Asymmetric Contests with Conditional Investments

Ron Siegel*

October 20, 2009

Abstract

This paper studies equilibrium behavior in a class of games that models asymmetric competitions with unconditional and conditional investments. Such competitions include lobbying settings, labor-market tournaments, and R&D races, among others. I provide an algorithm that constructs the unique equilibrium in these games, and apply it to study contests in which a fraction of each competitor's investment is sunk and the rest is paid only by the winners. Complete-information all-pay auctions are a special case. (*JEL* D44,D72,J41,L13,C71)

Rent-seeking scenarios, competitions for promotions, and research and development (R&D) races are examples of competitions in which participants expend resources regardless of whether they win or lose. Although similar in their "all-pay" feature, such competitions often differ in important dimensions. These include the type of technologies

---

*Department of Economics, Northwestern University, Evanston, IL 60208 (e-mail: r-siegel@northwestern.edu). I am indebted to Ed Lazear and Ilya Segal for their continuous guidance and encouragement. I thank Jeremy Bulow for many beneficial discussions. I thank Yuval Salant for valuable feedback throughout my work on this project. I thank three referees for their valuable comments. I thank Wouter Dessein, Chaim Fershtman, Julio González-Díaz, Matthew Jackson, Sunil Kumar, Jonathan Levin, George Mailath, Stephen Morris, Marco Ottaviani, Philip Reny, Kareen Rozen, Andrzej Skrzypacz, Bruno Strulovici, Balázs Szentes, Michael Whinston, Robert Wilson, Eyal Winter, and seminar participants at Chicago, Chicago GSB, Columbia, Cornell, CSIO-IDEI, Hebrew University, Northwestern, NYU Stern, Penn State, Stanford, Tel-Aviv, Toronto, UPenn, Washington (St. Louis), Yale, and Yonsei for helpful comments and suggestions.

employed in competition, the type of asymmetries among competitors, the degree to which expenditures are sunk, and the number of prizes.

Consider lobbying settings in which firms expend resources in order to influence a policy maker. If expenditures take the form of monetary outlays and all firms have the same cost of capital, then it is reasonable to think of firms' marginal cost of influence as linear. But if a firm has a reputational advantage and high costs of capital, its "lobbying technology" may be more reasonably thought of as having convex costs (relatively low marginal costs up to a moderate level of influence, due to the reputational advantage, and relatively high marginal costs at higher levels of influence, due to high costs of capital). Similarly, the lobbying technology of a firm that is relatively unknown and has low costs of capital may be characterized by concave costs. Lobbying restrictions, geographic location, or preferential treatment by the policy maker further impact firms' lobbying technologies, inducing additional non-linearities and asymmetries among firms. In some lobbying settings, firms also commit to expenditures conditional on a favorable outcome. For example, a firm may commit to building factories and creating new jobs if it is granted one of a limited number of manufacturing licenses. Thus, the specification of firms' lobbying technologies, the type of asymmetries among firms, the combination of unconditional and conditional investments, and the number of eventual "winners" may change according to the lobbying setting studied.

The same is true for other competitions with an all-pay feature. In a competition for promotions, workers often differ in their abilities, experience, and social skills. More than one prize may be awarded, in the form of a promotion to the next level in the hierarchy. In an R&D race, firms may differ in their research technologies, past R&D investments, and access to human capital. If firms compete by building prototypes, then all firms incur the cost of their prototypes, but the cost of developing the product is borne only by winning firms. In a political campaign, candidates typically differ in their constituencies, political savvy, and wealth. Only those elected have to make good on their promises, but all candidates expend resources during the campaign.

Clearly, the strategic interaction among competitors is influenced by these dimensions in which competitions differ, as are policy-relevant variables. In the context of lobbying, the amount of resources expended by competing firms and the efficiency of the outcome are affected by firms' cost of capital, legal restrictions, and other policy measures that impact firms' lobbying technologies. In a competition for promotions, the amount of effort employees exert depends on employees' characteristics, how they are evaluated, and the number and value of promotions. In an R&D race, the quality of competitors' products depends on the scoring rule used, whether subsidies are provided, and what part of the R&D investment is required before the winners are determined. As these examples demonstrate, understanding the relationship between the characteristics of a competition and its outcome is useful for making policy decisions and for contest design.

