Participation in Deterministic Contests

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Abstract

This paper considers participation in deterministic contests for \( m \geq 1 \) identical prizes, which employ an auction-like rule to determine the winners. In most papers that investigate such models, participation by precisely \( m + 1 \) players is associated with players having complete information about their opponents’ characteristics, and participation by more than \( m + 1 \) players is associated with players having incomplete information about their opponents’ characteristics. I show that incomplete information is in fact neither sufficient nor necessary for participation by more than \( m + 1 \) players.

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1 Introduction

Some real-world competitions induce active participation by many competitors, whereas others lead to active participation by only a few competitors, often only one more than the number of prizes. What determines the number of active participants in a competition for \( m \geq 1 \) identical prizes? More than \( m + 1 \) competitors may participate if, given their investments, there is exogenous uncertainty regarding the outcome. This is the case, for example, in Tullock’s (1980) lottery model, in which each player’s probability of winning is proportional to his share of the aggregate investments. When the outcome given investments is deterministic, as is the case in auction-like contest models, participation by more than \( m + 1 \) competitors occurs when competitors have private information about their valuations. This is the case, for example, in the classic symmetric independent private value all-pay auction (Krishna (2002)). In contrast, precisely the \( m + 1 \) competitors with the highest valuations participate in the complete-information all-pay auction when players’ valuations differ (Clark and Riis (1998)).\(^1\) Participation by only the \( m + 1 \) strongest players is also a hallmark of complete-information variants of the all-pay auction (examples include Che and Gale (1998) and González-Díaz (2010)). Thus, it is natural to ask whether equilibrium participation by more than \( m + 1 \) competitors in deterministic models of competition occurs, at least “generically,” if and only if competitors are uncertain about some characteristics of their rivals.

This paper includes two results, Propositions 1 and 2, which together show that the answer is “no.” The first result shows that for any number of players, prizes, and participants (larger than the number of prizes), there exist complete-information, deterministic contests with these number of players and prizes in which in any equilibrium precisely the specified number of players participate. This generalizes Siegel’s (2009) example in which three players participate in a complete-information contest for one prize. The result is robust in that it continues to hold when the contests are perturbed slightly. The second result identifies a large class of incomplete-information deterministic contests in which only the strongest \( m + 1 \) players participate. This class includes multiprize all-pay auctions in which players’ valuations are drawn from disjoint intervals. Intuitively, these results show that participation by many players does not stem from incomplete information per se, but arises when different players are known to have sufficient cost advantages in different regions of the competition.

\(^1\)Baye et al. (1996) showed that more than two competitors may participate when certain players’ valuations are identical. This participation result is not robust to slight changes in players’ valuations.
2 Complete-Information Contests

In a contest, $n$ players compete for $m$ homogeneous prizes, $0 < m < n$. The set of players \( \{1, \ldots, n\} \) is denoted by \( \mathcal{N} \). Every player $i$ chooses a score $s_i$ from \( \mathbb{R}_+ = [0, \infty) \). Given $s = (s_1, \ldots, s_n)$, where $s_i$ is player $i$’s chosen score, player $i$’s payoff is

\[
u_i(s) = P_i(s) V_i - c_i(s_i),\]

where $V_i > 0$ is player $i$’s valuation for a prize, $c_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ is player $i$’s continuous, strictly increasing cost function, with $c_i(0) = 0$ and $\lim_{s_i \rightarrow \infty} c_i(s_i) > V_i$, and $P_i : \mathbb{R}_+^n \rightarrow [0, 1]$ is player $i$’s probability of winning, which satisfies

\[
P_i(s) = \begin{cases} 0 & \text{if } s_j > s_i \text{ for } m \text{ or more players } j \neq i, \\ 1 & \text{if } s_j < s_i \text{ for } n - m \text{ or more players } j \neq i, \end{cases}\]

and $\sum_{j=1}^n P_j(s) = m$. When all costs are linear, we have an all-pay auction (Hillman and Samet (1987), Hillman and Riley (1989), Clark and Riis (1998)).\(^2\) The primitives of the contest are commonly known.

**Definition 1**

(i) Player $i$’s reach $r_i$ is the score whose cost equals the player’s valuation for a prize. That is, $r_i = c_i^{-1}(V_i)$. Re-index players in (any) decreasing order of their reach, so that $r_1 \geq r_2 \geq \ldots \geq r_n$.

