succcurlyeq \{z', z''\} \succcurlyeq y$ . It may be verified that, since  $\succcurlyeq$  is biseparable,  $c_{xEz'} E c_{z''Ey} \sim c_{xEz''} E c_{z'Ey}$ : that is, permuting  $z'$  and  $z''$  does not affect the decision maker’s preferences. We interpret this as indicating that the two inner outcomes  $z'$  and  $z''$  play a *symmetric* role in his evaluation of these bets.

Now notice that Eq. (3) may be rewritten as follows:  $c_{xEz} E c_{yEy} \sim c_{xEz} E c_{zEy}$ . In words, substituting  $z$  for the inner  $x$  and  $y$  in the ‘compound’ act on the left-hand side leaves the decision maker indifferent.

To summarize, the inner  $x$  and  $y$  in the compound bet  $c_{xEz} E c_{yEy}$  play a symmetric role in the evaluation of the latter, *and* they can both be replaced with  $z$  without changing the decision maker’s evaluation of the bet. Thus, Definition 4 may be seen as a preference version of the notion of a ‘ $\varphi$ -average.’

The following proposition (see the Appendix for a proof) provides the desired behavioral characterization of utility mixtures by showing that, if the decision maker’s preferences are biseparable, then preference averages and  $(1/2) : (1/2)$  utility mixtures coincide.

**Proposition 1** *Let  $\succcurlyeq$  be a biseparable preference. For each  $x, y \in X$  and each essential  $E \in \Sigma$ , a consequence  $z \in X$  is a preference average of  $x$  and  $y$  given  $E$  if and only if*

$$u(z) = \frac{1}{2}u(x) + \frac{1}{2}u(y).$$

*Hence, preference averages of  $x$  and  $y$  given  $E$  exist for every essential  $E \in \Sigma$ , do not depend either on the choice of  $E$  or on the normalization of  $u$ , and they form an indifference class.*

**Remark 2** The condition  $x \succcurlyeq z \succcurlyeq y$  in Definition 4 is necessary for Proposition 1 to hold. Consider, for example, the CEU preference on  $S = \{0, 1\}$  and  $X = \mathbb{R}$  with  $\rho(0) = 0.8$ ,  $\rho(1) = 0$ , and linear utility. Let  $E = \{0\}$ ,  $x = y = 1$ , and  $z = 2$ . Then  $xEy \sim c_{xEz} E c_{zEy}$  but  $u(z) \neq (1/2)u(x) + (1/2)u(y)$ .

Let  $w$  be a preference average of  $x$  and  $y$ . By Proposition 1,  $z$  is a preference average of  $x$  and  $w$  if and only if  $u(z) = (3/4)u(x) + (1/4)u(y)$ ; that is,  $z$  is a  $(3/4) : (1/4)$  utility mixture of  $x$  and  $y$ . This allows us to identify  $(3/4)x \oplus (1/4)y$  behaviorally. Proceeding along these lines and using a standard preference continuity argument, it is possible to identify behaviorally the  $\alpha : 1 - \alpha$  utility mixtures of  $x$  and  $y$ , for any  $\alpha \in [0, 1]$ .

Subjective mixtures of acts may then be defined pointwise, as usual. That is, given  $f, g \in \mathcal{F}$  and  $\alpha \in [0, 1]$ ,  $\alpha f \oplus (1 - \alpha)g$  is the act  $h \in \mathcal{F}$  defined by  $h(s) = \alpha f(s) \oplus (1 - \alpha)g(s)$  for any  $s \in S$ .

**Remark 3** Our construction does not require the full structure of biseparable preferences (see our (2001) for further details). Given an essential event  $E$ , a ‘local’ version of Proposition 1 holds for any preference  $\succcurlyeq$  that satisfies Eq. (2) on the set  $\mathcal{F}_E$ . Preference averages

thus derived may in general depend on the choice of  $E$ ; however, this is not the case if  $\succsim$  further satisfies the axiom of ‘certainty independence’ (axiom (i.a) of Prop. 5) with respect to preference averages constructed using some  $E$ . Thus, for most applications of interest, this weaker assumption suffices to obtain an appropriate mixture-set structure.

