Vector Expected Utility and Attitudes toward Variation

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Abstract

This paper analyzes a model of decision under ambiguity, deemed *vector expected utility* or VEU. According to the proposed model, an act $f:\Omega\to X$ is evaluated via the functional

$$V(f)=\int_\Omega u\circ f\,dp+A\left(\int_\Omega u\circ f\,dm\right),$$

where $u:X\to\mathbb{R}$ is a von Neumann-Morgenstern utility function, p is a *baseline probability* measure, $\int_\Omega u\circ f\,dm$ is a *adjustment vector* of finite or countably infinite dimension, whose i-th component is the Lebesgue integral $\int u\circ f\,dm_i$ of the real function $u\circ f$ with respect to a signed measure m_i on Ω, and the function A is symmetric about zero: $A(\varphi)=A(-\varphi)$. The signed measures $(m_i)_{0\le i<n}$ encode the possibility that ambiguity about certain events may (partially) "cancel out." The adjustment term $A(\int u\circ f\,dm)$ reflects the *variability* of the act f around its baseline expected utility $\int u\circ f\,dp$.

A behavioral characterization of the VEU model is provided. Furthermore, an updating rule for VEU preferences is proposed and characterized. The suggested updating rule facilitates the analysis of *sophisticated* dynamic choice with VEU preferences.

1 Introduction

The issue of ambiguity in decision-making has received considerable attention in recent years, both from a theoretical perspective and in applications to contract theory, information economics, finance, and macroeconomics. As Daniel Ellsberg [13] first observed, individuals may find it difficult to assign probabilities to certain events when available information is deemed scarce or unreliable. In these circumstances, agents may avoid taking actions whose ultimate outcomes depend crucially upon the realization of such ambiguous events, and instead opt for "safer" alternatives. Several decision models

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have been developed to accommodate these patterns of behavior: these models represent ambiguity via multiple priors (Gilboa and Schmeidler [25]; Ghirardato, Maccheroni and Marinacci [22]), non-additive beliefs (Schmeidler [48]), second-order probabilities (Klibanoff, Mukerjee and Marinacci [34]; Nau [41]; Ergin and Gul [17]), relative entropy (Hansen and Sargent [29]; Hansen, Sargent and Tallarini [30]), or variational methods (Maccheroni, Marinacci and Rustichini [37]).

This paper proposes a decision model that incorporates key insights from Ellsberg's original analysis, as well as from cognitive psychology and recent theoretical contributions on the behavioral implications of ambiguity. According to the proposed model, the individual evaluates uncertain prospects, or acts, by a process suggestive of *anchoring and adjustment* (Tversky and Kahneman [56]). The "anchor" is the expected utility of the prospect under consideration, computed with respect to a *baseline probability*; the "adjustment" depends upon its *variation* away from the anchor at states that the individual deems ambiguous. Formally, an act f, mapping each state $\omega \in \Omega$ to a consequence $x \in X$, is evaluated via the functional

$$V(f) = \int_\Omega u \circ f \, dp + A\left(\int_\Omega u \circ f \, dm\right). \tag{1}$$

In Eq. (1), $u : X \to \mathbb{R}$ is a von Neumann-Morgenstern utility function; p is a *baseline probability* on Ω; $\int_\Omega u \circ f \, dm$ is a *adjustment vector* of finite or countably infinite dimension, whose i-th component is the Lebesgue integral $\int u \circ f \, dm_i$ of the real function $u \circ f$ with respect to a signed measure m_i on Ω; and A is a symmetric function: $A(-\phi) = A(\phi)$ for every vector ϕ. I deem the proposed model *vector expected utility*, or VEU. The main result of this paper is a behavioral characterization of preferences that conform to the VEU model; an analysis of updating and dynamic choice for this family of preferences is also provided.

Three key features of the VEU representation are worth emphasizing. First, prospects are evaluated by means of a baseline prior, adjusted to account for ambiguity. Hillel Einhorn and Robin Hogarth [11, 12, 31] were the first to propose such an anchoring-and-adjustment strategy as a plausible approach to decisions under ambiguity. The cited papers explore the implications of this strategy in a series of experiments, dealing primarily with choice among binary lotteries. Ellsberg's seminal paper also suggests that, when faced with an ambiguous choice situation, "by compounding various probability judgments of various degrees of reliability, [the individual] can eliminate certain probability distributions over states of nature as 'unreasonable,' assign weights to others and *arrive at a composite 'estimated' distribution*" ([13], p. 661; italics added for emphasis). Additional contributions emphasizing the role of reference priors will be discussed in §5.1.

Second, decomposing the adjustment term in Eq. (1) into a suitable function $A(\cdot)$ and a collection $(m_i)_{0 \le i < n}$ of signed measures provides a direct, explicit representation of *eventwise complementarity*—a key behavioral feature of ambiguous events highlighted in the analysis of Larry Epstein and Jiankang Zhang [15]. To illustrate this notion and provide a simple application of the decision model of Eq. (1), consider Ellsberg's three-color urn experiment. A ball is to be drawn from an urn containing 30 red balls, and 60 blue and green balls; the proportion of blue vs. green balls is unknown. Denote by f_R, f_B, f_{RG}, f_{BG} the acts that yield \$10 if a red (resp. blue, red or green, blue or green) ball is drawn, and \$0 otherwise. As reported by Ellsberg, the modal preferences are

$$f_R \succ f_B \quad \text{and} \quad f_{RG} \prec f_{BG}. \tag{2}$$

Epstein and Zhang suggest that "[t]he intuition for this reversal is the complementarity between G and B—there is imprecision regarding the likelihood of B, whereas $\{B, G\}$ has precise probability $\frac{2}{3}$" ([15], p. 271). The proposed model enables a representation of the preferences in Eq. (2) that closely matches this interpretation: let p be uniform on the state space $\Omega = \{R, G, B\}$, assume w.l.o.g. that u is linear, and let m_0 be the signed measure given by

$$m_0(\{R\}) = 0, \quad m_0(\{B\}) = \frac{1}{3}, \quad m_0(\{G\}) = -\frac{1}{3}.$$

Finally, let $A(\phi) = -|\phi|$ for every $\phi \in \mathbb{R}$. Thus, in this example, $n = 1$: one-dimensional adjustment vectors suffice. The interpretation of the adjustment measure m_0 is as follows: since $A(m_0(\{G\})) = A(m_0(\{B\})$, G and B are "equally ambiguous"; however, $m_0(\{G\}) + m_0(\{B\}) = 0$, so their ambiguities "cancel out." This algebraic cancellation corresponds to Epstein and Zhang's notion of complementarity. It is then easily verified that $V(f_R) = \frac{10}{3}$, $V(f_B) = 0$, $V(f_{RG}) = \frac{10}{3}$ and $V(f_{BG}) = \frac{20}{3}$, consistently with the preferences in Eq. (2).[1]

Third, the symmetry property of the functional A (that is, the requirement that $A(\varphi) = A(-\varphi)$ for all vectors φ) supports the intuition that the adjustment applied to the baseline expected-utility (EU) evaluation of an act f is related to the *variability*, or dispersion, of the outcomes delivered by f at different states. In economic applications of decision models reflecting a concern of ambiguity, interesting patterns of behavior often arise out of the agents' desire to reduce outcome or utility variability (usually referred to as "hedging" or "utility smoothing"); for instance, see Bose, Ozdenoren and Pape [5], Epstein and Schneider [14], Ghirardato and Katz [21], or Mukerji [40]. Indeed, Schmeidler [48] suggests that "ambiguity aversion" can be defined as a preference for "smoothing or averaging utility distributions" [48, p. 582]; other authors have further investigated this and related hedging-based characterizations of ambiguity attitudes (for instance, Chateauneuf and Tallon [7]; Gilboa and Schmeidler [25]; Klibanoff [33]; Kopylov [36]; Maccheroni, Marinacci and Rustichini [37]). Thus, outcome or utility variability plays a key role in the evaluation of acts under ambiguity; the VEU representation makes this role explicit.

To elaborate, recall that virtually all classical measures of variability or dispersion for random variables, such as the variance and mean absolute deviation, the range[2], Gini's mean difference (cf. e.g.

[1] For instance, $V(f_{RG}) = 10 \cdot \frac{2}{3} - \left|10 \cdot 0 + 0 \cdot \frac{1}{3} + 10 \cdot \left(-\frac{1}{3}\right)\right| = \frac{20}{3} - \left|-\frac{10}{3}\right| = \frac{10}{3}$.

[2] As well as the interquantile range for continuous random variables.

Yitzhaki [58]), or the peakedness ordering (Bickel and Lehmann [4]), are *invariant to translation and sign changes*: for any constant c, the random variables Y and $c-Y$ are considered to be equally dispersed. Intuitively, these measures reflect the *extent* of deviations from a reference point, or across different states, rather than the direction of these deviations, or the location of the reference point itself.

In a decision setting à la Anscombe-Aumann [1], this invariance property may be translated as follows. Say that two acts f and $\bar{f}$ are *complementary* if their 50%:50% mixture is a constant act; it is easy to see that, for a suitable constant c, the utility profiles of f and $\bar{f}$ satisfy $u \circ \bar{f} = c - u \circ f$. Thus, complementary acts exhibit the same utility or outcome variability according to classical measures. Hence, if adjustments to the baseline evaluation of acts reflect their variability, then complementary acts should receive the same adjustment. The symmetry property of the functional A ensures that this is the case.[3] The main novel axiom in this paper, *Complementary Independence*, is chiefly responsible for this symmetry property.

One additional consequence of this property, and indeed of the Complementary Independence axiom, deserves special emphasis. Symmetry implies that adjustment terms cancel out when comparing two complementary acts using the VEU representation in Eq. (1); thus, the ranking of complementary acts is effectively determined by their baseline EU evaluation. Conversely, preferences over complementary acts uniquely identify the baseline prior: there is a unique probability p and a cardinally unique utility function u such that, for all complementary acts f and $\bar{f}$, $f \succcurlyeq \bar{f}$ iff $\int u \circ f \, dp \geq \int u \circ \bar{f} \, dp$. Thus, baseline priors have a simple behavioral interpretation in the present setting: *they provide a representation of the individual's preferences over complementary acts.* This implies that, under Complementary Independence, the baseline prior is behaviorally identified *independently of other elements of the VEU representation*; Sec. 4.4 elaborates on this point.

It is worth emphasizing that the functional representation in Eq. (1) is flexible enough to accommodate a broad range of attitudes towards ambiguity, while at the same time allowing for numerical and analytical tractability. The preferences in the preceding example display ambiguity aversion as defined by Schmeidler [48]; correspondingly, the adjustment function A is non-positive and concave. As will be shown in Sec. 4.1, a non-positive, but not necessarily concave adjustment function instead characterizes ambiguity aversion in the more general sense of Ghirardato and Marinacci [24]. Indeed, the VEU model can accommodate even more complex attitudes towards ambiguity; for instance, Sec. 4.5 provides a simple, tractable (in particular, differentiable) representation of VEU preferences that exhibit ambiguity appeal for small stakes and ambiguity aversion for large stakes—a pattern that has been documented in experiments (e.g. Hogarth and Einhorn [31], Koch and Schunk [35]).

This paper also proposes a possible *updating rule* for VEU preferences, and provides a behavioral

[3] To further elaborate, recall that the signed measures $(m_i)_{0 \leq i < n}$ in the VEU representation are assumed to satisfy $m_i(\Omega) = 0$; thus, if f and $\bar{f}$ are complementary, $\int u \circ \bar{f} \, dm = -\int u \circ f \, dm$. Since A is symmetric, $A\left(\int u \circ f \, dm\right) = A\left(\int u \circ \bar{f} \, dm\right)$.

characterization. Consider an individual with VEU preference, represented by a baseline prior p, a collection of signed measures $(m_i)_{0\le i<n}$, and a functional A as in Eq. (1). Then, under suitable assumptions, upon learning that an event E has occurred, the individual again holds VEU preferences; her baseline probability is the standard Bayesian update $p(\cdot|E)$ of p, she employs the same adjustment functional A, and the i-th updated signed measure $m_{E,i}$ is obtained from the corresponding measure m_i by letting

$$m_{E,i}(F)=m_i(F\cap E)+p(F|E)\cdot m_i(\Omega\setminus E)$$

for all events F. This characterization makes it possible to analyze sophisticated choice in dynamic decision problems using a *recursive* formulation: this observation is developed in Sec. 4.2.

The paper is organized as follows. Section 2 provides preliminary definitions and results. Section 3 presents the main characterization result. Section 4 contains additional results and examples. Finally, Section 5 discusses the related literature (§5.1), as well as additional features and extensions of the VEU representation (§5.2). All proofs, as well as additional technical results, are in the Appendix.

2 Preliminaries

2.1 Adjustment Tuples and Vectors

Consider a set Ω (the state space) and a sigma-algebra Σ of subsets of Ω (events). Adopt the following conventional notation: for any interval $\Gamma\subset\mathbb{R}$, $B_0(\Sigma,\Gamma)$ is the set of bounded, Σ-measurable simple functions on Ω taking values in Γ, and $B(\Sigma,\Gamma)$ is its sup-norm closure; if $\Gamma=\mathbb{R}$, these sets will be denoted simply as $B_0(\Sigma)$ and $B(\Sigma)$. The collection of bounded, countably additive measures on Σ, is denoted by $ca(\Sigma)$, whereas $ca_1(\Sigma)$ indicates the set of countably additive probability measures on (Ω,Σ).

As noted in the Introduction, the VEU representation employs collections of signed measures to encode adjustments to the baseline EU evaluation of acts. Such "adjustment measures" are normalized so as to reflect the fact that the empty event $\emptyset$ and the certain event Ω are not subject to ambiguity. These collections can be finite or countably infinite; in the latter case, adjustment measures are also required to be uniformly bounded and uniformly continuous. The following definition provides the details, and introduces additional, useful notation. Observe that, by Theorem 3.1, if the state space Ω is finite, then VEU preferences can always be represented using finitely many adjustment measures.

Definition 1 *An* ***adjustment tuple of size*** $n\in\mathbb{Z}_+\cup\{\infty\}$ *is a collection* $m=(m_i)_{0\le i<n}\subset ca(\Sigma)$ *such that*

1. $m_i(\Omega)=m_i(\emptyset)=0$ *for* $0\le i<n$*;*

2. *for every* $E\in\Sigma$ *there exists* $N(E)\in\mathbb{R}$ *such that* $|m_i(E)|<N(E)$ *for* $0\le i<n$*; and*

3. *for all sequences* $(E_k)_{k\geq 0}\subset\Sigma$ *with* $E_k\supset E_{k+1}$ *for all* $k\geq 0$ *and* $\bigcap_k E_k=\emptyset$, $\sup_{0\leq i<n}|m_i(E_k)|\to 0$.

Denote the set of adjustment tuples of size n *by* $M^n(\Sigma)$*. For every* $a\in B(\Sigma)$ *and* $m=(m_i)_{0\leq i<n}\in M^n(\Sigma)$*, define the* ***adjustment vector*** $\int a\,dm$ *by* $\int a\,dm=0$ *if* $n=0$*, and*

$$\int a\,dm=\left(\int a\,dm_i\right)_{0\leq i<n}$$

otherwise. For any interval $\Gamma\subset\mathbb{R}$*, the* ***range*** *of* m *and* Γ *is the set* $R_0(m,\Gamma)=\left\{\int a\,dm : a\in B_0(\Sigma,\Gamma)\right\}$.

Observation: adjustments as vector measures. Every adjustment tuple $m=(m_i)_{0\leq i<n}$ defines a countably additive set function $\hat{m}$ on Σ taking values in the Banach space ℓ^n_∞ of supnorm-bounded n-vectors; that is, $\hat{m}$ is an ℓ^n_∞–valued *vector measure* (cf. e.g. Dunford and Schwartz [10], §IV.10). Furthermore, for every function $a\in B(\Sigma)$, the (real-valued) vector $\int a\,dm$ defined above coincides with the vector integral of a with respect to $\hat{m}$ (cf. [10], pp. 322-323). In other words, $\int a\,dm$ may equivalently viewed as a collection of scalar integrals, or as the integral of a with respect to a vector-valued measure. This connection is made precise in Sec. A.1 of the Appendix; however, it is worth emphasizing that the results in this paper do not depend upon the mathematics of vector measures.

2.2 Decision Setting and VEU representation

Consider a convex set X of consequences (outcomes, prizes). As in Anscombe-Aumann [1], X could be the set of finite-support lotteries over some underlying collection of (deterministic) prizes, endowed with the usual mixture operation. Alternatively, the set X might be endowed with a subjective mixture operation, as in [6] or [23].

Next, let L_0 be the set of simple acts on the state space (Ω,Σ), i.e. the family of Σ-measurable functions from Ω to X with finite range. With the usual abuse of notation, denote by x the constant act assigning the consequence $x\in X$ to each $\omega\in\Omega$. The main object of interest is a preference relation $\succcurlyeq$ on L_0.

A precise definition of the VEU representation can now be provided. The following notation is useful: for a function $u:X\to\mathbb{R}$, $u(X)=\{u(x):x\in X\}$; also, 0_n denotes the zero vector in $\mathbb{R}^n$ $(0\leq n\leq\infty)$.

Definition 2 *A tuple* (u,p,n,m,A) *is a* ***VEU representation*** *of a preference relation* $\succcurlyeq$ *on* L_0 *if*

1. $u:X\to\mathbb{R}$ *is non-constant and affine,* $p\in ca_1(\Sigma)$*,* $n\in\mathbb{Z}_+\cup\{\infty\}$ *and* $m\in M^n(\Sigma)$*;*

2. $A:R_0(m,u(X))\to\mathbb{R}$ *satisfies*

   (a) *for all sequences* $(\varphi^k)_{k\geq 0}\subset R_0(m,u(X))$ *such that* $\sup_{0\leq i<n}|\varphi^k|\to 0$*,* $A(\varphi^k)\to 0$*;*

   (b) *for all* $\varphi\in R_0(m,u(X))$*,* $A(\varphi)=A(-\varphi)$*;*

3. *for all $a, b \in B_0(\Sigma, u(X))$, $a(\omega) \geq b(\omega)$ for all $\omega \in \Omega$ implies $\int a\, dp + A(\int a\, dm) \geq \int b\, dp + A(\int b\, dm)$;*

and, for every pair of acts $f, g \in L_0$,

$$f \succcurlyeq g \quad \Leftrightarrow \quad \int_\Omega u \circ f\, dp + A\left(\int_\Omega u \circ f\, dm\right) \geq \int_\Omega u \circ g\, dp + A\left(\int_\Omega u \circ g\, dm\right). \tag{3}$$

Condition 2(a) implies the normalization $A(0_n) = 0$ (take $\varphi^k = 0_n$ for all k): if all ambiguity about an act cancels out, then there is no adjustment to the baseline evaluation. Therefore, for general sequences converging to 0_n, this condition imposes supnorm-continuity at the origin. Condition 2(b) is the central **symmetry** assumption discussed in the Introduction (cf. in particular Footnote 3).

Condition 3 ensures monotonicity of the VEU representation. Simple examples show that monotonicity necessarily involves a joint restriction on p, m and A.[4] In many cases of interest, easy-to-check necessary and sufficient conditions can be provided: see Appendix A.2 for details.

It is useful to point out that the functional A, and hence the entire VEU representation, is *not* required to be positively homogeneous. This makes it possible to accommodate, for instance, members of the "variational preferences" family studied by Maccheroni, Marinacci and Rustichini [37] that satisfy the key symmetry requirement of this paper; furthermore, it enables differentiable specifications of the adjustment functional A, which would otherwise be precluded.

Finally, it is convenient to define a notion of "parsimonious" VEU representation. This is motivated by the decision-theoretic notion of "crisp acts" due to Ghirardato, Maccheroni and Marinacci [22]. Say that an act $f \in L_0$ is **crisp** if, for every $x \in X$ that satisfies $f \sim x$, and for every $g \in L_0$ and $\lambda \in (0, 1]$,

$$\lambda g + (1-\lambda)x \sim \lambda g + (1-\lambda) f. \tag{4}$$

That is, a crisp act "behaves like its certainty equivalent": in particular, as discussed in Ghirardato et al. [22], it does not provide a "hedge" against the ambiguity that influences any other act g.[5] Constant acts are obviously crisp; correspondingly, any VEU representation of the preference $\succcurlyeq$ assigns them the zero adjustment vector. Since crisp acts behave like constant acts, it seems desirable to ensure that their associated adjustment vector also be zero. This is the key requirement of the following definition.

Definition 3 *A VEU representation (u, p, n, m, A) of a preference relation $\succcurlyeq$ on L_0 is* ***sharp*** *if*

1. *for any crisp act $f \in L_0$, $\int u \circ f\, dm = 0_n$; and*

[4] Refer to the three-color-urn example in the Introduction, and let f'_B be a bet that yields 20 dollars if B obtains; since $A(\varphi) = -|\varphi|$, $A(\int f'_B\, dm) < A(\int f_B\, dm)$, even though $\int f'_B\, dm = \frac{20}{3} > \frac{10}{3} = \int f_B\, dm$. Taking $A(\varphi) = |\varphi|$ instead shows that no general assumption may be made regarding the direction of monotonicity for A alone.

[5] The present definition is weaker than the one provided by [22]: in particular, it allows for preferences that do not have a positively homogeneous representation. The two definitions are equivalent if positive homogeneity holds.

2. *if* (u', p', n', m', A') *is another VEU representation of* $\succcurlyeq$ *that satisfies Condition 1, then* $n' \geq n$.

As an immediate and intuitively appealing implication of Condition 1, note that, for an EU preference, all acts are crisp; thus, the unique sharp VEU representation of an EU preference features $n = 0$, i.e. an empty adjustment tuple.

It is sometimes convenient to employ VEU representations that are not sharp: see, for instance, the analysis of updating in Sec. 4.2. However, a notion of sharp representation provides a way to assess the *complexity* of the complementarity patterns that the individual perceives among ambiguous events. Example 1 in the following section provides a simple application of these ideas, and a geometric intuition.

3 Axiomatic Characterization of VEU preferences

It will be useful to assume that (Ω, Σ) is a standard Borel space (Kechris [32]): that is, Σ is the Borel sigma-algebra generated by a Polish topology τ on Ω. This is best viewed as a structural assumption on the sigma-algebra Σ: the generating topology τ plays no role in the analysis. All finite and countably infinite sets, as well as all Borel subsets of Euclidean n-space, are standard Borel spaces, as are many spaces of functions that arise in the theory of continuous-time stochastic processes.

Mixtures, or convex combinations of acts are taken pointwise: for every pair of acts $f, g \in L_0$ and for any $\alpha \in [0,1]$, $\alpha f + (1-\alpha) g$ is the act assigning the consequence $\alpha f(\omega) + (1-\alpha) g(\omega)$ to each state $\omega \in \Omega$.

Axioms 3.1–3.4 are standard:

Axiom 3.1 (Weak Order) $\succcurlyeq$ *is transitive and complete.*

Axiom 3.2 (Monotonicity) *For all acts* $f, g \in L_0$, $f(\omega) \succcurlyeq g(\omega)$ *for all* $\omega \in \Omega$ *implies* $f \succcurlyeq g$.

Axiom 3.3 (Continuity) *For all acts* $f, g, h \in L_0$, *the sets* $\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succcurlyeq h\}$ *and* $\{\alpha \in [0,1] : h \succcurlyeq \alpha f + (1-\alpha) g\}$ *are closed.*

Axiom 3.4 (Non-Degeneracy) *Not for all* $f, g \in L_0$, $f \succcurlyeq g$.

Next, a weak form of the Anscombe-Aumann [1] Independence axiom, due to Maccheroni, Marinacci and Rustichini [37], is assumed.

Axiom 3.5 (Weak Certainty Independence) *For all acts* $f, g \in L_0$, $x, y \in X$ *and* $\alpha \in (0,1)$: $\alpha f + (1-\alpha)x \succcurlyeq \alpha g + (1-\alpha)x$ *implies* $\alpha f + (1-\alpha)y \succcurlyeq \alpha g + (1-\alpha)y$.

Loosely speaking, preferences are required to be invariant to translations of utility profiles, but not to rescaling (note that the same weight α is employed when mixing with x and with y). As discussed in [37], this axiom weakens Gilboa and Schmeidler [25]'s *Certainty Independence*, which requires invariance to both translation and rescaling. Since Certainty Independence will be referenced below, it is useful to reproduce it here, even though it is *not* assumed in Theorem 3.1.

Axiom 3.5* (Constant-Act Independence) *For all acts $f, g \in L_0$, $h \in L_c$ and $\alpha \in (0,1)$: $f \succcurlyeq g$ implies $\alpha f + (1-\alpha)h \succcurlyeq \alpha g + (1-\alpha)h$.*

To ensure that all measures in the representation are countably additive, adopt the following axiom, which is in the spirit of Arrow [2].[6] A similar representation could be obtained without it, but it would not be possible to restrict attention to adjustment vectors of finite or countably-infinite dimension. To state the axiom, for every pair $x, y \in X$ and $E \in \Sigma$, denote by xEy the act that yields x at every state $\omega \in E$ and y elsewhere.

Axiom 3.6 (Monotone Continuity) *For all sequences $(A_k)_{k\geq 1} \subset \Sigma$ such that $A_k \supset A_{k+1}$ and $\bigcap_k A_k = \emptyset$, and all $x, y, z \in X$ such that $x \succ y \succ z$, there is $k \geq 1$ such that $zA_k x \succ y \succ xA_k z$.*

In order to state the novel axioms in this paper, a preliminary definition is required. Intuitively, it identifies pairs of acts whose utility profiles are "mirror images."

Definition 4 *Two acts $f, \bar{f} \in L_0$ are* ***complementary*** *if and only if, for any two states $\omega, \omega' \in \Omega$,*

$$\frac{1}{2}f(\omega) + \frac{1}{2}\bar{f}(\omega) \sim \frac{1}{2}f(\omega') + \frac{1}{2}\bar{f}(\omega').$$

If two acts $f, \bar{f} \in L_0$ are complementary, then $(f, \bar{f})$ is referred to as a ***complementary pair****.*

If preferences over X can be represented by a von Neumann-Morgenstern utility function $u(\cdot)$—which is the case under Axioms 3.1 through 3.5—then the *utility profiles* of the acts f and $\bar{f}$, denoted $u \circ f$ and $u \circ \bar{f}$ respectively, satisfy $u \circ \bar{f} = k - u \circ f$ for some constant $k \in \mathbb{R}$. Thus, complementarity is the preference counterpart of algebraic negation.

Notice that, if $(f, \bar{f})$ and $(g, \bar{g})$ are complementary pairs of acts, then, for any weight $\alpha \in [0,1]$, the mixtures $\alpha f + (1-\alpha)g$ and $\alpha \bar{f} + (1-\alpha)\bar{g}$ are themselves complementary.

The Complementary Independence axiom may now be formulated.

Axiom 3.7 (Complementary Independence) *For any two complementary pairs $(f, \bar{f})$ and $(g, \bar{g})$ in L_0, and all $\alpha \in [0,1]$: $f \succcurlyeq \bar{f}$ and $g \succcurlyeq \bar{g}$ imply $\alpha f + (1-\alpha)g \succcurlyeq \alpha \bar{f} + (1-\alpha)\bar{g}$.*

[6] Chateauneuf et al. [8] show that a similar (simpler) axiom delivers countable additivity of priors for α-maxmin preferences. Ghirardato et al. [22] obtain countable additivity for preferences that only satisfy the first five axioms in the text by imposing monotone continuity on a *derived* preference relation. Here, due to Axiom 3.7, it is possible to provide a simple axiom on the primitive preference $\succcurlyeq$, even if the latter is not a member of the α-maxmin family.