(ii) Player $m + 1$ is the marginal player.

(iii) The threshold $T$ of the contest is the reach of the marginal player: $T = r_{m+1}$.

A final requirement in the definition of a contest is that only the marginal player’s reach equals the threshold. In an all-pay auction, for example, a player’s reach is his valuation for a prize, the marginal player is the player with the $(m + 1)^{st}$ highest valuation, and the marginal player’s valuation is required to be different from those of the other players. The model of contests described here is a special case of Siegel’s (2009) all-pay contest model.

A player participates in an equilibrium of a contest if with positive probability he chooses positive scores (whose cost is positive).\(^3\)

**Proposition 1** For any $n$, $m$, and $k$ such that $n \geq k > m > 0$, there exist contests with $n$ players and $m$ prizes such that in any equilibrium precisely $k$ players participate.

\(^2\)In an all-pay auction, $c_i(s_i) = s_i$ and ties are resolved by randomizing uniformly.

\(^3\)A player wins a prize with positive probability if and only if he participates. Indeed, because participation is costly and choosing 0 is not, a participating player must win a prize with positive probability. In the other direction, the Tie Lemma in Siegel (2009) and the fact that 0 is the lowest possible score imply that a player who chooses 0 wins a prize with probability 0.
The proof of Proposition 1 is in the Appendix. Intuitively, Proposition 1 stems from the fact that many players participate when different players have local cost advantages in different regions. To see why this happens, consider a player whose cost in a certain interval of scores is much lower than those of the other players. Lemma 1 in the Appendix shows that at least two players choose scores in . If player does not participate, then his payoff is 0. But because other players choose scores in , by doing so they have to win with a probability that is sufficiently high to offset their costs. And because player ’s costs are much lower than those of the other players, player would obtain a positive payoff by choosing these scores in . Therefore, player must participate.

Baye et al. (1996) showed that many players may participate in certain complete-information, single-prize all-pay auctions in which many players have the same valuation for the prize. 4 Their finding differs from Proposition 1 in two ways. First, the auction games of Baye et al. (1996) have many equilibria, and in contrast to Proposition 1, the number of players that participate differs across equilibria, and equals two in some equilibria. Second, perturbing players’ valuations in an all-pay auction leads to a unique equilibrium, in which only the two players with the highest valuations participate. The contests constructed in the proof of Proposition 1 are robust to such perturbations: small changes in players’ cost functions or valuations do not change players’ participation.

3 Incomplete-Information Contests

Take a contest (as defined in Section 2), in which and are commonly known for every player . Players are indexed as in Definition 1. Now, add incomplete-information in the following way. Every player ’s valuation is , where is player ’s private information and is drawn from for some according to some distribution . Players’ cost functions remain commonly known. Each player, after observing his private information, chooses a score, and the winners are determined as in the complete-information case.

Proposition 2 If for some we have

\[ \frac{c_{m+1}(x)}{V_{m+1}} < \frac{c_i(x)}{V_i} \text{ for all } x > 0, \]  

then for small player does not participate in any equilibrium of the incomplete-information contest described above. In particular, if this condition holds for every player , then in any equilibrium only players may participate.

4These all-pay auctions do not meet the definition of a contest because players other than the marginal player have reaches that equal the threshold.
The proof of Proposition 2 is in the Appendix. The logic underlying Proposition 2 is as follows. The marginal player can be shown to have a payoff of 0 in any equilibrium when he has his lowest possible valuation. Suppose that some player \( i > m + 1 \) that satisfies the conditions of Proposition 1 participates, and consider a positive score chosen by player \( i \) in equilibrium. By choosing this score when he has his highest possible valuation, player \( i \) obtains a non-negative payoff, and therefore wins with a probability that is sufficiently high to offset his costs. Because the marginal player’s costs are strictly lower than those of player \( i \), the marginal player when he has his lowest possible type can obtain a positive payoff by choosing a score slightly higher than the highest score chosen by player \( i \), a contradiction.