## 2.2 Applications

It is simple to use the notion of subjective mixture to reformulate AA-style axiomatizations of popular models of decision making under uncertainty in a fully subjective environment. Axioms have the same interpretation as their ‘objective’ counterparts, and just as easily translate into assumptions on the functional representation of preferences. Some examples are provided here; we omit proofs, as they amount to restating the original arguments, after replacing objective AA-type mixtures with subjective mixtures.

We begin by showing the characterization of Schmeidler’s (1989) CEU model, which subsumes the classical SEU model. As it is well-known, the CEU model hinges on a weakening of the classical independence axiom which imposes the independence restriction only for acts that are ‘commonly monotonic.’ Formally,  $f, g \in \mathcal{F}$  are *comonotonic* if there are no  $s, s' \in S$  such that  $f(s) \succ f(s')$  and  $g(s') \succ g(s)$ . The following proposition states that a biseparable preference satisfies such ‘comonotonic independence’ if and only if it is represented by a Choquet integral (for a definition of the latter see, e.g., Schmeidler (1989)).

**Proposition 4** *Let  $\succsim$  be a biseparable preference. The following statements are equivalent:*

- (i) *For all comonotonic  $f, g, h \in \mathcal{F}$ , if  $f \succsim g$  then  $\alpha f \oplus (1 - \alpha) h \succsim \alpha g \oplus (1 - \alpha) h$  for all  $\alpha \in [0, 1]$ .*
- (ii) *For all  $f, g \in \mathcal{F}$ ,  $f \succsim g$  if and only if  $\int u(f) d\rho \geq \int u(g) d\rho$ .*

Strengthening (i) by asking that the implication hold for *every*  $f, g$  and  $h$  (‘independence’) yields an axiomatization of SEU corresponding to that of Anscombe and Aumann (1963). In particular,  $\rho$  is then a probability, and a Choquet integral with respect to a probability is a standard integral (in the sense of Savage (1954)).

Next, we offer a subjective axiomatization of Gilboa and Schmeidler’s MEU model (1989).

**Proposition 5** *Let  $\succsim$  be a biseparable preference. The following statements are equivalent:*

- (i) *For every  $f, g \in \mathcal{F}$ ,*
  - (a) *if  $f \succsim g$  then  $\alpha f \oplus (1 - \alpha)x \succsim \alpha g \oplus (1 - \alpha)x$  for all  $x \in X$  and  $\alpha \in [0, 1]$ ;*
  - (b) *if  $f \sim g$  then  $(1/2)f \oplus (1/2)g \succsim f$ .*
- (ii) *There exists a unique nonempty, weak\* compact and convex set  $\mathcal{D}$  of probabilities on  $\Sigma$  such that for all  $f, g \in \mathcal{F}$ ,*

$$f \succsim g \iff \min_{P \in \mathcal{D}} \int_S u(f) dP \geq \min_{P \in \mathcal{D}} \int_S u(g) dP.$$

Axiom (i.a) is a further generalization of the independence axiom, requiring only that a preference be unaffected by arbitrary mixtures with a *constant* act (observe that constants are comonotonic with respect to any act). Gilboa and Schmeidler call it ‘certainty independence,’ while they call ‘uncertainty aversion’ the hedging axiom (i.b).

An advantage of the extension of the AA approach outlined here is that the decision maker’s risk preferences are not required to be linear in objective probabilities when the latter are part of the model. Suppose that  $Z$  is a finite set of prizes and again let  $X = \Delta(Z)$ , the set of lotteries over  $Z$ . Then, given  $\zeta, \zeta' \in Z$  and the even-chance lottery  $[\zeta, 1/2; \zeta', 1/2]$ , it is possible that

$$\frac{1}{2}u(\zeta) + \frac{1}{2}u(\zeta') = u\left(\frac{1}{2}\zeta \oplus \frac{1}{2}\zeta'\right) \neq u\left(\left[\zeta, \frac{1}{2}; \zeta', \frac{1}{2}\right]\right);$$

