Abstract

This technical appendix contains the formal statement and proof of three results mentioned in the paper. Propositions 1 and 2 in this technical appendix are mentioned in Section 8.4 of the paper. Proposition 3 in this technical appendix is mentioned in Section 8.6 of the paper.
1 More general shocks

Section 8.4 of the paper presents results for a Ramsey problem with more general shocks. In the new Ramsey problem: $c_t^* = \varphi z_t$ replaces the original equation for efficient composite consumption, $p_{i,t}^* = p_t + \phi_c c_t + \phi_z z_t$ replaces the original equation for the profit-maximizing price, $s_{i,t} = z_t + \zeta_{i,t}$ replaces the original signal vector, and $\kappa = \frac{1}{2} \log_2 \left( \frac{\sigma_z^2 (t-1)}{\sigma_z^2} \right)$ replaces the original information flow constraint.

Formally, the new Ramsey problem reads:

$$\min_{\{G_t(L)\}_{t=0}} \sum_{t=0}^{\infty} \beta^t E \left[ (c_t - c_t^*)^2 + \frac{1}{T} \sum_{i=1}^{I} (p_{i,t} - p_t)^2 \right],$$

subject to

$$c_t^* = \varphi z_t,$$

$$c_t = m_t - p_t,$$

$$p_t = \frac{1}{T} \sum_{i=1}^{I} p_{i,t},$$

$$p_{i,t} = E \left[ p_{i,t}^* | \mathcal{I}_{i,t} \right],$$

$$p_{i,t}^* = p_t + \phi_c c_t + \phi_z z_t,$$

$$\mathcal{I}_{i,t} = \mathcal{I}_{i,-1} \cup \{s_{i,0}, s_{i,1}, \ldots, s_{i,t}\},$$

$$s_{i,t} = z_t + \zeta_{i,t},$$

$$z_t = \rho z_{t-1} + \nu_t,$$

and

$$m_t = G_t(L) \nu_t,$$

where $\delta > 0$, $\phi_c > 0$ and $\phi_z > 0$. The innovation $\nu_t$ follows a Gaussian white noise process. The noise terms $\zeta_{i,t}$ follow Gaussian white noise processes that are independent across firms and independent of the $\nu$ process. In the model with an endogenous signal precision, signal precision is given by the solution to:

$$\min_{(1/\sigma_i^2) \in \mathbb{R}_+} \left\{ \sum_{i=0}^{\infty} \beta^i \frac{\omega_i}{2} E_{i,-1} \left[ (p_{i,t} - p_{i,t}^*)^2 \right] + \frac{\mu}{1 - \beta^2} \kappa \right\},$$

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subject to
\[ p_{i,t} = E[p_{i,t}^* | I_{i,t}], \]  
and
\[ \kappa = \frac{1}{2} \log_2 \left( \frac{\sigma_z^2}{\sigma_{z|t}^2} \right), \]
where \( \omega > 0 \).

The next proposition is a generalization of Proposition 5 in the paper. It is a generalization because the desired markup is just a special case of a variable \( z_t \) with the property \( \varphi > -\frac{\phi_z}{\phi_c} \). In particular, for the desired markup, we have \( z_t = \lambda_t, \varphi = 0 \) and \( \phi_z = \phi_\lambda > 0 \).

**Proposition 1 (Endogenous signal precision)** Consider the Ramsey problem (1)-(13), where the signal precision \( \frac{1}{\omega^2} \) is given by the solution to problem (11)-(13). Suppose that \( \mu > 0, \sigma_z^2 > 0, \rho_z = 0 \) and \( \varphi > -\frac{\phi_z}{\phi_c} \). Consider policies of the form \( m_t = g_0 z_t \) and equilibria of the form \( p_t = \theta z_t \).

Define
\[ b \equiv \sqrt{\frac{\omega (\phi_c g_0 + \phi_z)^2 \sigma_z^2 \ln(2)}{\mu}}. \]

First, we characterize the set of equilibria at a given monetary policy \( g_0 \in \mathbb{R} \). Here \( \kappa^* \) denotes the equilibrium attention devoted to the variable \( z_t \). If and only if \( b \leq 1 \), there exists an equilibrium with
\[ \kappa^* = 0. \]

If and only if \( \phi_c \in (0, \frac{1}{2}] \) and \( b \in \left[ \sqrt{4 \phi_c (1 - \phi_c)}, 1 \right] \), there exists an equilibrium with
\[ \kappa^* = \log_2 \left( \frac{b - \sqrt{b^2 - 4 \phi_c (1 - \phi_c)}}{2 \phi_c} \right). \]

If and only if either \( \phi_c \in (0, \frac{1}{2}] \) and \( b \geq \sqrt{4 \phi_c (1 - \phi_c)} \) or \( \phi_c > \frac{1}{2} \) and \( b \geq 1 \), there exists an equilibrium with
\[ \kappa^* = \log_2 \left( \frac{b + \sqrt{b^2 - 4 \phi_c (1 - \phi_c)}}{2 \phi_c} \right). \]

The equilibrium price level, consumption level and price dispersion are given by
\[ p_t = \frac{(\phi_c g_0 + \phi_z) (1 - 2^{-2\kappa^*})}{1 - (1 - \phi_c)(1 - 2^{-2\kappa^*})} z_t, \]
\[ c_t = \left[ g_0 - \frac{(\phi_c g_0 + \phi_z) (1 - 2^{-2\kappa^*})}{1 - (1 - \phi_c)(1 - 2^{-2\kappa^*})} \right] z_t. \]
and
\[ E \left[ (p_{i,t} - p_t)^2 \right] = \frac{\mu}{\ln(2)} \left( 1 - 2^{-2\kappa^*} \right). \]

Second, we characterize optimal monetary policy. If \( \phi_c \in \left[ \frac{1}{2}, \infty \right) \), there exists a unique equilibrium for any monetary policy \( g_0 \in \mathbb{R} \) and the unique optimal monetary policy is
\[
g_0^* = \begin{cases} 
-\frac{\phi_z}{\phi_c} + \frac{\varphi}{\phi_c} \sqrt{\frac{\mu}{\omega \sigma^2 \ln(2)}} & \text{if } \frac{\omega (\phi_c \varphi + \phi_z)^2 \sigma^2 \ln(2)}{\mu} \leq 1 \\
-\frac{\phi_z}{\phi_c} + \frac{1}{\phi_c} \sqrt{\frac{\mu}{\omega \sigma^2 \ln(2)}} & \text{if } \frac{\omega (\phi_c \varphi + \phi_z)^2 \sigma^2 \ln(2)}{\mu} > 1 
\end{cases}
\]

At this policy, price setters in firms pay no attention to the variable \( z_t \), the price level does not respond to an innovation in \( z_t \), and there is no inefficient price dispersion.

**Proof.** See Appendix A. \( \blacksquare \)

The following proposition is a generalization of Proposition 6 in the paper. It is a generalization because the desired markup is just a special case of a variable \( z_t \) with the property \( \varphi > -\frac{\phi_c}{\phi_z} \).

**Proposition 2** (Endogenous signal precision) Consider the Ramsey problem (1)-(13), where the signal precision \( \frac{1}{\sigma_c^2} \) is given by the solution to problem (11)-(13). Suppose that \( \mu > 0 \), \( \sigma_z^2 > 0 \), \( \rho_z = 0 \) and \( \varphi > -\frac{\phi_z}{\phi_c} \). Consider policies of the form \( m_t = g_0 z_t \) and equilibria of the form \( p_t = \theta z_t \).