Axiom 3.7 is motivated by the intuition that VEU preferences rank acts according to a baseline EU evaluation, adjusted to reflect a concern for *utility or outcome variability around the baseline*. Observe first that, for EU preferences, $f \succcurlyeq \bar{f}$ and $g \succcurlyeq \bar{g}$ imply that $\alpha f + (1-\alpha)g \succcurlyeq \alpha \bar{f} + (1-\alpha)\bar{g}$ regardless of whether or not the acts under consideration are pairwise complementary; indeed, under Axioms 3.1—3.4, this property is equivalent to the standard Independence axiom, and hence characterizes EU preferences. Next, recall that complementary acts are "mirror images" of each other; therefore, as noted in the Introduction, virtually all classical measures of dispersion for random variables would attribute them the *same variability*. But, if adjustments reflect a concern for variability, complementary acts should then be subject to the *same adjustment*. Therefore, preferences over complementary acts cannot be driven by differences in their adjustments: *pairwise complementary acts are effectively ranked consistently with their baseline evaluation*. Axiom 3.7 then reflects an observable implication of the assumption that such baseline evaluations conform to EU.

A final axiom is required:

Axiom 3.8 (Complementary Translation Invariance) *For all complementary pairs $(f, \bar{f})$, and all $x, \bar{x} \in X$ with $f \sim x$ and $\bar{f} \sim \bar{x}$: $\frac{1}{2}f + \frac{1}{2}\bar{x} \sim \frac{1}{2}\bar{f} + \frac{1}{2}x$.*

Similarly to Complementary Independence, Axiom 3.8 captures a behavioral implication of the assumption that the adjustment applied to the baseline EU evaluation of complementary acts is the same. Observe first that, if the preference relation $\succcurlyeq$ is consistent with EU, the property in the Axiom holds regardless of whether or not f and $\bar{f}$ are complementary. Indeed, a stronger property holds for EU preferences: if the prizes x and $\bar{x}$ are "translated" in utility space by the *same* amount, thereby obtaining two new prizes $y, \bar{y}$, then it is also the case that $\frac{1}{2}f + \frac{1}{2}\bar{y} \sim \frac{1}{2}\bar{f} + \frac{1}{2}y$. If now the individual evaluates the complementary acts f and $\bar{f}$ by applying the same adjustment to their baseline evaluation, then the prizes $x \sim f$ and $\bar{x} \sim \bar{f}$ will differ from the *baseline* evaluation of f and $\bar{f}$ by the same utility shift, intuitively corresponding to the extent of their common adjustment. Furthermore, the mixtures $\frac{1}{2}f + \frac{1}{2}\bar{x}$ and $\frac{1}{2}\bar{f} + \frac{1}{2}x$ are ranked according to the individual's baseline preference, because they are complementary. Therefore, if baseline preferences are consistent with EU, the preceding observation implies that these mixtures should be indifferent, as required by the Axiom.

Complementary Translation Invariance should be viewed as less central to the characterization of VEU preferences than Complementary Independence (Axiom 3.7). Indeed, Axiom 3.8 is redundant in two important cases. First, Axiom 3.8 is implied by Axioms 3.1–3.5 and 3.7 if the utility function representing preferences over X is unbounded either above or below,[7] as is the case for the majority of

[7] A proof is available upon request. Also note that unboundedness of the utility function follows from well-known behavioral axioms: see e.g. [37].

monetary utility functions employed in applications. Second, regardless of the utility function, if preferences satisfy Axioms 3.1–3.4 and 3.5* (instead of Axiom 3.5), then it is trivial to verify that the indifference required by Axiom 3.8 holds *regardless* of whether or not f and $\bar{f}$ are complementary; in other words, Axiom 3.8 is automatically satisfied by all "invariant biseparable" preferences à la Ghirardato, Maccheroni and Marinacci [22].[8] Thus, Axiom 3.8 is *only* required to accommodate preferences that simultaneously violate Axiom 3.5* and are represented by a bounded utility function on X.[9]

The main result of this paper can now be stated.

Theorem 3.1 *Consider a preference relation $\succcurlyeq$ on L_0. The following statements are equivalent:*

(1) The preference relation $\succcurlyeq$ satisfies Axioms 3.1–3.8.

(2) $\succcurlyeq$ admits a sharp VEU representation (u, p, n, m, A).

(3) $\succcurlyeq$ admits a VEU representation (u, p, n, m, A).

In (2), if (u', p', n', m', A') is another VEU representation of $\succcurlyeq$, then $p' = p$, $u' = \alpha u + \beta$ for some $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$, and there exists a linear surjection $T : R_0(m', u'(X)) \to R_0(m, u(X))$ such that

$$\forall a' \in B_0(\Sigma, u'(X)), \qquad T\left(\int a'\, dm'\right) = \frac{1}{\alpha}\int a'\, dm \quad \text{and} \quad A'\left(\int a'\, dm'\right) = \alpha A\left(T\left(\int a'\, dm'\right)\right). \quad (5)$$

If (p, u', n', m', A') is sharp, then $n = n'$ and T is a bijection. Finally, if Ω is finite, then $n \le |\Omega| - 1$.

The primary message of Theorem 3.1 is the equivalence of (1) and (2): Axioms 3.1–3.8 are equivalent to the existence of a *sharp* VEU representation. However, as noted in Sec. 2.2, it is sometimes convenient to employ VEU representations that are not sharp. Theorem 3.1 ensures that the resulting preferences will still satisfy Axioms 3.1–3.8. To put it differently, if a preference admits a VEU representation, then it also admits a sharp VEU representations.

The second part of Theorem 3.1 indicates the uniqueness properties of the VEU representation. In particular, the baseline probability measure p is unique, and the adjustment tuple m and function A are unique up to transformations that preserve both the affine structure of the set $R_0(m, u(X))$ of adjustment vectors, as well as the actual adjustment associated with each element in that set.

To elaborate, recall that the role of the adjustment tuple m is to capture the patterns of "complementarity" among different events; for instance, if ambiguity about two events E and F cancels out, then

[8] This broad class includes for instance all multiple-priors, α-maximin, and Choquet-Expected Utility preferences.

[9] Imposing unbounded utility functions, or the full Certainty Independence axiom, seems too high a price to pay to dispense with Axiom 3.8, especially because none of the main results in this paper require these stronger assumptions.

$m(E)+m(F)=0.$[10] In order for another measure m' to capture the same complementarities as the measure m, it must be the case that also $m'(E)+m'(F)=0$. Similarly, complementarities among adjustment vectors associated with different acts must be preserved. The existence of a functional T with the properties listed in Theorem 3.1 ensures this. As the following example illustrates, this imposes considerable restrictions on transformations of a given adjustment that can be deemed inessential.

Example 1 Refer to the ambiguity-averse VEU preferences described in the Introduction in the context of the Ellsberg Paradox. In particular, recall that $\Omega=\{R,B,G\}$ and $m_0(\{R\})=0$, $m_0(\{B\})=-m_0(\{G\})=\frac{1}{3}$; the fact that the latter two adjustments have opposite signs indicates that ambiguity about B and G "cancels out." Now let $m=(m_0)$, so $n=1$; indeed, note that $R(m,u(X))$ is the entire real line.

Now consider a two-element tuple $m'\in M^2(\Sigma)$ and let $A'(\varphi)=-\sqrt{\varphi_1^2+\varphi_2^2}$ for all $\varphi\in\mathbb{R}^2$. Suppose there exists a map T as in Theorem 3.1. The fact that $A'=A\circ T$ implies that, in particular, $A'(m'(\{R\}))=A(T(m'(\{R\})))=A(m(\{R\}))=0$, so $m'(\{R\})=0$. Similarly, $T(m'(\{B,G\}))=m(\{B,G\})=0$, so $A'=A\circ T$ implies $A'(m'(\{B,G\}))=0$, and so $m'(\{B\})=-m'(\{G\})$. Finally, $A'(m'(\{B\}))=\frac{1}{3}=A'(m'(\{G\}))$.

In other words, m' encodes exactly the same information about B and G as m: the two events are equally ambiguous, but their ambiguities "cancel out". Of course, m does so in a more parsimonious way. This can also be seen geometrically: $m'(\{B\})$ and $m'(\{B\})$ are opposite points on a circle centered at the origin with radius equal to $\frac{1}{3}$, and $R(m',u(X))$ is a line through the origin. This intuitively suggests that ambiguity in the Ellsberg Paradox is really "one-dimensional", regardless of the particular vector representation one chooses.

4 Additional Results

4.1 Ambiguity Aversion

This section analyzes ambiguity aversion for VEU preferences. Two established definitions of this concept are considered: the first, due to Schmeidler [48], identifies ambiguity aversion with a preference for mixtures; the second, due to Ghirardato and Marinacci [24], captures a wider range of aversive attitudes towards ambiguity, and turns out to have a natural characterization for VEU preferences.

Begin with Schmeidler's classical axiom. Intuitively, an individual who is ambiguity-averse according to the proposed definition values mixtures because they *"smooth" utility profiles* (cf. Schmeidler [48, p. 582]; Klibanoff [33, p. 290]). This has an straightforward characterization for VEU preferences, stated below as a Corollary to the main representation result provided in Sec. 3.

Axiom 4.1 (Ambiguity Aversion) *For all acts $f,g\in L_0$ and $\alpha\in(0,1)$: $f\sim g$ implies $\alpha f+(1-\alpha)g\succcurlyeq g$.*

[10] Here and in the following, for any adjustment tuple m and event E, $m(E)=\int 1_E\,dm=(m_i(E))_{0\le i<n}$.

Corollary 4.1 *Consider a preference relation $\succcurlyeq$ on L_0 for which (1) in Theorem 3.1 holds, and let A be as in (2). Then $\succcurlyeq$ satisfies Axiom 4.1 if and only if A is non-positive and concave.*

Thus, as expected, Axiom 4.1 implies that the adjustment functional is non-positive and concave. However, for VEU preferences, it seems intuitive to associate non-positive, but not necessarily concave adjustment terms with a form of ambiguity aversion. It turns out that this notion is precisely captured by Ghirardato and Marinacci's "comparative" definition:

Definition 5 *Given two preference relations $\succcurlyeq_1$ and $\succcurlyeq_2$ on L_0, say that $\succcurlyeq_1$ is* ***comparatively ambiguity-averse*** *iff $\succcurlyeq_2$ is consistent with expected utility and, for all $f \in L_0$ and $x \in X$,*

$$f \succcurlyeq_1 x \quad \Rightarrow \quad f \succcurlyeq_2 x.$$

The reader is directed to [24] for a discussion of this definition. Finally, comparative ambiguity aversion can also be characterized using weaker forms of Axiom 4.1 for VEU preferences.

Axiom 4.2 (Simple Ambiguity Aversion) *For all complementary pairs $(f, \bar{f})$ and prizes $x, \bar{x} \in X$ such that $f \sim x$ and $\bar{f} \sim \bar{x}$: $\frac{1}{2}f + \frac{1}{2}\bar{f} \succcurlyeq \frac{1}{2}x + \frac{1}{2}\bar{x}$.*

Axiom 4.3 (Minimal Ambiguity Aversion) *For all complementary pairs $(f, \bar{f})$ with $f \sim \bar{f}$, $\frac{1}{2}f + \frac{1}{2}\bar{f} \succcurlyeq f$.*

Both axioms have the standard hedging interpretation, but concern complementary pairs, rather than arbitrary pairs of acts. Axiom 4.3 is related to Chateauneuf and Tallon's "diversification" property (see [7]). The main result of this subsection can now be stated.

Proposition 4.2 *Let $\succcurlyeq$ be a preference relation with VEU representation (u, p, n, m, A). Then the following statements are equivalent:*

(1) $\succcurlyeq$ is comparatively ambiguity-averse.

(2) $\succcurlyeq$ satisfies Axiom 4.2.

(3) For all $f \in L_0$, $A(\int u \circ f \, dm) \leq 0$.

If $u(X)$ is unbounded above or below, or if $\succcurlyeq$ satisfies Axiom 3.5, then (1)–(3) are equivalent to*

(4) $\succcurlyeq$ satisfies Axiom 4.3.

Example 2 Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $X = \mathbb{R}$, and a preference $\succcurlyeq$ with VEU representation $(u, p, 2, m, A)$, where $u(x) = x$, p is uniform, $m_0(\{\omega_1\}) = m_1(\{\omega_2\}) = -m_0(\{\omega_2\}) = -m_1(\{\omega_3\}) = \epsilon \in \left(0, \frac{1}{3}\right)$, and $A(\varphi) = -\min(|\varphi_0|, |\varphi_1|)$. It is easy to verify that $(u, p, 2, m, a)$ satisfies Def. 2; in particular, A is differentiable everywhere except at points where it takes the value 0, so Remark A.1 in the Appendix and the assumed restrictions on ϵ imply that the monotonicity requirement is met.

These preferences are comparatively ambiguity-averse by Proposition 4.2; however, they do not satisfy the standard Ambiguity Aversion axiom (i.e. Axiom 4.1): for instance, if $u \circ f = [1,0,0]$ and $u \circ g = [0,0,1]$ (obvious notation), then $f \sim g$, but $\frac{1}{2}f + \frac{1}{2}g \prec f$.

4.2 Updating

The theory developed so far only applies to one-period choice problems. This section proposes an updating rule for VEU preferences; sophisticated dynamic choice is briefly discussed in §4.3. Throughout this subsection, two binary relations on L_0 will be considered: $\succcurlyeq$ denotes the individual's *ex-ante* preferences, whereas $\succcurlyeq_E$ denotes her *preferences conditional upon the event* $E \in \Sigma$. To keep notation to a minimum, the event E will be fixed throughout.

As for conditional EU preferences, to ensure that updating is well-defined, it is necessary that the conditioning event E "matter" for the individual. This leads to the following standard requirement.

Axiom 4.4 (E is not null) *There exist acts $f, g \in L_0$ such that $f(\omega) = g(\omega)$ for all $\omega \notin E$ and $f \succ g$.*

Due to symmetry, the above requirement has a straightforward characterization for VEU preferences.

Remark 4.1 *Let $\succcurlyeq$ be a VEU preference, with baseline prior p. Then Axiom 4.4 holds iff $p(E) > 0$.*

As is the case for conditional EU preferences, it will be assumed throughout that the evaluation of acts upon learning that the event E has occurred does *not* depend upon the consequences that might have been obtained if, counterfactually, E had not obtained. This leads to the following, standard axiom.

Axiom 4.5 (Conditional Preference) *For all $f, g \in L_0$: if $f(\omega) = g(\omega)$ for all $\omega \in \Omega \setminus E$, then $f \sim_E g$.*

The main axiom of this section can be informally stated as follows: *if two acts have the same baseline evaluation both ex-ante and conditional upon E, and the outcomes they deliver differ from the baseline only on the event E, then their ex-ante and conditional ranking should be the same.* This is consistent with the proposed interpretation of VEU preferences. Consider an individual whose preferences are VEU both ex-ante and conditional on E. Upon learning that E has occurred, her evaluation of an act f may change for two reasons: the baseline EU evaluation of f may change, and outcome variability in states outside E no longer matters. But if one restricts attention to acts for which the baseline evaluation does *not* change upon conditioning on E, and which exhibit *no* variation away from the baseline at states outside E to begin with, it seems plausible to assume that the individual's evaluation of such acts will not change.

These special acts can be characterized by a behavioral condition that, once again, involves the notion of complementarity. Consider two complementary acts $h, \bar{h} \in L_0$ that are *constant on* $\Omega \setminus E$: that is,

$h(\omega) = h(\omega')$ and $\bar{h}(\omega) = \bar{h}(\omega')$ for all $\omega, \omega' \in \Omega \setminus E$. Suppose that, for any (hence all) $\omega \in \Omega \setminus E$,

$$\frac{1}{2}h + \frac{1}{2}\bar{h}(\omega) \sim \frac{1}{2}\bar{h} + \frac{1}{2}h(\omega). \tag{6}$$

If the preference relation $\succcurlyeq$ happens to be consistent with EU, then Eq. (6), together with complementarity, readily imply that $h \sim h(\omega)$ for any (hence all) $\omega \in \Omega \setminus E$.[11] This indicates that $h(\omega)$ is a certainty equivalent of h ex-ante. However, intuitively, $h(\omega)$ can also be viewed as a "conditional certainty equivalent" of h given E: since $h(\omega') = h(\omega)$ for all $\omega' \in \Omega \setminus E$, the ranking $h \sim h(\omega)$ suggests that receiving $h(\omega)$ for sure at states in E is just as good for the individual as allowing the act h to determine the ultimate prize she will receive conditional upon E.[12] Thus, for an EU preference, Eq. (6) implies that the act h has the same certainty equivalent both ex-ante and conditional upon E.

For general VEU preferences, the above intuition obviously does not apply: it may well be the case that $h \not\sim h(\omega)$ for $\omega \in \Omega \setminus E$. However, recall that Complementary Independence (Axiom 3.7) implies that *VEU preferences always rank complementary acts in accordance with their baseline EU evaluation.* Since the mixture acts in Eq. (6) are complementary, the above intuition *does apply* to the EU preference determined by the individual's baseline prior. One then concludes that, if Eq. (6) holds, then $h(\omega)$ is a *baseline certainty equivalent* of h, both ex-ante and conditional upon E; this is formally verified in the proof of Proposition 4.3. Furthermore, it is clear that h deviates from this baseline only at states in E. Thus, Eq. (6) identifies the class of acts that should be ranked consistently by prior and conditional VEU preferences.

Axiom 4.6 (Baseline-Variation Consistency)
For all complementary pairs $(f, \bar{f})$ and $(g, \bar{g})$ such that $f, \bar{f}, g, \bar{g}$ are constant on $\Omega \setminus E$ and, for every $\omega \in \Omega \setminus E$, $\frac{1}{2}f + \frac{1}{2}\bar{f}(\omega) \sim \frac{1}{2}\bar{f} + \frac{1}{2}f(\omega)$ and $\frac{1}{2}g + \frac{1}{2}\bar{g}(\omega) \sim \frac{1}{2}\bar{g} + \frac{1}{2}g(\omega)$: $f \succcurlyeq_E g$ if and only if $f \succcurlyeq g$.

Proposition 4.3 *Consider a preference relation $\succcurlyeq$ on L_0 having a VEU representation (u, p, n, m, A), an event $E \in \Sigma$, and another binary relation $\succcurlyeq_E$ on L_0. Assume that $\succcurlyeq_E$ is complete and transitive, and that Axiom 4.4 holds. Then the following are equivalent.*
(1) Axioms 4.5 and 4.6 hold;
(2) $\succcurlyeq_E$ has a VEU representation $\big(u, p(\cdot|E), n, m_E, A\big)$, where $m_E = (m_{i,E})_{0 \le i < n}$ satisfies

$$\forall F \in \Sigma,\ 0 \le i < n \qquad m_{i,E}(F) = m_i(E \cap F) + p(F|E)m_i(\Omega \setminus E). \tag{7}$$

[11] By complementarity, $\frac{1}{2}h + \frac{1}{2}\bar{h} \sim \frac{1}{2}h(\omega) + \frac{1}{2}\bar{h}(\omega)$; by Independence, combining this relation with Eq. (6) yields $\frac{1}{2}h + \frac{1}{2}k \sim \frac{1}{2}h(\omega) + \frac{1}{2}k$, with $k = \frac{1}{2}\bar{h} + \frac{1}{2}\bar{h}(\omega)$. Invoking Independence once more yields $h(\omega) \sim h$.

[12] Indeed, this condition may be used to characterize Bayesian updating for EU preferences, as well as prior-by-prior Bayesian updating for MEU preferences: see Pires [42].

In other words, under the proposed axioms, the updated preference is also VEU; its baseline probability is the Bayesian update of the prior, the functional A and utility u are unchanged, and the posterior adjustment tuple m_E is obtained from the prior tuple by Eq. (7). It should be noted that the resulting VEU representation is not necessarily sharp, even if the ex-ante representation is.

To gain some intuition for the updating rule in Eq. (7), consider a probability measure $\mu \in ca_1(\Sigma)$. Standard Bayesian updating on an event $E \in \Sigma$ may be viewed as a process whereby the mass $\mu(\Omega \setminus E)$ placed on the event that did not obtain is redistributed to states in E. In particular, since

$$\forall F \in \Sigma, \qquad \mu(F|E) = \mu(F \cap E) + \mu(F|E)\mu(\Omega \setminus E),$$

the Bayesian updating process can be seen as adding a fraction $\mu(F|E)$ of the mass $\mu(\Omega \setminus E)$ to the ex-ante probability mass of $F \cap E$. It should then be clear that Eq. (7) performs a similar operation, except that it adds fractions of the "mass" $m(\Omega \setminus E)$ to the "mass" $m(F \cap E)$.

The updating rule in Eq. (7) satisfies convenient and natural properties of conditional measures. Fix an adjustment tuple $m \in M^n(\Sigma)$; it is immediate to verify that $m_{i,E}(E) = 0$ for every index i: this is the conditional counterpart of the normalization property $m_i(\Omega) = 0$ for unconditional adjustment measures. Furthermore, a version of the "law of iterated conditioning" holds. Fix three events $E, F, G \in \Sigma$ such that $G \subset F \subset E$, and for all $i \in \{0, \ldots, n-1\}$, let $m_{i,E,F}$ be the signed measure obtained from $m_{i,E}$ by applying Eq. (7). Then $m_{i,E,F}(G) = m_{i,F}(G)$ for all indices i. That is: conditioning on E first, then conditioning the resulting measure on F yields the same tuple of signed measures as conditioning on F directly. This property is shared by some, but not all updating rules for known decision models under ambiguity: for instance, the "maximum-likelihood" rule for multiple-priors preferences (cf. Gilboa and Schmeidler [26]) violates it.

4.3 Recursion: An Example

The conditional preferences derived in Proposition 4.3 only satisfy a weak form of dynamic consistency. Thus, the proposed updating rule must be complemented with a criterion, such as *consistent planning* (Strotz [54]), to resolve possible conflicts between the ex-ante and ex-post evaluation of future choices. However, the updating rule axiomatized in the preceding section allows for a *recursive* formulation of the consistent-planning problem. This section sketches the basic idea, and then illustrates it by means of a simple example; a full treatment is left for future work.

It is immediate to verify that, if $\mathscr{F} \subset \Sigma$ is a finite partition of Ω, and for every $E \in \mathscr{F}$ the tuple m_E is defined as in Eq. (7),

$$\int a\, dm_i = \sum_{E \in \mathscr{F}} \int_E a\, dm_{i,E} + m_i(E) \cdot \int a\, dp(\cdot|E). \tag{8}$$

Notice that the integrals in the r.h.s. are, respectively, the adjustment vector and baseline for the function a conditional upon each event E. In other words, the adjustment vector $\int a\, dm$ can be obtained from the conditional vectors $\int_E a\, dp(\cdot|E)$ and $\int_E a\, dm_E$ for all $E \in \mathscr{F}$, just like the baseline $\int a\, dp$ can be obtained from the conditional baseline integrals.

This suggests a recursive approach to the solution of dynamic decision problems via consistent planning. Loosely speaking, the conditional integrals $\int_E a\, dp$ and $\int_E a\, dm_{i,E}$ are part of the VEU "value function" obtained by solving the one-step-ahead problem; these can be plugged into the current-period problem, as is the case for EU preferences. Example 3 below illustrates this approach in a very simple setting with two decision epochs.

To conclude, note that, if there are more than two decision epochs, posterior adjustment measures can be constructed in two equivalent ways, because the proposed updating rule satisfies the law of iterated conditioning. It is of course possible to fix a prior adjustment tuple on the entire state space, and derive from it the relevant time-t conditional adjustment tuples by applying Eq. (7). However, alternatively, adjustment tuples can be constructed by iterated one-step-ahead conditioning, which may be especially convenient if uncertainty has a Markov structure.

Example 3 Consider a 3-period ($t = 0, 1, 2$) consumption-savings problem. The agent's initial endowment is $w_0 > 0$, and her per-period utility is $v(x) = \frac{x^{1-\gamma}}{1-\gamma}$, with $\gamma > 1$ for definiteness; consumption streams are evaluated by discounting at the rate $\delta \in (0,1)$. The output of the sole productive activity in the economy is characterized by a rate of return that can be either r_H or $r_L < r_H$. At the beginning of periods $t = 0, 1$, given her current wealth w_t, the agent sets the quantity s_t to be saved, and consumes the rest; then the rate of return r_t is realized, and the period ends. At the beginning of the following period, $w_{t+1} = s_t r_t$. At time $t = 2$ there is no decision to be made: the agent consumes the entire output.

To describe the possible realizations of the production process at times 0 and 1, let $\Omega = \{(r_0, r_1) : r_0, r_1 \in \{r_H, r_L\}\}$. Assume a uniform baseline p, and an adjustment given by

$$n = 1, \quad m_0(\{(r_H, r_H)\}) = m_0(\{(r_L, r_L)\}) = \epsilon = -m_0(\{(r_H, r_L)\}) = -m_0(\{(r_L, r_H)\}) \quad \text{and} \quad A(\xi) = -|\xi|$$

with $0 \le \epsilon \le \frac{1}{4}$. This is one of the simplest possible specifications for the problem under consideration; it is inspired by the analysis of dilation in Seidenfeld and Wasserman [49]. Intuitively, the agent does not perceive any ambiguity about the *marginal* probability of high or low returns in either period; however, there is ambiguity about the correlation of outcomes.

Letting $H_t = \{(r_0, r_1) : r_t = r_H\}$ and similarly for L_t, and applying Eq. (7), we get $m_{0,H_0}(H_1) = \epsilon = -m_{0,H_0}(L_1)$ and similarly $m_{0,L_0}(H_1) = -\epsilon = -m_{0,L_0}(L_1)$. Thus, after observing the realization of time-0 production, time-1 returns become ambiguous. If $r_0 = r_H$ and wealth equals w_1, the agent solves

$$\max_{0 \le s_1 \le w_1} v(w_1 - s_1) + \delta \left\{ \frac{1}{2} v(r_H s_1) + \frac{1}{2} v(r_L s_1) \right\} - \left| \delta \left[\epsilon v(r_H s_1) - \epsilon v(r_L s_1) \right] \right| \tag{9}$$

(here and in the following, the term corresponding to the current payoff does not appear in the adjustment term, because it is constant and so is assigned a zero adjustment "vector"). Since v is increasing, the adjustment $\delta\epsilon[v(r_H s_1) - v_L s_1)]$ is positive, so the agent will behave as if the probability of high output was $\frac{1}{2} - \epsilon$; then, simple (and standard) manipulations show that

$$\begin{aligned} s_1(w_1) &= \alpha_1 w_1, & (10)\\ V_1^p(w_1) &\equiv v(w_1 - s_1) + \delta\left\{\frac{1}{2}v(r_H s_1) + \frac{1}{2}v(r_L s_1)\right\} = \beta_1 v(w_1), & (11)\\ V_{1,H}^m(w_1) &\equiv \delta\left[\epsilon v(r_H s_1) - \epsilon v(r_L s_1)\right] = \beta_1^m v(w_1), & (12) \end{aligned}$$

where the constants α_1, β_1 and β_1^m depend upon the parameters $\gamma, \delta, \epsilon, r_L, r_H$ but not on w_1. The results in Eqs. (10) and (11) are well-known; note however that a similarly convenient expression is obtained for the adjustment "vector" $V_{1,H}^m(w_1)$. Finally, if $r_0 = r_L$, the agent solves a problem similar to that in Eq. (9), except that the adjustment vector is $V_{1,L}^m(w_1) \equiv \delta[-\epsilon v(r_H s_1) + \epsilon v(r_L s_1)] = -V_{1,H}^m(w_1)$; this clearly leads to the same solution and baseline utility.