As an application of Proposition 2, take a complete-information all-pay auction in which the marginal player’s valuation differs from those of all other players, and add some incomplete information as specified above. Proposition 2 shows that for \( \delta > 0 \) that is not too large players \( m + 2, \ldots, n \) do not participate in any equilibrium of the incomplete-information contest. The proof of Proposition 2 shows that this is true for any \( \delta < \min \left\{ \frac{V_{m} - V_{m+1}}{2}, \frac{V_{m+1} - V_{m+2}}{2} \right\} \).
A Appendix - Proofs of Propositions 1 and 2

A.1 Notation and Existing Results

The following three results refer to complete-information contests, and are immediate corollaries of results in Siegel (2009). I use these results in the proofs of Propositions 1 and 2 below. The first result characterizes players’ equilibrium payoffs in terms of their power, where player $i$’s power $w_i$ is his payoff if he chooses the threshold and wins: $w_i = V_i - c_i (T)$.

**Theorem 1** In any equilibrium of a contest, the expected payoff of every player equals the maximum of his power and 0.

In addition to giving a closed-form formula for players’ equilibrium payoffs, Theorem 1 shows that players $1, \ldots, m$ have positive expected payoffs, and players $m + 1, \ldots, n$ have expected payoffs of 0.

The second result provides a sufficient condition for players $m + 2, \ldots, n$ not to participate in any equilibrium.

**Theorem 2** If the normalized cost function of the marginal player is lower than that of player $i > m + 1$, that is

$$\frac{c_{m+1} (x)}{V_{m+1}} < \frac{c_i (x)}{V_i} \text{ for all } x > 0,$$

then player $i$ does not participate in any equilibrium (of the complete-information contest). In particular, if this condition holds for all players $m + 2, \ldots, n$, then in any equilibrium only players $1, \ldots, m + 1$ may participate.

Consider an equilibrium profile of cumulative distribution functions (CDFs) $G = (G_1, \ldots, G_n)$, where $G_i (x)$ is the probability that player $i$ chooses a score no higher than $x$.

**Lemma 1** In any equilibrium $G$ of a contest, (1) $G$ is continuous on $(0, T)$, and (2) every score in $(0, T)$ is a best response for at least two players.

The following result, which is a version of the Threshold Lemma of Siegel (2009), describes another property of every equilibrium.

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5 The results follow, respectively, from Theorem 1, Theorem 2, and Lemma 1 in Siegel (2009).
Threshold Lemma In any equilibrium $G$ of a contest, players $1, \ldots, m+1$ choose scores up to the threshold, that is $G_i(x) < 1$ for every $x < T$ and $i \leq m+1$.

Proof. Recall that, by definition, the reach of every player $m+2, \ldots, n$ is lower than the threshold, so these players do not choose scores up to the threshold. By Theorem 1, the payoff of every player $i \leq m+1$ is equal to his power. Suppose some player $i \leq m+1$ did not choose scores up to the threshold. Then every other player $j \leq m+1$ could win with certainty by choosing a score lower than the threshold, which would give him a payoff higher than his power.

With a slight abuse of notation, when an equilibrium is specified denote by $P_i(x)$ player $i$’s probability of winning a prize when he chooses $x$ and the other players play their equilibrium strategies, and by $u_i(x)$ player $i$’s utility when he chooses $x$ and the other players play their equilibrium strategies, so $u_i(x) = P_i(x)V_i - c_i(x)$.

A.2 Proof of Proposition 1

The proof proceeds in three stages: (1) for any $n > 1$, I construct a contest with $n$ players and one prize, in which all players participate, (2) for any $m > 1$, I extend this contest by adding $m-1$ prizes and $m-1$ players, such that all players participate, and (3) I add any number of players who do not participate. The value of a prize is set to 1; distinct valuations are achieved by multiplying a player’s valuations and cost function by any positive constant. This does not change the contest’s set of equilibria, since a player’s strategic behavior is invariant to positive affine transformations of his Bernoulli utility function.