If \( \phi_c \in (0, \frac{1}{2}) \), there exist multiple equilibria for all \( g_0 \in [\hat{g}_0, \bar{g}_0] \) where
\[
\hat{g}_0 = -\frac{\phi_z}{\phi_c} + \frac{4\phi_c (1 - \phi_c)}{\phi_c} \sqrt{\frac{\mu}{\omega \sigma^2 \ln(2)}},
\]

and
\[
\bar{g}_0 = -\frac{\phi_z}{\phi_c} + \frac{1}{\phi_c} \sqrt{\frac{\mu}{\omega \sigma^2 \ln(2)}}.
\]

If \( \frac{\omega (\phi_c \varphi + \phi_z)^2 \sigma^2 \ln(2)}{\mu} < 4\phi_c (1 - \phi_c) \), the best policy among all \( g_0 \in \mathbb{R} \) that yield a unique equilibrium is \( g_0^* = \varphi \). If \( \frac{\omega (\phi_c \varphi + \phi_z)^2 \sigma^2 \ln(2)}{\mu} \geq 4\phi_c (1 - \phi_c) \), the best policy among all \( g_0 \in \mathbb{R} \) that yield a unique equilibrium is a \( g_0 \) marginally below \( \bar{g}_0 \). At this policy, price setters in firms pay no attention to the variable \( z_t \), the price level does not respond to an innovation in \( z_t \), and there is no inefficient price dispersion.

**Proof.** See Appendix A. \( \blacksquare \)

In summary, when \( \mu > 0 \) and \( \rho_z = 0 \), complete price stabilization is optimal in response to any variable \( z_t \) with the property \( \varphi > -\frac{\phi_z}{\phi_c} \). At the optimal monetary policy, price setters pay no attention to the variable \( z_t \).
2 Monetary transaction frictions

Section 8.6 of the paper presents results for a Ramsey problem with monetary transaction frictions. In the new Ramsey problem, the original equation for the profit-maximizing price is replaced by

\[ p_{i,t}^* = p_t + \phi_c c_t - \phi_a a_t + \phi_\lambda \lambda_t \]

\[ + \frac{1}{1 + \frac{1 - \alpha}{\alpha} \frac{1 + \lambda}{\lambda}} E_t \left[ \gamma (c_{t+1} - c_t) + (p_{t+1} - p_t) \right]. \]

Formally, the new Ramsey problem reads:

\[
\min_{\{F_t(L), G_t(L)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \mathbb{E} \left[ \left( c_t - c_t^* \right)^2 + \delta \frac{1}{T} \sum_{i=1}^{I} (p_{i,t} - p_t)^2 \right],
\]

subject to

\[ c_t^* = \frac{\phi_a a_t}{\phi_c}, \] (15)

\[ c_t = m_t - p_t, \] (16)

\[ p_t = \frac{1}{T} \sum_{i=1}^{I} p_{i,t}, \] (17)

\[ p_{i,t} = \mathbb{E} \left[ p_{i,t}^* | \mathcal{I}_{i,t} \right], \] (18)

\[ p_{i,t}^* = p_t + \phi_c c_t - \phi_a a_t + \phi_\lambda \lambda_t + \frac{1}{1 + \frac{1 - \alpha}{\alpha} \frac{1 + \lambda}{\lambda}} E_t \left[ \gamma (c_{t+1} - c_t) + (p_{t+1} - p_t) \right]. \] (19)

\[ \mathcal{I}_{i,t} = \mathcal{I}_{i,t-1} \cup \{ s_{i,0}, s_{i,1}, \ldots, s_{i,t} \}, \] (20)

\[ s_{i,t} = \begin{pmatrix} a_t + \eta_{i,t} \\ \lambda_t + \zeta_{i,t} \end{pmatrix}, \] (21)

\[ a_t = \rho_a a_{t-1} + \varepsilon_t, \] (22)

\[ \lambda_t = \rho_\lambda \lambda_{t-1} + \nu_t, \] (23)

and

\[ m_t = F_t(L) \varepsilon_t + G_t(L) \nu_t, \] (24)
where

\[ \phi_c = \frac{\psi + \gamma + \frac{1-\alpha}{\alpha}}{1 + \frac{1-\alpha}{\alpha (1+\Lambda)}} > 0, \] (25)

\[ \phi_a = \frac{\psi + \frac{1}{\alpha}}{1 + \frac{1-\alpha}{\alpha (1+\Lambda)}} > 0, \] (26)

\[ \phi_\lambda = \frac{\Lambda}{1 + \frac{1-\alpha}{\alpha (1+\Lambda)}} > 0, \] (27)

\[ \delta = \frac{\frac{1+\Lambda}{1+\alpha} (1+\frac{1}{\alpha})^2}{\gamma - 1 + \frac{1}{\alpha (1+\psi)}} > 0. \] (28)

The innovations \( \varepsilon_t \) and \( \nu_t \) follow independent Gaussian white noise processes. The noise terms \( \eta_{i,t} \) and \( \zeta_{i,t} \) follow independent Gaussian white noise processes that are independent across firms and independent of the \( \varepsilon \) process and the \( \nu \) process. In the model with an endogenous signal precision, signal precision is given by the solution to:

\[
\min_{(1/\sigma_\eta^2, 1/\sigma_\zeta^2) \in \mathbb{R}^2_+} \left\{ \sum_{t=0}^{\infty} \beta^t \omega \frac{E_{i,-1}}{2} \left[ \left( p_{i,t} - p_{i,t}^* \right)^2 \right] + \frac{\mu}{1 - \beta} \kappa \right\},
\] (29)

subject to

\[ p_{i,t} = E \left[ p_{i,t}^* | \mathcal{I}_{i,t} \right], \] (30)

and

\[
\frac{1}{2} \log_2 \left( \frac{\sigma_{\eta|t-1}^2}{\sigma_{\eta|t}^2} \right) + \frac{1}{2} \log_2 \left( \frac{\sigma_{\zeta|t-1}^2}{\sigma_{\zeta|t}^2} \right) = \kappa,
\] (31)

where

\[
\omega = C^{-\gamma} \frac{WL_i}{P} \frac{1+\Lambda}{\alpha} \left( 1 + \frac{1-\alpha}{\alpha (1+\Lambda)} \right) > 0.
\] (32)

The next proposition characterizes the equilibrium allocation as a function of monetary policy in the case of \( \rho_\Lambda = 0 \).

**Proposition 3** (Endogenous signal precision) Consider the Ramsey problem (14)-(32), where the signal precisions \( 1/\sigma_\eta^2 \) and \( 1/\sigma_\zeta^2 \) are given by the solution to problem (29)-(32). Suppose that \( \mu > 0, \sigma_\eta^2 > 0, \rho_\lambda = 0 \) and \( \sigma_\varepsilon^2 = a_{-1} = 0 \). Consider policies of the form \( G_t(L) \nu_t = g_0 \nu_t \) and
equilibria of the form $p_t = \theta \lambda_t$. Define

$$b \equiv \sqrt{\frac{\omega \left( \hat{\phi}_c g_0 + \phi \lambda \right)^2 \sigma^2 \ln (2)}}.$$

$$\hat{\phi}_c \equiv \phi_c - \frac{\gamma}{1 + \frac{\lambda}{\alpha} + \Lambda}.$$

$$\tilde{\phi}_c \equiv \phi_c + \frac{1 - \gamma}{1 + \frac{\lambda}{\alpha} + \Lambda}.$$

Assume that $\psi > 0$ or $\alpha \in (0, 1)$. This implies $\hat{\phi}_c > 0$ and $\tilde{\phi}_c > 0$. We now characterize the set of equilibria at a given monetary policy $g_0 \in \mathbb{R}$. Here $\kappa^*$ denotes equilibrium attention devoted to the desired markup. If and only if $b \leq 1$, there exists an equilibrium with

$$\kappa^* = 0.$$

If and only if $\hat{\phi}_c \in (0, \frac{1}{2}]$ and $b \in \left[ \sqrt{4\hat{\phi}_c \left( 1 - \hat{\phi}_c \right)}, 1 \right]$, there exists an equilibrium with

$$\kappa^* = \log_2 \left( \frac{b - \sqrt{b^2 - 4\tilde{\phi}_c \left( 1 - \tilde{\phi}_c \right)}}{2\tilde{\phi}_c} \right).$$

If and only if either $\tilde{\phi}_c \in (0, \frac{1}{2}]$ and $b \geq \sqrt{4\tilde{\phi}_c \left( 1 - \tilde{\phi}_c \right)}$ or $\tilde{\phi}_c > \frac{1}{2}$ and $b \geq 1$, there exists an equilibrium with

$$\kappa^* = \log_2 \left( \frac{b + \sqrt{b^2 - 4\tilde{\phi}_c \left( 1 - \tilde{\phi}_c \right)}}{2\tilde{\phi}_c} \right).$$