that is, the decision maker’s preference over  $\Delta(Z)$  is nonlinear in probabilities. For instance, let  $v : Z \rightarrow \mathbb{R}$  and  $P$  be a probability on  $\Sigma$ . Consider a decision maker whose preferences over  $\Delta(Z)$  are represented by  $u(x) = \int_Z v(\zeta) d(x(\zeta))^2$  and whose preferences over  $\mathcal{F}$  are represented by  $V(f) = \int_S u(f(s)) d(P(s))^2$ , where both integrals are taken in the sense of Choquet. Because of the nonlinearity of  $u(\cdot)$ , this decision maker does not satisfy the axioms of Schmeidler (1989), whereas he satisfies the axioms given above for the CEU model.

The notion of subjective mixture can also be employed to model phenomena that are not necessarily related to the presence of ambiguity. For example, we can provide (along the lines of Ozdenoren (2002)) a subjective mixture version of the models based on Kreps (1979), such as Kreps (1992), Nehring (1999), Dekel, Lipman and Rustichini (2001) and Gul and Pesendorfer (2001).

### 3 A Subjective Axiomatization of Bewley’s Model

Finally, we provide a subjective foundation to Bewley’s (2002) model of choice with incomplete preferences. For this purpose, we add further requirements on the sets  $S$  and  $X$  and modify our basic assumption on preferences, since biseparability implies completeness of  $\succsim$ .

**Structural Assumption** *The set  $X$  is a connected and compact topological space with topology  $\tau$ . The set  $S$  is finite.*

**Preference Assumption** *There exists an essential  $E \in \Sigma$  such that the restriction of  $\succsim$  to  $\mathcal{F}_E$  has a SEU representation with a  $\tau$ -continuous utility index  $u : X \rightarrow \mathbb{R}$ .*

The conditions on  $X$  and  $S$  are analogous to those in Bewley (2002). For an axiomatization of the Preference Assumption see, e.g., Chew and Karni (1994).

The convexity of  $u(X)$  is now a consequence of the connectedness of  $X$  and of the continuity of  $u$ . It follows that every act  $f \in \mathcal{F}_E$  has a certainty equivalent (clearly, this need not be true of acts outside  $\mathcal{F}_E$ ), so that we can apply Definition 4 to define preference averages, and thus define subjective mixtures of acts. Such mixtures will in general enjoy all the properties stated in Proposition 1 except for the possibility of dependence of the notion of mixture on the event  $E$ . As we prove later, such dependence is excluded by the following axioms.

We require that  $\succsim$  satisfies four axioms in addition to the Preference Assumption. The first axiom allows for incompleteness, the second is a statewise dominance condition.

**Axiom B 1 (Preorder)**  $\succsim$  is reflexive and transitive.

**Axiom B 2 (Dominance)** For every  $f, g \in \mathcal{F}$ , if  $f(s) \succsim g(s)$  for every  $s \in S$  then  $f \succsim g$ .

The third axiom is a standard continuity requirement (cf. Dubra, Maccheroni and Ok (2001)). The topology  $\tau$  on  $X$  induces the product topology on the set  $X^S$  of all functions from  $S$  into  $X$ . In this topology, a net  $\{f_\alpha\}_{\alpha \in D} \subseteq \mathcal{F}$  converges to  $f \in \mathcal{F}$  if and only if  $f_\alpha(s) \xrightarrow{\tau} f(s)$  for all  $s \in S$  (hence the name of *pointwise convergence topology*).

**Axiom B 3 (Continuity)** Let  $\{f_\alpha\}_{\alpha \in D} \subseteq \mathcal{F}$  and  $\{g_\alpha\}_{\alpha \in D} \subseteq \mathcal{F}$  be nets that converge pointwise to  $f \in \mathcal{F}$  and  $g \in \mathcal{F}$  respectively. If  $f_\alpha \succsim g_\alpha$  for all  $\alpha \in D$ , then  $f \succsim g$ .

Finally, we have the independence axiom, which is stronger than the versions in Propositions 4 and 5 as it applies to arbitrary mixtures of (not necessarily comonotonic) acts.