To investigate this relationship, I analyze equilibrium behavior in a *contest* model that accommodates a wide range of technologies, general asymmetries, a combination of unconditional and conditional investments, and one or more identical prizes. The analysis makes it possible to calculate players' effort and output, the degree of rent dissipation, and the level of efficiency of the outcome. It also makes it possible to compare these variables across different specifications of contests. This provides a framework in which competitions can be studied and various policy and design issues can be addressed.

In a contest, each player chooses a "score," and the players with the highest scores obtain one prize each. In a lobbying setting, for example, a player's score represents the influence he achieves over the policy maker. Conditional on winning or losing, a player's payoff decreases continuously with his chosen score. This formulation allows for a high degree of heterogeneity among players. In addition, the difference in a player's payoff between winning and losing may depend on his chosen score. This accommodates a combination of unconditional and conditional investments.[1] The primitives of the model are commonly known, capturing players' knowledge of the asymmetries among them. This helps interpret players' payoffs as "economic rents," in contrast to "information rents" that arise in models of competition with private information.[2] Contests are defined in Section 1.

[1] It also accommodates player-specific risk attitudes, and player- and score-dependent valuations for a prize.

[2] Examples of such models include those studied by Benny Moldovanu and Aner Sela (2001, 2006), Todd Kaplan et al. (2002), and Sergio Parreiras and Anna Rubinchik (2009). These models are related to the

Section 2 presents a constructive characterization of the unique equilibrium for a large class of single-prize and multiprize contests. When there are two players and one prize, players' equilibrium strategies are given by a simple closed-from formula.[3] For the more complicated case of $m > 1$ prizes, I provide an algorithm that constructs the unique equilibrium when there are $m + 1$ players. The main component of the algorithm is a dynamic "supply function" of the hazard rates of players' strategies as a function of score. The construction relies on knowledge of players' equilibrium payoffs, which is provided by the payoff result of Ron Siegel (2009).[4] A property of the equilibrium is that players may choose scores from multiple intervals. This happens when players have relative cost advantages at different scores. The participation result of Siegel (2009) provides a sufficient condition for the algorithm to construct the unique equilibrium when there are more than $m + 1$ players.

The second part of the paper applies these results to investigate the effects of conditional investments when competitors differ in their costs of employing a common underlying production technology. In this class of *simple contests*, each player is characterized by his cost coefficient and valuation for a prize. Every player pays a positive fraction $\alpha \leq 1$ of the cost associated with his chosen score. This cost is the product of the player's cost coefficient and a common cost function that represents the common production technology. The remaining fraction $1-\alpha$ of the cost is paid only by the winners of the $m \geq 1$ prizes. For example, in R&D races in which firms compete by building prototypes, the contest designer

extensive literature on auctions, both in assuming incomplete information and in how prizes are allocated. The contest model presented here uses an auction-like allocation rule, but postulates complete information. Consequently, the analysis necessitates different techniques than those used in the auction literature, and also leads to qualitatively different results. Other models of competition postulate a probabilistic relation between competitors' efforts and prize allocation. See Gordon Tullock (1980) and Edward Lazear and Sherwin Rosen (1981). For a comprehensive treatment of the literature on competitions with sunk investments, see Shmuel Nitzan (1994) and Kai A. Konrad (2007).

[3] This generalizes the results of Yeon-Koo Che and Ian Gale (2006) and Kaplan and David Wettstein (2006), who considered two-player contests with ordered cost functions and no conditional investments.

[4] Siegel's (2009) model is more general (he allows for players' payoffs to weakly decrease in score), but he does not solve for equilibrium.

often specifies the degree of functionality that a prototype must demonstrate relative to the actual product. Each firm then chooses the quality of the product it will produce if it wins, and pays its cost of building the corresponding prototype. This cost is a fraction α of the total cost of developing the product. Only the winning firms pay the additional fraction $1 - \alpha$ of the total cost to complete the development the product. The higher the degree of functionality required by the contest designer, the higher the fraction α of the development cost that firms incur as unconditional investments.

I show that simple contests have a unique equilibrium, in which every player chooses a score from an interval. Moreover, after normalizing each player's efficiency by dividing his cost coefficient by his valuation for a prize, I show that the equilibrium strategies of more efficient players first-order stochastically dominate those of less efficient players, and that more efficient players win prizes more often than less efficient players. As α approaches 0, the most efficient players obtain a prize with near certainty. When players differ only in their valuations for a prize, as α approaches 0 the allocation of prizes becomes efficient and expenditures are maximized.