The equilibrium price level, composite consumption and price dispersion are given by

$$p_t = \frac{\left( \hat{\phi}_c g_0 + \phi \lambda \right) \left( 1 - 2^{-2\kappa^*} \right)}{1 - \left( 1 - \hat{\phi}_c \right) \left( 1 - 2^{-2\kappa^*} \right)} \lambda_t,$$

$$c_t = \left[ g_0 - \frac{\left( \hat{\phi}_c g_0 + \phi \lambda \right) \left( 1 - 2^{-2\kappa^*} \right)}{1 - \left( 1 - \hat{\phi}_c \right) \left( 1 - 2^{-2\kappa^*} \right)} \right] \lambda_t,$$

and

$$E \left[ (p_{t,t} - p_t)^2 \right] = \frac{\mu}{\omega} \frac{\mu}{\ln (2)} \left( 1 - 2^{-2\kappa^*} \right).$$

**Proof.** See Appendix B.
A Proof of Propositions 1 and 2

Steps 1-5: Identical to steps 1-5 in the proof of Proposition 5 in the paper, apart from small changes in notation: $z_t, \phi_z, \sigma_z^2, s_{i,t}$ and $\sigma^2_{z|x}$ replace $\lambda_t, \phi_\lambda, \sigma_\lambda^2, s_{\lambda,i,t}$ and $\sigma^2_{\lambda|x}$, respectively. There are two reasons why this part of the proof does not change. First, steps 1-5 in the proof of Proposition 5 characterize the equilibrium for a given monetary policy. Therefore, this part of the proof does not depend on the central bank’s objective and the equation for efficient composite consumption. Second, steps 1-5 in the proof of Proposition 5 do not rely on particular values for $\phi_\lambda$ and $\sigma^2_\lambda$. They only require that these two parameters are strictly positive.

Step 6: Optimal monetary policy has to satisfy $g_0 \geq -\frac{\phi_z}{\phi_c}$. Having characterized the set of rational expectations equilibria of the form $p_t = \theta z_t$ for given monetary policy $g_0$, we now derive results concerning optimal monetary policy. We begin by showing that optimal monetary policy has to satisfy $g_0 \geq -\frac{\phi_z}{\phi_c}$. The proof is as follows. First, at the monetary policy $g_0 = -\frac{\phi_z}{\phi_c}$ we have $b = 0$ and thus the unique rational expectations equilibrium of the form $p_t = \theta z_t$ is a zero attention equilibrium, implying that price dispersion equals zero and inefficient consumption variance equals

$$E\left[(c_t - c^*_t)^2 \right] = \left(-\frac{\phi_z}{\phi_c} - \varphi\right)^2 \sigma_z^2.$$  

Second, consider a monetary policy $g_0 < -\frac{\phi_z}{\phi_c}$. Price dispersion at a monetary policy $g_0 < -\frac{\phi_z}{\phi_c}$ is weakly larger than price dispersion at the monetary policy $g_0 = -\frac{\phi_z}{\phi_c}$ because price dispersion is always weakly larger than zero. Furthermore, inefficient consumption variance at a monetary policy $g_0 < -\frac{\phi_z}{\phi_c}$ is strictly larger than inefficient consumption variance at the monetary policy $g_0 = -\frac{\phi_z}{\phi_c}$. This result follows from the fact that inefficient consumption variance at an equilibrium is given by

$$E\left[(c_t - c^*_t)^2 \right] = \left[g_0 - \frac{(\phi_c g_0 + \phi_z)\left(1 - 2 - \frac{2\phi^*}{\phi_c}\right)}{1 - (1 - \phi_c)(1 - 2 - 2\phi^*)} - \varphi\right]^2 \sigma_z^2,$$

and for all $g_0 < -\frac{\phi_z}{\phi_c}$, the condition $\varphi > -\frac{\phi_z}{\phi_c}$ implies

$$g_0 - \frac{(\phi_c g_0 + \phi_z)\left(1 - 2 - \frac{2\phi^*}{\phi_c}\right)}{1 - (1 - \phi_c)(1 - 2 - 2\phi^*)} - \varphi < -\frac{\phi_z}{\phi_c} - \varphi < 0.$$  

In summary, a monetary policy $g_0 < -\frac{\phi_z}{\phi_c}$ yields weakly larger price dispersion and strictly larger inefficient consumption variance than the monetary policy $g_0 = -\frac{\phi_z}{\phi_c}$. Hence, a monetary policy $g_0 < -\frac{\phi_z}{\phi_c}$ cannot be optimal.

Step 7: Optimal monetary policy when $\phi_c \geq \frac{1}{2}$. First, when $\phi_c \geq \frac{1}{2}$, there exists a unique rational expectations equilibrium of the form $p_t = \theta z_t$ for any monetary policy $g_0 \in \mathbb{R}$: if the variable
$b \in [0, 1]$ then $\kappa^* = 0$ is the unique equilibrium; if the variable $b = 1$ then $\kappa^* = \log_2 (x_H) = 0$ is the unique equilibrium; and if the variable $b > 1$ then $\kappa^* = \log_2 (x_H)$ is the unique equilibrium, where $x_H$ is defined as 

$$x_H = \frac{b + \sqrt{b^2 - 4\phi_c (1 - \phi_c)}}{2\phi_c}. \quad (33)$$

See step 3. Second, in the derivation of optimal monetary policy we can focus on $g_0 \geq -\frac{\phi_c}{\phi_c}$. See step 6. Furthermore, the definition of the variable $b$ implies that $b$ is an increasing function of $g_0$ for all $g_0 \geq -\frac{\phi_c}{\phi_c}$. Define $\bar{g}_0$ as the value of $g_0 \in \left[-\frac{\phi_c}{\phi_c}, \infty\right]$ at which $b = 1$. Formally,

$$\bar{g}_0 = -\frac{\phi_c}{\phi_c} + \frac{1}{\phi_c} \sqrt{\frac{\mu}{\omega \sigma^2 \ln (2)}}.$$