Stage 1. Fix $n \geq 2$. For $n = 2$, let

$$c_i(x) = \begin{cases} 
4x\alpha (1 - \gamma) & \text{if } x < \frac{1}{4} \\
\alpha (1 - \gamma) + 2\alpha \gamma (x - \frac{1}{4}) & \text{if } \frac{1}{4} \leq x \leq \frac{3}{4} \\
\alpha + 8(x - \frac{2}{3})(\beta) & \text{if } \frac{3}{4} \leq x \leq \frac{7}{8} \\
\alpha + \beta + 8i(x - \frac{2}{3})(1 - \alpha - \beta) & \text{if } x > \frac{7}{8}
\end{cases}$$

for $i \in \{1,2\}$, $\alpha \in (0,1)$ close to 1, and small $\beta, \gamma > 0$ such that $\alpha + \beta < 1$. The Threshold Lemma shows that both players choose scores up to $r_2 = T > \frac{7}{8}$, so both players participate.

For $n > 2$, let $K = n - 2$ and partition $\left[\frac{1}{4}, \frac{3}{4}\right]$ to $2K$ segments $L_l, l \in [1 \ldots 2K]$ of length $\frac{1}{4K}$, so that $L_l = \left[\frac{3}{4} - \frac{l}{4K}, \frac{3}{4} - \frac{l-1}{4K}\right] = \left[L^l_1, L^l_2\right]$. Define $c_1(\cdot)$ and $c_2(\cdot)$ as in the case $n = 2$. For $k \in 3, \ldots, n$, the idea is to define player $k$’s cost function so that (i) player $k$ does not choose scores higher than $L^l_2$ and (ii) if player $k$ does not participate in an hypothesized equilibrium, then he can obtain a positive payoff by choosing scores in the interior of $L^l_2$, contradicting the equilibrium hypothesis. I achieve (i) by having player $k$’s costs increase sufficiently fast in the interval $L_{2(k-2)-1}$, and (ii) by having player $k$’s costs in the interval $L_{2(k-2)}$ be low relative
to those of players 1, . . . , k − 1 and using (i) applied to players k + 1, . . . , n to show that players
k + 1, . . . , n do not choose scores in L_{2(k−2)}. Formally, for k ∈ {3, . . . , n}, define \( c_k(·) \) in the
following way, using \( c_1(·) , . . . , c_{k−1}(·) \). Let \( q_k = \min_{l≤k−1} c_l \left( L_{2(k−2)}^b \right) \in (0, 1) \), and
\[
c_k(x) = \begin{cases} 
4x(q_k^3 - \gamma_{nk}) & \text{if } x < \frac{1}{4} \\
(q_k^3 - \gamma_{nk}) + \frac{\gamma_{nk}}{r_{2(k−2)} - \frac{3}{4}}(x - \frac{1}{4}) & \text{if } \frac{1}{4} ≤ x ≤ L_{2(k−2)}^t \\
q_k^3 + 4K(1 - q_k^3)(x - L_{2(k−2)}^t) & \text{if } x > L_{2(k−2)}^t 
\end{cases}
\]
for \( \gamma_{nk} > 0 \) such that \( (q_k^3 - \gamma_{nk}) > 0 \). Note that \( r_k = L_{2(k−2)}^t - \frac{3}{4} \), so we have a contest with
\( T = r_2 \), and players 3, . . . , n have negative powers. The Threshold Lemma shows that players 1
and 2 choose scores up to the threshold and therefore participate in every equilibrium.

I now show that players 3, . . . , n also participate in every equilibrium. Suppose, in contradiction,
that there is an equilibrium in which player \( k > 2 \) does not participate. This equilibrium is
clearly an equilibrium of the reduced contest without player \( k \). Consider this reduced contest
and the corresponding reduced equilibrium \( \tilde{G} \). Since the reach of every player \( k + 1, . . . , n \) is
at most \( L_{2(k−2)}^b \), these players have no best responses above \( L_{2(k−2)}^b \), and therefore choose scores
above \( L_{2(k−2)}^b \) with probability 0. Recall that every score in \((0, T)\) is a best response for at
least two players, and let \( i_x, j_x < k \) be two distinct players for whom the score \( x \) in the interior
of \( L_{2(k−2)} \) is a best response. Since players have non-negative equilibrium payoffs, it must be
that \( \Pi_{r<k, r\neq i_x} \tilde{G}_r(x) ≥ q_k \) and similarly for \( j_x \). Thus, \( \Pi_{r<k} \tilde{G}_r(x) ≥ q_k^2 \) for every score in
the interior of \( L_{2(k−2)} \). Since \( c_k(x) ≤ q_k^3 < \Pi_{r<k} \tilde{G}_r(x) \) for every \( x \) in \( L_{2(k−2)} \), player \( k \) can obtain a
positive payoff under \( \tilde{G} \) in the original contest by choosing a score in the interior of \( L_{2(k−2)} \), a
contradiction.