**Axiom B 4 (Independence)** For every  $f, g, h \in \mathcal{F}$  and every  $\alpha \in [0, 1]$ ,

$$f \succsim g \implies \alpha f \oplus (1 - \alpha) h \succsim \alpha g \oplus (1 - \alpha) h.$$

The characterization result can now be stated (see the Appendix for a proof).

**Proposition 6** Suppose that  $X$  and  $S$  satisfy the Structural Assumption, and that  $\succsim$  satisfies the Preference Assumption. The preference  $\succsim$  satisfies axioms B1–B4 if and only if there exist a unique nonempty, closed and convex set  $\mathcal{C}$  of probabilities on  $\Sigma$  such that for all  $f, g \in \mathcal{F}$ ,

$$f \succsim g \iff \int_S u(f) dP \geq \int_S u(g) dP \quad \text{for all } P \in \mathcal{C}. \quad (4)$$

By Eq. (4) and the Preference Assumption, the set  $\mathcal{C}$  satisfies  $P(E) = P'(E)$  for all  $P, P' \in \mathcal{C}$ . It follows that the notion of subjective mixture is independent of the essential event used to construct it: If  $F \neq E$  is essential and  $\succsim$  is complete on  $\mathcal{F}_F$ , then  $x F y \sim c_{xFz} F c_{zFy}$  if and only if  $x E y \sim c_{xEz} E c_{zEy}$ . Hence,  $F$  induces the same preference averages as  $E$ .

## 4 The Related Literature

Ever since AA's seminal paper (1963), it is well-understood that a mixture operation on the choice set makes it possible to obtain simple and intuitive axiomatizations of preferences under uncertainty. To the best of our knowledge, ours is the first attempt to provide a generalization of AA-style axiomatics and techniques that applies to a wide range of preference models; previous contributions focus on specific preference models.

The paper closest to this is Casadesus-Masanell, Klibanoff and Ozdenoren (2000a), which provides an axiomatization of the MEU model in a Savage-style setting. Their axioms employ

standard sequences (a tool from measurement theory) to identify acts whose utility profiles are convex combinations of the utility profiles of other acts. Thus, although these authors do not explicitly define a mixture operation, their approach is similar in spirit to ours. Standard sequences are more complex constructs than our notion of preference average, so the statements and interpretations of their axioms are more involved than those in (i) of Prop. 5. However, their characterization result is essentially equivalent to Proposition 5.

Other contributions employ an alternative notion of subjective mixture of acts for axiomatic purposes, without deriving from it a mixture-set structure with the algebraic properties we look for. Such a mixture notion is due to Gul (1992), and may be briefly described as follows. Given an event  $A$ , call the certainty equivalent of the bet  $x A y$  the ‘ $A$ -preference average’ of  $x$  and  $y$ . Then, define the *direct ( $A$ -) mixture* of two acts the act whose payoff in state  $s$  is the  $A$ -preference average of the outcomes assigned to  $s$  by the original acts.<sup>8</sup>

Gul introduces direct mixtures to provide an axiomatization of SEU in a finite state space. He assumes the existence of an ‘ethically neutral’ event  $A$  such that the preference over bets on  $A$  have an SEU representation; in our notation, such that  $\rho(A) = \rho(A^c) = 1/2$ . He then uses  $A$ -mixtures to formulate an independence axiom that implies the non-comonotonic form of axiom (i) discussed after Proposition 4. seen as a special case

In a companion to the paper discussed above, Casadesus *et al.* (2000b) use direct mixtures to obtain a different axiomatization of the MEU model. They assume the existence of an event  $A$  such that the bets on  $A$  have a SEU representation (i.e.,  $\rho(A) + \rho(A^c) = 1$ ), and use  $A$ -mixtures for their key axioms. While their hedging axiom is analogous to axiom (i.b) of Prop. 5, their certainty independence axiom is significantly different from axiom (i.a). The preferences they describe are a subset of those described by Proposition 5.