Hence, if $g_0 \in \left[-\frac{\phi_c}{\phi_c}, \bar{g}_0\right]$ then $b \in [0, 1]$ and $\kappa^* = 0$ is the unique equilibrium; if $g_0 = \bar{g}_0$ then $b = 1$ and $\kappa^* = \log_2 (x_H) = 0$ is the unique equilibrium; and if $g_0 > \bar{g}_0$ then $b > 1$ and $\kappa^* = \log_2 (x_H)$ is the unique equilibrium. For the rest of this proof it is important that the condition $\varphi > -\frac{\phi_c}{\phi_c}$ implies that in the case of $\frac{\omega (\phi_c \varphi + \phi_i)^2 \sigma^2 \ln (2)}{\mu} \leq 1$ we have $\varphi \leq \bar{g}_0$ while in the case of $\frac{\omega (\phi_c \varphi + \phi_i)^2 \sigma^2 \ln (2)}{\mu} > 1$ we have $\varphi > \bar{g}_0$. Third, consider the case of $\phi_c \geq \frac{1}{2}$ and $\frac{\omega (\phi_c \varphi + \phi_i)^2 \sigma^2 \ln (2)}{\mu} \leq 1$. Then, at the policy $g_0 = \varphi$, we have $g_0 \leq \bar{g}_0$ and thus $\kappa^* = 0$ is the unique equilibrium, implying that price dispersion equals zero, $E \left[(p_{i,t} - p_t)^2\right] = 0$, and inefficient consumption variance equals zero, $E \left[(c_t - \bar{c}^*_t)^2\right] = 0$. Thus, in the case of $\phi_c \geq \frac{1}{2}$ and $\frac{\omega (\phi_c \varphi + \phi_i)^2 \sigma^2 \ln (2)}{\mu} \leq 1$ the policy $g_0 = \varphi$ attains the efficient allocation. In addition, any monetary policy $g_0 \neq \varphi$ does not attain the efficient allocation. If the equilibrium at the monetary policy $g_0 \neq \varphi$ is an equilibrium with $\kappa^* = 0$ then inefficient consumption variance is strictly positive while if the equilibrium at the monetary policy $g_0 \neq \varphi$ is an equilibrium with $\kappa^* > 0$ then price dispersion is strictly positive. Hence, in the case of $\phi_c \geq \frac{1}{2}$ and $\frac{\omega (\phi_c \varphi + \phi_i)^2 \sigma^2 \ln (2)}{\mu} \leq 1$ the unique optimal monetary policy is $g_0^* = \varphi$. Fourth, consider the case of $\phi_c \geq \frac{1}{2}$ and $\frac{\omega (\phi_c \varphi + \phi_i)^2 \sigma^2 \ln (2)}{\mu} > 1$. We derive the optimal monetary policy in this case by showing that the monetary policy minimizing the central bank’s objective among all $g_0 \in \left[-\frac{\phi_c}{\phi_c}, \bar{g}_0\right]$ is $g_0 = \bar{g}_0$ and the monetary policy minimizing the central bank’s objective among all $g_0 \in [\bar{g}_0, \infty)$ is $g_0 = \bar{g}_0$. Combining these results yields that in the case of $\phi_c \geq \frac{1}{2}$ and $\frac{\omega (\phi_c \varphi + \phi_i)^2 \sigma^2 \ln (2)}{\mu} > 1$ the unique optimal monetary policy among all $g_0 \in \mathbb{R}$ is $g_0 = \bar{g}_0$. The details are as follows. For all $g_0 \in \left[-\frac{\phi_c}{\phi_c}, \bar{g}_0\right]$, $\kappa^* = 0$ is the unique equilibrium, implying that price dispersion equals zero and inefficient consumption variance equals $E \left[(c_t - \bar{c}^*_t)^2\right] = (g_0 - \varphi)^2 \sigma^2$. Furthermore, $\frac{\omega (\phi_c \varphi + \phi_i)^2 \sigma^2 \ln (2)}{\mu} > 1$ and $g_0 \leq \bar{g}_0$ implies $g_0 \leq \bar{g}_0 < \varphi$. Thus, $E \left[(c_t - \bar{c}^*_t)^2\right]$ is
decreasing in $g_0$. Hence, in the case of $\phi_c \geq \frac{1}{2}$ and $\frac{\omega(\phi_c^2 + \phi_z^2)\ln(2)}{\mu} > 1$ the monetary policy minimizing objective (1) among all monetary policies $g_0 \in [-\frac{\phi_z}{\phi_c}, \bar{g}_0]$ is $g_0 = \bar{g}_0$. Next, for all $g_0 \in [\bar{g}_0, \infty)$, $\kappa^* = \log_2 (x_H)$ is the unique equilibrium. Let us study price dispersion and inefficient consumption variance in an equilibrium with $\kappa^* = \log_2 (x_H)$. Equilibrium price dispersion equals

$$E \left[ (p_{i,t} - p_t)^2 \right] = \frac{\mu}{\ln(2)} \left( 1 - 2^{-2\kappa^*} \right).$$

See step 5. Since equilibrium price dispersion is strictly increasing in $\kappa^*$, $\kappa^* = \log_2 (x_H)$, $x_H$ is strictly increasing in the variable $b$, and $b$ is strictly increasing in $g_0$ for all $g_0 \geq \bar{g}_0$, we have that equilibrium price dispersion is strictly increasing in $g_0$ for all $g_0 \geq \bar{g}_0$. Let us turn to inefficient consumption variance. We first collect a few equilibrium relationships. Equilibrium consumption is given by

$$c_t = \left[ g_0 - \frac{(\phi_z g_0 + \phi_z)}{1 - (1 - \phi_c)(1 - 2^{-2\kappa^*})} \right] z_t.$$

See step 5. Furthermore, in an equilibrium with $\kappa^* = \log_2 (x_H)$, the following equation holds

$$\kappa^* = \frac{1}{2} \log_2 \left( \frac{\omega(\phi_z g_0 + \phi_z)^2}{[1 - (1 - \phi_c)(1 - 2^{-2\kappa^*})] \sigma_z^2 \ln(2)} \right).$$

See step 3. Rearranging the last equation using $g_0 \geq -\frac{\phi_z}{\phi_c}$ yields

$$\frac{\phi_z g_0 + \phi_z}{1 - (1 - \phi_c)(1 - 2^{-2\kappa^*})} = \sqrt{\frac{\mu}{\omega \sigma_z^2 \ln(2)}} \cdot 2^{\kappa^*}.$$

Substituting the last equation into the equation for equilibrium consumption yields

$$c_t = \left[ g_0 - \sqrt{\frac{\mu}{\omega \sigma_z^2 \ln(2)}} \left( 2^{\kappa^*} - 2^{-\kappa^*} \right) \right] z_t. \quad (34)$$

Furthermore, solving the definition of the variable $b$ for $g_0$ using $g_0 \geq -\frac{\phi_z}{\phi_c}$ yields

$$g_0 = -\frac{\phi_z}{\phi_c} + \frac{b}{\phi_c} \sqrt{\frac{\mu}{\omega \sigma_z^2 \ln(2)}}. \quad (35)$$

Substituting equation (35), $\kappa^* = \log_2 (x_H)$ and equation (33) into equation (34) yields

$$c_t = \left[ -\frac{\phi_z}{\phi_c} + \sqrt{\frac{\mu}{\omega \sigma_z^2 \ln(2)}} \left( \frac{b}{\phi_c} - \frac{b + \sqrt{b^2 - 4\phi_c (1 - \phi_c)}}{2\phi_c} + \frac{1}{2\phi_c} \right) \right] z_t.$$
Rearranging the last equation yields
\[ c_t = \left[ -\frac{\phi_z}{\phi_c} + \sqrt{\frac{\mu}{\omega \sigma_z^2 \ln(2)}} \cdot \frac{2}{b + \sqrt{b^2 - 4\phi_c (1 - \phi_c)}} \right] z_t, \] (36)

Hence, when \( g_0 \geq \bar{g}_0 \), inefficient consumption variance equals
\[ E \left[ (c_t - c_t^*)^2 \right] = \left[ -\frac{\phi_z}{\phi_c} + \sqrt{\frac{\mu}{\omega \sigma_z^2 \ln(2)}} \cdot \frac{2}{b + \sqrt{b^2 - 4\phi_c (1 - \phi_c)}} - \varphi \right]^2 \sigma_z^2. \] (37)

The term in square brackets in equation (37) is strictly decreasing in \( b \) for all \( b \geq 1 \). Furthermore, in the case of \( \phi_c \geq \frac{1}{2} \) and \( \omega(\phi_c + \phi_c)^2 \sigma_z^2 \ln(2) > 1 \), the condition \( \varphi > -\frac{\phi_z}{\phi_c} \) implies that the term in square brackets in equation (37) is strictly negative at \( b = 1 \). Thus, in the case of \( \phi_c \geq \frac{1}{2}, \varphi > -\frac{\phi_z}{\phi_c} \) and \( \omega(\phi_c + \phi_c)^2 \sigma_z^2 \ln(2) > 1 \), inefficient consumption variance is strictly increasing in \( g_0 \) for all \( g_0 \geq \bar{g}_0 \). Hence, in the case of \( \phi_c \geq \frac{1}{2}, \varphi > -\frac{\phi_z}{\phi_c} \) and \( \omega(\phi_c + \phi_c)^2 \sigma_z^2 \ln(2) > 1 \), both price dispersion and inefficient consumption variance are strictly increasing in \( g_0 \) for all \( g_0 \geq \bar{g}_0 \), implying that the monetary policy minimizing the central bank’s objective (1) among all \( g_0 \in [\bar{g}_0, \infty) \) is \( g_0 = \bar{g}_0 \).