This proves that every player participates in every equilibrium of the \( n \)-player, single-prize
contest constructed. Denote this contest by \( C \) and its threshold by \( T_c \).

Stage 2. The idea is to add an equal number of prizes and players, such that these additional
players only choose scores higher than \( \frac{7}{8} \). To achieve this, the additional players’ power is set
high enough to guarantee that if an additional player has a best response of at most \( \frac{7}{8} \) then
some original player’s CDF at \( \frac{7}{8} \) is so high that another original player obtains too high a payoff.
The additional players participate, but because they do not choose scores lower than \( \frac{7}{8} \), their
addition does not affect existing players’ behavior and participation at scores lower than \( \frac{7}{8} \), so
all players participate.

Formally, recall that \( r_i ≤ \frac{3}{4} \) for \( i ≥ 3 \), and that in equilibrium every score in \((0, T_c)\) is a best
response for at least two players. Thus, in any equilibrium \( \tilde{G} \) we have
\[
G_1 \left( \frac{7}{8} \right) = c_2 \left( \frac{7}{8} \right) , G_2 \left( \frac{7}{8} \right) = c_1 \left( \frac{7}{8} \right) + w_1.
\tag{2}
\]

Since players 1 and 2 choose scores up to the threshold \( T_c > \frac{7}{8} \), we have \( G_1 \left( \frac{7}{8} \right) , G_2 \left( \frac{7}{8} \right) < 1 \). Choose \( \varepsilon \in \left( \max \{ G_1 \left( \frac{7}{8} \right) , G_2 \left( \frac{7}{8} \right) \} , 1 \right) \), and note that (2) implies that \( \varepsilon \) is independent of the
specific equilibrium $G$. Now add $m - 1$ prizes and $m - 1$ players $i$ with $c_i(T_c) < (1 - \varepsilon)^2$. The threshold of the new $m$-prize, $(n + m - 1)$-player contest is $T_c$, the powers of the $n$ original players do not change, and those of the new players exceed $1 - (1 - \varepsilon)^2$. For the remainder of the proof “player 1” and subscript 1 refer to player 1 of the contest $C$, and similarly “player 2” and subscript 2.

Choose an equilibrium $\tilde{G}$ of the new contest. The Threshold Lemma shows that players 1 and 2 participate, as do the new players. To show that the $n - 2$ remaining players also participate, I first show that $\tilde{G}_i \left( \frac{7}{8} \right) = 0$ for every new player $i$. For this, it suffices to demonstrate that $\tilde{s}_{inf}$, the infimum of the union of the best-response sets of the new players, is higher than $\frac{7}{8}$. Suppose $\tilde{s}_{inf} \leq \frac{7}{8}$, and note that $\tilde{s}_{inf} > 0$ (otherwise one of the new players would have a payoff of 0, as in the Zero Lemma of Siegel (2009)). Since $\tilde{s}_{inf} \in (0, \frac{7}{8})$ and there are no atoms in $(0, T_c)$, by definition of $\tilde{s}_{inf}$ we have $u_j(\tilde{s}_{inf}) = P_j(\tilde{s}_{inf}) - c_j(\tilde{s}_{inf})$ for some new player $j$. Also, $P_j(\tilde{s}_{inf}) \leq 1 - \left( 1 - \tilde{G}_1(\tilde{s}_{inf}) \right) \left( 1 - \tilde{G}_2(\tilde{s}_{inf}) \right)$: if player $j$ wins a prize, it must be that player 1, player 2, or both chose a score lower than $\tilde{s}_{inf}$, since all other new players choose scores higher than $\tilde{s}_{inf}$ with probability 1. Since $c_1 \left( \frac{7}{8} \right) + w_1 < \varepsilon$ (from (2) and the definition of $\varepsilon$), we have $\tilde{G}_2 \left( \frac{7}{8} \right) < \varepsilon$, otherwise player 1 could obtain in $\tilde{G}$ more than his power by choosing $\frac{7}{8}$. Similarly, $\tilde{G}_1 \left( \frac{7}{8} \right) < \varepsilon$. Thus, if $\tilde{s}_{inf} \leq \frac{7}{8}$ then
\[
1 - (1 - \varepsilon)^2 < u_j(\tilde{s}_{inf}) < P_j(\tilde{s}_{inf}) \leq 1 - \left( 1 - \tilde{G}_1(\tilde{s}_{inf}) \right) \left( 1 - \tilde{G}_2(\tilde{s}_{inf}) \right) \leq 1 - (1 - \varepsilon)^2
\]
a contradiction. Consequently, $\tilde{s}_{inf} > \frac{7}{8}$ and $\tilde{G}_i \left( \frac{7}{8} \right) = 0$ for every new player $i$.