It is now possible to contrast the direct approach to sophisticated choice and the recursive approach suggested by Eq. (8). In the direct approach, time-0 savings s_0 are determined by maximizing the entire ex-ante VEU functional, substituting for the optimal time-1 choice as a function of savings in time 0 and the realized rate of return. That is, using $w_1 = r_0 s_0$ and so $s_1 = \alpha_1 r_0 s_0$, one must solve

$$\begin{aligned} \max_{0 \le s_0 \le w_0} v(w_0 - s_0) \quad &+ \sum_{(r_0, r_1)\in\Omega} \frac{1}{4}\left[\delta v((1-\alpha_1) r_0 s_0) + \delta^2 v(r_1 \alpha_1 r_0 s_0)\right] - & (13)\\ &- \left|\sum_{(r_0, r_1)\in\Omega} m(\{r_0, r_1\})\left[\delta v((1-\alpha) r_0 s_0) + \delta^2 v(r_1 \alpha_1 r_0 s_0)\right]\right|. \end{aligned}$$

Taking a recursive approach instead, invoking Eq. (8) and taking care to discount appropriately, at time $t = 0$ the agent solves

$$\max_{0 \le s_0 \le w_0} v(w_0 - s_0) + \delta\left\{\frac{1}{2}V_1^p(r_H s_0) + \frac{1}{2}V_1^p(r_L s_0)\right\} - \left|\delta\left[V_{1,H}^m(r_H s_0) + V_{1,L}^m(r_L s_0)\right]\right|,$$

which, substituting for V_1^p, $V_{1,H}^m$ and $V_{1,L}^m$, yields

$$\max_{0 \le s_0 \le w_0} v(w_0 - s_0) + \delta\left\{\frac{1}{2}\beta_1 v(r_H s_0) + \frac{1}{2}\beta_1 v(r_L s_0)\right\} - \left|\delta\left[\beta_1^m v(r_H s_0) - \beta_1^m v(r_L s_0)\right]\right|. \quad (14)$$

Even in this very simple example, the objective function in Eq. (14) is slightly easier to analyze than Eq. (13); also, Eq. (14) shows that the time-0 problem is structurally analogous to the time-1 problem, except that the "discount factor" is $\delta\beta_1$. Arguing as in the time-1 problem, the agent behaves as if the probability of high output was $\frac{1}{2} + \frac{\beta_1^m}{\beta_1}$;[13] optimal time-0 savings are then given by $s_0 = \alpha_0 w_0$, where α_0 has an expression analogous to α_1.

[13] It turns out that $\beta_1^m < 0$ for $\gamma > 1$; since v is increasing, $|\delta\beta_1^m[v(r_H s_0) - v(r_L s_0)]| = -\delta\beta_1^m[v(r_H s_0) - v(r_L s_0)]$. Also, in the parameter range under consideration, $\frac{1}{2} + \beta_1^m \in (0,1)$.

4.4 Complementary Independence for Other Decision Models

This section investigates the implications of the Complementary Independence axiom for certain well-known preference models. It will be shown that this axiom makes it possible to identify a baseline prior from behavioral primitives, independently of the functional representation of preferences. Furthermore, the specific role of this prior in the different models under consideration will be clarified.[14]

Begin with what are perhaps the two best-known models of decision under ambiguity: the maxmin-expected utility (MEU) or multiple-priors model of Gilboa and Schmeidler [25], and Schmeidler's [48] Choquet-expected utility (CEU) model. A MEU preference is characterized by a utility function u and a weak* closed, convex set $C \subset ba_1(\Sigma)$ of probability charges; the representing functional can be written as $I \circ u : L_0 \to \mathbb{R}$, where $I(a) = \min_{q \in C} \int a\, dq$ for all $a \in B_0(\Sigma)$. Also recall that a *capacity* is a set function $v : \Sigma \to [0,1]$ such that (1) $v(\emptyset) = 0$ and $v(\Omega) = 1$, and (2) $A, B \in \Sigma$ and $A \subset B$ imply $v(A) \leq v(B)$. A CEU preference is represented by the functional $I_v \circ u : L_0 \to \mathbb{R}$, where I_v is the Choquet integral with respect to the capacity v (see [48]).

Preferences conforming to these models satisfy Axioms 3.1–3.4 and Certainty Independence (Axiom 3.5*), which is stronger than Axiom 3.5. MEU preferences additionally satisfy Axiom 4.1, Ambiguity Aversion; CEU preferences satisfy a stronger independence axiom, deemed Comonotonic Independence.

Proposition 4.4 (Complementary Independence for MEU and CEU preferences)

(1) A MEU preference $\succcurlyeq$ satisfies Axiom 3.7 if and only if there is $p \in C$ such that, for all $q \in C$, $2p - q \in C$ (that is, p is the barycenter of C).

(2) A CEU preference $\succcurlyeq$ satisfies Axiom 3.7 if and only if there is $p \in ba_1(\Sigma)$ such that, for all $E \in \Sigma$, $v(E) + [1 - v(\Omega \setminus E)] = 2p(E)$.

In (1) and (2), $p \in ba_1(\Sigma)$ is the unique probability charge that satisfies $f \succcurlyeq \bar{f} \Leftrightarrow \int u \circ f\, dp \geq \int u \circ \bar{f}\, dp$ for all complementary pairs $(f, \bar{f})$, where u is the utility function in the MEU or CEU representation of $\succcurlyeq$.

Thus, for both MEU and CEU preferences, Complementary Independence identifies a baseline prior that, as in the VEU model, represents preferences over complementary acts. Observe that the set function $E \mapsto 1 - v(\Omega \setminus E)$ is also a capacity, sometimes denoted $\bar{v}$ and referred to as the *dual* of the capacity v. Thus, Complementary Independence corresponds to the property that $\frac{1}{2}v + \frac{1}{2}\bar{v} = p$.

Ghirardato, Maccheroni and Marinacci [22] provide a general representation for the family of preferences that satisfy the MEU axioms *minus* Ambiguity Aversion. As a preliminary step in the proof of the

[14]The models considered here are consistent with the axioms in Sec. 3, but the results provided here do not rely on this.

main characterization result (Theorem 3.1), the present paper extends the Ghirardato *et al.* representation to preferences that satisfy the weaker Axiom 3.5 in lieu of Certainty Independence. The interested reader is referred to Sec. B.2 in the Appendix.

Turn now to the *variational preferences* characterized by Maccheroni, Marinacci and Rustichini [37]: given a utility function $u : X \to \mathbb{R}$,

$$f \succcurlyeq g \quad \Leftrightarrow \quad \min_{q \in ba_1(\Sigma)} \left(\int u \circ f \, dq + c^*(q) \right) \geq \min_{q \in ba_1(\Sigma)} \left(\int u \circ g \, dq + c^*(q) \right),$$

where, denoting by x_f a certainty equivalent of the act f for every $f \in L_0$, the function $c^* : ba_1(\Sigma) \to \mathbb{R}_+ \cup \{\infty\}$ is defined by

$$c^*(q) = \sup_{f \in L_0} \left(u(x_f) - \int u \circ f \, dq \right).$$

Maccheroni et al. [37] show that variational preferences are characterized by Axioms 3.1–3.4 and Ambiguity Aversion (Axiom 4.1). The following result shows that Complementary Independence corresponds to a natural symmetry property of the "cost function" c^*, and again identifies a unique baseline prior.

Proposition 4.5 (Complementary Independence for Variational Preferences) *Let $\succcurlyeq$ be a variational preference, and assume that the utility function u is unbounded either above or below. Then $\succcurlyeq$ satisfies Axiom 3.7 if and only if there exists $p \in ba_1(\Sigma)$ such that*

$$\forall q \in ba_1(\Sigma), \qquad 2p - q \in ba_1(\Sigma) \Rightarrow c^*(q) = c^*(2p - q) \text{ and } 2p - q \notin ba_1(\Sigma) \Rightarrow c^*(q) = \infty.$$

In particular, $c^*(p) = 0$. Finally, p is the unique probability charge such that, for all complementary pairs $(f, \bar{f})$, $f \succcurlyeq \bar{f} \Leftrightarrow \int u \circ f \, dp \geq \int u \circ \bar{f} \, dp$.

The reader is referred to [37] for a discussion of the unboundedness assumption.

4.5 More Examples

4.5.1 Variation and distance-based adjustments

As can be expected in light of the discussion in the Introduction, a natural class of VEU preferences is obtained by adopting one of the standard *measures of dispersion* as the adjustment function A. This subsection discusses interesting special cases, corresponding to the combination of specific dispersion measures with specific adjustment tuples.

Assume first, for simplicity, that Ω is finite, let $\Sigma = 2^\Omega$, and write $\Omega = \{\omega_0, \ldots, \omega_n\}$; correspondingly, identify $ba_1(\Sigma)$ with $\Delta(\Omega) \equiv \{p \in \mathbb{R}^{n+1}_+ : \sum_{i=0}^n p_i = 1\}$, the unit simplex in $\mathbb{R}^{n+1}$. Fix a strictly positive baseline probability $p \in \Delta(\Omega)$. Grant and Kaji [27] consider the preference functional $I : B_0(\Sigma) \to \mathbb{R}$ defined by

$$\forall a \in \mathbb{R}^{n+1} = B_0(\Sigma), \quad I(a) = \sum_{i=0}^{n} p_i a_i - \epsilon \sqrt{\sum_{i=0}^{n} p_i \left(a_i - \sum_{j=0}^{n} p_i a_j \right)^2} \equiv \mathrm{E}_p(a) - \epsilon \sigma_p(a);$$

they show that I is monotonic provided $\min_i p_i \geq \frac{\epsilon^2}{1+\epsilon^2}$, in which case the resulting preferences are consistent with the MEU model. It is easy to see that these preferences also satisfy Complementary Independence, and are thus also VEU preferences; in particular, a corresponding adjustment tuple $m \in M^n(\Sigma)$ and function $A : \mathbb{R}^{n+1} \to \mathbb{R}$ are given by $m_i(\{\omega_i\}) = 1 - p_i$ and $m_i(\{\omega_j\}) = -p_j$ for all i and $j \neq i$, and $A(\varphi) = -\epsilon \sqrt{\sum_{i=0}^{n} p_i \varphi_i^2}$.

One shortcoming of the "mean–standard-deviation preferences" defined above is that they necessarily violate monotonicity when the state space Ω is infinite (cf. Grant and Kaji [27]). If instead dispersion is measured by the *Gini mean difference* (Yitzhaki [58]; Yitzhaki and Olkin [59]), monotonicity is preserved. In particular, for an arbitrary measurable space (Ω, Σ), the functional $I : B_0(\Sigma) \to \mathbb{R}$ defined by

$$I(a) = \mathrm{E}_p[a] - \frac{1}{2}\theta \int\int |a(\omega) - a(\omega')|\, p(d\omega')\, p(d\omega)$$

is monotonic for all $\theta \in [0,1]$ (cf. [59], Eqs. 3.1 and 3.2), and characterizes well-defined VEU preferences. If Ω is finite, the adjustment term can be represented via the adjustment tuple $m = (m_{ij})_{i \neq j} \in M^{n(n+1)}(\Sigma)$ such that $m_{ij}(\{\omega_i\}) = 1 = -m_{ij}(\{\omega_j\})$ and $m_{ij}(\{\omega_k\}) = 0$ for all $k \neq i, j$, together with the function $A : \mathbb{R}^{n(n+1)} \to \mathbb{R}$ such that $A(\varphi) = -\frac{1}{2}\theta \sum_{i=0}^{n} \sum_{j \neq i} p_i p_j |\varphi_{ij}|$.

A different class of ambiguity-averse VEU preferences can be constructed via *distance functions*. Again assume that Ω is finite, fix a baseline probability $p \in \Delta(\Omega)$, and consider a constant $\epsilon > 0$ such that $p_i \equiv p(\{\omega_i\}) > \epsilon$ for every $i = 0, \ldots, n$. Now define an adjustment tuple $m \in M^n(\Omega)$ and a corresponding adjustment function $A : \mathbb{R}^{n+1} \to \mathbb{R}$ by $m_i(\{\omega_i\}) = \frac{n}{n+1}\epsilon$ and $m_i(\{\omega_j\}) = -\frac{1}{n+1}\epsilon$ for all i and $j \neq i$, and $A(\varphi) = -\sqrt{\sum_{i=0}^{n} \varphi_i^2}$; then, for every vector $a \in B_0(\Sigma, u(X)) = \mathbb{R}^{n+1}$,

$$I(a) \equiv \int a\, dp + A\left(\int a\, dm \right) = \sum_{i=0}^{n} a_i p_i - \epsilon \sqrt{\sum_{i=0}^{n} \left(a_i - \frac{1}{n+1} \sum_{j=0}^{n} a_j \right)^2}. \tag{15}$$

Notice that the VEU preferences defined by (u, p, n, m, A) are differentiable everywhere except "at certainty." It turns out that the resulting preferences are also consistent with the MEU decision model:

Remark 4.2 *For every $a \in \mathbb{R}^{n+1}$, $I(a) = \min_{q \in C} \int a\, dq$, where $C = \left\{ q \in \Delta(\Omega) : \sqrt{\sum_{i=0}^{n} (q_i - p_i)^2} \leq \epsilon \right\}$.*

4.5.2 Ambiguity Attitudes and Outcome Size

In the examples considered so far, the individual's ambiguity attitudes are qualitatively the same for all acts under consideration: in particular, for a VEU decision-maker with baseline prior p and utility function u who is ambiguity-averse in the sense of Def. 5, the certainty equivalent of *every* act f is not greater than its "unambiguous" certainty equivalent $u^{-1}\left(\int u\circ f\,dp\right)$. However, there is evidence that some individuals might be ambiguity-*seeking* when contemplating small-stake bets, and ambiguity-averse when considering large bets. For instance, this is documented in recent experimental work by Koch and Schunk [35]; earlier evidence along the same lines can be found in Hogarth and Einhorn [31].[15] Incidentally, an analogous pattern has been broadly documented in the setting of risky choice: subjects display risk-seeking attitudes when stakes are low, and risk-averse behavior for larger bets (Prelec and Lowenstein [43] deem this the "peanuts effect").

I now indicate a possible VEU representation of this preference pattern. The cited experiments deal with draws from an urn of unknown composition, so it is enough to consider the state space $\Omega=\{\omega,\omega'\}$; for simplicity, the baseline probability p will be taken to be uniform, and utility will be assumed linear. It is sufficient to consider a scalar adjustment $m\in M^1(\Omega)$; finally, fix a real number $t\geq 1$ and let

$$A(\varphi)=\log\left(1+t^2\right)-\frac{1}{2}\log\left(1+(\varphi+t)^2\right)-\frac{1}{2}\log\left(1+(\varphi-t)^2\right).$$

The function A is positive for $\varphi\in\left(-\sqrt{2(t^2-1)},\sqrt{2(t^2-1)}\right)$, and has a unique (positive) maximum at either side of zero; for values of φ outside this interval, it becomes negative. Notice that this function is differentiable everywhere, and its derivative lies between -1 and 1; by Remark A.1 in the Appendix, the VEU preferences characterized by $(u,p,1,m,A)$ will be monotonic if $p(E)\geq|m(E)|$ for all events E.

Take $t=3$ and $m(\{\omega\})=-m(\{\omega'\})=0.2$, and consider the acts f,g such that

$$f(\omega)=10,\quad g(\omega)=1000,\quad f(\omega')=g(\omega')=0.$$

Intuitively, f corresponds to a "small" bet on the event that ω occurs, whereas g is a "large" bet on the same event. The "unambiguous" certainty equivalents of these acts are, respectively, $\int f\,dp=5$ and $\int g\,dp=500$; the certainty equivalent of f is approximately $5.33>5$, whereas the certainty equivalent of g is $491.71<500$. Thus, this individual displays ambiguity-seeking preferences for the small bet f, and ambiguity-averse behavior for the large bet g.[16]

[15] Specifically, Table 4 in [31] shows that the fraction of subjects who display ambiguity-averse preferences in the experiments under consideration increases with outcome size; the subsequent discussion on p. 798 indicates that subjects not classified as ambiguity-averse are to be considered ambiguity-seeking. Hence, there a fraction of subjects must display ambiguity-seeking preferences for small stakes, and switch to ambiguity-averse behavior for large stakes.

[16] More generally, whenever $g(\omega)>20$, the certainty equivalent of g is smaller than its unambiguous certainty equivalent 500; and whenever $f(\omega)<20$, the certainty equivalent of f is greater than 5.

5 Discussion

5.1 Related Literature

In the context of choice under risk, Quiggin and Chambers [44, 45] analyze models featuring an exogenously given, objective reference probability p. Under suitable assumptions, a random variable y is evaluated according to the difference between its expectation $\mathrm{E}_p(y)$ with respect to p, and a "risk index" $\rho(y)$—a representation that is clearly reminiscent of the VEU representation.[17]

Similar functional forms also appear in the social-choice literature. A classic result due to Roberts [46] characterizes social-welfare functionals that evaluate a profile $u_1, \dots, u_I$ of utility imputations according to the form $\bar{u} - g(u_1 - \bar{u}, \dots, u_I - \bar{u})$, where $\bar{u} = \frac{1}{I}\sum_i u_i$. Ben-Porath and Gilboa [3] characterize orderings over income distributions that can be represented in what is essentially a special case of the VEU functional, with the uniform distribution as reference probability. Incidentally, the adjustment part of the representation in [3] has an interesting interpretation in terms of adjustment tuples (cf. Def. 1).

While these contributions are not directly relevant to choice under uncertainty, it is worth emphasizing that representations similar to the one proposed here have proved effective in a variety of settings.

The literature on model uncertainty, initiated by Lars Hansen, Thomas Sargent and coauthors (see e.g. [29, 30]), also prominently features a reference prior; the focus in this literature is largely on applications to macroeconomics and finance, rather than on behavioral foundations. An interesting axiomatization has recently been provided by Strzalecki [55]; see also Wang [57].

A recent paper by Grant and Polak [28] provides a "primal representation" of Maccheroni et al.'s variational preferences model [37] in a finite-states setting, and generalizes it by relaxing translation invariance (monotonicity and ambiguity aversion are also weakened). The representation Grant and Polak propose is related to the ones in Quiggin and Chambers [45] and Roberts [46]: each act f is evaluated by aggregating a "reference expected utility" term $\mathrm{E}_p[u \circ f]$, where p denotes a suitable reference prior, and an "ambiguity index" $\rho(\cdot)$ that depends upon the utility differences $u(f(\omega_i)) - \mathrm{E}_p[u \circ f]$ in each state ω_i. Grant and Polak show that variational preferences aggregate these two components additively, whereas relaxing translation invariance leads to more general aggregators.

In comparison with the VEU representation proposed here, the reference prior in [28] is defined by a geometric, rather than behavioral condition, and is not unique in general. More precisely, in the space of utility profiles, the prior p in [28] corresponds to a hyperplane supporting the individual's indifference curves at a point on the certainty line. Decision models featuring a kink at certainty (e.g. MEU, CEU or invariant biseparable preferences) allow for multiple supporting hyperplanes, and hence multiple reference priors as defined in [28]. One way to ensure uniqueness is to assume that indifference curves

[17] See also Epstein [16] and Safra and Segal [47].

are “flat” or smooth at certainty; but, in this case, the prior p only reflects (indeed, under smoothness, approximates) local behavior around the certainty line. The baseline prior in the VEU representation is instead uniquely identified by the individual’s preferences over complementary acts. Hence, *every* act contributes to the behavioral identification of the baseline prior in the VEU representation; conversely, the baseline prior provides a behaviorally significant contribution to the VEU evaluation of *every* act.

Furthermore, Grant and Polak maintain a form of ambiguity aversion, which is required for the existence of a supporting hyperplane at certainty; the VEU representation instead allows for arbitrary ambiguity attitudes. Finally, the ambiguity index ρ in [28] is not invariant to sign changes; the VEU adjustment functional A instead satisfies this invariance property, which supports the intuition that adjustments to baseline evaluations reflect outcome variability, or dispersion.

On the other hand, the analysis of VEU preferences provided in this paper does assume translation invariance (cf. Axiom 3.5); however, see §5.2 below.

Decision models that incorporate a reference prior have also been analyzed in environments where the objects of choice either consists of, or include sets of probabilities. In Stinchcombe [53], Gajdos, Tallon and Vergnaud [19] and Gajdos, Hayashi, Tallon and Vergnaud [20], the reference prior is characterized as the Steiner point of the set of probabilities under consideration. In Gajdos, Tallon and Vergnaud [18] and Wang [57], each object of choice explicitly indicates the reference prior. The present paper complements the analysis in these contributions by offering a characterization of a decision model featuring a baseline prior in a fully subjective environment.

Kopylov [36] axiomatizes a special case of maxmin-expected utility preferences, where the characterizing set of priors is generated by ϵ-contamination: that is, it takes the form $\{(1-\epsilon)p+\epsilon q : q \in \Delta\}$, where p serves as a reference prior and Δ is a set of “contaminating” probability measures. While the prior p is endogenously derived, the set Δ must be specified exogenously.

Finally, recall that, for any capacity (non-additive set function) v, the Moebius inverse μ of v, is a set function with the property that $v(E)$ can be obtained as the sum of $\mu(F)$, for all $F \subset E$. The quantity $\mu(F)$ is interpreted as the “weight of evidence” supporting F, independently of its subsets; see Shafer [50] for details. In a somewhat “dual” fashion, the adjustment measures $(m_i)_{0\le i<n}$ in the VEU model represents the interaction patterns among ambiguous events; loosely speaking, it indicates how evidence about one event can be combined with evidence about another.[18]

5.2 Additional Features and Extensions

Probabilistic Sophistication. It is possible to construct examples of non-EU VEU preferences that are probabilistically sophisticated in the sense of Machina and Schmeidler [38]. A precise characterization

[18] I thank Peter Wakker for pointing out this connection.

of probabilistic sophistication for VEU preferences is left for future work; however, Sec. B.10 in the Appendix provides a simple, related result that sheds further light on the central role of baseline probabilities in the VEU model.

Specifically, given a preference relation $\succcurlyeq$ on L_0, define the *induced likelihood ordering* $\succcurlyeq_\ell \subset \Sigma \times \Sigma$ by

$$\forall E, F \in \Sigma, \qquad E \succcurlyeq_\ell F \quad \Leftrightarrow \quad xEy \succcurlyeq xFq \quad \text{for all } x, y \in X \text{ with } x \succ y.$$

Proposition B.15 in Sec. B.10 shows that, if $\succcurlyeq$ is a VEU preference, then the induced likelihood ordering is represented by a (convex-ranged) probability measure μ if and only if μ is the baseline prior for $\succcurlyeq$.

Translation-invariance. Because they satisfy the Weak Certainty Independence axiom 3.5, VEU preferences are invariant to "translation in utility space"; in the language of Grant and Polak [28], they display "constant absolute ambiguity aversion," as do, for instance, MEU, CEU, variational and invariant-biseparable preferences.

It should be emphasized that this is solely a consequence of Axiom 3.5: in particular, the key novel axiom in the characterization of the VEU representation, namely Complementary Independence (Axiom 3.7, is consistent with departures from translation invariance. Consider an "aggregator" function $W : \mathbb{R}^2 \to \mathbb{R}$, strictly increasing in both arguments.[19] Also let u, p, m and A be as in the VEU representation. Then one may consider preferences defined by

$$\forall f, g \in L_0 \qquad f \succcurlyeq g \quad \Leftrightarrow \quad W\left(\int u \circ f\, dp, A\left(\int u \circ f\, dm\right)\right) \geq W\left(\int u \circ g\, dp, A\left(\int u \circ g\, dm\right)\right);$$

thus, the representation considered in this paper corresponds to the aggregator $W(x, y) = x + y$. It is then clear that Axiom 3.7 holds for such preferences: to elaborate, if the acts f and $\bar{f}$ are complementary, $A(\int u \circ f\, dm) = A(\int u \circ \bar{f}\, dm)$, and therefore the ranking of f and $\bar{f}$ is still determined by their baseline expected utilities because W is strictly increasing; this immediately implies the claim.

Therefore, it may be possible to relax Axiom 3.5 to characterize a version of the VEU representation that does not necessarily satisfy "constant absolute ambiguity aversion." Notice that the proposed generalized VEU representation would still feature sign- and translation-invariant *adjustments* $A(\int u \circ f\, dm)$, and hence would be fully consistent with the variability interpretation described in this paper.[20] Such an extension is left to future work.

Finite adjustment tuples. In applications, it is convenient to consider representations featuring a finite number of adjustment tuples. Theorem 3.1 shows that this is without loss of generality, if the state space is finite. For general state spaces, the approach in Siniscalchi [52] provides one way to guarantee by means of behavioral axioms that finitely many adjustment measures are sufficient.

[19] That is, such that $x \geq x'$ and $y > y'$, or $x > x'$ and $y \geq y'$, both imply $W(x, y) > W(x', y')$.

[20] Axiom 3.8 may impose restrictions on the aggregator W. However, recall from Sec. 3 that this axiom may be dropped by ensuring that utility is unbounded, as in Maccheroni, Marinacci and Rustichini [37] or Grant and Polak [28].

A Appendix: Miscellanea

A.1 Adjustment tuples and vector measures

Consider an adjustment tuple $m \in M^n(\Sigma)$, $0 < n \leq \infty$, and let $\hat{m} : \Sigma \to \mathbb{R}^n$ be defined as in Sec. 2.1. By assumption, for every $E \in \Sigma$, there is $N(E) < \infty$ such that $\sup_i |m_i(E)| \leq N(E)$; hence, one can view $\hat{m}$ as a map from Σ to ℓ^n_∞, the set of supnorm-bounded real sequences of n terms (which of course coincides with $\mathbb{R}^n$ if $n < \infty$). Moreover, by the Nicodym boundedness theorem (cf. Dunford and Schwartz [10], §IV.9.8), this implies that $\sup\{|m_i(E)| : 0 \leq i < n, E \in \Sigma\} < \infty$; in turn, this implies that $\hat{m}$ is a vector measure with bounded *semi-variation* (cf. [10], §IV.10.3-4): that is, letting Π denote the set of all finite Σ-measurable partitions of Ω,

$$\|\hat{m}\|(\Omega) \equiv \sup_{\pi \in \Pi,\, \epsilon_1, \epsilon_2, \ldots \in [-1,1]} \sup_{0 \leq i < n} \left| \sum_{A_k \in \pi} \epsilon_k \hat{m}(A_k) \right| < \infty.$$

Integration with vector measures of bounded semivariation can be defined as in the development of the Lebesgue theory. For every simple function $a = \sum_{k=1}^K \alpha_k 1_{E_k} \in B_0(\Sigma)$, define

$$\int a \, d\hat{m} = \sum_{k=1}^K \alpha_k \hat{m}(E_k).$$

Next, for any $a \in B(\Sigma)$, consider a sequence $(a_k)_{k \geq 1} \subset B_0(\Sigma)$ such that $a_k \to a$ in the sup norm; then let

$$\int a \, d\hat{m} = \lim_{k \to \infty} \int a_k \, d\hat{m},$$

where the limit is taken w.r.to the ℓ_∞ norm. It is simple to verify that these definitions are well-posed (for $a \in B_0(\Sigma)$, the integral is the same for any representation of a as a linear combination of indicator functions; and for $a \in B(\Sigma)$, the integral is the same for all approximating sequences of simple functions).