Chew and Karni (1994) generalize Gul (1992) in two respects. First, they characterize CEU preferences (their preferences coincide with those described by Proposition 4). Second, they show that for this purpose it is enough to use direct  $A$ -mixtures with respect to an arbitrary essential event  $A$ ; i.e., the bets on/against  $A$  need not have an SEU representation. As a consequence, their comonotonic independence axiom is quite dissimilar from ours.<sup>9</sup>

Machina (2001) differs from all the papers discussed so far: he defines ‘almost-objective’ events in a Savage-style setting. Assuming that the state space has a Euclidean structure and that preferences satisfy an ‘event smoothness’ condition, he constructs sequences of events that in the limit are treated ‘as if’ they had an ‘objective’ (agreed upon) probability. He investigates the properties of such almost-objective events and suggests constructing almost-objective mixtures of acts. The latter differ from our subjective mixtures, in that they mix events rather than prizes, and hence do not average (or even require the existence of) utility profiles state-by-state.

Finally, we note that there exist characterizations of SEU that are similar in spirit to AA’s original result, but do not postulate the existence of lotteries with objective probabilities. In particular, Pratt, Raiffa and Schleifer (1964) assume that the state space is the

---

<sup>8</sup>Formally, the  $A$ -mixture of acts  $f$  and  $g$  is the act  $f A g$  defined by  $f A g(s) = c_{f(s) A g(s)}$  for every  $s \in S$ .

<sup>9</sup>Furthermore, it can be shown that this generalization actually makes it impossible to construct a mixture-set structure over the set  $\mathcal{F}$  that preserves the isomorphism with convex combinations of utilities.

Cartesian product of a finite set  $E$  of ‘real-world elementary events’ and the set  $[0, 1] \times [0, 1]$ , the collection of outcomes of a ‘hypothetical experiment.’ They show that such a device considerably simplifies the development of SEU. However, they do not use it to derive a notion of mixture, and their analysis does not employ the vector-space techniques that distinguish the AA approach.

*Dipartimento di Statistica e Matematica Applicata, Università di Torino, Piazza Arbarello 8, 10122 Torino, Italy, and ICER, Torino; ghirardato@econ.unito.it,*

*IMQ, Università Bocconi, Viale Isonzo 25, 20135 Milano, Italy, and ICER, Torino; fabio.maccheroni@uni-bocconi.it,*

*Dipartimento di Statistica e Matematica Applicata, Università di Torino, Piazza Arbarello 8, 10122 Torino, Italy, and ICER, Torino; massimo@econ.unito.it,*

and

*Department of Economics, Northwestern University, 2001 Sheridan Road, Evanston, IL 60208, and Economics Dept., Princeton University; marciano@northwestern.edu.*

APPENDIX

**Proof of Prop. 1:** Let  $E \in \Sigma$  be essential and  $x, y \in X$  be such that  $x \succ y$  (the argument for the case  $y \succ x$  is symmetric). A consequence  $z$  is such that

$$x \succ z \succ y \quad \text{and} \quad x E y \sim c_{xEz} E c_{zEy} \tag{5}$$

iff  $u(x) \geq u(z) \geq u(y)$  and  $V(x E y) = V(c_{xEz} E c_{zEy})$ . Setting  $r = \rho(E) \in (0, 1)$ , we obtain  $V(x E y) = u(x)r + u(y)(1 - r)$ ,  $V(x E z) = u(x)r + u(z)(1 - r)$ , and  $V(z E y) = u(z)r + u(y)(1 - r)$ , so that, in particular,  $c_{xEz} \succ c_{zEy}$ . Using these equations, we have

$$\begin{aligned} V(c_{xEz} E c_{zEy}) &= u(c_{xEz})r + u(c_{zEy})(1 - r) = V(x E z)r + V(z E y)(1 - r) \\ &= [u(x)r + u(z)(1 - r)]r + [u(z)r + u(y)(1 - r)](1 - r) \\ &= u(x)r^2 + u(y)(1 - r)^2 + 2u(z)r(1 - r). \end{aligned}$$