Therefore, on $[0, \frac{3}{4}]$ only the $n$ original players compete for one prize (all other players choose scores in $[0, \frac{3}{4}]$ with probability 0). The participation arguments from stage (2) applied to the restriction of $\tilde{G}$ to $[0, \frac{3}{4}]$ show that the remaining $n - 2$ players also participate in $\tilde{G}$.

**Stage 3.** By Theorem 2, any additional player with costs higher than those of the marginal player does not participate in any equilibrium.

### A.3 Proof of Proposition 2

The proof consists of three lemmas, and uses the following notation. Given a contest, let
\[
r_i^{min} = c_i^{-1}(V_i - \delta) < r_i \quad \text{and} \quad r_i^{max} = c_i^{-1}(V_i + \delta) > r_i,
\]
which are well defined for small enough $\delta > 0$. Because the marginal player’s reach is distinct from that of the other players, for small enough $\delta$ and $i \neq m + 1$ we have
\[
\left[ r_i^{min}, r_i^{max} \right] \cap \left[ r_{m+1}^{min}, r_{m+1}^{max} \right] = \emptyset.
\]

If $\tilde{G}_2 \left( \frac{7}{8} \right) \geq \varepsilon$, then player 1’s payoff would be at least
\[
\tilde{G}_2 \left( \frac{7}{8} \right) - c_1 \left( \frac{7}{8} \right) \geq \varepsilon - c_1 \left( \frac{7}{8} \right) > c_1 \left( \frac{7}{8} \right) + w_1 - c_1 \left( \frac{7}{8} \right) = w_1.
\]
Choose $\delta > 0$ small enough so that $r_{i}^{\min}$ and $r_{i}^{\max}$ are defined for every player $i$ and (3) holds for every player $i \neq m + 1$. Choose an equilibrium of the incomplete-information contest, and for every player $j$ and $\varepsilon_{j}$ in $[-\delta, \delta]$ denote by $BR_{j,\varepsilon_{j}}$ player $j$’s set of best responses when his valuation is $V_{j} + \varepsilon_{j}$.

**Lemma 2** For every player $j < m + 1$, every $\varepsilon_{j}$ in $[-\delta, \delta]$, and every $x$ in $BR_{j,\varepsilon_{j}}$, player $j$’s payoff when his valuation is $V_{j} + \varepsilon_{j}$ and he chooses $x$ is positive if all other players play their equilibrium strategies.

**Proof.** Because the marginal player’s reach is distinct from that of every other player and (3) holds, $r_{j}^{\min} > r_{k}^{\max}$ for every $k \geq m + 1$. And because in equilibrium no player $k \geq m + 1$ chooses scores higher than $r_{k}^{\max}$ with positive probability, player $j$ can win with certainty by choosing any score in $(r_{m+1}^{\max}, r_{j}^{\min})$. Doing so gives player $j$ a positive payoff for any $\varepsilon_{j}$ in $[-\delta, \delta]$, so choosing a best response also gives him a positive payoff. ■

**Lemma 3** For any $j \geq m + 1$, when player $j$’s valuation is $V_{j} - \delta$ he cannot get a positive payoff at any score.