Combining results yields that in the case of \( \phi_c \geq \frac{1}{2} \), \( \varphi > -\frac{\phi_z}{\phi_c} \) and \( \omega(\phi_c + \phi_c)^2 \sigma_z^2 \ln(2) > 1 \) the unique optimal monetary policy among all policies \( g_0 \in \mathbb{R} \) is \( g_0 = \bar{g}_0 \). At this policy, price setters in firms pay no attention to the variable \( z_t \), the price level does not respond to an innovation in \( z_t \), and there is no inefficient price dispersion. This completes the proof of Proposition 1.

**Step 8: Optimal monetary policy when \( \phi_c \in (0, \frac{1}{2}) \).** First, when \( \phi_c \in (0, \frac{1}{2}) \), there exist multiple rational expectations equilibria of the form \( p_t = \theta z_t \) for some monetary policies \( g_0 \in \mathbb{R} \).

We will use the following results below. If \( b \in \left[ 0, \sqrt{4\phi_c (1 - \phi_c)} \right] \) then \( \kappa^* = 0 \) is an equilibrium. If \( b \geq 1 \) then \( \kappa^* = \log_2 (x_H) \) is an equilibrium. In addition, if \( b \in \left[ 0, \sqrt{4\phi_c (1 - \phi_c)} \right] \) or \( b > 1 \) there exists a unique equilibrium, whereas if \( b \in \left[ \sqrt{4\phi_c (1 - \phi_c)}, 1 \right] \) there exist multiple equilibria. See step 4. Second, in the derivation of optimal monetary policy we can focus on \( g_0 \geq -\frac{\phi_z}{\phi_c} \). See step 6. Furthermore, the definition of the variable \( b \) implies that \( b \) is strictly increasing in \( g_0 \) for all \( g_0 \geq -\frac{\phi_z}{\phi_c} \). Define \( \tilde{g}_0 \) as the value of \( g_0 \in \left[ -\frac{\phi_z}{\phi_c}, \infty \right) \) at which \( b = \sqrt{4\phi_c (1 - \phi_c)} \). Define \( \bar{g}_0 \) as the value of \( g_0 \in \left[ -\frac{\phi_z}{\phi_c}, \infty \right) \) at which \( b = 1 \). Formally,
\[ \tilde{g}_0 = -\frac{\phi_z}{\phi_c} + \frac{\sqrt{4\phi_c (1 - \phi_c)}}{\phi_c} \sqrt{\frac{\mu}{\omega \sigma_z^2 \ln(2)}}, \]
and
\[ \bar{g}_0 = -\frac{\phi_z}{\phi_c} + \frac{1}{\phi_c} \sqrt{\frac{\mu}{\omega \sigma_z^2 \ln(2)}}. \]
For all \( g_0 \in \left[ -\frac{\phi_x}{\phi_c}, \hat{g}_0 \right] \), \( \kappa^* = 0 \) is the unique equilibrium. For all \( g_0 \in [\hat{g}_0, \bar{g}_0] \), there exist multiple equilibria. For all \( g_0 > \bar{g}_0 \), \( \kappa^* = \log_2 (x_H) \) is the unique equilibrium. Note that the conditions
\[
\frac{\omega (\phi_c, \varphi + \phi_c)^2 \sigma^2 \ln(2)}{\mu} < 4\phi_c (1 - \phi_c) \quad \text{and} \quad \varphi > -\frac{\phi_x}{\phi_c}
\]
imply \( \varphi < \hat{g}_0 \). Third, consider the case of \( \phi_c \in (0, \frac{1}{2}) \),
\[
\frac{\omega (\phi_c, \varphi + \phi_c)^2 \sigma^2 \ln(2)}{\mu} < 4\phi_c (1 - \phi_c) \quad \text{and} \quad \varphi > -\frac{\phi_x}{\phi_c}.
\]
At the monetary policy \( g_0 = \varphi \), we have \( g_0 < \hat{g}_0 \) and thus \( \kappa^* = 0 \) is the unique equilibrium, implying that price dispersion equals zero and inefficient consumption variance equals zero. Thus, when \( \phi_c \in (0, \frac{1}{2}) \),
\[
\frac{\omega (\phi_c, \varphi + \phi_c)^2 \sigma^2 \ln(2)}{\mu} < 4\phi_c (1 - \phi_c) \quad \text{and} \quad \varphi > -\frac{\phi_x}{\phi_c},
\]
the monetary policy \( g_0 = \varphi \) attains the efficient allocation as the unique equilibrium allocation. Furthermore, any monetary policy \( g_0 \neq \varphi \) does not attain the efficient allocation. If the equilibrium at the monetary policy \( g_0 \neq \varphi \) is an equilibrium with \( \kappa^* = 0 \) then inefficient consumption variance is strictly positive, while if the equilibrium at the monetary policy \( g_0 \neq \varphi \) is an equilibrium with \( \kappa^* > 0 \) then price dispersion is strictly positive. Hence, when \( \phi_c \in (0, \frac{1}{2}) \),
\[
\frac{\omega (\phi_c, \varphi + \phi_c)^2 \sigma^2 \ln(2)}{\mu} < 4\phi_c (1 - \phi_c) \quad \text{and} \quad \varphi > -\frac{\phi_x}{\phi_c},
\]
the unique optimal monetary policy is \( g_0^* = \varphi \).

Fourth, consider the case of \( \phi_c \in (0, \frac{1}{2}) \),
\[
\frac{\omega (\phi_c, \varphi + \phi_c)^2 \sigma^2 \ln(2)}{\mu} \geq 4\phi_c (1 - \phi_c) \quad \text{and} \quad \varphi > -\frac{\phi_x}{\phi_c}.
\]
We now show that in this case: (i) the value of the central bank’s objective (1) at an equilibrium with \( \kappa^* = 0 \) is strictly decreasing and continuous in \( g_0 \) for all \( g_0 \in \left[ -\frac{\phi_x}{\phi_c}, \hat{g}_0 \right] \), (ii) the value of the central bank’s objective (1) at an equilibrium with \( \kappa^* = \log_2 (x_H) \) is strictly increasing in \( g_0 \) for all \( g_0 \geq \hat{g}_0 \), and (iii) the equilibrium with \( \kappa^* = 0 \) at \( g_0 = \hat{g}_0 \) yields a strictly smaller value of the central bank’s objective (1) than the equilibrium with \( \kappa^* = \log_2 (x_H) \) at \( g_0 = \bar{g}_0 \).

For all \( g_0 \in \left[ -\frac{\phi_x}{\phi_c}, \hat{g}_0 \right] \), \( \kappa^* = 0 \) is an equilibrium. In a zero attention equilibrium, price dispersion equals zero and inefficient consumption variance equals
\[
E \left[ (c_t - c_t^*)^2 \right] = (g_0 - \varphi)^2 \sigma^2.
\]
Furthermore, \( \phi_c \in (0, \frac{1}{2}) \),
\[
\frac{\omega (\phi_c, \varphi + \phi_c)^2 \sigma^2 \ln(2)}{\mu} \geq 4\phi_c (1 - \phi_c) \quad \text{and} \quad g_0 \leq \hat{g}_0 \) implies \( g_0 \leq \hat{g}_0 \leq \varphi \). Hence, in the case of \( \phi_c \in (0, \frac{1}{2}) \),
\[
\frac{\omega (\phi_c, \varphi + \phi_c)^2 \sigma^2 \ln(2)}{\mu} \geq 4\phi_c (1 - \phi_c) \quad \text{and} \quad \varphi > -\frac{\phi_x}{\phi_c}.
\]
the value of the central bank’s objective (1) at an equilibrium with \( \kappa^* = 0 \) is strictly decreasing and continuous in \( g_0 \) for all \( g_0 \in \left[ -\frac{\phi_x}{\phi_c}, \hat{g}_0 \right] \). Next, for all \( g_0 \geq \hat{g}_0 \), \( \kappa^* = \log_2 (x_H) \) is an equilibrium. Equilibrium price dispersion equals
\[
E \left[ (p_{t,t} - p_{t})^2 \right] = \frac{\mu}{\ln(2)} \left( 1 - 2^{-2\kappa^*} \right).
\]
See step 5. Since equilibrium price dispersion is strictly increasing in \( \kappa^* \), \( \kappa^* = \log_2 (x_H) \), \( x_H \) is strictly increasing in the variable \( b \), and \( b \) is strictly increasing in \( g_0 \) for all \( g_0 \geq \hat{g}_0 \), we have that equilibrium price dispersion is strictly increasing in \( g_0 \) for all \( g_0 \geq \hat{g}_0 \). Let us turn to inefficient consumption variance. The same derivation as in step 7 yields that, when \( g_0 \geq -\frac{\phi_x}{\phi_c} \), in an
equilibrium with $\kappa^* = \log_2 (x_H)$ inefficient consumption variance equals

$$E \left[ (c_t - c^*_t)^2 \right] = \left[ -\frac{\phi_z}{\phi_c} + \sqrt{\frac{\mu}{\omega \sigma_z^2 \ln(2)}} b + \sqrt{b^2 - 4\phi_c(1 - \phi_c)} - \varphi \right]^2 \sigma_z^2. \tag{39}$$