Thus, a consequence  $z$  satisfies Eq. (5) iff  $u(x) \geq u(z) \geq u(y)$  and

$$u(x)r + u(y)(1 - r) = u(x)r^2 + u(y)(1 - r)^2 + 2u(z)r(1 - r).$$

The last equation is in turn equivalent to

$$u(z) = \frac{u(x) + u(y)}{2} \tag{6}$$

(which also implies  $u(x) \geq u(z) \geq u(y)$ ). Since  $u(X)$  is convex, for all  $x \succ y$  there exists a  $z \in X$  such that Eq. (6) is satisfied. The other statements follow immediately. *Q.E.D.*

**Proof of Prop. 6:** Necessity is straightforward. We prove sufficiency. Let  $u$  be the utility from the Preference Assumption. Since  $u$  is nonconstant (recall that  $E$  is essential), we can assume  $u(X) = [0, 1]$ . Define a binary relation on  $B([0, 1])$ , the class of the simple  $\Sigma$ -measurable functions  $S \rightarrow [0, 1]$ , as follows: For all  $f, g \in \mathcal{F}$ ,  $u(f) \succsim u(g)$  iff  $f \succcurlyeq g$ . Then:

- $\succsim$  is well-defined, reflexive and transitive: Assume  $u(f) = u(f')$  and  $u(g) = u(g')$ . By monotonicity of  $\succcurlyeq$ ,  $f \sim f'$  and  $g \sim g'$ , so  $f \succcurlyeq g$  iff  $f' \succcurlyeq g'$ . Reflexivity and transitivity follow from the analogous properties of  $\succcurlyeq$ .
- $\succsim$  is isotonic:  $u(f(s)) \geq u(g(s))$  for all  $s \in S$  implies  $u(f) \succsim u(g)$  (monotonicity of  $\succcurlyeq$ ).
- $\succsim$  is continuous: Assume that  $u(f_\alpha) \rightarrow u(f)$ ,  $u(g_\alpha) \rightarrow u(g)$  and  $u(f_\alpha) \succsim u(g_\alpha)$ , hence  $f_\alpha \succcurlyeq g_\alpha$ , for all  $\alpha \in D$ . Considering subnets, assume that  $f_\beta \rightarrow f'$  and  $g_\beta \rightarrow g'$  (recall that  $\mathcal{F} = X^S$  is compact). By axiom B3  $f' \succcurlyeq g'$ , so  $u(f') \succsim u(g')$ . Finally,  $u(f') = u(f)$  and  $u(g') = u(g)$ . In fact,  $u(f_\beta) \rightarrow u(f')$  (continuity of  $u$ ) and  $u(f_\beta) \rightarrow u(f)$  (since  $u(f_\alpha) \rightarrow u(f)$ ), and similar considerations hold for  $u(g_\beta)$ . Hence  $u(f) \succsim u(g)$ .
- $\succsim$  is independent: Let  $f, g, h \in \mathcal{F}$  and suppose that  $u(f) \succsim u(g)$ . Then  $f \succcurlyeq g$  implies  $\alpha f \oplus (1 - \alpha)h \succcurlyeq \alpha g \oplus (1 - \alpha)h$  for all  $\alpha \in [0, 1]$ , by axiom B4. This implies

$$\begin{aligned} \alpha u(f) + (1 - \alpha)u(h) &= u(\alpha f \oplus (1 - \alpha)h) \\ &\succsim u(\alpha g \oplus (1 - \alpha)h) = \alpha u(g) + (1 - \alpha)u(h). \end{aligned}$$

The statement in the proposition now follows from a standard result, whose proof we omit:

**Lemma 7**  $\succsim$  is a nontrivial, independent, continuous and isotonic preorder on  $B([0, 1])$  if and only if there exists a unique, nonempty, closed and convex set  $C$  of probability measures on  $\Sigma$  such that

$$\varphi \succsim \psi \iff \int_S \varphi dP \geq \int_S \psi dP \quad \text{for all } P \in C.$$

Q.E.D.