The limit of the equilibria as α approaches 0 is an equilibrium of the corresponding "discriminatory-price auction," in which all investments are conditional on winning.[5] A special case is the complete-information first-price auction. When α equals 1, all investments are unconditional. If, in addition, the common production technology is linear, we have a complete-information all-pay auction (henceforth: all-pay auction).[6] The all-pay auction, along with its variants, has been used extensively to model rent-seeking and lobbying activities (Arye L. Hillman and Dov Samet (1987), Hillman and John G. Riley (1989), Michael R. Baye, Dan Kovenock, and Casper de Vries (1993), Julio González-Díaz (2009)), competitions for a monopoly position (Tore Ellingsen (1991)), waiting in line (Clark and

[5] In this equilibrium, no player chooses weakly dominated strategies with positive probability, and the equilibrium is robust to the tie-breaking rule. This provides a selection criterion among the continuum of equilibria of the discriminatory-price auction, which are not payoff equivalent.

[6] The equilibrium uniqueness result applied to all-pay auctions corrects that of Derek J. Clark and Christian Riis (1998). They constructed an equilibrium for the multiprize all-pay auction and claimed it was unique. Their proof of uniqueness relied on showing that in any equilibrium the best response set of each player is an interval. Their proof of this latter claim was incorrect.

Riis (1998)), sales (Hal Varian (1980)), R&D races (Partha Dasgupta (1986)), competitions for multiple prizes (Clark and Riis (1998) and Yasar Barut and Kovenock (1998)), the effect of lobbying caps (Che and Gale (1998, 2006) and Kaplan and Wettstein (2006)), and R&D races with endogenous prizes (Che and Gale (2003)). Simple contests generalize all-pay auctions by allowing for non-linear costs and accommodating both unconditional and conditional investments.

The Appendix contains proofs of results from Section 2. An Online Appendix contains an example, the specification of the contest depicted in Figure 2, and proofs of results from Section 3.

1 The Model

In a *contest*, n players compete for m homogeneous prizes, $0 < m < n$. The set of players $\{1, \ldots, n\}$ is denoted by $\mathcal{N}$. Players compete by each choosing a score from $\mathbb{R}_+ = [0, \infty)$, simultaneously and independently. Each of the m players with the highest scores wins one prize. In case of a relevant tie, any procedure may be used to allocate the tie-related prizes among the tied players.

Given scores $\mathbf{s} = (s_1, \ldots, s_n)$, where s_i is player i's chosen score, *player i's payoff* is

$$u_i(\mathbf{s}) = P_i(\mathbf{s}) v_i(s_i) - (1 - P_i(\mathbf{s})) c_i(s_i),$$

where $v_i : \mathbb{R}_+ \to \mathbb{R}$ is *player i's valuation for winning*, $c_i : \mathbb{R}_+ \to \mathbb{R}$ is *player i's cost of losing*, and $P_i : \mathbb{R}_+^n \to [0, 1]$ is *player i's probability of winning*, which satisfies

$$P_i(\mathbf{s}) = \begin{cases} 0 & \text{if } s_j > s_i \text{ for } m \text{ or more players } j \neq i; \\ 1 & \text{if } s_j < s_i \text{ for } n - m \text{ or more players } j \neq i; \\ \text{any value in } [0,1] & \text{otherwise,} \end{cases}$$

such that $\sum_{j=1}^{n} P_j(\mathbf{s}) = m$.

Note that a player's probability of winning depends on all players' scores, but his valuation for winning and cost of losing depend only on his chosen score. The primitives of the contest are commonly known.

A player's score represents his overall performance, which may depend more than one factor. For example, in a lobbying setting a firm's score may be a function of the amount of money the firm invests, its reputation, and its political connections. In a competition for promotions, a worker's score may be a function of the number of hours worked, tenure, and social skills. In an R&D race in which a firm's score is the quality of the prototype it produces, this quality may be a function of various performance measures. The score represents an optimal choice of the different factors as one number according to which players are evaluated. Differences among players are captured by their different valuations for winning and costs of losing.

I consider contests that meet the following Assumptions B1-B3.