The term in square brackets in equation (39) is strictly decreasing in $b$ for all $b \geq 1$. Furthermore, in the case of $\phi_c \in (0, \frac{1}{2})$, $\frac{\omega(\phi_c \varphi + \phi_c)^2 \sigma_z^2 \ln(2)}{\mu} \geq 4\phi_c(1 - \phi_c)$ and $\varphi > -\frac{\phi_z}{\phi_c}$, the term in square brackets in equation (39) is strictly negative at $b = 1$. Hence, in the case of $\phi_c \in (0, \frac{1}{2})$, $\frac{\omega(\phi_c \varphi + \phi_c)^2 \sigma_z^2 \ln(2)}{\mu} \geq 4\phi_c(1 - \phi_c)$ and $\varphi > -\frac{\phi_z}{\phi_c}$, price dispersion and inefficient consumption variance at an equilibrium with $\kappa^* = \log_2 (x_H)$ are strictly increasing in $g_0$ for all $g_0 \geq \bar{g}_0$, implying that the value of the central bank’s objective (1) at an equilibrium with $\kappa^* = \log_2 (x_H)$ is strictly increasing in $g_0$ for all $g_0 \geq \bar{g}_0$.

Finally, we compare the equilibrium with $\kappa^* = 0$ at $g_0 = \hat{g}_0$ to the equilibrium with $\kappa^* = \log_2 (x_H)$ at $g_0 = \bar{g}_0$. At $g_0 = \bar{g}_0$ we have $b = 1$ and thus $\kappa^* = \log_2 (x_H) = \log_2 \left( \frac{1}{\phi_c} - 1 \right) > 0$ in the case of $\phi_c \in (0, \frac{1}{2})$. See equation (33). Thus, price dispersion is strictly smaller in the zero attention equilibrium at $g_0 = \bar{g}_0$ than in the equilibrium with $\kappa^* = \log_2 (x_H)$ at $g_0 = \bar{g}_0$. See equation (38).

Furthermore, inefficient consumption variance is strictly smaller in the zero attention equilibrium at $g_0 = \bar{g}_0$ than in the equilibrium with $\kappa^* = \log_2 (x_H)$ at $g_0 = \bar{g}_0$ because the conditions $\phi_c \in (0, \frac{1}{2})$, $\frac{\omega(\phi_c \varphi + \phi_c)^2 \sigma_z^2 \ln(2)}{\mu} \geq 4\phi_c(1 - \phi_c)$ and $\varphi > -\frac{\phi_z}{\phi_c}$ imply

$$(\hat{g}_0 - \varphi)^2 \sigma_z^2 < \left[ -\frac{\phi_z}{\phi_c} + \sqrt{\frac{\mu}{\omega \sigma_z^2 \ln(2)}} b + \sqrt{1 - 4\phi_c(1 - \phi_c)} - \varphi \right]^2 \sigma_z^2. \tag{40}$$

Hence, in the case of $\phi_c \in (0, \frac{1}{2})$, $\frac{\omega(\phi_c \varphi + \phi_c)^2 \sigma_z^2 \ln(2)}{\mu} \geq 4\phi_c(1 - \phi_c)$ and $\varphi > -\frac{\phi_z}{\phi_c}$, the equilibrium with $\kappa^* = 0$ at $g_0 = \hat{g}_0$ yields a strictly smaller value of the central bank’s objective (1) than the equilibrium with $\kappa^* = \log_2 (x_H)$ at $g_0 = \bar{g}_0$. In summary, in the case of $\phi_c \in (0, \frac{1}{2})$, $\frac{\omega(\phi_c \varphi + \phi_c)^2 \sigma_z^2 \ln(2)}{\mu} \geq 4\phi_c(1 - \phi_c)$ and $\varphi > -\frac{\phi_z}{\phi_c}$, we have the following four results: (i) there exists a unique rational expectations equilibrium of the form $p_t = \theta z_t$ for all $g_0 \in \left[ -\frac{\phi_z}{\phi_c}, \hat{g}_0 \right]$ and $g_0 > \bar{g}_0$, while there exist multiple rational expectations equilibria of the form $p_t = \theta z_t$ for all $g_0 \in \left[ \hat{g}_0, \bar{g}_0 \right]$, (ii) the value of objective (1) at an equilibrium with $\kappa^* = 0$ is strictly decreasing and continuous in $g_0$ for all $g_0 \in \left[ -\frac{\phi_z}{\phi_c}, \hat{g}_0 \right]$, (iii) the value of objective (1) at an equilibrium with $\kappa^* = \log_2 (x_H)$ is strictly increasing in $g_0$ for all $g_0 \geq \bar{g}_0$, and (iv) the equilibrium with $\kappa^* = 0$ at $g_0 = \hat{g}_0$ yields a strictly smaller value of objective (1) than the equilibrium with $\kappa^* = \log_2 (x_H)$ at $g_0 = \bar{g}_0$. Hence, in the case of $\phi_c \in (0, \frac{1}{2})$, $\frac{\omega(\phi_c \varphi + \phi_c)^2 \sigma_z^2 \ln(2)}{\mu} \geq 4\phi_c(1 - \phi_c)$ and $\varphi > -\frac{\phi_z}{\phi_c}$, the best the central bank can do among all monetary policies $g_0 \in \mathbb{R}$ if the central bank wants to obtain a unique equilibrium.
of the form \( p_t = \theta z_t \) is to choose a \( g_0 \) marginally below \( \hat{g}_0 \). At this policy, price setters in firms pay no attention to the variable \( z_t \), the price level does not respond to an innovation in \( z_t \), and there is no inefficient price dispersion. This completes the proof of Proposition 2.

B Proof of Proposition 3

Step 1: Characterizing equilibrium attention by two equations. We begin by rewriting the equation for the profit-maximizing price (19). Substituting the cash-in-advance constraint (16), \( a_t = 0 \), the monetary policy \( m_t = g_0 \lambda_t \), and \( p_t = \theta \lambda_t \) into the equation for the profit-maximizing price (19) yields

\[
p^*_{i,t} = \left[ (1 - \phi_c) \theta + \phi_c g_0 + \phi \lambda \right] \lambda_t + \frac{\gamma g_0 + \frac{1}{1 + \frac{1 - \gamma}{1 - \alpha \Lambda}} E_t \left[ \lambda_{t+1} - \lambda_t \right]}{1 + \frac{1 - \gamma}{1 - \alpha \Lambda}}.
\]

Using the fact that \( \lambda_t \) is i.i.d. over time yields

\[
p^*_{i,t} = \left[ \left( 1 - \tilde{\phi}_c \right) \theta + \hat{\phi}_c g_0 + \phi \lambda \right] \lambda_t,
\]

(41)

where

\[
\tilde{\phi}_c \equiv \phi_c - \frac{\gamma}{1 + \frac{1 - \gamma}{1 - \alpha \Lambda}}.
\]

\[
\hat{\phi}_c \equiv \phi_c + \frac{1 - \gamma}{1 + \frac{1 - \gamma}{1 - \alpha \Lambda}}.
\]