## References

- ANSCOMBE, F. J., AND R. J. AUMANN (1963): "A Definition of Subjective Probability," *Annals of Mathematical Statistics*, 34, 199–205.
- BEWLEY, T. (2002): "Knightian Decision Theory: Part I," *Decisions in Economics and Finance*, 25(2), 79–110, (first version 1986).
- CASADESUS-MASANELL, R., P. KLIBANOFF, AND E. OZDENOREN (2000a): "Maxmin Expected Utility over Savage acts with a Set of Priors," *Journal of Economic Theory*, 92, 33–65.
- (2000b): "Maxmin Expected Utility through Statewise Combinations," *Economics Letters*, 66, 49–54.
- CHEW, S. H., AND E. KARNI (1994): "Choquet Expected Utility with a Finite State Space: Commutativity and Act-Independence," *Journal of Economic Theory*, 62, 469–479.
- DEKEL, E., B. L. LIPMAN, AND A. RUSTICHINI (2001): "Representing Preferences with a Unique Subjective State Space," *Econometrica*, 69(4), 891–934.
- DUBRA, J., F. MACCHERONI, AND E. OK (2001): "Expected Utility Theory without the Completeness Axiom," *Journal of Economic Theory*, p. forthcoming.
- FISHBURN, P. C. (1970): *Utility Theory for Decision Making*. Wiley, New York.
- (1982): *The Foundations of Expected Utility*. Reidel, Dordrecht.
- GHIRARDATO, P., F. MACCHERONI, M. MARINACCI, AND M. SINISCALCHI (2001): "A Subjective Spin on Roulette Wheels," Social Science Working Paper 1127, Caltech.
- GHIRARDATO, P., AND M. MARINACCI (2001): "Risk, Ambiguity, and the Separation of Utility and Beliefs," *Mathematics of Operations Research*, 26, 864–890.
- GILBOA, I., AND D. SCHMEIDLER (1989): "Maxmin Expected Utility with a Non-Unique Prior," *Journal of Mathematical Economics*, 18, 141–153.
- GUL, F. (1991): "A Theory of Disappointment Aversion," *Econometrica*, 59, 667–686.

- (1992): “Savage’s Theorem with a Finite Number of States,” *Journal of Economic Theory*, 57, 99–110.
- GUL, F., AND W. PESENDORFER (2001): “Temptation and Self-Control,” *Econometrica*, 69(6), 1403–1435.
- KREPS, D. M. (1979): “A Representation Theorem for “Preference for Flexibility”,” *Econometrica*, 47(3), 565–577.
- (1992): “Static Choice in the Presence of Unforeseen Contingencies,” in *Economic analysis of markets and games*, ed. by P. Dasgupta, D. Gale, O. Hart, and E. Maskin, pp. 258–281. MIT Press, Cambridge, MA.
- LUCE, R. D. (2000): *Utility of Gains and Losses: Measurement-Theoretical and Experimental Approaches*. Lawrence Erlbaum, London.
- MACHINA, M. J. (2001): “Almost-Objective Uncertainty,” Mimeo, UC San Diego.
- MACHINA, M. J., AND D. SCHMEIDLER (1992): “A More Robust Definition of Subjective Probability,” *Econometrica*, 60, 745–780.
- (1995): “Bayes without Bernoulli: Simple Conditions for Probabilistically Sophisticated Choice,” *Journal of Economic Theory*, 67, 106–128.
- NAKAMURA, Y. (1990): “Subjective Expected Utility with Non-Additive Probabilities on Finite State Spaces,” *Journal of Economic Theory*, 51, 346–366.
- NEHRING, K. (1999): “Preference for Flexibility in a Savage Framework,” *Econometrica*, 67, 101–119.
- OZDENOREN, E. (2002): “Completing the State Space with Subjective States,” *Journal of Economic Theory*, 105, 531–539.
- PRATT, J. W., H. RAIFFA, AND R. SCHLAIFER (1964): “The Foundations of Decision under Uncertainty: An Elementary Exposition,” *Journal of the American Statistical Association*, 59(306), 353–375.
- SAVAGE, L. J. (1954): *The Foundations of Statistics*. Wiley, New York.
- SCHMEIDLER, D. (1989): “Subjective Probability and Expected Utility without Additivity,” *Econometrica*, 57, 571–587.
- VON NEUMANN, J., AND O. MORGENSTERN (1947): *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, second edn.