The equality $\int a \, d\hat{m} = \left(\int a \, dm_i\right)_{0 \leq i < n}$ is true by definition for $a \in B_0(\Sigma)$. Furthermore, since $\hat{m}$ has bounded semivariation,

$$\infty > \sup_{\pi \in \Pi(E),\, \epsilon_1, \epsilon_2, \ldots \in [-1,1]} \sup_i \left| \sum_{A_k \in \pi} \epsilon_k m_i(A_k) \right| \geq \sup_{\pi \in \Pi(E),\, \epsilon_1, \epsilon_2, \ldots \in [-1,1]} \left| \sum_{A_k \in \pi} \epsilon_k m_i(A_k) \right| \geq$$

$$\geq \sup_{\pi \in \Pi(E)} \left| \sum_{A_k \in \pi} \mathrm{sgn}(m_i(A_k)) m_i(A_k) \right| = \sup_{\pi \in \Pi(E)} \sum_{A_k \in \pi} |m_i(A_k)|,$$

where $\mathrm{sgn}(x)$ equals $-1, 0, 1$ iff x is negative, zero, or positive respectively, and the last expression is the total variation $v(m_i, \Omega)$ of the scalar, signed measure m_i on the set Ω. Hence, $v(m_i, \Omega) \leq \|\hat{m}\|(\Omega) < \infty$ for all $0 \leq i < n$. Consequently, for every $b \in B(\Sigma)$, $\left|\int b \, dm_i\right| < \|b\| \cdot \|m\|(\Omega)$. But this implies that, if $a_k \to a$ in the supremum norm and every a_k is simple, $\left|\int a_k \, dm_i - \int a \, dm_i\right| \leq \|a_k - a\| \cdot \|\hat{m}\| \to 0$, and hence $\int a_k \, dm_i \to \int a \, dm_i$, uniformly in i, which implies that $\int a \, d\hat{m} = \left(\int a \, dm_i\right)_{0 \leq i < n}$.

A.2 Conditions for Monotonicity

Remark A.1 *If a tuple (u, p, n, m, A) satisfies Conditions 1 and 2 in Def. 2, $n < \infty$, and A is continuous on $R_0(m, u(X))$ and differentiable on $R_0(m, u(X)) \setminus A^{-1}(0)$, then it satisfies Condition 3 if and only if $p(E) + \sum_{0 \leq i < n} \frac{\partial A}{\partial \varphi_i}(\varphi) m_i(E) \geq 0$ for all $\varphi \notin A^{-1}(0)$.*

Proof: It is easy to see that Condition 3 is equivalent to the following requirement: for all $a \in B_0(\Sigma, u(X))$, $E \in \Sigma$ and $\epsilon > 0$ such that $a + \epsilon 1_E \in B_0(\Sigma, u(X))$:

$$\epsilon p(E) + A(\int a\, dm + \epsilon m(E)) - A(\int a\, dm) \geq 0. \tag{16}$$

For any $\varphi \in R_0(m, u(X))$, if $A(\varphi) = 0$ or $\varphi = \int a\, dm$ and $a + 1_E \epsilon \in B_0(\Sigma, u(X))$ for some $\epsilon > 0$, Eq. (16) readily implies the condition in the Remark; if $A(\varphi) \neq 0$, $\varphi = \int a\, dm$, but $a + 1_E \epsilon \notin B_0(\Sigma, u(X))$ for any $\epsilon > 0$, then let $F = \{\omega : a(\omega) = \max\ u(X)\}$; since a takes up finitely many distinct values, it must be the case that $F \neq \emptyset$. In this case, consider the sequence (a_k) given by $a_k = a - 1_F \frac{1}{k}$; for k sufficiently large, $a_k \in B_0(\Sigma, u(X))$, $A(\int a_k\, dm) \neq 0$, and there is $\epsilon_k > 0$ such that $a_k + 1_E \epsilon_k \in B_0(\Sigma, u(X))$. Then $p(E) + \sum_{0 \leq i < n} \frac{\partial A}{\partial \varphi_i}(\int a_k\, dm) m_i(E) \geq 0$ for all large k, and the claim follows by continuity of the partial derivatives $\frac{\partial A}{\partial \varphi_i}$.

Now suppose the condition in the Remark holds, and fix $a, E, \epsilon > 0$ such that $a, a + 1_E \epsilon \in B_0(\Sigma, u(X))$; to simplify the notation, write $\varphi_\eta = \int a\, dm + \eta m(E)$ for all $\eta \in [0, \epsilon]$.

Consider first the case $A(\varphi_0) = 0$. Let $\epsilon_0 = \sup\{\eta \in [0, \epsilon] : A(\varphi_\eta) = 0\}$. If $\epsilon_0 = 0$, then $A(\varphi_\eta)$ is differentiable for all $\eta \in (0, \epsilon)$, and

$$\epsilon p(E) + A(\varphi_\epsilon) - A(\varphi_0) = 0 \cdot p(E) + A(\varphi_0) - A(\varphi_0) + \int_0^\epsilon \left[p(E) + \sum_{0 \leq i < n} \frac{\partial}{\partial \varphi_i} A(\varphi_\eta) m_i(E) \right] d\eta \geq 0, \tag{17}$$

as required. If $\epsilon_0 > 0$, then by continuity $A(\varphi_{\epsilon_0}) = 0 = A(\varphi_0)$, so

$$\epsilon_0 p(E) + A(\varphi_{\epsilon_0}) - A(\varphi_0) = \epsilon_0 p(E) \geq 0. \tag{18}$$

Thus, in particular, if $\epsilon_0 = \epsilon$, Eq. (16) holds. If instead $\epsilon_0 < 1$, then one can repeat the preceding argument with $a' = \int a\, dm + \epsilon_0 1_E$ and $\epsilon' = \epsilon - \epsilon_0$ in lieu of a and ϵ; by assumption $A(\int a'\, dm + \eta m(E)) \neq 0$ for all $\eta \in (0, \epsilon')$, so the argument just given implies that $(\epsilon - \epsilon_0) p(E) + A(\varphi_\epsilon) - A(\varphi_{\epsilon_0}) \geq 0$; together with Eq. (18), this implies that Eq. (16) holds in this case as well.

Consider now the case $A(\varphi_0) > 0$. Let $\epsilon_1 = \sup\{\eta \in [0, \epsilon] : A(\varphi_\eta) \neq 0\}$. By continuity of A, $\epsilon_1 > 0$; thus, integrating on $(0, \epsilon_1)$ as in Eq. (17) yields $\epsilon_1 p(E) + A(\varphi_{\epsilon_1}) - A(\varphi_0) \geq 0$. If $\epsilon_1 = \epsilon$ the proof is complete. Otherwise, note that, by continuity of A, $A(\varphi_{\epsilon_1}) = 0$. Letting $a' = a + \epsilon_1 1_E$ and $\epsilon' = \epsilon - \epsilon_1$ in lieu of a and ϵ, and applying the argument given above yields $(\epsilon - \epsilon_1) p(E) + A(\varphi_\epsilon) - A(\varphi_{\epsilon'}) \geq 0$; together with $\epsilon_1 p(E) + A(\varphi_{\epsilon_1}) - A(\varphi_0) \geq 0$, this implies that Eq. (16) holds. ■

Remark A.2 *If a tuple (u, p, n, m, A) satisfies Conditions 1 and 2 in Def. 2 and A is concave and positively homogeneous, then (u, p, n, m, A) satisfies Condition 3 if and only if $p(E) + A(m(E)) \geq 0$ for all $E \in \Sigma$.*

Proof: Since A is positively homogeneous, it has a unique positively homogeneous extension to $R_0(m, \mathbb{R})$ given by $A(\int \alpha a\, dm) = \alpha A(\int a\, dm)$ for all $\alpha > 0$ and $a \in B_0(\Sigma, u(X))$. Hence, $A(\int a\, dm)$ is well-defined for all $a \in B_0(\Sigma)$, and A is also concave on this domain. It follows that, for all $\varphi, \psi \in R_0(m, \mathbb{R})$, $A(\varphi) = A(\psi + (\varphi - \psi)) = 2A(\frac{1}{2}\psi + \frac{1}{2}(\varphi - \psi)) \geq 2\frac{1}{2}A(\psi) + 2\frac{1}{2}A(\varphi - \psi)$, so $A(\varphi - \psi) \leq A(\varphi) - A(\psi)$.

Now suppose that $p(E) + A(m(E)) \geq 0$ for all $E \in \Sigma$, and consider $a, b \in B_0(\Sigma, \mathbb{R})$ with $a(\omega) \geq b(\omega)$ for all ω. Then $a - b \in B_0(\Sigma, \mathbb{R})$, and since $a(\omega) - b(\omega) \geq 0$ for all ω, concavity and homogeneity, together with linearity and monotonicity of $\int \cdot\, dp$, imply that $\int(a-b)\,dp + A(\int(a-b)\,dm) \geq 0$. But the argument given above implies that $A(\int(a-b)\,dm) \leq A(\int a\,dm) - A(\int b\,dm)$, so $\int a\,dp + A(\int a\,dm) \geq \int b\,dp + A(\int b\,dm)$. The other direction is immediate. ■

A.3 Examples

Proof of Remark 4.2: Fix a non-constant $a \in \mathbb{R}^{n+1}$; since both the functional I and the MEU functional are constant-linear, it is enough to consider vectors a such that $\sum_i a_i = 0$. Now consider the problem $\min_{q \in C} \int a\,dq$; the constraint set can be written as $C = \left\{q \in \mathbb{R}^{n+1}_+ : \sum_i (q_i - p_i)^2 \leq \epsilon^2, \sum_i q_i = 1\right\}$, so the Lagrangian is

$$\sum_i a_i q_i + \lambda\left(\sum_i (q_i - p_i)^2 - \epsilon^2\right) + \mu(1 - \sum_i q_i)$$

with $\lambda \geq 0$, $\mu \in \mathbb{R}$ and $q_i \geq 0$ for all i. Differentiating with respect to q_i and equating to 0 yields $a_i + 2\lambda(q_i - p_i) - \mu = 0$ (the assumption that $p_i > \epsilon$ implies that $q_i > 0$ at the optimum, so the first-order condition must hold with equality). Assuming further that $\lambda > 0$, summing over all i, and invoking the constraint $\sum_i q_i = 1$ yields $\mu = 0$, and so $q_i = p_i - \frac{1}{2\lambda} a_i$. Now using the other constraint (which must hold with equality by standard arguments) yields $\sum_i \frac{1}{4\lambda^2} a_i^2 = \epsilon^2$, so $2\lambda = \frac{1}{\epsilon}\sqrt{\sum_i a_i^2}$, which is indeed strictly positive because a is non-constant; thus, finally

$$q_i = p_i - \epsilon \frac{a_i}{\sqrt{\sum_{j=0}^n a_j^2}} \quad \Longrightarrow \quad \sum_{i=0}^n a_i q_i = \sum_{i=0}^n a_i p_i - \epsilon \sum_{i=0}^n a_i \cdot \frac{a_i}{\sqrt{\sum_{j=0}^n a_j^2}} = \sum_{i=0}^n a_i p_i - \epsilon \sqrt{\sum_{i=0}^n a_i^2} = I(a). \blacksquare$$

B Appendix: Proofs

B.1 Additional Notation and Preliminaries on Niveloids

Throughout this Appendix, if Ω is endowed with a topology, the set of continuous real functions on Ω will be denoted by $C(\Omega)$. Furthermore, $ba(\Sigma)$ and $ba_1(\Sigma)$ indicate, respectively, the set of finitely additive measures and the set of charges (finitely additive probabilities) on (Ω, Σ); recall that $ba(\Sigma)$ is isometrically isomorphic to the norm dual of $B_0(\Sigma)$ and $B(\Sigma)$, and similarly, if Ω is a compact metric space, $ca(\Sigma)$ is isometrically isomorphic to the norm dual of $C(\Omega)$. Recall that the $\sigma(ba(\Sigma, B(\Sigma))$ and $\sigma(ba(\Sigma), B_0(\Sigma))$ topologies coincide on $ba_1(\Sigma)$, the set of probability charge; they are referred to as the weak* topology.

Furthermore, if $\Gamma \subset \mathbb{R}$ is a non-empty, non-singleton interval, denote by $B_0(\Sigma, \Gamma)$, $B(\Sigma, \Gamma)$ and $C(\Sigma, \Gamma)$ the restrictions of $B_0(\Sigma)$, $B(\Sigma)$ and $C(\Sigma)$ to functions taking values in Γ. Then the weak* topology on $ba_1(\Sigma)$ also coincides with the $\sigma(ba(\Sigma), B_0(\Sigma, \Gamma))$ and $\sigma(ba(\Sigma), B(\Sigma, \Gamma))$ topologies.

The indicator function of an event $E \in \Sigma$ will be denoted by 1_E. Inequalities between two elements a, b of $B_0(\Sigma)$, $B(\Sigma)$ or $C(\Omega)$ are interpreted pointwise: $a \geq b$ means that $a(\omega) \geq b(\omega)$ for all $\omega \in \Omega$.

Let $\Phi \subset B(\Sigma)$ be convex. A functional $I : \Phi \to \mathbb{R}$ is a *niveloid* iff $I(a) - I(b) \leq \sup(a - b)$ for all $a, b \in \Phi$; it is *normalized* if $I(\gamma 1_\Omega) = \gamma$ for all $\gamma \in \mathbb{R}$ such that $\gamma 1_\Omega \in \Phi$; *monotonic* iff, for all $a, b \in \Phi$, $a \geq b$ implies $I(a) \geq I(b)$; *constant-mixture invariant* iff, for all $a \in \Phi$, $\alpha \in (0,1)$, and $\gamma \in \mathbb{R}$ with $\gamma 1_\Omega \in \Phi$, $I(\alpha a + (1-\alpha)\gamma) = I(\alpha a) + (1-\alpha)\gamma$;

vertically invariant iff $I(a+\gamma)=I(a)+\gamma$ for all $a\in\Phi$ and $\gamma\in\mathbb{R}$ such that $a+\gamma\in\Phi$; and *affine* iff, for all $a,b\in\Phi$ and $\alpha\in(0,1)$, $I(\alpha a+(1-\alpha)b)=\alpha I(a)+(1-\alpha)I(b)$. Maccheroni, Marinacci and Rustichini [37] (MMR henceforth) demonstrated the usefulness of niveloids in decision theory, and established certain useful results reviewed below.

If $\Phi=B_0(\Sigma)$ or $\Phi=B(\Sigma)$, then a functional $I:\Phi\to\mathbb{R}$ is *positively homogeneous* iff, for all $a\in\Phi$ and $\alpha\geq 0$, $I(\alpha a)=\alpha I(a)$; *c-additive* iff $I(a+\alpha)=I(a)+\alpha$ for all $\alpha\in\mathbb{R}_+$ and $a\in\Phi$; *additive* iff $I(a+b)=I(a)+I(b)$ for all $a,b\in\Phi$; *c-linear* iff it is c-additive and positively homogeneous; and *linear* iff it is additive and positively homogeneous.

The following useful results on niveloids are due to or reviewed in MMR. In particular, item 6 provides a first representation for preferences satisfying the basic axioms considered here, except for the symmetry requirements.

Proposition B.1 (MMR) *Let Γ be an interval such that $0\in\mathrm{int}(\Gamma)$ and $I:B_0(\Sigma,K)\to\mathbb{R}$.*

1. *If I is a niveloid, it is is supnorm, hence Lipschitz continuous.*
2. *If $I:B_0(\Sigma,K)\to\mathbb{R}$ is a niveloid, then it has a (minimal) niveloidal extension to $B(\Sigma)$.*
3. *I is a niveloid iff it is monotonic and constant-mixture invariant.*
4. *If I is constant-mixture invariant, then it is vertically invariant.*
5. *If I is vertically invariant, then it has a unique, vertically invariant extension $\hat{I}$ to $B_0(\Sigma,\Gamma)+\mathbb{R}\equiv\{a+1_\Omega\gamma : a\in B_0(\Sigma,\Gamma),\gamma\in\Gamma\}$.*
6. *A preference $\succcurlyeq$ on L_0 satisfies Axioms 3.1–3.4 if and only if there is a non-constant, affine function $u:X\to\mathbb{R}$ and a normalized niveloid $I:B_0(\Sigma,u(X))\to\mathbb{R}$ such that $f\succcurlyeq g$ iff $I(u\circ f)\geq I(u\circ g)$.*

The following uniqueness result is straightforward:

Corollary B.2 *If I,u and I',u' provide two representations of $\succcurlyeq$ as per the last point of Prop. B.1, then $u'=\alpha u+\beta$ (with $\alpha>0$) and $I'(\alpha a)=\alpha I(a)$ for all $a\in B(\Sigma,u(X))$.*

Proof: Since I and I' are normalized, standard results imply that $u'=\alpha u+\beta$ for some $\alpha>0$ and $\beta\in\mathbb{R}$. Next, for every $a\in B(\Sigma,\Gamma)$, let $f\in L_0$ be such that $u\circ f=a$ and $x\sim f$: thus, since I and I' are normalized, $u(x)=I(u\circ f)=I(a)$ and similarly $u'(x)=I'(u'\circ f)$, i.e. $\alpha u(x)+\beta=I'(\alpha u\circ f+\beta)$, so $\alpha u(x)=I'(\alpha u\circ f)$ by vertical invariance [the requirement that $\alpha u\circ f+\beta\in B_0(\Sigma,u'(X))$ is trivially satisfied, as $\alpha u\circ f+\beta=u'\circ f\in B_0(\Sigma,u'(X))$]. But then $\alpha I(a)=I'(\alpha a)$, as required. [Note that this is consistent with normalization: $\alpha I(\gamma 1_\Omega)=\alpha\gamma$ and $I'(\alpha\gamma 1_\Omega)=\alpha\gamma$.] ■

B.2 A generalized α-MEU representation

This subsection extends the characterization results of Ghirardato, Maccheroni and Marinacci [22] (GMM henceforth) to allow for the weakened c-Independence axiom adopted here (Axiom 3). This entails replicating and often modifying their arguments and conclusions, so as not to rely upon positive homogeneity of I. The main objective is to represent a normalized niveloid $I : B_0(\Sigma,\Gamma)$ in the form

$$I(a) = \gamma(a) \min_{q\in C} \int a\,dq + [1-\gamma(a)] \max_{q\in C} \int a\,dq, \tag{19}$$

where $C \subset ba_1(\Sigma)$ is weak* closed and convex, and $\gamma : B_0(\Sigma,\Gamma) \to [0,1]$ is such that $\gamma(a) = \gamma(b)$ whenever there is $\delta \in \mathbb{R}$ such that $\int a\,dq = \int b\,dq + \delta$ for all $q \in C$. As the construction carried out below demonstrates, the set C can be viewed as providing a representation of "unambiguous preferences," as is the case in the original setting adopted by GMM. Also, while C can no longer be identified with the Clarke differential of I at 0, its support functional has a similar interpretation as the Clarke (lower) derivative of I in GMM. This turns out to be sufficient for the purposes of constructing the VEU representation.

The first step in this construction is to define and characterize an "unambiguous" ordering on utility profiles. Note that GMM first define an "unambiguous preference relation" $\succcurlyeq^*$ on acts, and then translate that into an ordering over their utility profiles; since this paper does not focus on the interpretation of $\succcurlyeq^*$, a more direct route is taken; the techniques, however, are similar.

Lemma B.3 *Let $\Gamma \subset \mathbb{R}$ be a non-singleton interval and $I : B_0(\Sigma,\Gamma) \to \mathbb{R}$ be a niveloid. Define a binary relation $\succeq$ on $B_0(\Sigma,\Gamma)$ by*

$$\forall a,b \in B_0(\Sigma,\Gamma), \qquad a \succeq b \quad \Leftrightarrow \quad \forall \alpha \in (0,1], c \in B_0(\Sigma,\Gamma) : I(\alpha a + (1-\alpha)c) \geq I(\alpha b + (1-\alpha)c).$$

Then there is a unique, weak compact and convex set $C \subset ba_1(\Sigma)$ such that $a \succeq b$ iff $\int a\,dq \geq \int b\,dq$ for all $q \in C$.*

Note: write "$a \simeq b$" for "$a \succeq b$ and $b \succeq a$."

Proof: It will be shown that $\succeq$ is a monotonic, conic, continuous and non-trivial preorder; the result then follows from Proposition A.2 in GMM. The arguments closely mimic Prop. 4 in GMM.

$\succeq$ is monotonic: if $a(\omega) \geq b(\omega)$ for all ω, then also $\alpha a(\omega) + (1-\alpha)c(\omega) \geq \alpha b(\omega) + (1-\alpha)c(\omega)$ for all $\alpha \in (0,1]$. Since I is monotonic, $I(\alpha a + (1-\alpha)c) \geq I(\alpha b + (1-\alpha)c)$, i.e. $a \succeq b$.

$\succeq$ is reflexive: follows from monotonicity.

$\succeq$ is transitive: if $a \succeq b$ and $b \succeq c$, then for all $\alpha \in (0,1]$ and all d, $I(\alpha a + (1-\alpha)d) \geq I(\alpha b + (1-\alpha)d) \geq I(\alpha c + (1-\alpha)d)$, so $a \succeq c$.

$\succeq$ is conic (i.e. independent): if $\alpha \in (0,1)$, then, for all $\beta \in (0,1]$, note that $\beta[\alpha a + (1-\alpha)c] + (1-\beta)d = \beta\alpha a + (1-\beta\alpha)[\frac{\beta(1-\alpha)}{1-\beta\alpha}c + \frac{1-\beta}{1-\beta\alpha}d]$ and similarly for b. Thus, $a \succeq b$ implies, in particular, that

$$I(\beta[\alpha a + (1-\alpha)c] + (1-\beta)d) = I\left(\beta\alpha a + (1-\beta\alpha)\left[\frac{\beta(1-\alpha)}{1-\beta\alpha}c + \frac{1-\beta}{1-\beta\alpha}d\right]\right) \geq$$
$$\geq I\left(\beta\alpha b + (1-\beta\alpha)\left[\frac{\beta(1-\alpha)}{1-\beta\alpha}c + \frac{1-\beta}{1-\beta\alpha}d\right]\right) = I(\beta[\alpha b + (1-\alpha)c] + (1-\beta)d)$$

for all $\beta \in (0,1]$, so $\alpha a + (1-\alpha)c \succeq \alpha b + (1-\alpha)c$. The case $\alpha = 1$ is trivial.

$\succeq$ is continuous: if $a^n \to a$ and $b^n \to b$ in $B_0(\Sigma,\Gamma)$, then for every α and c, $I(\alpha a^n + (1-\alpha)c) \to I(\alpha a + (1-\alpha)c)$ and similarly for b because I is supnorm-continuous, and the claim follows.

Finally, $\succeq$ is nontrivial: consider $\gamma,\gamma' \in \Gamma$ such that $\gamma > \gamma'$; by monotonicity, $\gamma \succeq \gamma'$, and if also $\gamma \preceq \gamma'$, then $I(\alpha\gamma+(1-\alpha)c) \le I(\alpha\gamma'+(1-\alpha)c)$ for all $c \in B_0(\Sigma,\Gamma)$ and $\alpha \in (0,1]$: for $\alpha=1$, one obtains $I(\gamma) \le I(\gamma')$, but by vertical invariance $I(\gamma)=I(\gamma'+(\gamma-\gamma'))=I(\gamma')+\gamma-\gamma' > I(\gamma')$: contradiction. Thus, $\gamma \succeq \gamma'$ but not $\gamma' \succeq \gamma$, as required. ■

Recall that a niveloid $I : B_0(\Sigma,\Gamma) \to \mathbb{R}$ has a minimal extension to a niveloid $\hat{I} : B(\Sigma) \to \mathbb{R}$. The next Lemma shows that (1) the extension is unique on $B(\Sigma,\Gamma)$, and (2) if $\succeq$ is extended to $B(\Sigma,\Gamma)$, then it is represented by the set C as in Lemma B.3.

Lemma B.4 *Assume that $0 \in \text{int}(\Gamma)$. Then a niveloid $I : B_0(\Sigma,\Gamma) \to \mathbb{R}$ has a unique niveloidal extension to $B(\Sigma,\Gamma)$, denoted by $\hat{I}$. Furthermore, define the relation $\hat{\succeq}$ on $B(\Omega,\Sigma)$ by*

$$\forall a,b \in B(\Sigma,\Gamma), \qquad a \hat{\succeq} b \quad \Leftrightarrow \quad \forall \alpha \in (0,1], c \in B(\Sigma,\Gamma) : \hat{I}(\alpha a+(1-\alpha)c) \ge \hat{I}(\alpha b+(1-\alpha)c).$$

Then $\hat{\succeq}$ extends $\succeq$ in Lemma B.3, and $a \hat{\succeq} b$ iff $\int a\,dq \ge \int b\,dq$ for all $q \in C$, where C is as in that Lemma.

Proof: Let $\tilde{I}$ be the (minimal) niveloidal extension of I to $B(\Sigma)$; its restriction to $B(\Sigma,\Gamma)$, denoted $\hat{I}$, is an extension of I to $B(\Sigma,\Gamma)$ (i.e. it is a niveloid on the latter set). Furthermore, suppose there is another extension $I' : B(\Sigma,\Gamma) \to \mathbb{R}$ of I, and take $a \in B(\Sigma,\Gamma)$. There is a sequence $(a^k) \subset B_0(\Sigma,\Gamma)$ such that $a^k \to a$; since both $\hat{I}$ and I' are supnorm continuous, $\hat{I}(a)=\lim_k \hat{I}(a^k)=\lim_k I'(a^k)=I'(a)$; thus, the niveloidal extension of I to $B(\Sigma,\Gamma)$ is unique.

Turn to the relation $\hat{\succeq}$. Note first that, if $\hat{I}(\lambda a+(1-\lambda)c) \ge \hat{I}(\lambda b+(1-\lambda)c)$ for all $c \in B_0(\Sigma,\Gamma)$, then this is true also for all $c \in B(\Sigma,\Gamma)$, because $\hat{I}$ is continuous. This implies that $\hat{\succeq}$ extends $\succeq$.

Suppose that $a,b \in B(\Sigma,\Gamma)$ and $a \hat{\succeq} b$, so $\hat{I}(\lambda a+(1-\lambda)c) \ge \hat{I}(\lambda b+(1-\lambda)c)$ for all $\lambda \in (0,1]$ and $c \in B_0(\Sigma,\Gamma)$. Now note that, for every $c \in B_0(\Sigma,\Gamma)$ and $\lambda \in (0,1]$, $\lambda(\frac{1}{2}a)+(1-\lambda)c = \frac{1}{2}\lambda a+(1-\frac{1}{2}\lambda)\frac{1-\lambda}{1-\frac{1}{2}\lambda}c$, and $\frac{1-\lambda}{1-\frac{1}{2}\lambda}c \in B_0(\Sigma,\Gamma)$; similarly for b. Therefore, $\frac{1}{2}a \hat{\succeq} \frac{1}{2}b$. Also, since $0 \in \text{int}(\Gamma)$, $\sup \frac{1}{2}a(\Omega) < \sup \Gamma$ and $\inf \frac{1}{2}b(\Omega) > \inf \Gamma$, so there are sequences $(a^k),(b^k) \subset B_0(\Sigma,\Gamma)$ such that $a^k \downarrow \frac{1}{2}a$ and $b^k \uparrow \frac{1}{2}b$;[21] for such sequences, by monotonicity of $\hat{I}$, one has $I(\lambda a^k+(1-\lambda)c)=\hat{I}(\lambda a^k+(1-\lambda)c) \ge \hat{I}(\lambda\frac{1}{2}a+(1-\lambda)c) \ge \hat{I}(\lambda\frac{1}{2}b+(1-\lambda)c) \ge \hat{I}(\lambda b^k+(1-\lambda)c)=I(\lambda b^k+(1-\lambda)c)$ for all $\lambda \in (0,1]$ and $c \in B_0(\Sigma,\Gamma)$, and therefore $\int a^k\,dq \ge \int b^k\,dq$ for all $q \in C$. Thus, also $\int \frac{1}{2}a\,dq \ge \int \frac{1}{2}b\,dq$, i.e. $\int a\,dq \ge \int b\,dq$ for all $q \in C$.