Since the profit-maximizing price is given by equation (41) and the desired markup follows a white noise process, the attention problem of firm \( i \) reads

\[
\min_{\kappa \in \mathbb{R}_+} \left\{ \frac{\omega}{2} E \left[ \left( p_{i,t} - p^*_{i,t} \right)^2 \right] + \mu \kappa \right\},
\]

subject to

\[
p^*_{i,t} = \left[ \left( 1 - \tilde{\phi}_c \right) \theta + \hat{\phi}_c g_0 + \phi \lambda \right] \lambda_t,
\]

\[
p_{i,t} = E \left[ p^*_{i,t} | s_{\lambda,i,t} \right],
\]

\[
s_{\lambda,i,t} = \lambda_t + \zeta_{i,t},
\]

and

\[
\frac{1}{2} \log_2 \left( \frac{\sigma^2_{\lambda}}{\sigma^2_{\lambda s_{\lambda}}} \right) = \kappa.
\]
Substituting the constraints into the objective, the attention problem of firm \( i \) can be expressed as

\[
\min_{\kappa \in \mathbb{R}_+} \left\{ \frac{\omega}{2} \left[ \left( 1 - \tilde{\phi}_c \right) \theta + \hat{\phi}_c g_0 + \phi_\lambda \right]^2 \sigma^2 \lambda 2^{-2\kappa} + \mu \kappa \right\}. \tag{42}
\]

The solution to this attention problem is

\[
\kappa^* = \begin{cases} 
\frac{1}{2} \log_2 \left( \frac{\omega \left[ (1 - \tilde{\phi}_c) \theta + \hat{\phi}_c g_0 + \phi_\lambda \right]^2 \sigma^2 \lambda \ln(2)}{\mu} \right) & \text{if } \frac{\omega \left[ (1 - \tilde{\phi}_c) \theta + \hat{\phi}_c g_0 + \phi_\lambda \right]^2 \sigma^2 \lambda \ln(2)}{\mu} \geq 1 \\
0 & \text{otherwise}
\end{cases} \tag{43}
\]

The price set by firm \( i \) in period \( t \) then equals

\[
p_{i,t} = \left( 1 - \tilde{\phi}_c \right) \theta + \hat{\phi}_c g_0 + \phi_\lambda \left( 1 - 2^{-2\kappa^*} \right) \lambda_t, \tag{44}
\]

where

\[
\frac{\sigma^2_\lambda}{\sigma^2_\zeta} = 2^{2\kappa^*} - 1. \tag{46}
\]

The price level in period \( t \) equals

\[
p_t = \left( 1 - \tilde{\phi}_c \right) \theta + \hat{\phi}_c g_0 + \phi_\lambda \left( 1 - 2^{-2\kappa^*} \right) \lambda_t. \tag{47}
\]

Thus, the set of rational expectations equilibria of the form \( p_t = \theta \lambda_t \) is given by the solutions to the following two equations:

\[
\theta = \left( 1 - \tilde{\phi}_c \right) \theta + \hat{\phi}_c g_0 + \phi_\lambda \left( 1 - 2^{-2\kappa^*} \right), \tag{48}
\]

and

\[
\kappa^* = \begin{cases} 
\frac{1}{2} \log_2 \left( \frac{\omega \left[ (1 - \tilde{\phi}_c) \theta + \hat{\phi}_c g_0 + \phi_\lambda \right]^2 \sigma^2 \lambda \ln(2)}{\mu} \right) & \text{if } \frac{\omega \left[ (1 - \tilde{\phi}_c) \theta + \hat{\phi}_c g_0 + \phi_\lambda \right]^2 \sigma^2 \lambda \ln(2)}{\mu} \geq 1 \\
0 & \text{otherwise}
\end{cases} \tag{49}
\]

Equation (48) determines \( \theta \) (the responsiveness of the price level to the desired markup) as a function of \( \kappa^* \) (equilibrium attention), while equation (49) determines \( \kappa^* \) as a function of \( \theta \). Solving equation (48) for \( \theta \) yields

\[
\theta = \frac{\hat{\phi}_c g_0 + \phi_\lambda}{1 - \left( 1 - \tilde{\phi}_c \right) \left( 1 - 2^{-2\kappa^*} \right)}. \tag{50}
\]
The set of rational expectations equilibria of the form $p_t = \theta \lambda_t$ for given monetary policy $g_0$ consists of the pairs $(\kappa^*, \theta)$ that solve equations (49)-(50).

**Step 2: Zero attention equilibrium.** We now study under which conditions there exists a solution to equations (49)-(50) with the property $\kappa^* = 0$. We call this a zero attention equilibrium. It follows from equation (50) that $\kappa^* = 0$ implies $\theta = 0$. Furthermore, it follows from equation (49) that at $\theta = 0$ we have $\kappa^* = 0$ if and only if

$$\frac{\omega \left( \hat{\phi}_c g_0 + \phi_\lambda \right)^2 \sigma^2_\lambda \ln(2)}{\mu} \leq 1.$$  \hspace{1cm} (51)

Thus, there exists a rational expectations equilibrium of the form $p_t = \theta \lambda_t$ with $\kappa^* = 0$ if and only if condition (51) is satisfied. Note that the central bank can always ensure the existence of a zero attention equilibrium by making the term $\left( \hat{\phi}_c g_0 + \phi_\lambda \right)^2$ sufficiently small through an appropriate choice of $g_0$.

**Step 3: Interior attention equilibrium.** Next we study under which conditions there exists a solution to equations (49)-(50) with the property $\kappa^* = \frac{1}{2} \log_2 \left( \frac{\omega \left( \hat{\phi}_c g_0 + \phi_\lambda \right)^2 \sigma^2_\lambda \ln(2)}{\mu} \right)$. We call this an interior attention equilibrium because in such an equilibrium the non-negativity constraint $\kappa \geq 0$ in the firms' attention problem is not binding. Substituting equation (50) into equation (52) yields

$$\kappa^* = \frac{1}{2} \log_2 \left( \frac{\omega \left( \hat{\phi}_c g_0 + \phi_\lambda \right)^2 \sigma^2_\lambda \ln(2)}{\mu} \right).$$  \hspace{1cm} (52)

Rearranging the last equation yields a quadratic equation in $2\kappa^*$:

$$\tilde{\phi}_c \left( 2^{\kappa^*} \right)^2 - \sqrt{\frac{\omega \left( \hat{\phi}_c g_0 + \phi_\lambda \right)^2 \sigma^2_\lambda \ln(2)}{\mu}} 2^{\kappa^*} + 1 - \tilde{\phi}_c = 0.$$  \hspace{1cm} (54)

Defining $x \equiv 2^{\kappa^*}$, the last equation can be written as

$$\tilde{\phi}_c x^2 - \sqrt{\frac{\omega \left( \hat{\phi}_c g_0 + \phi_\lambda \right)^2 \sigma^2_\lambda \ln(2)}{\mu}} x + 1 - \tilde{\phi}_c = 0.$$  \hspace{1cm} (55)
An interior attention equilibrium has to satisfy this quadratic equation as well as: \( x \in \mathbb{R} \) and \( x \geq 1 \).

Define
\[
b \equiv \sqrt{\frac{\omega \left( \phi_c g_0 + \phi_L \right)^2 \sigma_\lambda^2 \ln(2)}{\mu}}.
\] (56)

The quadratic equation (55) has two solutions:
\[
x_H = \frac{b + \sqrt{b^2 - 4\phi_c (1 - \phi_c)}}{2\phi_c},
\] (57)
and
\[
x_L = \frac{b - \sqrt{b^2 - 4\phi_c (1 - \phi_c)}}{2\phi_c}.
\] (58)

We now check whether these two solutions to the quadratic equation (55) satisfy: \( x \in \mathbb{R} \) and \( x \geq 1 \).

First, consider the case of \( \tilde{\phi}_c \in (0, \frac{1}{2}] \). Then \( x_H \) and \( x_L \) are real if and only if \( b \geq \sqrt{4\phi_c (1 - \tilde{\phi}_c)} \).