**Proof.** Consider a set $J$ of any $m + 1$ players, and let $x^{\inf} = \inf \cup_{j \in J, \varepsilon_{j} \in [-\delta, \delta]} BR_{j,\varepsilon_{j}}$. By definition of $x^{\inf}$, there is a player $j$ in $J$ and a sequence of scores $x_{j}^{l} \downarrow_{l \to \infty} x^{\inf}$ so that every $x_{j}^{l}$ in the sequence is a best response for player $j$ for some valuation $V_{j} + \varepsilon_{j}$. Along this sequence, player $j$ wins with a probability that approaches 0 (this follows from arguments similar to the one used in the proofs of the Tie Lemma and the Zero Lemma in Siegel (2009)). Therefore, along this sequence player $j$ has a payoff that approaches at most 0. This implies that player $j$ cannot get a positive payoff by choosing some score $x$ when his valuation is $V_{j} - \delta$, otherwise by choosing $x$ player $j$ would get a positive payoff bounded away from 0 regardless of his valuation, contradicting the definition of the sequence $x_{j}^{l}$. The previous lemma implies that $j \geq m + 1$, and because $J$ was a set of any $m + 1$ players the result follows. ■

Suppose that for some player $i > m + 1$ we have

$$\frac{c_{m+1}(x)}{V_{m+1} - \delta} < \frac{c_{i}(x)}{V_{i} + \delta} \text{ for all } x > 0.$$

(4)

**Lemma 4** If player $i$ participates, then player $m + 1$ can obtain a positive payoff when his valuation is $V_{m+1} - \delta$.

**Proof.** Suppose that player $i$ participates (chooses positive scores with positive probability), and let $x_{i}^{\sup} = \sup \cup_{\varepsilon_{i} \in [-\delta, \delta]} BR_{i,\varepsilon_{i}}$. Player $i$ obtains a non-negative payoff if his valuation is $V_{i} + \delta$ and he chooses $x_{i}^{\sup}$, i.e.,

$$P_{i}(x_{i}^{\sup})(V_{i} + \delta) - c_{i}(x_{i}^{\sup}) \geq 0.$$

(5)
Otherwise, by choosing $x^\text{sup}_i$ player $i$ would obtain a negative payoff bounded away from 0 regardless of his valuation, and by continuity of $c_i$ the same would be true for scores slightly below $x^\text{sup}_i$, contradicting the definition of $x^\text{sup}_i$ (recall that every player can obtain at least 0 by choosing 0 regardless of his valuation). Now, by choosing $x^\text{sup}_i + \gamma$ for some $\gamma > 0$, player $m+1$ wins a prize with a weakly higher probability than player $i$ does when he chooses $x^\text{sup}_i$ (player $m+1$ beats player $i$ with probability 1, and beats players $\mathcal{N}\setminus\{i, m+1\}$ at least as often as player $i$ does). From (4) we have

$$\frac{c_i(x)}{V_i + \delta} - \frac{c_{m+1}(x)}{V_{m+1} - \delta} = \alpha > 0,$$

so for some small $\gamma > 0$ we have

$$P_i(x^\text{sup}_i) - \frac{c_i(x^\text{sup}_i)}{V_i + \delta} = P_i(x^\text{sup}_i) - \frac{c_{m+1}(x^\text{sup}_i)}{V_{m+1} - \delta} - \alpha < P_i(x^\text{sup}_i) - \frac{c_{m+1}(x^\text{sup}_i + \gamma)}{V_{m+1} - \delta} - \frac{\alpha}{2}$$

$$\leq P_{m+1}(x^\text{sup}_i + \gamma) - \frac{c_{m+1}(x^\text{sup}_i + \gamma)}{V_{m+1} - \delta} - \frac{\alpha}{2} < P_{m+1}(x^\text{sup}_i + \gamma) - \frac{c_{m+1}(x^\text{sup}_i + \gamma)}{V_{m+1} - \delta}.$$ (6)

From (5) and (6) we have

$$P_i(x^\text{sup}_i) - \frac{c_i(x^\text{sup}_i)}{V_i + \delta} \geq 0 \Rightarrow P_{m+1}(x^\text{sup}_i + \gamma) - \frac{c_{m+1}(x^\text{sup}_i + \gamma)}{V_{m+1} - \delta} > 0$$

$$\Rightarrow P_{m+1}(x^\text{sup}_i + \gamma)(V_{m+1} - \delta) - c_{m+1}(x^\text{sup}_i + \gamma) > 0.$$ 

If player $i$ participates, then Lemmas 3 and 4 lead to a contradiction. Therefore, (4) implies that player $i$ does not participate, and because (1) implies (4) for small enough $\delta > 0$, we are done.
References