Conversely, suppose that $\int a\,dq \ge \int b\,dq$ for all $q \in C$. Fix $\alpha \in (0,1)$; clearly, $\int(\alpha a)\,dq \ge \int(\alpha b)\,dq$ for all $q \in C$, and $\sup\ \alpha a, \inf\ \alpha b \in (\inf \Gamma, \sup \Gamma)$. Consider sequences $(a^k),(b^k)$ such that $a^k \downarrow \alpha a$ and $b^k \uparrow \alpha b$. Then $\int a^k\,dq \ge \int b^k\,dq$ for all $q \in C$, so $\hat{I}(\lambda a^k+(1-\lambda)c)=I(\lambda a^k+(1-\lambda)c) \ge I(\lambda b^k+(1-\lambda)c)=\hat{I}(\lambda b^k+(1-\lambda)c)$ for all λ and $c \in B_0(\Sigma,\Gamma)$, and by continuity $\hat{I}(\lambda(\alpha a)+(1-\lambda)c) \ge \hat{I}(\lambda(\alpha b)+(1-\lambda)c)$. Letting $\alpha \to 1$ and again invoking continuity shows that $a \succeq b$. ■

The following result identifies a useful vertical-invariance property relating $\hat{I}$ and the set C.

Lemma B.5 *In the setting of Lemma B.4, if $a,b \in B(\Sigma,\Gamma)$ and, for some $\delta \in \mathbb{R}$, $\int a\,dq=\int b\,dq+\delta$ for all $q \in C$, then $\hat{I}(a)=\hat{I}(b)+\delta$.*

[21]Suppose $\inf\ c(\Omega) > \inf\ \Gamma$ and $\sup c(\Omega) < \sup\ \Gamma$. For each $k=0,1,\dots$ and $\ell=0,\dots,2^k$, let $\gamma_{\ell,k}=\inf\ c(\Omega)+\frac{\ell}{2^k}[\sup\ c(\Omega)-\inf\ c(\Omega)]$; then let $a^k(\omega)=\min_{\ell=0,\dots,k}\{\gamma_{\ell,k} : c(\omega) \le \gamma_{\ell,k}\}$ and $b^k(\omega)=\max_{\ell=0,\dots,k}\{\gamma_{\ell,k} : c(\omega) \ge \gamma_{\ell,k}\}$. Then $a^k \downarrow c$ and $b^k \uparrow c$.

Proof: Assume first that $\inf\ b(\Omega), \sup\ b(\Omega) \in \text{int}(\Gamma)$. Then there exists $\alpha \in (0,1)$ such that $b + \alpha\delta \in B(\Sigma,\Gamma)$. For all $k \geq 0$, let $a^k = [1-(1-\alpha)^k]a + (1-\alpha)^k b$. Then $a^k \in B(\Sigma,\Gamma)$ for all $k \geq 0$; furthermore,

$$(1-\alpha)a^k + \alpha a = (1-\alpha)[1-(1-\alpha)^k]a + (1-\alpha)^{k+1}b + \alpha a = [1-(1-\alpha)^{k+1}]a + (1-\alpha)^{k+1}b = a^{k+1}.$$

Claim: for all k, $a^k + \alpha(1-\alpha)^k\delta \in B(\Sigma,\Gamma)$ and $a^{k+1} \hat{\simeq} a^k + \alpha(1-\alpha)^k\delta$.

Proof: For $k=0$, $a^0 + \alpha(1-\alpha)^0\delta = b + \alpha\delta \in B(\Sigma,\Gamma)$ by the choice of δ; furthermore, for all $q \in C$, $\int a^1\, dq = \int [(1-\alpha)b + \alpha a]\, dq = (1-\alpha)\int b\, dq + \alpha \int a\, dq = (1-\alpha)\int b\, dq + \alpha\int b\, dq + \alpha\delta = \int b\, dq + \alpha\delta = \int [a^0 + \alpha(1-\alpha)^0\delta]\, dq$, so $a^1 \simeq a^0 + \alpha(1-\alpha)^0\delta$. By induction, for $k > 0$,

$$(1-\alpha)[a^{k-1} + \alpha(1-\alpha)^{k-1}\delta] + \alpha a = (1-\alpha)a^{k-1} + \alpha a + \alpha(1-\alpha)^k\delta = a^k + \alpha(1-\alpha)^k\delta;$$

thus, $a^k + \alpha(1-\alpha)^k\delta \in B(\Sigma,\Gamma)$ because $a, a^{k-1} + \alpha(1-\alpha)^{k-1}\delta \in B(\Sigma,\Gamma)$; furthermore, if $a^k \hat{\simeq} a^{k-1} + \alpha(1-\alpha)^{k-1}\delta$, then also

$$a^{k+1} = (1-\alpha)a^k + \alpha a \hat{\simeq} (1-\alpha)[a^{k-1} + \alpha(1-\alpha)^{k-1}\delta] + \alpha a = a^k + \alpha(1-\alpha)^k\delta$$

because $\hat{\simeq}$ is a conic preorder.[22]

The claim implies that, for all $k \geq 1$, $\hat{I}(a^k) = \hat{I}(a^{k-1} + \alpha(1-\alpha)^{k-1}\delta) = \hat{I}(a^{k-1}) + \alpha(1-\alpha)^{k-1}\delta$, where the second equality follows from vertical invariance; thus,

$$\hat{I}(a^k) = \hat{I}(b) + \alpha\delta\sum_{\ell=0}^{k-1}(1-\alpha)^\ell = \hat{I}(b) + \alpha\delta\frac{1-(1-\alpha)^k}{\alpha} = \hat{I}(b) + \delta[1-(1-\alpha)^k].$$

Since $a^k \to a$ and $\hat{I}$ is continuous, the result follows.

If b is arbitrary, for $k \geq 0$, let $a^k = \frac{k}{k+1}a$ and $b^k = \frac{k}{k+1}b$, so in particular $b^k(\Omega) \subset \text{int}(\Gamma)$; furthermore, for every $k \geq 0$ and $q \in C$, $\int a^k\, dq = \frac{k}{k+1}\int a\, dq = \frac{k}{k+1}\int b\, dq + \frac{k}{k+1}\delta = \int b^k\, dq + \frac{k}{k+1}\delta$, and it has just been shown that then $\hat{I}(a^k) = \hat{I}(b^k) + \frac{k}{k+1}\delta$. Since $a^k \to a$ and $b^k \to b$, continuity implies that $\hat{I}(a) = \hat{I}(b) + \delta$. ■

With $I, \hat{I}$ and C as in Lemmata B.3 and B.4, define, for every $a \in B(\Sigma,\Gamma)$:

$$\begin{aligned}
C(a) &= \left\{\int a\, dq : q \in C\right\}, \quad C_{\min}(a) = \min C(a), \quad C_{\max}(a) = \max C(a) \\
I_0(a) &= \inf_{b \in B_0(\Sigma,\Gamma), \lambda \in (0,1]} \frac{1}{\lambda}\left[\hat{I}(\lambda a + (1-\lambda)b) - I((1-\lambda)b)\right] \\
\hat{I}_0(a) &= \inf_{b \in B(\Sigma,\Gamma), \lambda \in (0,1]} \frac{1}{\lambda}\left[\hat{I}(\lambda a + (1-\lambda)b) - \hat{I}((1-\lambda)b)\right] \\
I^0(a) &= \sup_{b \in B_0(\Sigma,\Gamma), \lambda \in (0,1]} \frac{1}{\lambda}\left[\hat{I}(\lambda a + (1-\lambda)b) - I((1-\lambda)b)\right] \\
\hat{I}^0(a) &= \sup_{b \in B(\Sigma,\Gamma), \lambda \in (0,1]} \frac{1}{\lambda}\left[\hat{I}(\lambda a + (1-\lambda)b) - \hat{I}((1-\lambda)b)\right].
\end{aligned}$$

Clearly (take $\lambda = 1$) $I_0(a) \leq I(a) \leq I^0(a)$; also, if I is positive homogeneous, then $I_0(a) = \inf_{b \in B_0(\Sigma,\Gamma)} \hat{I}(a+b) - I(a)$ etc., which, as shown in GMM, is the (lower) Clarke derivative of $\hat{I}$ at 0. Finally, notice that I_0 and $\hat{I}_0$ differ in that the "perturbation" b is chosen from $B_0(\Sigma,\Gamma)$ and $B(\Sigma,\Gamma)$ respectively.

The following Lemma provides the sought-after GMM-type representation of $\hat{I}$.

[22] The proof is identical to that given in Lemma B.3; alternatively, it follows from the representation provided in Lemma B.4.

Lemma B.6 *Let $I, \hat{I}$ and C be as in Lemmata B.3 and B.4. For all $a \in B(\Sigma,\Gamma)$, $C_{\min}(a) = I_0(a) = \hat{I}_0(a)$ and $C_{\max}(a) = I^0(a) = \hat{I}^0(a)$. Furthermore, there is a function $\gamma : B(\Sigma,\Gamma) \to \mathbb{R}$ such that*

$$\forall a \in B(\Sigma,\Gamma), \qquad \hat{I}(a) = \gamma(a) C_{\min}(a) + [1-\gamma(a)] C_{\max}(a), \tag{20}$$

where $\gamma(a)$ is uniquely defined whenever $C_{\min}(a) < C_{\max}(a)$. If $a, b \in B(\Sigma,\Gamma)$ and $\delta \in \mathbb{R}$ are such that $\int a\, dq = \int b\, dq + \delta$ and $C_{\min}(a) < C_{\max}(a)$, then $\gamma(a) = \gamma(b)$.

Proof: Observe first that, for every $\epsilon > 0$, if $\frac{1}{\lambda}[\hat{I}(\lambda a + (1-\lambda)b) - \hat{I}((1-\lambda)b) \in [\hat{I}_0(a), \hat{I}_0(a)+\epsilon)$ for some $b \in B(\Sigma,\Omega)$, then there is $b' \in B_0(\Sigma,\Omega)$ such that $\frac{1}{\lambda}[\hat{I}(\lambda a + (1-\lambda)b') - \hat{I}((1-\lambda)b') \in [\hat{I}_0(a), \hat{I}_0(a)+\epsilon)$ as well; thus, $I_0 = \hat{I}_0$, and similarly $I^0 = \hat{I}^0$.

Observe first that $I_0, C_{\min}, I^0, C_{\max}$ are monotonic functionals. Now assume that $\inf\ a(\Omega, \sup\ a(\Omega) \in \text{int}(\Gamma)$. Then, by monotonicity, $C_{\min}(a), C_{\max}(a) \in \Gamma$, and mimicking GMM's Lemma B.4, observe that $C_{\min}(a) \preceq a$: hence, for all $\lambda \in (0,1]$ and all $b \in B_0(\Sigma,\Gamma)$, $\hat{I}(\lambda a + (1-\lambda)b) \geq \hat{I}(\lambda C_{\min}(a) + (1-\lambda)b) = \lambda C_{\min}(a) + \hat{I}((1-\lambda)b)$, where the equality follows by vertical invariance, and this implies that $C_{\min}(a) \leq I_0(a)$.

Conversely, by definition $I_0(a) \leq \frac{1}{\lambda}[I(\lambda a + (1-\lambda)b) - I((1-\lambda)b)]$ for all $\lambda \in (0,1]$ and $b \in B_0(\Sigma,\Gamma)$, i.e. $\lambda I_0(a) + I((1-\lambda)b) \leq I(\lambda a + (1-\lambda)b)$, i.e. $I(\lambda I_0(a) + (1-\lambda)b) \leq I(\lambda a + (1-\lambda)b)$ by vertical invariance. By monotonicity of I_0, $I_0(a) \in \Gamma$, so $I_0(a) \preceq a$, which implies that $I_0(a) \leq \int a\, dq$ for all $q \in C$, and hence that $I_0(a) \leq C_{\min}(a)$.

Now note that $C_{\min}$ is positively homogeneous; furthermore, $\alpha I_0(a) \leq I_0(\alpha a)$ for all $\alpha \in (0,1)$ and $a \in B(\Sigma,\Gamma)$. To see this, suppose that, for some $\alpha \in (0,1)$, $\epsilon > 0$, $\lambda \in (0,1]$ and $b \in B_0(\Sigma,\Gamma)$ are such that $\frac{1}{\lambda}[\hat{I}(\lambda(\alpha a) + (1-\lambda)b) - I((1-\lambda)b)] \leq I_0(\alpha a) + \epsilon$; but $b' = \frac{1-\lambda}{1-\lambda\alpha} b \in B(\Sigma,\Gamma)$, and so $I_0(a) \leq \frac{1}{\lambda\alpha}[\hat{I}(\lambda\alpha a + (1-\lambda\alpha)b') - I((1-\lambda\alpha)b')] = \frac{1}{\lambda\alpha}[\hat{I}(\lambda(\alpha a) + (1-\lambda)b) - I((1-\lambda)b)] \leq \frac{1}{\alpha}[I_0(\alpha a) + \epsilon]$. Thus, $\alpha I_0(a) \leq I_0(\alpha a)$, as claimed. Finally, for an arbitrary $a \in B(\Sigma,\Gamma)$, let $a^k = \frac{k}{k+1} a$ for $k \geq 0$; then $\inf\ a^k(\Omega), \sup\ a^k(\Omega) \in \Gamma$, so $I_0(a^k) = C_{\min}(a^k)$ for all k; by the claim just proved, for $k \geq 1$, $I_0(a) \leq \frac{k+1}{k} I_0(a^k) = \frac{k+1}{k} C_{\min}(a^k) = C_{\min}(a)$. Suppose the inequality is strict; then there is $b \in B_0(\Sigma,\Gamma)$ and $\lambda \in (0,1]$ such that $\frac{1}{\lambda}[\hat{I}(\lambda a + (1-\lambda)b) - I((1-\lambda)b] = C_{\min}(a) - \epsilon$ for some $\epsilon > 0$. Let k be such that $|C_{\min}(a) - C_{\min}(a^k)| < \frac{\epsilon}{2}$ and

$$\left| \frac{1}{\lambda}[\hat{I}(\lambda a + (1-\lambda)b) - I((1-\lambda)b] - \frac{1}{\lambda}[\hat{I}(\lambda a^k + (1-\lambda)b) - I((1-\lambda)b] \right| < \frac{\epsilon}{2}.$$

Then $C_{\min}(a^k) = I_0(a^k) \leq \frac{1}{\lambda}[\hat{I}(\lambda a^k + (1-\lambda)b) - I((1-\lambda)b] < C_{\min}(a^k)$, a contradiction. Thus, $I_0(a) = C_{\min}(a)$ for all $a \in B(\Sigma,\Gamma)$. The argument for I^0 and $C_{\max}$ is analogous.

Turn now to the representation of $\hat{I}$. Let $\gamma(a) = \frac{\hat{I}(a) - C_{\max}(a)}{C_{\min}(a) - C_{\max}(a)}$ if $C_{\min}(a) < C_{\max}(a)$, and define $\gamma(a)$ arbitrary otherwise. Then it is clear that Eq. (20) obtains. Furthermore, if a, b, δ are as in the statement, then $C_{\min}(a) = C_{\min}(b) + \delta$ and $C_{\max}(a) = C_{\max}(b) + \delta$; furthermore, by Lemma B.5, $\hat{I}(a) = \hat{I}(b) + \delta$. Thus,

$$\gamma(a) = \frac{\hat{I}(a) - C_{\max}(a)}{C_{\min}(a) - C_{\max}(a)} = \frac{\hat{I}(b) + \delta - C_{\max}(b) - \delta}{C_{\min}(b) + \delta - C_{\max}(b) - \delta} = \frac{\hat{I}(b) - C_{\max}(b)}{C_{\min}(b) - C_{\max}(b)} = \gamma(b),$$

as required. ■

Finally, a characterization of crisp acts (cf. §2.2) is provided.

Lemma B.7 *Consider a preference $\succcurlyeq$ that has a niveloidal representation (I, u), with $0 \in \text{int}(u(x))$. For any act $f \in L_0$, the following are equivalent:*

(1) f is crisp;

(2) $u \circ f \simeq I(u \circ f)$;

(3) $\min_{q \in C} \int u \circ f \, dq = \max_{q \in C} \int u \circ f \, dq$, where C is as in Lemma B.3.

Proof: (1) ⇒ (2): by assumption, $f \sim x$ implies $\lambda f + (1-\lambda) g \sim \lambda x + (1-\lambda) g$ for all $g \in L_0$ and $\lambda \in [0,1]$; that is, $I(\lambda u \circ f + (1-\lambda) u \circ g) = I(\lambda u(x) + (1-\lambda) u \circ g)$ for all $\lambda \in [0,1]$. Since $f \sim x$ implies $I(u \circ f) = u(x)$, the claim follows.

(2) ⇒ (1): by Lemma B.3, $u \circ f \simeq I(u \circ f)$ implies that $\int u \circ f \, dq = I(u \circ f)$ for all $q \in C$, and the claim follows.

(3) ⇒ (1): by Lemma B.6, $C_{\min}(u \circ f) \le I(u \circ f) \le C_{\max}(u \circ f)$; by assumption, $C_{\min}(u \circ f) = C_{\max}(u \circ f)$, so $I(u \circ f) = \int u \circ f \, dq$ for all $q \in C$. Thus, if $x \sim f$, then $u(x) = \int u \circ f \, dq$ for all $q \in C$. But this implies that $u \circ f \simeq u(x)$, i.e. for all $g \in L_0$ and $\lambda \in [0,1]$, $I(\lambda u \circ f + (1-\lambda) u \circ g) = I(\lambda u(x) + (1-\lambda) u \circ g)$, i.e. $\lambda f + (1-\lambda) g \sim \lambda x + (1-\lambda) g$. Hence, f is crisp. ■

B.3 Functional Characterizations of Complementary Independence

This subsection provides the key steps in the characterization of VEU preferences. The starting point is the "niveloidal representation" of $\succcurlyeq$ provided by Part 6, which was shown to also have a GMM-type formulation in §B.2. It will first be shown that Axioms 3.8 and 3.7 hold if and only if a "baseline linear functional" J can be defined. Then, the linearity of the functional J is related to the GMM-type representation of I: specifically, it is shown to correspond to (1) symmetry of the set C of probabilities in Eq. (19) around the probability p that represents J, and (2) a symmetry property of the weight function $\gamma(\cdot)$.

Lemma B.8 *Let $\succcurlyeq$ be represented by I, u as in Prop. B.1, and assume wlog that $0 \in \text{int}(u(X))$. Define a functional $J : B_0(\Sigma, u(X)) \to \mathbb{R}$ by letting, for all $a \in B_0(\Sigma, u(X))$ and $\gamma \in \mathbb{R}$ with $\gamma - a \in B_0(\Sigma, u(X))$, $J(a) = \frac{1}{2}\gamma + \frac{1}{2} I(a) - \frac{1}{2} I(\gamma - a)$. Then J is a well-defined, normalized niveloid; furthermore, $\succcurlyeq$ satisfies Axioms 3.8 and 3.7 if and only if J is affine; in this case, J has a unique, normalized and positive linear extension to $B(\Sigma)$.*

Corollary B.9 (Extension to $B(\Sigma, u(X))$) *The unique niveloidal extension $\hat{I}$ of I to $B(\Sigma, u(X))$ satisfies $J(a) = \frac{1}{2}\gamma + \frac{1}{2}\hat{I}(a) - \frac{1}{2}\hat{I}(\gamma - a)$ for all $a, \gamma - a \in B(\Sigma, u(X))$.*

Proof: J as above is well-defined: first, for every $a \in B_0(\Sigma, u(X))$, if $\gamma = \inf_\Omega a + \sup_\Omega a$, then $\gamma - a = \sup_\Omega a - [a - \inf_\Omega a] \in B_0(\Sigma, u(X))$; furthermore, if $\gamma, \gamma' \in \mathbb{R}$ are such that $\gamma - a, \gamma - a' \in B_0(\Sigma, u(X))$, then $\gamma - a = (\gamma' - a) + (\gamma - \gamma')$, so vertical invariance of I implies that $I(\gamma - a) = I(\gamma' - a) + \gamma - \gamma'$, and so $\frac{1}{2}\gamma - \frac{1}{2} I(\gamma - a) = \frac{1}{2}\gamma - \frac{1}{2} I(\gamma' - a) - \frac{1}{2}(\gamma - \gamma') = \frac{1}{2}\gamma' - \frac{1}{2} I(\gamma' - a)$, as required. Next, J is normalized: if $\gamma \in u(X)$, then $\gamma - \gamma = 0 \in u(X)$, so $J(\gamma) = \frac{1}{2}\gamma + \frac{1}{2} I(\gamma) - \frac{1}{2} I(\gamma - \gamma) = \frac{1}{2}\gamma + \frac{1}{2}\gamma + 0 = \gamma$, because I is normalized and $0 \cdot 1_\Omega \in B_0(\Sigma, u(X))$. Finally, J is a niveloid: for $a, b \in B_0(\Sigma, u(X))$, if $\alpha, \beta \in u(X)$ are such that $\alpha - a, \beta - b \in B_0(\Sigma, u(X))$, then

$$\begin{aligned} 2[J(a) - J(b)] &= \alpha + I(a) - I(\alpha - a) - \beta - I(b) + I(\beta - b) \le \\ &\le (\alpha - \beta) + \sup_\Omega (a - b) + \sup_\Omega (\beta - b - \alpha + a) = 2 \sup_\Omega (a - b). \end{aligned}$$

Turn now to Axioms 3.8 and 3.7.

First, it will be shown that $\succcurlyeq$ satisfies Axiom 3.8 if and only if $J(\frac{1}{2}a)=\frac{1}{2}J(a)$ for all $a\in B_0(\Sigma,u(X))$. Fix $f,\bar f,x,\bar x$ as in Axiom 3.8 and let $a\in B_0(\Sigma,u(X))$ and $\gamma\in\mathbb{R}$ be such that $a=u\circ f$ and $\gamma-a=u\circ\bar f$; then $\frac{1}{2}f+\frac{1}{2}\bar x\sim\frac{1}{2}\bar f+\frac{1}{2}x$ iff $I(\frac{1}{2}a+\frac{1}{2}u(\bar x))=I(\frac{1}{2}\bar f+\frac{1}{2}u(x))$; by vertical invariance [note that $\frac{1}{2}a,\frac{1}{2}(\gamma-a)\in B_0(\Sigma,u(X))$] and the properties of $x,\bar x$, this equals

$$I(\frac{1}{2}a)+\frac{1}{2}I(\gamma-a)=I(\frac{1}{2}(\gamma-a))+\frac{1}{2}I(a).$$

By the definition of J, rearranging terms, this holds iff $J(\frac{1}{2}a)+\frac{1}{4}\gamma=\frac{1}{2}[J(a)+\frac{1}{2}\gamma]$, i.e. $J(\frac{1}{2}a)=\frac{1}{2}J(a)$. Thus, if J has this property, then Axiom 3.8 holds. Conversely, for any $a\in B_0(\Sigma,u(X))$, there is $f\in L_0$ such that $u\circ f=a$, and as noted in the first part of this proof, one can find $\gamma\in\mathbb{R}$ with $\gamma-a\in B_0(\Sigma,u(X))$; again, there will be $\bar f\in L_0$ with $u\circ\bar f=\gamma-a$, so that $f,\bar f$ are complementary: if Axiom 3.8 holds, the argument just given shows that $J(\frac{1}{2}a)=\frac{1}{2}J(a)$.

Now assume that J is affine; then, in particular, for all $a\in B_0(\Sigma,u(X))$, $J(\frac{1}{2}a)=J(\frac{1}{2}a+\frac{1}{2}\cdot 0)=\frac{1}{2}J(a)+\frac{1}{2}J(0)=\frac{1}{2}J(a)$, and, as shown above, in this case Axiom 3.8 holds. Next, consider $(f,\bar f)$, $(g,\bar g)$ and α as in Axiom 3.7. Let $a=u\circ f$, $b=u\circ g$, and let $z,z'\in\mathbb{R}$ be such that $\frac{1}{2}u(f(\omega))+\frac{1}{2}u(\bar f(\omega))=z$, $\frac{1}{2}u(g(\omega))+\frac{1}{2}u(\bar g(\omega))=z'$ for all ω; finally, let $\bar a=2z-a$ and $\bar b=2z'-b$, so $\bar a=u\circ\bar f$ and $\bar b=u\circ\bar g$. Then $f\succcurlyeq\bar f$ and $g\succcurlyeq\bar g$ imply $I(a)\ge I(\bar a)=I(2z-a)$, so $J(a)\ge z+\frac{1}{2}I(a)-\frac{1}{2}(2z-a)=z$; similarly, $J(b)\ge z'$. If J is affine, then $J(\alpha a+(1-\alpha)b)=\alpha J(a)+(1-\alpha)J(b)\ge[\alpha z+(1-\alpha)z']$, so

$$\begin{aligned}&I(\alpha a+(1-\alpha)b)-I(\alpha\bar a+(1-\alpha)\bar b)=I(\alpha a+(1-\alpha)b)-I(\alpha[2z-a]+(1-\alpha)[2z'-b])=\\&=I(\alpha a+(1-\alpha)b)-I(2[\alpha z+(1-\alpha)z']-\alpha a-(1-\alpha)b)=2J(\alpha a+(1-\alpha)b)-2[\alpha z+(1-\alpha)z']\ge 0.\end{aligned}$$

where the last equality follows from the definition of J. Thus, $\alpha f+(1-\alpha)g\succcurlyeq\alpha\bar f+(1-\alpha)\bar g$, i.e. Axiom 3.7 holds.

Conversely, assume that Axioms 3.8 and 3.7 hold. The argument given above shows that $J(\frac{1}{2}a)=\frac{1}{2}J(a)$ for all $a\in B_0(\Sigma,u(X))$; it will now be shown that $J(\frac{1}{2}a+\frac{1}{2}b)=\frac{1}{2}J(a)+\frac{1}{2}J(b)$ for all $a,b\in B_0(\Sigma,u(X))$.

Since $0\in\mathrm{int}(u(X))$, there is $\delta>0$ such that $[-\delta,\delta]\subset u(X)$. Assume first that $\|a\|,\|b\|\le\frac{1}{2}\delta$; this implies that (a) $a,b,-a,-b\in B_0(\Sigma,u(X))$, and furthermore (b) $a-J(a),b-J(b),J(a)-a,J(b)-b\in B_0(\Sigma,u(X))$, because monotonicity of J implies that $J(a),J(b)\in[-\frac{1}{2}\delta,\frac{1}{2}\delta]$. Let $f,g,\bar f,\bar g\in L_0$ be such that $a-J(a)=u\circ f$, $b-J(b)=u\circ g$, $J(a)-a=u\circ\bar f$ and $J(b)-b=u\circ\bar g$. Clearly, $(f,\bar f)$ and $(g,\bar g)$ are complementary pairs; furthermore, applying the definition of J with $\gamma=0$, $J(a-J(a))=\frac{1}{2}I(a-J(a)-\frac{1}{2}I(J(a)-a)$ and similarly $J(b-J(b))=\frac{1}{2}I(b-J(b))-\frac{1}{2}I(J(b)-b)$; finally, by vertical invariance of J, $J(a-J(a))=J(a)-J(a)=0$ and similarly $J(b-J(b))=0$. Thus, $f\sim\bar f$ and $g\sim\bar g$, so Axiom 3.7 implies that $\frac{1}{2}f+\frac{1}{2}g\sim\frac{1}{2}\bar f+\frac{1}{2}\bar g$. It follows that $I(\frac{1}{2}[a-J(a)]+\frac{1}{2}[b-J(b)])=I(\frac{1}{2}[J(a)-a]+\frac{1}{2}[J(b)-b])$, or $J(\frac{1}{2}[a-J(a)]+\frac{1}{2}[b-J(b)])=0$; but by vertical invariance of J, this is equivalent to $J(\frac{1}{2}a+\frac{1}{2}b)=\frac{1}{2}J(a)+\frac{1}{2}J(b)$, as claimed.