At \( b = \sqrt{4\tilde{\phi}_c (1 - \tilde{\phi}_c)} \), we have \( x_H = x_L = \sqrt{\frac{1}{\tilde{\phi}_c} - 1} \geq 1 \). Furthermore, \( x_H \) is increasing in \( b \) and thus \( x_H \geq 1 \) for all \( b \geq \sqrt{4\phi_c (1 - \tilde{\phi}_c)} \), whereas \( x_L \) is decreasing in \( b \) and \( x_L \geq 1 \) for all \( b \in \left[ \sqrt{4\phi_c (1 - \tilde{\phi}_c)}, 1 \right] \). Hence, if \( \tilde{\phi}_c \in (0, \frac{1}{2}] \), then \( x_H \) is an interior attention equilibrium so long as \( b \geq \sqrt{4\phi_c (1 - \tilde{\phi}_c)} \), while \( x_L \) is an interior attention equilibrium so long as \( b \in \left[ \sqrt{4\phi_c (1 - \tilde{\phi}_c)}, 1 \right] \).

Second, consider the case of \( \tilde{\phi}_c \in (\frac{1}{2}, 1] \). Again \( x_H \) and \( x_L \) are real if and only if \( b \geq \sqrt{4\phi_c (1 - \tilde{\phi}_c)} \).

At \( b = \sqrt{4\tilde{\phi}_c (1 - \tilde{\phi}_c)} \), we have \( x_H = x_L = \sqrt{\frac{1}{\tilde{\phi}_c} - 1} < 1 \). Furthermore, \( x_H \) is increasing in \( b \) and \( x_H \geq 1 \) for all \( b \geq 1 \), whereas \( x_L \) is non-increasing in \( b \) and thus \( x_L \leq 1 \) for all \( b \geq \sqrt{4\phi_c (1 - \tilde{\phi}_c)} \).

Hence, if \( \tilde{\phi}_c \in (\frac{1}{2}, 1] \), then \( x_H \) is an interior attention equilibrium so long as \( b \geq 1 \), while \( x_L \) is not an interior attention equilibrium. Finally, consider the case of \( \tilde{\phi}_c > 1 \). Then \( x_H \) and \( x_L \) are real for all \( b \geq 0 \). At \( b = 0 \), we have \( x_H = \sqrt{1 - \frac{1}{\tilde{\phi}_c}} < 1 \) and \( x_L = -\sqrt{1 - \frac{1}{\tilde{\phi}_c}} < 0 \). Furthermore, \( x_H \) is increasing in \( b \) and \( x_H \geq 1 \) for all \( b \geq 1 \), whereas \( x_L < 0 \) for all \( b \geq 0 \). Hence, if \( \tilde{\phi}_c > 1 \), then \( x_H \) is an interior attention equilibrium so long as \( b \geq 1 \), while \( x_L \) is not an interior attention equilibrium. In summary, if and only if either \( \tilde{\phi}_c \in (0, \frac{1}{2}] \) and \( b \geq \sqrt{4\phi_c (1 - \tilde{\phi}_c)} \) or \( \tilde{\phi}_c > \frac{1}{2} \) and \( b \geq 1 \), then \( x_H \) is an interior attention equilibrium. In addition, if and only if \( \tilde{\phi}_c \in (0, \frac{1}{2}] \) and \( b \in \left[ \sqrt{4\phi_c (1 - \tilde{\phi}_c)}, 1 \right] \), then \( x_L \) is an interior attention equilibrium.
Step 4: Uniqueness and multiplicity of equilibria. When $\tilde{\phi}_c \geq \frac{1}{2}$, there exists a unique rational expectations equilibrium of the form $p_t = \theta \lambda_t$ for any monetary policy $g_0 \in \mathbb{R}$. In particular, if $b \in [0,1)$ then $\kappa^* = 0$ is the unique equilibrium; if $b = 1$ then $\kappa^* = \log_2 (x_H) = 0$ is the unique equilibrium; and if $b > 1$ then $\kappa^* = \log_2 (x_H)$ is the unique equilibrium. By contrast, when $\tilde{\phi}_c \in (0, \frac{1}{2})$, there exist multiple rational expectations equilibria of the form $p_t = \theta \lambda_t$ for some monetary policies $g_0 \in \mathbb{R}$. In particular, if $b \in \left[0, \sqrt{4\tilde{\phi}_c (1 - \tilde{\phi}_c)}\right)$ then $\kappa^* = 0$ is the unique equilibrium; if $b = \sqrt{4\tilde{\phi}_c (1 - \tilde{\phi}_c)}$ then $\kappa^* = 0$ and $\kappa^* = \log_2 (x_L) = \log_2 (x_H) = \log_2 \left(\sqrt{\frac{1}{\tilde{\phi}_c} - 1}\right)$ are equilibria; if $b \in \left(\sqrt{4\tilde{\phi}_c (1 - \tilde{\phi}_c)}, 1\right)$ then $\kappa^* = 0$, $\kappa^* = \log_2 (x_L)$ and $\kappa^* = \log_2 (x_H)$ are equilibria, where $x_L$ is decreasing in $b$ and $x_H$ is increasing in $b$; if $b = 1$ then $\kappa^* = \log_2 (x_L) = 0$ and $\kappa^* = \log_2 (x_H) = \log_2 \left(\frac{1}{\tilde{\phi}_c} - 1\right)$ are equilibria; and if $b > 1$ then $\kappa^* = \log_2 (x_H)$ is the unique equilibrium. See steps 2 and 3.

Step 5: Price dispersion and consumption variance. We now derive expressions for price dispersion and consumption variance at an equilibrium. First, we derive expressions for individual prices and the price level. Substituting equations (46) and (50) into equation (45) yields

$$p_{i,t} = \left(\tilde{\phi}_c g_0 + \phi_\lambda\right) \left(1 - 2^{-2\kappa^*}\right) \left(\lambda_t + \zeta_{i,t}\right).$$

Substituting equation (50) into equation (47) yields

$$p_t = \left(\tilde{\phi}_c g_0 + \phi_\lambda\right) \left(1 - 2^{-2\kappa^*}\right) \lambda_t.$$  \hfill (59)

Second, we derive a simple expression for price dispersion at an equilibrium. Consider the case of an equilibrium with $\kappa^* > 0$. An equilibrium with $\kappa^* > 0$ is an interior attention equilibrium and in an interior attention equilibrium equation (53) holds. Equations (59) and (60) imply

$$E \left[ (p_{i,t} - p_t)^2 \right] = \left[ \left(\tilde{\phi}_c g_0 + \phi_\lambda\right) \left(1 - 2^{-2\kappa^*}\right) \right]^2 \sigma_\zeta^2,$$

Substituting equation (53) into the last equation yields

$$E \left[ (p_{i,t} - p_t)^2 \right] = \frac{\mu_\phi}{\sigma_\lambda^2 \ln (2)} \left(1 - 2^{-2\kappa^*}\right)^2 \sigma_\zeta^2.$$
Furthermore, substituting equation (46) into the last equation and rearranging yields

\[ E \left[ (p_{i,t} - p_t)^2 \right] = \frac{\mu}{\ln(2)} \left( 1 - 2^{-\kappa^*} \right). \tag{61} \]

Next consider the case of an equilibrium with \( \kappa^* = 0 \). Equation (61) holds again because in an equilibrium with \( \kappa^* = 0 \) we have \( E \left[ (p_{i,t} - p_t)^2 \right] = 0 \). In summary, in any equilibrium, price dispersion is given by equation (61). It follows that equilibrium price dispersion is an increasing function of equilibrium attention. Third, we derive an expression for consumption variance at an equilibrium. Substituting the monetary policy \( m_t = g_0 \lambda_t \) and the equation for the price level (60) into the cash-in_advance constraint (16) yields

\[ c_t = \left[ g_0 - \left( \phi_c g_0 + \phi_{\lambda} \right) \left( 1 - 2^{-\kappa^*} \right) \right] \lambda_t. \tag{62} \]

The first term in square brackets in equation (62) equals the response of nominal spending to the desired markup, while the second term in square brackets in equation (62) equals the response of the price level to the desired markup. The difference between the two determines the response of composite consumption to the desired markup. This completes the proof of Proposition 3.