Now, for arbitrary $a,b\in B_0(\Sigma,u(X))$, there is an integer $K>0$ such that $2^{-K}\|a\|,2^{-K}\|b\|\le\frac{1}{2}\delta$. Then the argument just given shows that $J(\frac{1}{2}(2^{-K}a)+\frac{1}{2}(2^{-K})b)=\frac{1}{2}J(2^{-K}a)+\frac{1}{2}J(2^{-K}b)$; but it was shown above that, for all $c\in B_0(\Sigma,u(X))$, $J(\frac{1}{2}c)=\frac{1}{2}J(c)$, and so it follows that

$$J\left(\frac{1}{2}a+\frac{1}{2}b\right)=2^K J\left(2^{-K}\left(\frac{1}{2}a+\frac{1}{2}b\right)\right)=2^K\frac{1}{2}J(2^{-K}a)+2^K\frac{1}{2}J(2^{-K}b)=\frac{1}{2}J(a)+\frac{1}{2}J(b).$$

This implies that $J(\alpha a+(1-\alpha)b)=\alpha J(a)+(1-\alpha)J(b)$ for all dyadic rationals $\alpha=k2^{-K}$, with $k\in\{0,\dots,K\}$ for some integer $K>0$.[23] But since these are dense in $[0,1]$ and J is supnorm-continuous, J is affine. The extension

[23] The claim is easily established by induction on K.

of J to $B(\Sigma)$ is now standard.

Finally, to prove the Corollary, if $a, \gamma - a \in B(\Sigma,\Gamma)$, there is a sequence $(a^k) \subset B_0(\Sigma,\Gamma)$ such that $a^k \to a$ and, for all k, $\min_\Omega a^k > \inf_\Omega a$ and $\max_\Omega a^k < \sup_\Omega a$;[24] thus, $\gamma - a^k(\omega) \le \gamma - \min a^k < \gamma - \inf a = \gamma + \sup(-a) = \sup(\gamma - a)$, and similarly $\gamma - a^k(\omega) > \inf(\gamma - a)$. It follows that $\gamma - a^k \in B_0(\Sigma, u(X))$ for all k, and so $J(a^k) = \frac{1}{2}\gamma + \frac{1}{2}I(a^k) - \frac{1}{2}I(\gamma - a^k) = \frac{1}{2}\gamma + \frac{1}{2}\hat{I}(a^k) - \frac{1}{2}\hat{I}(\gamma - a^k)$, so the claim follows by continuity of $\hat{I}$. ■

Next, the implications of the linearity of J for the GMM representation of I are investigated. The following notation is convenient: let $\mathscr{C} = \{Q \in B^*(\Sigma) : \exists q \in C \text{ s.t. } \forall a \in B(\Sigma), T(a) = \int a\, dq\}$. Also, say that $\mathscr{C}$ is *symmetric* around some $\hat{Q} \in B^*(\Sigma)$ iff, for every $Q \in \mathscr{C}$, $2\hat{Q} - Q \in \mathscr{C}$ (which implies that $\hat{Q} \in \mathscr{C}$ as well).

Lemma B.10 *In the setting of Lemma B.8, the functional J is affine on $B_0(\Sigma, u(X))$ if and only if $\mathscr{C}$ is symmetric around J and $\gamma(a) = \gamma(\alpha - a)$ for all $a \in B_0(\Sigma, u(X))$ such that $C_{\min}(a) < C_{\max}(a)$ and $\alpha \in \mathbb{R}$ such that $\alpha - a \in B_0(\Sigma, u(X))$.*

Corollary B.11 (Extension to $B(\Sigma, u(X))$) *The functional $\gamma(\cdot)$ also satisfies $\gamma(a) = \gamma(\alpha - a)$ whenever $a, \alpha - a \in B(\Sigma, u(X))$, $\alpha \in \mathbb{R}$, and $C_{\min}(a) < C_{\max}(a)$.*

Corollary B.12 (Symmetry of C) *If J is linear, let $p \in ba_1(\Sigma)$ be such that $J(a) = \int a\, dp$ for all $a \in B(\Sigma)$. Then C is symmetric around p: if $q \in C$, then $2p - q \in C$.*

Proof: Let $Q_b^{\min} \in \arg\min_{Q \in \mathscr{C}} Q(b)$ and $Q_b^{\max} \in \arg\max_{Q \in \mathscr{C}} Q(b)$ for any $b \in B_0(\Sigma)$. Two preliminary observations will be useful.

Claim 1: regardless of whether or not $\mathscr{C}$ is symmetric, $Q_b^{\min}(b) = Q_{\beta - b}^{\max}(b)$ for all $\beta \in \mathbb{R}$.

Claim 2: if $\mathscr{C}$ is symmetric around some $\hat{Q} \in \mathscr{C}$, then $Q_b^{\min}(b) + Q_b^{\max}(b) = 2\hat{Q}(b)$. To prove this claim, note that, by definition, $Q_b^{\min}(b) \le Q(b)$ for all $Q \in \mathscr{C}$. Now fix one such Q. Then, in particular, $Q_b^{\min}(b) \le 2\hat{Q}(b) - Q(b)$, because $2\hat{Q} - Q \in \mathscr{C}$. Hence $2\hat{Q}(b) - Q_b^{\min}(b) \ge Q(b)$: that is, for all $Q \in \mathscr{C}$, $2\hat{Q}(b) - Q_b^{\min}(b) \ge Q(b)$. Thus, $Q_b^{\max}(b) = 2\hat{Q}(b) - Q_b^{\min}(b)$.

Now, for necessity, suppose that $a, \alpha - a \in B_0(\Sigma, u(X))$ and calculate:

$$\begin{aligned} J(a) &= \frac{1}{2}\alpha + \frac{1}{2}\gamma(a)Q_a^{\min}(a) + \frac{1}{2}[1-\gamma(a)]Q_a^{\max}(a) - \\ &- \frac{1}{2}\gamma(\alpha - a)Q_{\alpha-a}^{\min}(\alpha - a) - \frac{1}{2}[1-\gamma(\alpha-a)]Q_{\alpha-a}^{\max}(\alpha - a) = \\ &= \frac{1}{2}\gamma(a)Q_a^{\min}(a) + \frac{1}{2}[1-\gamma(a)]Q_a^{\max}(a) - \frac{1}{2}\gamma(a)Q_{\alpha-a}^{\min}(-a) - \frac{1}{2}[1-\gamma(a)]Q_{\alpha-a}^{\max}(-a) = \\ &= \gamma(a)\left[\frac{1}{2}Q_a^{\min}(a) + \frac{1}{2}Q_{\alpha-a}^{\min}(a)\right] + [1-\gamma(a)]\left[\frac{1}{2}Q_a^{\max}(a) + \frac{1}{2}Q_{\alpha-a}^{\max}(a)\right]. \end{aligned}$$

Now Claim 1 implies that each term in square brackets equals $\frac{1}{2}Q_a^{\min}(a) + \frac{1}{2}Q_a^{\max}(a)$; since furthermore $\mathscr{C}$ is symmetric around some $\hat{Q} \in \mathscr{C}$, Claim 2 implies that the expressions in square brackets equal $\hat{Q}(a)$, and necessity follows.

[24]For each $k = 0, 1, \ldots$ and $\ell = 0, \ldots, 2^k$, let $\alpha_{\ell,k} = \inf\ a(\Omega) + \frac{\ell}{2^k}[\sup\ a(\Omega) - \inf\ a(\Omega)]$; also define $\bar{a} = \frac{1}{2}[\inf\ a(\Omega) + \sup\ a(\Omega)]$. Finally, let $a^k(\omega) = \frac{1}{k+1}\bar{a} + \frac{k}{k+1}\min_{\ell=0,\ldots,k}\{\alpha_{\ell,k} : a(\omega) \le \alpha_{\ell,k}\}$.

Now turn to sufficiency. For every $a \in B_0(\Sigma, u(X))$, let $\mu(a) = \min\ a(\Omega) + \max\ a(\Omega)$, so $\mu(a) - a \in B_0(\Sigma, u(X))$ and $J(a) = \frac{1}{2}\mu(a) + \frac{1}{2}I(a) - \frac{1}{2}I(\alpha_a - a)$.

Claim 3: for every $a \in B_0(\Sigma, u(X))$, $J(a) = \frac{1}{2}\mu(a) + \frac{1}{2}I_0(a) - \frac{1}{2}I_0(\mu(a) - a)$.

Three algebraic facts are key to the proof of Claim 3. First, for all $\lambda \in (0,1)$, $\mu(\lambda a) = \lambda\mu(a)$; second, although in general $\mu(\lambda a + (1-\lambda)b) \neq \lambda\mu(a) + (1-\lambda)\mu(b)$, it is nevertheless the case that $[\lambda\mu(a) + (1-\lambda)\mu(b)] - [\lambda a + (1-\lambda)b] = \lambda(\mu(a) - a) + (1-\lambda)(\mu(b) - b) \in B_0(\Sigma, u(X))$. Third, $\mu(\mu(a) - a) = \mu(a)$. Combining these facts, one obtains

$$[\lambda\mu(a) + (1-\lambda)\mu(b)] - \lambda[\mu(a) - a] - (1-\lambda)b \in B_0(\Sigma, u(X)).$$

As a consequence, the definition of J implies that

$$\begin{aligned} 2J\Big(\lambda[\mu(a)-a] + (1-\lambda)b\Big) &= [\lambda\mu(a) + (1-\lambda)\mu(b)] + I\Big(\lambda[\mu(a)-a] + (1-\lambda)b\Big) - \\ &- I\Big([\lambda\mu(a) + (1-\lambda)\mu(b)] - \lambda[\mu(a)-a] - (1-\lambda)b\Big). \end{aligned} \tag{21}$$

Similar, but simpler calculations yield

$$2J\Big((1-\lambda)b\Big) = (1-\lambda)\mu(b) + I\Big((1-\lambda)b\Big) - I\Big((1-\lambda)\mu(b) - (1-\lambda)b\Big). \tag{22}$$

One can then calculate:

$$\begin{aligned} -I_0\big(\mu(a)-a\big) &= -\inf_{b\in B_0(\Sigma,u(X)),\lambda\in(0,1]} \frac{1}{\lambda}\Big[I\Big(\lambda[\mu(a)-a] + (1-\lambda)b\Big) - I\Big((1-\lambda)b\Big)\Big] = \\ &= -\inf_{b\in B_0(\Sigma,u(X)),\lambda\in(0,1]} \frac{1}{\lambda}\Big[2J\Big(\lambda(\mu(a)-a) + (1-\lambda)b\Big) + \\ &+ I\Big([\lambda\mu(a) + (1-\lambda)\mu(b)] - \lambda[\mu(a)-a]) - (1-\lambda)b\Big) - \lambda\mu(a) - (1-\lambda)\mu(b) - \\ &- 2J\Big((1-\lambda)b\Big) - I\Big((1-\lambda)\mu(b) - (1-\lambda)b\Big) + (1-\lambda)\mu(b)\Big] = \\ &= 2J(a) - \mu(a) - \inf_{b\in B_0(\Sigma),\lambda\in(0,1]} \frac{1}{\lambda}\Big[I\Big(\lambda a + (1-\lambda)[\mu(b)-b]\Big) - I\Big((1-\lambda)[\mu(b)-b]\Big) = \\ &= 2J(a) - \mu(a) - I_0(a). \end{aligned}$$

The second equality follows by using Eqs. (21) and (22) to substitute for $I(\lambda[\mu(a)-a] + (1-\lambda)b)$ and $I((1-\lambda)b)$. The third equality follows by canceling the terms $(1-\lambda)\mu(b)$, using the fact that J is linear, then canceling one of the terms $\mu(a)$ and finally simplifying and rewriting the arguments of the two functionals I. Finally, the last equality follows by noting that $b \in B_0(\Sigma, u(X))$ if and only if $\mu(b) - b \in B_0(\Sigma, u(X))$.

Claim 3 implies that $\mathscr{C}$ is symmetric around J. To see this, pick $Q \in \mathscr{C}$, so $Q(a) \geq I_0(a) = C_{\min}(a)$ for all $a \in B_0(\Sigma, u(X))$; then, for all a,

$$2J(a) - Q(a) = \mu(a) + I_0(a) - I_0(\mu(a) - a) - Q(a) \leq \mu(a) - I_0(\mu(a) - a),$$

or equivalently $I_0(\mu(a) - a) \leq 2J(\mu(a) - a) - Q(\mu(a) - a)$ for all a; since, again, $a \in B_0(\Sigma, u(X))$ iff $\mu(a) - a \in B_0(\Sigma, u(X))$, this is also equivalent to

$$\forall a \in B_0(\Sigma, u(X)), \quad I_0(a) \leq 2J(a) - Q(a).$$

Now recall that $I_0 = C_{\min}$, and standard separation results[25] imply that

$$C = \left\{ q \in ba_1(\Sigma) : \forall a \in B_0(\Sigma, u(X)), \int a\, dq \geq C_{\min}(a) \right\};$$

then, it follows that $2J - Q \in \mathscr{C}$. Since $\mathscr{C}$ is convex and non-empty, this also implies that it must contain J as well.

Finally, with $Q_a^{\min}$, etc. defined as above, and letting $\alpha = \mu(a)$ for simplicity,

$$\begin{aligned}
2J(a) &= \alpha + \gamma(a) Q_a^{\min}(a) + [1-\gamma(a)] Q_a^{\max}(a) - \gamma(\alpha - a) Q_{\alpha-a}^{\min}(\alpha - a) - [1 - \gamma(\alpha - a)] Q_{\alpha-a}^{\max}(\alpha - a) = \\
&= \alpha + \gamma(a) Q_a^{\min}(a) + [1-\gamma(a)] Q_a^{\max}(a) - \gamma(\alpha - a) Q_a^{\max}(\alpha - a) - [1 - \gamma(\alpha - a)] Q_a^{\min}(\alpha - a) = \\
&= [\gamma(a) + 1 - \gamma(\alpha - a)] Q_a^{\min}(a) + [1 - \gamma(a) + \gamma(\alpha - a)] Q_a^{\max}(a) = \\
&= [Q_a^{\min}(a) + Q_a^{\max}(a)] + [\gamma(a) - \gamma(\alpha - a)][Q_a^{\min}(a) - Q_a^{\max}(a)] = \\
&= 2J(a) + [\gamma(a) - \gamma(\alpha - a)][Q_a^{\min}(a) - Q_a^{\max}(a)],
\end{aligned}$$

where the last equality follows from the fact that $\mathscr{C}$ is symmetric around J (see Claim 2 at the beginning of this proof). If $Q_a^{\min}(a) < Q_a^{\max}(a)$, equality can only obtain if $\gamma(a) = \gamma(\alpha - a)$, as claimed.

To prove the first Corollary, suppose $a, \alpha - a \in B(\Sigma, \Gamma)$ for some $\alpha \in \mathbb{R}$ and consider a sequence $(a^k) \subset B_0(\Sigma, u(X))$ such that $a_k \to a$ and, for each k, $\inf\ a(\Omega) < \min\ a^k(\Omega) \leq \max\ a^k(\Omega) < \sup\ a(\Omega)$. Then, as in the proof of the Corollary to Lemma B.8, $\alpha - a^k \in B_0(\Sigma, u(X))$, so that $\gamma(a) = \frac{\hat{I}(a) - C_{\max}(a)}{C_{\min}(a) - C_{\max}(a)} = \lim_{k\to\infty} \frac{I(a^k) - C_{\max}(a^k)}{C_{\min}(a^k) - C_{\max}(a^k)} = \lim_{k\to\infty} \gamma(a^k) = \lim_{k\to\infty} \gamma(\alpha - a^k) = \lim_{k\to\infty} \frac{I(\alpha - a^k) - C_{\max}(\alpha - a^k)}{C_{\min}(\alpha - a^k) - C_{\max}(\alpha - a^k)} = \frac{\hat{I}(\alpha - a) - C_{\max}(\alpha - a)}{C_{\min}(\alpha - a) - C_{\max}(\alpha - a)} = \gamma(\alpha - a)$. The second Corollary is straightforward. ■

B.4 Monotone Continuity

Assume that Γ is non-singleton. A functional $H : B_0(\Sigma, \Gamma) \to \mathbb{R}$ is *monotonely continuous* iff, for every $\alpha, \beta, \gamma \in \Gamma$ with $\alpha > \beta > \gamma$ and every sequence of events $(A_k) \subset \Sigma$ such that $A_k \supset A_{k+1}$ for all n and $\cap A_k = \emptyset$, there is k such that $H(\alpha - (\alpha - \gamma) 1_{A_k}) > \beta > H(\gamma + (\alpha - \gamma) 1_{A_k})$—or, abusing the notation for binary acts, $H(\gamma A_k \alpha) > \beta > H(\alpha A_k \gamma)$.

Continue to focus on the representation I, u of $\succcurlyeq$; assume wlog that $0 \in \mathrm{int}(u(X))$. Clearly, $\succcurlyeq$ satisfies Axiom 3.6 iff I is monotonely continuous. This property will now be characterized in terms of the functional J defined in Lemma B.8, and the set C defined in Lemma B.3. One implication will be that C consists of countably additive measures; Lemma B.14 will establish a useful consequence of this fact: it is possible to restrict attention to a *countable* subset of measures.

Lemma B.13 *The following statements are equivalent:*

(1) I is monotonely continuous;

(2) For every decreasing sequence $(A_k) \subset \Sigma$ such that $\bigcap A_k = \emptyset$, $J(1_{A_k}) \to 0$;

[25] Call C' the set in the r.h.s.; clearly, $C \subset C'$, so suppose there is $q \in C' \setminus C$. Since C is weak* closed and $\{q\}$ is weak* compact, by a version of the Separating Hyperplane Theorem (See Megginson, Theorem 2.2.28) there is a weak* continuous $T : ba(\Sigma) \to \mathbb{R}$ such that $Tq < \inf_{q' \in C} Tq'$; furthermore, $Tq' = \int a\, dq'$ for some $a \in B_0(\Sigma)$ (cf. Megginson, Prop. 2.6.4), and it can be assumed wlog that $a \in B_0(\Sigma, u(X))$. But then $\int a\, dq < C_{\min}(a)$, a contradiction.

If the functional J is linear, then (2) above is also equivalent to

(3) for every decreasing sequence $(A_k)\subset\Sigma$ such that $\bigcap A_k=\emptyset$, and for every $\epsilon>0$, there is k such that $q(A_k)<\epsilon$ for all $q\in C$.

As noted above, (3) implies in particular that every $q\in C$ is a probability measure.

Proof: (1) $\Rightarrow$ (2): let $\alpha\in u(X)$ be such that $\alpha>0$ and $-\alpha\in u(X)$. For every $\epsilon\in(0,\alpha)$, there is k' such that $\epsilon>I(\alpha 1_{A_{n'}})$ and k'' such that $I(\alpha(1-1_{A_{k''}}))>\alpha-\epsilon$ (take $\gamma=0$ and $\beta=\epsilon,\alpha-\epsilon$ in the definition of monotone continuity). Letting $k=\max(k',k'')$, so $A\subset A_{k'}$ and $A\subset A_{k''}$, by monotonicity both $\epsilon>I(\alpha 1_{A_k})$ and $I(\alpha(1-1_{A_k}))>\alpha-\epsilon$ hold; furthermore, since $-\alpha\in u(X)$, vertical invariance of I implies that $I(\alpha(1-1_{A_k}))=\alpha+I(-\alpha 1_{A_k})>\alpha-\epsilon$, i.e. $\epsilon>-I(-\alpha 1_{A_k})$. Hence, $\epsilon>\frac{1}{2}I(\alpha 1_{A_k})-\frac{1}{2}I(-\alpha 1_{A_k})=J(\alpha 1_{A_k})$. To sum up, if $\eta\geq 1$, then monotonicity implies that $J(1_{A_k})\leq\eta$ for all k; and for $\eta\in(0,1)$, taking $\epsilon=\eta\alpha$ yields k such that $J(1_{A_k})=\frac{1}{\alpha}J(\alpha 1_{A_k})<\frac{1}{\alpha}\epsilon=\eta$, as required.

(2) $\Rightarrow$ (1): Fix $\alpha,\beta,\gamma\in u(X)$ with $\alpha>\beta>\gamma$; then there is k' such that $J(\gamma+(\alpha-\gamma)1_{A_{k'}})<\gamma+\frac{1}{2}(\beta-\gamma)$. Let $\mu=\alpha+\gamma$. so $\mu-\gamma-(\alpha-\gamma)1_{A'_k}=\alpha-(\alpha-\gamma)1_{A_{k'}}\in B_0(\Sigma,u(X))$: then, by the definition of J,

$$\gamma+\frac{1}{2}(\beta-\gamma)>\frac{1}{2}\mu+\frac{1}{2}I(\gamma+(\alpha-\gamma)1_{A_{k'}})-\frac{1}{2}I(\mu-\gamma-(\alpha-\gamma)1_{A_{k'}});$$

substituting for μ and simplifying this reduces to

$$\frac{1}{2}\beta>\frac{1}{2}\alpha+\frac{1}{2}I(\gamma+(\alpha-\gamma)1_{A_{k'}})-\frac{1}{2}I(\alpha-(\alpha-\gamma)1_{A_{k'}})\geq\frac{1}{2}I(\gamma+(\alpha-\gamma)1_{A_{k'}}),$$

where the inequality follows from monotonicity of I, as $\alpha-(\alpha-\gamma)1_{A_{k'}}\leq\alpha$. Thus, $\beta>I(\gamma+(\alpha-\gamma)1_{A_{k'}})$. Similarly, there is k'' such that $J(\alpha-(\alpha-\gamma)1_{A_{k''}})>\alpha-\frac{1}{2}(\alpha-\beta)$, i.e.

$$\alpha-\frac{1}{2}(\alpha-\beta)<\frac{1}{2}\mu+\frac{1}{2}I(\alpha-(\alpha-\gamma)1_{A_{k''}})-\frac{1}{2}I(\mu-\alpha+(\alpha-\gamma)1_{A_{k''}}),$$

and again substituting for μ and simplifying yields

$$\frac{1}{2}\beta<\frac{1}{2}\gamma+\frac{1}{2}I(\alpha-(\alpha-\gamma)1_{A_{k''}})-\frac{1}{2}I(\gamma+(\alpha-\gamma)1_{A_{k''}})\leq\frac{1}{2}I(\alpha-(\alpha-\gamma)1_{A_{k''}}),$$

because $\gamma+(\alpha-\gamma)1_{A_{k''}}\geq\gamma$. Thus, $I(\alpha-(\alpha-\gamma)1_{A_{k''}})>\beta$. Therefore, by monotonicity, $k=\max(k',k'')$ satisfies $I(\alpha-(\alpha-\gamma)1_{A_k})>\beta>I(\gamma+(\alpha-\gamma)1_{A_k})$, as required.

Turning now to the final statement, note that $J\in\mathscr{C}$ by Lemma B.10, so (3) clearly implies (2). In the opposite direction, fix a sequence as in (2) and (3), let $p\in C$ represent J, and choose $\epsilon>0$; then there is k such that $J(1_{A_k})=p(A_k)<\frac{1}{2}\epsilon$. If now $q(A_k)\geq\epsilon$ for some $q\in C$, then $q'=2p-q$ satisfies $q'(A_k)<\epsilon-\epsilon=0$, and Lemma B.10 implies that $q'\in C$: but this contradicts the fact that C consists of probability charges. Therefore, $q(A_k)<\epsilon$ for all $q\in C$, as claimed. ■

Lemma B.14 *Suppose that Ω is a compact metric space and Σ is its Borel sigma-algebra. Let $C\subset ca_1(\Sigma)$ be $\sigma(ca(\Sigma),B(\Sigma))$-compact and symmetric around some $p\in C$. Then:*

(i) for every $\epsilon>0$ there is $\delta>0$ such that, for every $E\in\Sigma$, $p(E)<\delta$ implies $q(E)<\epsilon$ for all $q\in C$.

(ii) there exists a countable subset $D\subset C$ such that, for every $a\in B(\Sigma)$, $q\in C$, and $\epsilon>0$, there is $q'\in D$ with $|\int a\,dq-\int a\,dq'|<\epsilon$.

Proof: Since $\sigma(ca(\Sigma), B(\Sigma)) \subset \sigma(ca(\Sigma), C(\Omega))$, C is also $\sigma(ca(\Sigma), C(\Omega))$-compact. Since the $\sigma(ca(\Sigma), C(\Omega))$ topology is metrizable, there is a countable $\sigma(ca(\Sigma), C(\Omega))$-dense subset D of C.

Notice that, for every $E \in \Sigma$ and $q \in C$, $q(E) \leq 2p(E)$. To see this, suppose that $q(E) > 2p(E)$ for some E and q. Since C is symmetric around p, $q' = 2p - q \in C$; but then $q'(E) = 2p(E) - q(E) < 0$, which contradicts the assumption that $C \subset ca^1(\Sigma)$. Part (i) then follows immediately.

Now consider $a \in B(\Sigma)$, $q \in C \setminus D$, and $\epsilon > 0$. If $\|a\| = 0$, so $a = 0$, there is nothing to prove; thus, assume $\|a\| > 0$. By (i), there is $\delta > 0$ such that $q''(E) < \frac{\epsilon}{6\|a\|}$ for all $q'' \in C$ and $E \in \Sigma$ such that $p(E) < \delta$. By Lusin's Theorem on measurable functions (Kechris [32, Thm. 17.12]), there exists $b \in C(\Omega)$ such that $p(\{\omega : b(\omega) \neq a(\omega)\}) < \delta$; moreover, b can be chosen so that $\|b\| \leq \|a\|$.[26] Therefore, for all $q'' \in C$, $q''(\{\omega : b(\omega) \neq a(\omega)\}) < \frac{\epsilon}{6\|a\|}$, so

$$\left|\int_\Omega a\, dq'' - \int_\Omega b\, dq''\right| = \left|\int_{\{\omega: a(\omega)\neq b(\omega)\}} (a-b)\, dq''\right| \leq 2\|a\| \frac{\epsilon}{6\|a\|} = \frac{\epsilon}{3}.$$

Finally, by assumption, there is a sequence $\{q^n\} \subset D$ such that $q^n \to q$ in the $\sigma(ca(\Sigma), C(\Omega))$ topology. Hence, there is n such that $|\int b\, dq^n - \int b\, dq| < \frac{\epsilon}{3}$. Thus,

$$\left|\int a\, dq - \int a\, dq^n\right| \leq \left|\int a\, dq - \int b\, dq\right| + \left|\int b\, dq - \int b\, dq^n\right| - \left|\int b\, dq^n - \int a\, dq^n\right| < \epsilon.$$

■

Note: the preceding result does not actually require symmetry: the existence of $p \in C$ as in (i) follows from a theorem of Bartle, Dunford and Schwartz (cf. [9, Corollary 6, p. 14]).

B.5 Proof of Theorem 3.1

Recall that, for any vector measure m on Σ and interval $\Gamma \subset \mathbb{R}$, $R_0(m,\Gamma) = \left\{\int a\, dm : a \in B_0(\Sigma,\Gamma)\right\}$. It is convenient to extend this to images of functions in $B(\Sigma,\Gamma)$: let $R(m,\Gamma) = \left\{\int a\, dm : a \in B(\Sigma,\Gamma)\right\}$. Clearly, $R(m,\Gamma)$ is the closure of $R_0(m,\Gamma)$.[27] It is clear that (2) implies (3) in Theorem 3.1; thus, focus on the non-trivial implications.

B.5.1 (3) implies (1)

For all $a \in B_0(\Sigma, u(X))$, let $J_p(a) = \int a\, dp$ and $I(a) = J_p(a) + A(\int a\, dm)$; thus, for all $f, g \in L_0$, $f \succcurlyeq g$ iff $I(u \circ f) \geq I(u \circ g)$. It is easy to verify that I is constant-mixture invariant and normalized (because $m(\Omega) = 0$ and $A(0) = 0$); furthermore, by part 3 of Def. 2, it is monotonic, and hence a niveloid by Prop. B.1. This implies that $\succcurlyeq$ satisfies the first five axioms in (1). Furthermore, for all $a \in B(\Sigma,\Gamma)$, letting $\mu \in u(X)$ be such that $\mu - a \in B_0(\Sigma, u(X))$,

$$J(a) \equiv \frac{1}{2}\mu + \frac{1}{2}\hat{I}(a) - \frac{1}{2}\hat{I}(\mu - a) = \frac{1}{2}\mu + \frac{1}{2}J_p(a) + \frac{1}{2}A\left(\int a\, dm\right) - \frac{1}{2}J_p(\mu - a) - \frac{1}{2}A\left(\int (\mu - a)\, dm\right) = J_p(a),$$

[26] If $\|b\| > \|a\|$, consider the function b' such that $b'(\omega) = \max(-\|a\|, \min(\|a\|, b(\omega)))$ for every $\omega \in \Omega$.

[27] If $\varphi \in R(m,\Gamma)$ then there is $a \in B(\Sigma,\Gamma)$ and $(a_k)_{k\geq 0} \subset B_0(\Sigma,\Gamma)$ such that $\varphi = \int a\, dm$ and $a_k \to a$, so $\int a_k\, dm_i \to \varphi$ uniformly in i; hence, $R(m,\Gamma) \subset \mathrm{cl}\, R_0(m,\Gamma)$. The other inclusion is obvious.

because again $m(\Omega) = 0$ and $A(\phi) = A(-\phi)$ for all $\phi \in R(m,\Gamma)$; thus, the functional J defined in Lemma B.8 coincides with J_p, and hence it is affine; thus, $\succcurlyeq$ satisfies Axioms 3.7 and 3.8 as well. It remains to be shown that I is monotonely continuous. To see this, fix a sequence $(A_k) \subset \Sigma$ decreasing to $\emptyset$, and note that, for $\alpha > \beta > \gamma$ in Γ,

$$I(\gamma+(\alpha-\gamma)1_{A_k}) = \gamma+(\alpha-\gamma)p(A_k)+A((\alpha-\gamma)m(A_k)), \quad I(\gamma+(\alpha-\gamma)1_{A_k}) = \alpha-(\alpha-\gamma)p(A_k)+A(-(\alpha-\gamma)m(A_k)).$$

Since p is countably additive and the m_i's are uniformly countably additive, $p(A_k) \to 0$ and $\sup_i |m_i(A_k)| \to 0$; furthermore, since $A(\cdot)$ is supnorm-continuous at 0_n, $A(m(A_k)) \to 0$ and $A(-m(A_k)) \to 0$. Hence, $I(\gamma+(\alpha-\gamma)1_{A_k}) \downarrow \gamma$ and $I(\gamma+(\alpha-\gamma)1_{A_k}) \uparrow \alpha$, which implies that, for some k, $I(\gamma+(\alpha-\gamma)1_{A_k}) > \beta > I(\gamma+(\alpha-\gamma)1_{A_k})$. Thus, I is monotonely continuous, so $\succcurlyeq$ satisfies Axiom 3.6.

B.5.2 (1) implies (2)

Since (Ω,Σ) is standard Borel, it is sufficient to establish the claim under the additional assumption that Ω is a compact metric space and Σ is its Borel sigma-algebra.[28]

Since $\succcurlyeq$ satisfies Axioms 3.1–3.5, it admits a non-degenerate niveloidal representation I, u by Proposition B.1; furthermore, it is wlog to assume that $0 \in \text{int}(u(X))$. Moreover, since $\succcurlyeq$ satisfies Axioms 3.7 and 3.8, the functional J defined in Lemma B.10 is affine on $B_0(\Sigma, u(X))$; finally, since $\succcurlyeq$ satisfies Axiom 3.6, I is monotonely continuous. In the following, $\hat{I}$ will denote the unique niveloidal extension of I to $B(\Sigma, u(X))$.

Let C be the set of probability charges delivered by Lemma B.3. Let p be the probability charge representing J, so by Lemma B.10, $p \in C$ and C is symmetric around p. By Lemma B.13, $C \subset ca(\Sigma)$, and furthermore for every sequence (A_k) decreasing to $\emptyset$, $\sup_{q\in C} q(A_k) \to 0$. Finally, let $D \subset C$ be the set delivered by (ii) of Lemma B.14.

Note that, by Lemma B.4, for all $b, b' \in B(\Sigma, u(X))$, $\int b\,dq = \int b'\,dq$ for all $q \in C$ implies that $\hat{I}(b) = \hat{I}(b')$. I claim that, for all $b, b' \in B(\Sigma, u(X))$, $\int b\,dq = \int b'\,dq$ for all $q \in D$ implies $\int b\,dq = \int b'\,dq$ for all $q \in C$, or, equivalently, that $\int b\,dq = 0$ for all $q \in D$ implies that $\int b\,dq = 0$ for all $q \in C$. Consider $b \in B(\Sigma, u(X))$ and $q \in C \setminus D$; assume that $\int b\,dq' = 0$ for all $q' \in D$. Suppose there is $q \in C$ with $|\int b\,dq| = \epsilon > 0$: then (ii) in Lemma B.14 yields $q' \in D$ with $|\int b\,dq - \int b\,dq'| < \epsilon$, so $|\int b\,dq'| > 0$, a contradiction.

Now define a collection $m = (m_i)_{0\le i<\infty}$ of signed measures by letting $m_i(E) = q_i(E) - p(E)$ for every $E \in \Sigma$ and $i \in \{0, 1, \ldots\}$, where $q_0, q_1, \ldots$ is an enumeration of D. Clearly, for each i, $m_i \in ca_1(\Sigma)$, $m_i(\emptyset) = m_i(\Omega) = 0$, and $|m_i(E)| \le |q_i(E) - p(E)| \le 2$. Moreover, let $(A_k)_{k\ge0} \subset \Sigma$ be a sequence decreasing to $\emptyset$; since, as noted above, $\sup_{q\in C} q(A_n) \to 0$,

$$\sup_i |m_i(A_k)| = \sup_i |q_i(A_k) - p(A_k)| \le \sup_i q_i(A_k) + p(A_k) \to 0,$$

i.e. $m_i(A_k) \to 0$ uniformly in i; thus, m is an adjustment tuple as per Def. 1.

Next, suppose that $a, b \in B_0(\Sigma, u(X))$ are such that $\int a\,dm = \int b\,dm$; therefore, for all $q \in D$, $\int a\,dq = \int b\,dq + \delta$, where $\delta = \int a\,dp - \int b\,dp$. By Lemma B.5, this implies that $\hat{I}(a) = \hat{I}(b) + \delta$. Furthermore, suppose that $\alpha - a, \beta - b \in$

[28] Since (Ω,Σ) is standard Borel, there exists a Borel isomorphism $\sigma : \Omega \to K$ (cf. Kechris [32], p.90), where K is compact metric and endowed with the Borel sigma-algebra; if Ω is countable, let $K = \Omega$, endowed with the Fort topology (fix a point $\omega \in \Omega$ and declare $E \subset \Omega$ to be open if either (i) $\omega \notin E$ or (ii) $\omega \in E$ and $\Omega \setminus E$ is finite). Let L_0^σ be the set of simple, Borel-measurable functions from K to X, and define a binary relation $\succcurlyeq^\sigma$ on L_0^σ by letting $f \circ \sigma^{-1} \succcurlyeq^\sigma g \circ \sigma^{-1}$ iff $f \succcurlyeq g$ for all $f, g \in L_0$. Then it is clear that $\succcurlyeq$ satisfies any one of the axioms in Sec. 3 if and only if $\succcurlyeq^\sigma$ does, too. Finally, if $\succcurlyeq^\sigma$ has a VEU representation $(u, p^\sigma, n, m^\sigma, A)$, then $\succcurlyeq$ has a VEU representation $(u, p^\sigma \circ \sigma, n, m^\sigma \circ \sigma, A)$.

$B_0(\Sigma, u(X))$ for $\alpha, \beta \in \mathbb{R}$; then

$$\int [\alpha - a]\, dq = \alpha - \int a\, dq = \alpha - \int b\, dq - \delta = \int [\beta - b]\, dq + \alpha - \beta - \delta$$

for all $q \in C$, and therefore $\hat{I}(\alpha - a) = \hat{I}(\beta - b) + \alpha - \beta - \delta$. Hence,

$$\begin{aligned}\hat{I}(a) - J(a) &= \hat{I}(a) - \frac{1}{2}\alpha - \frac{1}{2}\hat{I}(a) + \frac{1}{2}\hat{I}(\alpha - a) = \frac{1}{2}\hat{I}(a) + \frac{1}{2}\hat{I}(\alpha - a) - \frac{1}{2}\alpha = \\ &= \frac{1}{2}\hat{I}(b) + \frac{1}{2}\delta + \frac{1}{2}\hat{I}(\beta - b) + \frac{1}{2}(\alpha - \beta) - \frac{1}{2}\delta - \frac{1}{2}\alpha = \frac{1}{2}\hat{I}(b) + \frac{1}{2}\hat{I}(\beta - b) - \frac{1}{2}\beta = \hat{I}(b) - J(b).\end{aligned}$$

Therefore, it is possible to define a functional $A : R(m, u(X)) \to \mathbb{R}$ by letting $A(\phi) = \hat{I}(a) - J(a)$ for every $\phi \in \ell_\infty$ such that $\phi = \int a\, dm$ for some $a \in B(\Sigma, u(X))$. Also, the map $a \mapsto J(a) + A(\int a\, dm)$ coincides with $\hat{I}$, and hence is monotonic by assumption, as required by Def. 2.

It will now be shown that A is supnorm-continuous at $0 \in \mathbb{R}^\infty$. Observe first that $\int 0_\Omega\, dm = 0 \in \ell_\infty$, where $0_\Omega(\omega) = 0$ for all ω; thus, $A(0) = \hat{I}(0) - J(0) = 0$. Moreover, if $\phi = \int a\, dm$ and $\alpha - a \in B(\Sigma, u(X))$, then $A(\phi) = \frac{1}{2}I(a) + \frac{1}{2}I(\alpha - a) - \frac{1}{2}\alpha$. Now, using the GMM-like representation of $\hat{I}$ in Lemma B.6, and the fact that $\gamma(a) = \gamma(\alpha - a) \equiv \gamma$,

$$\begin{aligned}\hat{I}(a) &= \gamma \min_{q\in C} \int a\, dq + (1-\gamma) \max_{q \in C} \int a\, dq = \gamma \inf_{q\in D} \int a\, dq + (1-\gamma) \sup_{q\in D} \int a\, dq = \\ &= \gamma \inf_i \int a\, d(m_i + p) + (1-\gamma) \sup_i \int a\, d(m_i + p) = J(a) + \gamma \inf_i \phi_i + (1-\gamma) \sup_i \phi_i\end{aligned}$$

and similarly for $\hat{I}(\alpha - a)$, we get

$$\begin{aligned}2A(\phi) &= \hat{I}(a) + \hat{I}(\alpha - a) - \alpha = \\ &= J(a) + \gamma \inf_i \phi_i + (1-\gamma) \sup_i \phi_i + J(\alpha - a) + \gamma \inf_i (-\phi_i) + (1-\gamma) \sup_i (-\phi_i) - \alpha = \\ &= \gamma \inf_i \phi_i + (1-\gamma) \sup_i \phi_i - \gamma \sup_i \phi_i - (1-\gamma) \inf_i \phi_i = \\ &= (2\gamma - 1) \inf_i \phi_i + (1 - 2\gamma) \sup_i \phi_i = (2\gamma - 1)[\inf_i \phi_i - \sup_i \phi_i].\end{aligned}$$

Since $\gamma \in [0,1]$, it follows that, for all $\phi \in R(m, u(X))$,

$$\inf_i \phi_i - \sup_i \phi_i \le 2A(\phi) \le \sup_i \phi_i - \inf_i \phi_i.$$

If $\phi^n \to 0$ in ℓ_∞, then $\inf_i \phi_i^n \to 0$ and $\sup_i \phi_i^n \to 0$, so $A(\phi^n) \to 0 = A(0)$, as required.[29]

Thus, (u, p, ∞, m, A) is a VEU representation of $\succcurlyeq$. It remains to be shown that there is a sharp VEU representation. Suppose that $f \in L_0$ is crisp and $f \sim x$. By Lemma B.7, $\int u \circ f\, dq = u(x)$ for all $q \in C$; since $p \in C$, also $\int u \circ f\, dp = u(x)$. Hence, $\int u \circ f\, dm_i = \int u \circ f\, dq_i - \int u \circ f\, dp = 0$ for all $i = 0, 1, \ldots$, as required in (1) of Def. 3. Since there is at least one representation (u, p, ∞, m, A) that satisfies this property, it is clear that there is one adjustment that satisfies (2) in that definition, i.e. a sharp VEU representation.

Turn now to the case of Ω finite. Each signed measure $m_0, m_1, \ldots$ can be viewed as a point in the finite-dimensional space $\mathbb{R}^\Omega$. Let n' be the maximal number of linearly independent coordinate measures; note that,

[29] For every $\epsilon > 0$ there is $n(\epsilon)$ such that $|\phi_i^n| < \epsilon$ for all $n \ge n(\epsilon)$; hence, for such n, $\epsilon \ge \sup_i \phi_i^n \ge \inf_i \phi_i^n \ge \epsilon$.

due to the normalization $m_i(\Omega)=0$, $n' \le |\Omega|-1$. By relabeling, one can assume w.l.o.g. that $m_0,\dots,m_{n'-1}$ are a maximal collection of linearly independent measures. Let $m'=(m_0,\dots,n_{n'-1})$ and define $A': R(m',u(X))\to\mathbb{R}$ by $A'(\int a\,dm')=A(\int a\,dm)$ for every $a\in B(\Sigma,u(X))$. This definition is well-posed: if $a,b\in B(\Sigma,u(X))$ satisfy $\int a\,dm'=\int b\,dm'$, i.e. $\int a\,dm_j=\int b\,dm_j$ for $j=0,\dots,n'-1$, then also $\int a\,dm_i=\int b\,dm_i$ for all $i\ge n'$, because each such m_i must be a linear combination of $m_0,\dots,m_{n'-1}$. Also, clearly, $m'\in M_{n'}(\Sigma)$, and it is easy to check that $A'(0)=0$, A' is symmetric, and $a\mapsto\int a\,dp+A'(\int a\,dm')$ is monotonic; furthermore, if $f\in L_0$ is crisp, then it was shown above that $\int u\circ f\,dm=0$, so a fortiori $\int u\circ f\,dm'=0$. I now claim that A' is continuous at 0.

For every $i\ge 0$, let $\alpha_i=(\alpha_{i0},\dots,\alpha_{in'-1})\in\mathbb{R}^{n'}$ be the unique vector such that $m_i=\sum_{j=0}^{n'-1}\alpha_{ij}m_j$. Clearly, for every $\omega\in\Omega$, $|m_i(\omega)|\le|q_i(\omega)|+|p(\omega)|\le 2$; hence, $\{m_1,m_2,\dots\}$ is a bounded subset of $\mathbb{R}^\Omega$. Moreover, for every $j=0,\dots,n'-1$ the "coordinate functional" A_j associating with each point of the form $\mu=\sum_{j=0}^{n'-1}\alpha_j m_j$ the coefficient α_j is bounded (cf. Megginson [39], Thm. 1.4.12). These two facts imply that $\bar\alpha\equiv\max_{j=0,\dots,n'-1}\sup_{i\ge0}|\alpha_{ij}|<\infty$. Thus, consider a sequence $\{\varphi^k\}\subset R(m',u(X))$ such that $\varphi^k\to 0$, and for each k, let $a^k\in B(\Sigma,u(X))$ be such that $\int a^k\,dm'=\varphi^k$. Then, for all $i\ge 0$,

$$\left|\int a^k\,dm_i\right|=\left|\sum_{j=0}^{n'-1}\alpha_{ij}\int a^k\,dm_j\right|\le\sum_{j=0}^{n'-1}|\alpha_{ij}|\left|\varphi^k_j\right|\le\bar\alpha\sum_{j=0}^{n'-1}\left|\varphi^k_j\right|.$$

Thus, $\varphi^k\to0$ implies that $\int a^k\,dm_i\to0$ *uniformly* in $i\ge0$. Then $\lim_{k\to\infty}A'(\varphi^k)=\lim_{k\to\infty}A(\int a^k\,dm)=0$, as required. Therefore, (u,p,n',m',A') is VEU representation that satisfies (1) in Def. 3, and such that $n'\le|\Omega|-1$. It follows that, as claimed, any minimal VEU representation will employ an adjustment tuple of size at most n'.

Observation: the above proof of (1) ⇒ (2) actually constructs a representation of the extension $\hat I$ of I to $B(\Sigma,u(X))$. Therefore, as usual, the VEU representation (u,p,n,m,A) applies to the unique continuous extension of $\succcurlyeq$ to $L(\succcurlyeq)$, the class of $\succcurlyeq$-bounded, Σ-measurable acts.

B.5.3 Uniqueness

By standard arguments, $u'=\alpha u+\beta$ for some $\alpha,\beta\in\mathbb{R}$ with $\alpha>0$; consequently, $\psi\in R_0(m,u(X))$ if and only if $\alpha\psi\in R_0(m',u(X))$; the constant β can be disregarded, as $m(\Omega)=0$ and $m'(\Omega)=0$. Next, for every $a\in B_0(\Sigma,u(X))$, let $I(a)=\int a\,dp+A(\int a\,dm)$; define I' similarly using the second VEU representation. By Cor. B.2, $\alpha I(a)=I'(\alpha a)$ for every $a\in B_0(\Sigma,u(X))$; hence, if J and J' are the corresponding functionals defined as in Lemma B.8, their extension to $B(\Sigma)$ coincides, and so $p=p'$; hence,

$$\alpha A\left(\int a\,dm\right)=\alpha I(a)-\alpha J(a)=I'(\alpha a)-J'(\alpha a)=A'(\alpha a) \tag{23}$$

for all $a\in B_0(\Sigma,u(X))$. Now, to define a suitable linear surjection $T: R_0(m',u'(X))\to R_0(m,u(X))$, suppose that $\int\alpha a\,dm'=\int\alpha b\,dm'$ for $a,b\in B_0(\Sigma,u(X))$; clearly, for all $\psi'\in R_0(m',u'(X))$ and $\lambda\in(0,1]$, $A'((1-\lambda)\psi'+\lambda\int\alpha(a-b)\,dm')=A'((1-\lambda)\psi'+0)=A'((1-\lambda)\psi')$. Now pick $\psi\in R_0(m,u(X))$, so there is $c\in B_0(\Sigma,u(X))$ such that $\int c\,dm=\psi$;

then, for all $\lambda \in (0,1]$,

$$A\left((1-\lambda)\psi + \lambda \int (a-b)\,dm\right) = A\left(\int [(1-\lambda)c + \lambda(a-b)]\,dm\right) = \frac{1}{\alpha}A'\left(\int \alpha[(1-\lambda)c + \lambda(a-b)]\,dm'\right) =$$
$$= \frac{1}{\alpha}A'\left(\int \alpha(1-\lambda)c\,dm'\right) = A\left(\int (1-\lambda)c\,dm\right) = A(\psi);$$

the second and fourth equalities follow from Eq. (23). In particular, $A(\int (a-b)\,dm) = 0$. Now let $\gamma \in u(x)$ be such that $\gamma - b \in B_0(\Sigma, u(x))$; then there is $f \in L_0$ such that $\frac{1}{2}a + \frac{1}{2}(\gamma - b) = u \circ f$. Now $\int u \circ f\,dm = \frac{1}{2}\int (a-b)\,dm$, so $A(\int u \circ f\,dm) = 0$, and if $f \sim x \in X$, then $u(x) = \int u \circ f\,dp$. Therefore, for every $g \in L_0$ and $\lambda \in (0,1]$,

$$I(u\circ[(1-\lambda)g + \lambda f]) = (1-\lambda)\int u\circ g\,dp + \lambda \int u \circ f\,dp + A\left((1-\lambda)\int u\circ g\,dm + \lambda\int u\circ f\,dm\right) =$$
$$= (1-\lambda)\int u\circ g\,dp + \lambda u(x) + A\left((1-\lambda)\int u\circ g\,dm + \lambda\frac{1}{2}\int (a-b)\,dm\right) =$$
$$= (1-\lambda)\int u\circ g\,dp + \lambda u(x) + A\left(\left(1-\frac{1}{2}\lambda\right)\frac{1-\lambda}{1-\frac{1}{2}\lambda}\int u\circ g\,dm + \frac{1}{2}\lambda\int (a-b)\,dm\right) =$$
$$= (1-\lambda)\int u\circ g\,dp + \lambda u(x) + A\left(\left(1-\frac{1}{2}\lambda\right)\frac{1-\lambda}{1-\frac{1}{2}\lambda}\int u\circ g\,dm\right) =$$
$$= (1-\lambda)\int u\circ g\,dp + \lambda u(x) + A\left((1-\lambda)\int u\circ g\,dm\right) = I(u\circ[(1-\lambda)g + \lambda x]).$$

The third equality is justified by noting that, for any $\gamma \in u(X)$, $\frac{1-\lambda}{1-\frac{1}{2}\lambda}u\circ g + \frac{\frac{1}{2}\lambda}{1-\frac{1}{2}\lambda}\gamma \in B_0(\Sigma, u(X))$, and of course $\int \gamma\,dm = 0$. Thus, the act f is crisp, and applying (1) in Def. 3 to (u,p,n,m,A), $\int (a-b)\,dm = 2\int u\circ f\,dm = 0$. Thus, we can define T by letting $T(\int \alpha a\,dm') = \int a\,dm$ for all $a \in B_0(\Sigma, u(X))$. That T is affine is immediate, as is the fact that T is onto. Finally, if $\varphi' = \int \alpha a\,dm'$, then $A(T(\varphi')) = A(T(\int \alpha a\,dm')) = A(\int a\,dm) = \frac{1}{\alpha}A'(\int \alpha a\,dm') = \frac{1}{\alpha}A'(\varphi')$, where the second equality follows from the definition of T, and the third from Eq. (23): thus, $A = \frac{1}{\alpha}A' \circ T$. Finally, if $(u', p'n', m', A')$ is also sharp, then $n = n'$; furthermore, assume that $\int a\,dm = \int b\,dm$: then, since also (u', p', n', m', A') satisfies (1) in Def. 3, the argument used above to show that T is well-defined can be employed to show that $\int a\,dm = \int b\,dm$ implies $\int \alpha a\,dm' = \int \alpha b\,dm'$, so T is a bijection.

B.6 Ambiguity Aversion

Proof of Corollary 4.1

If $\succcurlyeq$ satisfies Ambiguity Aversion, then I is concave (cf. MMR, p. 28); in particular, if $a, \gamma - a \in B_0(\Sigma, u(X))$, $\frac{1}{2}\gamma = I(\frac{1}{2}a + \frac{1}{2}(\gamma - a)) \geq \frac{1}{2}I(a) + \frac{1}{2}I(\gamma - a) = \frac{1}{2}\int a\,dp + \frac{1}{2}A(\int a\,dm) + \frac{1}{2}\gamma - \frac{1}{2}\int a\,dp + \frac{1}{2}A(\int (\gamma - a)\,dm) = \frac{1}{2}\gamma + A(\int a\,dm)$, and so A is non-positive. Finally, A is clearly also concave.

Conversely, suppose that A is concave (hence, also non-positive). Then I is concave, so for all $f, g \in L_0$ with $f \sim g$, $I(u\circ[\lambda f + (1-\lambda)g]) \geq I(u \circ \lambda f)$.

Proof of Proposition 4.2

(3) $\Rightarrow$ (1) is immediate (consider the EU preference determined by p and u). To see that (3) $\Leftrightarrow$ (2), note that, if $f, \bar{f}$ are complementary, with $\frac{1}{2}f + \frac{1}{2}\bar{f} \sim z \in X$, $f \sim x$ and $\bar{f} \sim \bar{x}$, then $\frac{1}{2}f + \frac{1}{2}\bar{f} \succcurlyeq \frac{1}{2}x + \frac{1}{2}\bar{x}$ iff $u(z) \geq$

$\frac{1}{2}\int u\circ f\,dp+\frac{1}{2}A(\int u\circ f\,dm)+\frac{1}{2}\int u\circ\bar f\,dp+\frac{1}{2}A(\int u\circ\bar f\,dm)=u(z)+A(\int u\circ f\,dm)$, because $\int u\circ\bar f\,dm=-\int u\circ f\,dm$ and A is symmetric; hence, the required ranking obtains iff $A(\int u\circ f\,dm)\le 0$.

Turn now to (1) ⇒ (3). Suppose that $\succcurlyeq$ is more ambiguity-averse than some EU preference relation $\succcurlyeq'$. By Corollary B.3 in [22], one can assume that $\succcurlyeq'$ is represented by the non-constant utility u on X. Arguing by contradiction, suppose that there is $f\in L_0$ such that $A(\int u\circ f\,dm)>0$. Let $\gamma\in\mathbb{R}$ be such that $\gamma-u\circ f\in B_0(\Sigma,u(X))$, and $\bar f\in L_0$ such that $u\circ\bar f=\gamma-u\circ f$. Then $A(\int u\circ\bar f\,dm)=A(\int u\circ f\,dm)>0$; furthermore, $\frac{1}{2}u\circ f+\frac{1}{2}u\circ\bar f=u\circ(\frac{1}{2}f+\frac{1}{2}\bar f)=\frac{1}{2}\gamma$, which implies $A(\int u\circ(\frac{1}{2}f+\frac{1}{2}\bar f)\,dm)=A(\frac{1}{2}\gamma m(\Omega))=A(0)=0$. If now $f\sim x$ and $\bar f\sim\bar x$ for $x,\bar x\in X$, then $\frac{1}{2}u(x)+\frac{1}{2}u(\bar x)=\frac{1}{2}\gamma+A(\varphi)>\frac{1}{2}\gamma$, so $\frac{1}{2}x+\frac{1}{2}\bar x\succ\frac{1}{2}f+\frac{1}{2}\bar f$. Now let $z\in X$ be such that $\frac{1}{2}f(\omega)+\frac{1}{2}\bar f(\omega)\sim z$ for all ω; then $\frac{1}{2}x+\frac{1}{2}\bar x\succ z$, so $\frac{1}{2}x+\frac{1}{2}\bar x\succ' z$. But $f\sim x$ and $\bar f\sim\bar x$ imply $f\succcurlyeq' x$ and $\bar f\succcurlyeq'\bar x$, and since $\succcurlyeq'$ is an EU preference, $\frac{1}{2}f+\frac{1}{2}\bar f\succcurlyeq'\frac{1}{2}x+\frac{1}{2}\bar x$; hence, $z\succcurlyeq'\frac{1}{2}x+\frac{1}{2}\bar x$, a contradiction.

To see that (3) ⇔ (4), consider first the following *Claim*: for a complementary pair $(f,\bar f)$ such that $f\sim\bar f$, $\frac{1}{2}f+\frac{1}{2}\bar f\sim z\succcurlyeq f$ iff $A(\int u\circ f\,dm)\le 0$. To prove this claim, let $\frac{1}{2}f+\frac{1}{2}\bar f\sim z\in X$: then, since $f\sim\bar f$ and these acts have the same adjustments, $\int u\circ f\,dp=\int u\circ\bar f\,dp$, so both integrals equal $u(z)$. Therefore, $\frac{1}{2}f+\frac{1}{2}\bar f\sim z\succcurlyeq f$ if and only if $u(z)\ge u(z)+A(\int u\circ f\,dm)=\int u\circ f\,dp+A(\int u\circ f\,dm)$.

The Claim immediately shows that (3) implies (4). For the converse, assume that Axiom 4.3 and consider the cases (a) $\succcurlyeq$ satisfies C-Independence or (b) $u(X)$ is unbounded. In case (a), then I is positively homogeneous, so if $\varphi=\int a\,dm$ for some $a\in B(\Sigma,u(X))$ and $\alpha>0$, then $A(\alpha\varphi)=\hat I(\alpha a)-J(\alpha a)=\alpha[\hat I(a)-J(a)]=\alpha A(\varphi)$: that is, A is also positively homogeneous. In this case, it is wlog to assume that $u(X)\supset[-1,1]$ and prove the result for $f\in L_0$ such that $\|u\circ f\|\le\frac{1}{3}$. This ensures the existence of $\bar f\in L_0$ such that $u\circ\bar f=-u\circ f$, as well as $g,\bar g\in L_0$ such that $u\circ g=u\circ f-\int u\circ f\,dp$ and $u\circ\bar g=u\circ\bar f-\int u\circ\bar f\,dp=-u\circ g$. By construction, $(g,\bar g)$ are complementary and $g\sim\bar g$, because $\int u\circ g\,dp=\int u\circ\bar g\,dp=0$. The above Claim implies that $A(\int u\circ f\,dm)=A(\int u\circ g\,dm)\le 0$, as required.

In case (b), suppose $u(X)$ is unbounded below (the other case is treated analogously). Consider $f\in L_0$ and construct $\bar f\in L_0$ such that $u\circ\bar f=\min\ u\circ f(\Omega)+\max\ u\circ f(\Omega)-f$. Then $f,\bar f$ are complementary. If $f\sim\bar f$, then the Claim suffices to prove the result. Otherwise, let $\delta=\int u\circ f\,dp-\int u\circ\bar f\,dp$. If $\delta>0$, consider $f'\in L_0$ such that $u\circ f'=u\circ f-\delta$: then $\int u\circ f'\,dp=\int u\circ\bar f\,dp$ and $f',\bar f$ are complementary, so $f'\sim\bar f$ and the Claim implies that $A(\int u\circ f\,dm)=A(\int u\circ f'\,dm)\le 0$. If instead $\delta<0$, consider f' such that $u\circ f'=\bar f-\delta$, so again $f\sim f'$ and the Claim can be invoked to yield the required conclusion.

B.7 Updating

For $a,b\in B_0(\Sigma,u(X))$, denote by aEb the element of $B_0(\Sigma,u(X))$ that coincides with a on E and with b elsewhere.

Proof of Remark 4.1.

Only if: it will be shown that, for any event $E\in\Sigma$, $p(E)=0$ implies $I(a)=I(b)$ for all $a,b\in B_0(\Sigma,u(X))$ such that $a(\omega)=b(\omega)$ for $\omega\notin E$. To see this, assume wlog that $I(a)\ge I(b)$, and let $\alpha=\max\{\max\ a(\Omega),\max\ b(\Omega)\}$ and $\beta=\min\{\min\ a(\Omega),\min\ b(\Omega)\}$. Then monotonicity implies that $I(\alpha Ea)\ge I(a)\ge I(b)\ge I(\beta Eb)=I(\beta Ea)$. Thus, it is sufficient to show that $I(\alpha Ea)=I(\beta Ea)$. This is immediate if $\alpha=\beta$, so assume $\alpha>\beta$. Since $p(E)=0$, $\int\alpha Ea\,dp=\int_{\Omega\setminus E}a\,dp=\int\beta Ea\,dp$, so if $I(\alpha Ea)>I(\beta Ea)$, it must be the case that $A(\int\alpha Ea\,dm)>A(\int\beta Ea\,dm)$.

Letting $\gamma=\alpha+\beta$, as usual $\gamma-\alpha Ea, \gamma-\beta Ea \in B_0(\Sigma, u(X))$; now

$$\begin{aligned} I(\gamma-\alpha Ea) &= \int [\gamma-\alpha Ea]\, dp + A\left(\int [\gamma-\alpha Ea]\, dm\right) = \int_{\Omega\setminus E} [\gamma-a]\, dp + A\left(-\int \alpha Ea\, dm\right) = \\ &= \int_{\Omega\setminus E} [\gamma-a]\, dp + A\left(\int \alpha Ea\, dm\right) > \int_{\Omega\setminus E} [\gamma-a]\, dp + A\left(\int \beta Ea\, dm\right) = \\ &= \int_{\Omega\setminus E} [\gamma-a]\, dp + A\left(\int [\gamma-\beta Ea)\, dm\right) = \int [\gamma-\beta Ea]\, dp + A\left(\int [\gamma-\beta Ea)\, dm\right) = I(\gamma-\beta Ea), \end{aligned}$$

which is a violation of monotonicity, as $\gamma-\alpha=\beta<\alpha=\gamma-\beta$.

If: suppose that $p(E)>0$, and fix $x, y \in X$ with $x \succ y$. If $xEy \succ y$, we are done. Otherwise, note that $xEy \sim y$, i.e. $[u(x)-u(y)]p(E)+A([u(x)-u(y)]m(E))=0$, implies $A([u(x)-u(y)]m(\Omega\setminus E)) = A([u(x)-u(y)]m(E)) = -[u(x)-u(y)]p(E)$; hence,

$$\begin{aligned} I(u\circ yEx) &= u(y)+p(\Omega\setminus E)[u(x)-u(y)]+A([u(x)-u(y)]m(\Omega\setminus E)) = \\ &= u(y)+p(\Omega\setminus E)[u(x)-u(y)]-[u(x)-u(y)]p(E) = u(y)+[u(x)-u(y)][p(\Omega\setminus E)-p(E)] < u(x), \end{aligned}$$

because $p(\Omega\setminus E)-p(E)=1-2p(E)<1$ as $p(E)>0$. Thus, $x\succ yEx$, and again Axiom 4.4 holds. ■

Proof of Proposition 4.3. Since E is not null, $p(E)>0$, so $p(\cdot|E)$ is well-defined. Begin with the following

Claim: If $(f,\bar f)$ is a complementary pair, f and $\bar f$ are constant on $\Omega\setminus E$, and for all $\omega\in\Omega\setminus E$, then

$$\frac{1}{2}f+\frac{1}{2}\bar f(\omega)\sim\frac{1}{2}\bar f+\frac{1}{2}f(\omega)$$

holds if and only if $u(f(\omega))=\int u\circ f\, dp=\int_E u\circ f\, dp(\cdot|E)$ for all $\omega\in\Omega\setminus E$.

Proof of the Claim: Let $\gamma\in\mathbb{R}$ be such that $\frac{1}{2}\gamma=\frac{1}{2}u(f(\omega))+\frac{1}{2}u(\bar f(\omega))$ for all $\omega\in\Omega$; also let $\alpha=u(f(\omega))$ and $\beta=u(\bar f(\omega))$ for any (hence all) $\omega\in\Omega\setminus E$. Then $u\circ\bar f=\gamma-u\circ f$ and $\beta=\gamma-\alpha$; thus, for $\omega\in\Omega\setminus E$,

$$\begin{aligned} I\left(u\circ\left(\frac{1}{2}f+\frac{1}{2}\bar f(\omega)\right)\right) &= \frac{1}{2}\int u\circ f\, dp+\frac{1}{2}\beta+A\left(\frac{1}{2}\int u\circ f\, dm\right)= \\ &= \frac{1}{2}\int u\circ f\, dp+\frac{1}{2}\gamma-\frac{1}{2}\alpha+A\left(\frac{1}{2}\int u\circ f\, dm\right) \quad \text{and} \\ I\left(u\circ\left(\frac{1}{2}\bar f+\frac{1}{2}f(\omega)\right)\right) &= \frac{1}{2}\int u\circ \bar f\, dp+\frac{1}{2}\alpha+A\left(\frac{1}{2}\int u\circ \bar f\, dm\right)= \\ &= \frac{1}{2}\gamma-\int u\circ f\, dp+\frac{1}{2}\alpha+A\left(\frac{1}{2}\int u\circ f\, dm\right), \end{aligned}$$

where the last equality uses the fact that $\int u\circ\bar f\, dm=-\int u\circ f\, dm$ and A is symmetric. Hence, $\frac{1}{2}f+\frac{1}{2}\bar f(\omega)\sim\frac{1}{2}\bar f+\frac{1}{2}f(\omega)$ holds if and only if $\alpha=\int u\circ f\, dp$. Furthermore, $\int u\circ f\, dp=\int_E u\circ f\, dp+\alpha p(\Omega\setminus E)$, so it follows that $\alpha=\int_E u\circ f\, dp(\cdot|E)$ as well.

Next, note that the tuple of set functions $m_E = (m_{i,E})_{0\le i<n}$ defined by Eq. (7) is easily seen to be an element of $M^n_\infty(\Sigma)$; in particular, for all $F \in \Sigma$, $|m_{i,E}(F)| \le |m_i(F\cap E)| + |m_i(\Omega\setminus E)| \le N(F\cap E) + N(\Omega\setminus E)$ for all i, and if $(F_k)\downarrow\emptyset$, then $\sup_{0\le i<n}|m_{i,E}(F)| \le \sup_{0\le i<n}|m_i(F_k\cap E)| + p(F_k|E)N(\Omega\setminus E)\to 0$. Furthermore, $m_E(E) = m(E) + p(E|E)m(\Omega\setminus E) = 0$ as well. Note also that, for all $a \in B(\Sigma)$, $\int_E a\, dm_E = \int_E a\, dm + \int_E a\, dp(\cdot|E)m(\Omega\setminus E)$; this is immediate for indicator functions, holds by linearity on $B_0(\Sigma)$, and extends to $B(\Sigma)$ by continuity.

The tuple (n, m_E, A) is an adjustment for $p(\cdot|E)$; to verify monotonicity, observe that, for $a, b \in B_0(\Sigma, u(X))$, $a \ge b$ implies that $\int a\, dp(\cdot|E) \ge \int b\, dp(\cdot|E)$, and hence $aE\left(\int_E a\, dp(\cdot|E)\right) \ge bE\left(\int_E b\, dp(\cdot|E)\right)$. Since (n, m, A) is an adjustment for p, $\int aE\left(\int_E a\, dp(\cdot|E)\right) dp + A(\int aE\left(\int_E a\, dp(\cdot|E)\right) dm) \ge \int bE\left(\int_E b\, dp(\cdot|E)\right) dp + A(\int bE\left(\int_E b\, dp(\cdot|E)\right) dm)$, i.e. equivalently $\int_E a\, dp(\cdot|E) + A(\int_E a\, dm_E) \ge \int_E b\, dp(\cdot|E) + A(\int_E b\, dm_E)$, as required.

Now suppose (1) holds. Fix $f, g, \bar f, \bar g \in L_0$ as in Axiom 4.6. By the Claim, $u\circ f(\omega) = \int_E u\circ f\, dp(\cdot|E) = \int u\circ f\, dp$ and $u\circ g(\omega) = \int_E u\circ g\, dp(\cdot|E) = \int u\circ g\, dp$ for all $\omega \in \Omega\setminus E$. Then the Axiom implies that $f \succcurlyeq_E g$ iff fEg, i.e. iff

$$\begin{aligned}
&\int u\circ f\, dp + A\left(\int u\circ f\, dm\right) \ge \int u\circ g\, dp + A\left(\int u\circ g\, dm\right)\\
\Leftrightarrow &\int_E u\circ f\, dp(\cdot|E) + A\left(\int_E u\circ f\, dm + m(\Omega\setminus E)\int_E u\circ f\, dp(\cdot|E)\right)\\
&\ge \int_E u\circ g\, dp(\cdot|E) + A\left(\int_E u\circ g\, dm + m(\Omega\setminus E)\int_E u\circ g\, dp(\cdot|E)\right)\\
\Leftrightarrow &\int_E u\circ f\, dp(\cdot|E) + A\left(\int_E u\circ f\, dm_E\right) \ge \int_E u\circ g\, dp(\cdot|E) + A\left(\int_E u\circ g\, dm_E\right).
\end{aligned}$$

If now $f, g \in L_0$ are arbitrary, let $x, y \in X$ be such that $u(x) = \int_E u\circ f\, dp(\cdot|E)$ and $u(y) = \int_E u\circ g\, dp(\cdot|E)$. Notice that then $\int u\circ fEx\, dp = \int_E u\circ fEx\, dp(\cdot|E) = u(x)$, and similarly for gEy. Finally, let f', g' be such that (fEx, f') and (gEy, g') are complementary; notice that this requires that f', g' be constant on $\Omega\setminus E$. Then, by the Claim, the acts fEx, f', gEy, g' satisfy all the assumptions of Axiom 4.6, and the preceding argument just given shows that then $fEx \succcurlyeq_E gEy$ iff $\int_E u\circ f\, dp(\cdot|E) + A(\int_E u\circ f\, dm_E) \ge \int_E u\circ g\, dp(\cdot|E) + A(\int_E u\circ g\, dm_E)$. But by Axiom 4.5, $fEx \succcurlyeq_E gEy$ iff $f \succcurlyeq_E g$, so (2) holds.

In the opposite direction, assume that (2) holds. It is then immediate that Axiom 4.5 is satisfied. Now assume that $f, g, \bar f, \bar g$ are as in Axiom 4.6. Then the Claim shows that $u(f(\omega)) = \int_E u\circ f\, dp(\cdot|E)$ and $u(g(\omega)) = \int_E u\circ g\, dp(\cdot|E)$ for all $\omega \in \Omega\setminus E$, so $\int_E u\circ f\, dp(\cdot|E) + A(\int_E u\circ f\, dm_E) = p(E)\int_E u\circ f\, dp(\cdot|E) + p(\Omega\setminus E)u(f(\omega)) + A(\int u\circ f\, dm + u(f(\omega))m(\Omega\setminus E)) = \int u\circ f\, dp + A(\int u\circ f\, dm)$, and similarly $\int_E u\circ g\, dp(\cdot|E) + A(\int_E u\circ g\, dm_E) = \int u\circ g\, dp + A(\int u\circ g\, dm)$, so that Axiom 4.6 holds. ■

Conclude by verifying that the "law of iterated conditioning" holds: with notation as in §4.2,

$$\begin{aligned}
m_{i,E,F}(G) &= m_{i,E}(G) + p(G|F)m_{i,E}(\Omega\setminus F) = m_{i,E}(G) - p(G|F)m_{i,E}(F) =\\
&= m_i(G) - p(G|E)m_i(E) - p(G|F)m_i(F) + p(G|F)p(F|E)m_i(E) =\\
&= m_i(G) - p(G|F)m_i(F) = m_{i,F}(G).
\end{aligned}$$

B.8 Proof of Proposition 4.4

(1) follows immediately from Lemma B.10. For (2), notice that the Choquet integral is positively homogeneous; hence, I has a unique extension from $B_0(\Sigma, u(X))$ to $B_0(\Sigma)$, and $J(a) = \frac{1}{2}I(a) - \frac{1}{2}I(-a)$ for all $a \in B_0(\Sigma)$. If $\succcurlyeq$ satisfies Complementary Independence, then, using the VEU representation, $I(1_E) = p(E) + A(m(E))$ and $I(-1_E) = -p(E) + A(-m(E)) = -p(E) + A(m(E))$, so $I(1_E) - I(-1_E) = 2p(E)$. On the other hand, using the CEU representation, $I_v(E) = v(E)$ and $I_v(-1_E) = -[1 - v(\Omega \setminus E)]$; since $I = I_v$, the claim follows. In the opposite direction, suppose that $a = \sum_{k=1}^K \alpha_k 1_{E_k}$ for a partition $E_1, \dots, E_K$ of Ω and numbers $\alpha_1 < \alpha_2 < \dots < \alpha_K$. Then $I_v(a) = \sum_{k=1}^K \alpha_k \left[v\left(\cup_{\ell=k}^K E_\ell\right) - v\left(\cup_{\ell=k+1}^K E_\ell\right)\right]$ and similarly, invoking the condition in the Proposition,

$$I_v(-a) = \sum_{k=1}^K (-\alpha_k)\left[v\left(\cup_{\ell=1}^k E_\ell\right) - v\left(\cup_{\ell=1}^{k-1} E_\ell\right)\right] =$$
$$= \sum_{k=1}^K (-\alpha_k)\left[2p\left(\cup_{\ell=1}^k E_\ell\right) - 1 + v\left(\cup_{\ell=k+1}^K E_\ell\right) - 2p\left(\cup_{\ell=1}^{k-1} E_\ell\right) + 1 - v\left(\cup_{\ell=k}^K E_\ell\right)\right] = -2\sum_{k=1}^K \alpha_k p(E_k) + I_v(a),$$

and so $\frac{1}{2}I(a) - \frac{1}{2}I(-a) = J(a)$, where J is the linear functional represented by p. The claim now follows from Lemma B.8.

B.9 Proof of Proposition 4.5

The preference $\succcurlyeq$ has a niveloidal representation I_{c^*}, u, where $I_c(a) = \min_{q \in ba_1(\Sigma)} \int a\, dq + c^*(q)$. For conciseness, say that c^* is *symmetric around* $p \in ba_1(\Sigma)$ iff it satisfies the condition in Prop. 4.5. By Lemma B.8, Axiom 3.7 holds iff the functional J defined by $J(a) = \frac{1}{2}\gamma + \frac{1}{2}I_{c^*}(a) - \frac{1}{2}I_{c^*}(\gamma - a)$ is affine. Thus it suffices to show that J is affine iff c^* is symmetric around p.

Suppose that c^* is symmetric around p. Consider a complementary pair $(f, \bar{f})$, and let $z \in X$ be such that $\frac{1}{2}f(\omega) + \frac{1}{2}\bar{f}(\omega) \sim z$; thus, $a \equiv u \circ f = 2u(z) - u \circ \bar{f} \equiv \gamma - u \circ \bar{f}$. Now let $q^* \in \arg\min_{q \in ba_1(\Sigma)} \int a\, dq + c^*(q)$; since clearly $c^*(q^*) < \infty$, $2p - q^* \in ba_1(\Sigma)$ and $c^*(q^*) = c^*(2p - q^*)$. Now, for all $q \in ba_1(\Sigma)$ such that $2p - q \in ba_1(\Sigma)$,

$$\int (\gamma - a)\, d(2p - q) + c^*(2p - q) = \gamma - 2\int a\, dp + \int a\, dq + c^*(q) \geq$$
$$\geq \gamma - 2\int a\, dp + \int a\, dq^* + c^*(q^*) = \int (\gamma - a)\, d(2p - q^*) + c^*(2p - q^*).$$

Since any $q \in ba_1(\Sigma)$ such that $2p - q \in ba_1(\Sigma)$ can obviously be written as $q = 2p - [2p - q]$, and all other $q \in ba_1(\Sigma)$ have $c^*(q) = \infty$, it follows that $I_{c^*}(\gamma - a) = \gamma - 2\int a\, dp + \int a\, dq^* + c^*(2p - q^*) = \gamma - 2\int a\, dp + I_{c^*}(a)$; therefore, $J(a) = \frac{1}{2}\gamma + \frac{1}{2}I_{c^*}(a) - \frac{1}{2}I_{c^*}(\gamma - a) = \int a\, dp$, i.e. J is affine and represented by p.

In the opposite direction, suppose that $\gamma + \frac{1}{2}I_{c^*}(a) - \frac{1}{2}I_{c^*}(\gamma - a) = \int a\, dp$ for all $a, \gamma - a \in B_0(\Sigma)$; also, for every $f \in L_0$, let $m_f \in X$ be such that $u(m_f) = \frac{1}{2}\min_{\omega \in \Omega} u(f(\omega)) + \frac{1}{2}\max_{\omega \in \Omega} u(f(\omega))$, and recall that $u(x_f) = I_{c^*}(u \circ f)$.

For every $q \in ba_1(\Sigma)$ such that $2p - q \in ba_1(\Sigma)$,

$$c^*(2p-q) = \sup_{f \in L_0} u(x_f) - \int u \circ f \, d(2p-q) = -2\int u \circ f \, dp + \sup_{f \in L_0} I_{c^*}(u \circ f) - \int (-u \circ f)\, dq =$$
$$= -2\int u \circ f \, dp + \sup_{f \in L_0} 2\int u \circ f \, dp + I_{c^*}(2u(m_f) - u \circ f) - 2u(m_f) - \int (-u \circ f)\, dq =$$
$$= \sup_{f \in L_0} I_{c^*}(2u(m_f) - u \circ f) - \int [2u(m_f) - u \circ f]\, dq = \sup_{f \in L_0} I_{c^*}(u \circ f) - \int u \circ f \, dq = c^*(q);$$

the last step follows because, for every $f \in L_0$, there is $\bar{f} \in L_0$ such that $u \circ \bar{f} = 2u(m_f) - u \circ f$, and therefore computing the supremum over $f \in L_0$ is the same as computing it over the complementary acts $\bar{f}$ constructed from each $f \in L_0$ in this way. If instead $2p - q \notin ba_1(\Sigma)$ but $c^*(q) < \infty$, the above calculations still show that

$$\sup_{f \in L_0} u(x_f) - \int u \circ f \, d(2p-q) = c^*(q) < \infty.$$

Now $2p(\Omega) - q(\Omega) = 1$, so there must be $E \in \Sigma$ such that $2p(E) - q(E) < 0$. Therefore,

$$\sup_{f \in L_0} u(x_f) - \int u \circ f \, d(2p-q) = \sup_{f \in L_0} I_{c^*}(u \circ f) - \int u \circ f \, d(2p-q) \geq$$
$$\geq \sup_{\alpha,\beta \in u(X): \alpha > \beta} I_{c^*}(\beta + (\alpha-\beta)1_E) - \int [\beta + (\alpha-\beta)1_E]\, d(2p-q) =$$
$$= \sup_{\alpha,\beta \in u(X): \alpha > \beta} I_{c^*}(\beta + (\alpha-\beta)1_E) - \beta - (\alpha-\beta)[2p(E) - q(E)] \geq$$
$$\geq \sup_{\alpha,\beta \in u(X): \alpha > \beta} \beta - \beta - (\alpha-\beta)[2p(E) - q(E)] = \infty$$

which contradicts $c^*(q) < \infty$. The second equality follows from the fact that $2p(\Omega) - q(\Omega) = 1$, and the second inequality follows from monotonicity of I_{c^*}; the final equality uses the fact that $u(X)$ is unbounded and $2p(E) - q(E) < 0$. ■

B.10 Probabilistic Sophistication for VEU preferences

An induced likelihood ordering $\succcurlyeq_\ell$ is *represented* by a probability $\mu \in ca_1(\Sigma)$ iff, for all $E, F \in \Sigma$, $E \succcurlyeq_\ell F$ iff $\mu(E) \geq \mu(F)$. Finally, a probability measure μ is *convex-ranged* iff, for every event $E \in \Sigma$ such that $\mu(E) > 0$, and for every $\alpha \in (0,1)$, there exists $A \in \Sigma$ such that $A \subset E$ and $\mu(A) = \alpha\mu(E)$.

Proposition B.15 *Fix a VEU preference relation $\succcurlyeq$ and let $p \in ca_1(\Sigma)$ be the corresponding baseline probability. If the induced likelihood ordering $\succcurlyeq_\ell$ is represented by a convex-ranged probability measure $\mu \in ca_1(\Sigma)$, then $\mu = p_1$.*

Proof: Fix $x, y \in X$ with $x \succ y$. Since the ranking of bets xEy is represented by μ and also by the map defined by $E \mapsto u(x)p(E) + u(y)p(E^c) + A([u(x) - u(y)]m(E))$, there exists an increasing function $g : [0,1] \to [u(y), u(x)]$ such that $u(x)p(E)+u(y)p(E^c)+A([u(x)-u(y)]m(E))=g(\mu(E))$ for all events E [this function g will in general depend upon x and y, but this is inconsequential]. Hence, recalling that $m(\Omega\setminus E)=-m(E)$ and A is symmetric,

$$g(\mu(E))-g(1-\mu(E))=[u(x)-u(y)](2p(E)-1) \tag{24}$$

for all events $E\in\Sigma$. Since g is increasing, so is the map $\gamma\mapsto g(\gamma)-g(1-\gamma)$; thus, $\mu(E)=\mu(F)$ if and only if $p(E)=p(F)$. Now, since μ is convex-ranged, for any integer n there exists a partition $\{E^n_1,\dots,E^n_n\}$ of Ω such that $\mu(E^n_j)=\frac{1}{n}$ for all $j=1,\dots,n$; correspondingly, $p(E^n_j)=p(E^n_k)$ for all $j,k\in\{1,\dots,n\}$, and therefore $p(E^n_j)=\frac{1}{n}$ for all $j=1,\dots,n$. This implies that, for every event E such that $\mu(E)$ is rational, $p(E)=\mu(E)$.

To extend this equality to arbitrary events, note that, for every event E such that $\mu(E)>0$ and number $r<\mu(E)$, since μ is convex-ranged, there exists $L\subset E$ such that $\mu(L)=\frac{r}{\mu(E)}\mu(E)=r$. Similarly, for every event E such that $\mu(E)<1$ and number $r>\mu(E)$, there exists an event $U\supset E$ such that $\mu(U)=r$: to see this, note that $\mu(\Omega\setminus E)>0$ and $1-r<\mu(\Omega\setminus E)$, so there exists $L\subset\Omega\setminus E$ such that $\mu(L)=1-r$; hence, $U=\Omega\setminus L$ has the required properties.

Now consider sequences of rational numbers $\{\ell_n\}_{n\geq 0}\subset[0,1]$ and $\{u_n\}_{n\geq 0}\subset[0,1]$ such that $\ell_n\uparrow\mu(E)$ and $u_n\downarrow\mu(E)$; by the preceding argument, for every $n\geq 1$ there exist sets $L_n\subset E\subset U_n$ such that $\mu(L_n)=\ell_n$ and $\mu(U_n)=u_n$. It was shown above that $p(L_n)=\mu(L_n)$ and $p(U_n)=\mu(U_n)$; moreover, $L_n\subset E\subset U_n$ implies that $p(L_n)\leq p(E)\leq p(U_n)$. Therefore, $p(E)=\mu(E)$, as required. ■

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