Assumption B1 *v_i and $-c_i$ are continuous and strictly decreasing.*

Assumption B2 *$v_i(0) > 0$ and $\lim_{s_i \to \infty} v_i(s_i) < c_i(0) = 0$.*

Assumption B2 says that prizes are valuable, and that sufficiently high scores are prohibitively costly. Assumption B1 captures the all-pay component of contests. It is not satisfied by complete-information first-price auctions, for example, since a player pays nothing if he loses, so $c_i \equiv 0$. But assumption B1 *is* satisfied when an all-pay element is introduced, e.g., when every bidder pays some positive fraction of his bid whether he wins or not, and only the winner pays the balance of his bid. Assumptions B1 and B2 are depicted in Figure 1.

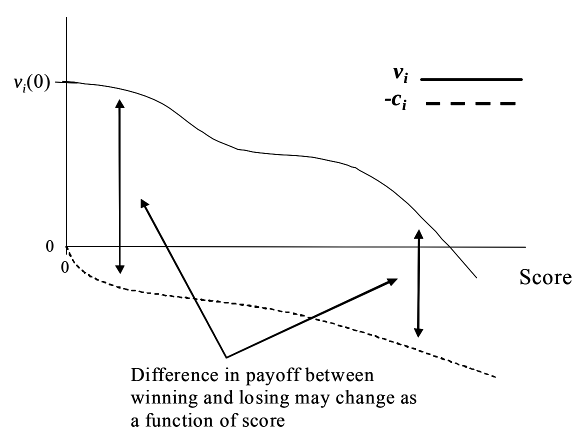

Figure 1: Assumptions B1 and B2

As Figure 1 shows, the payoff difference between winning and losing may depend on the player's chosen score. This accommodates a combination of unconditional and conditional investments, as shown in Section 3. A non-constant payoff difference can also be used to model different risk attitudes and score-dependent prize values.

In a *separable contest*, this payoff difference is constant, so $v_i(s_i) = V_i - c_i(s_i)$ and $u_i(\mathbf{s}) = P_i(\mathbf{s}) V_i - c_i(s_i)$, where $V_i = v_i(0) > 0$. The value V_i can be thought of as player i's valuation for a prize, which does not depend on his chosen score, and $c_i(s_i)$ can be thought of as player i's cost of choosing score s_i, which does not depend on whether he wins or loses. All investments are unconditional, and players are risk neutral. Figure 2 in Section 2 depicts a separable contest with non-linear costs. Separable contests with linear costs are all-pay auctions (Hillman and Samet (1987), Hillman and Riley (1989), Clark and Riis (1998)).[7]

Assumption B3, which completes the description of contests, uses the following definition.

Definition 1 (i) *Player i's* reach r_i *is the score at which his valuation for winning is* 0*. That is,* $r_i = v_i^{-1}(0)$*. Re-index players in (any) decreasing order of their reach, so that* $r_1 \geq r_2 \geq \ldots \geq r_n$*.*

(ii) *Player* $m+1$ *is the* marginal player*.*

(iii) *The* threshold T *of the contest is the reach of the marginal player:* $T = r_{m+1}$*.*

Assumption B3 states that the reach of the marginal player is different from that of the other players.

Assumption B3 Only the marginal player's reach is equal to the threshold, so $r_i \neq T$ for every player $i \neq m+1$.

In a separable contest, a player's reach is the score whose cost is the player's valuation

[7] In an all-pay auction, $v_i(s_i) = V_i - s_i$, $c_i(s_i) = s_i$, and ties are resolved by randomizing uniformly, where V_i is bidder i's valuation for a prize.

for a prize. For example, in an all-pay auction a player's reach is his valuation for a prize, so Assumption B3 is met if the marginal player's valuation for a prize is different from those of the other players.

The model of contests described here is a special case of Siegel's (2009) all-pay contest model. The main difference is that he allows for weakly decreasing v_i and $-c_i$.[8]

1.1 Existing Results

This subsection lists four results, Theorem 1, Theorem 2, Lemma 1, and Lemma 2, which I use in solving for equilibrium. They are immediate corollaries of results in Siegel (2009).[9] The first result characterizes players' equilibrium payoffs (without solving for equilibrium), and uses the following definition.

Definition 2 *Players i's* power w_i *is his valuation for winning at the threshold. That is,* $w_i = v_i(T)$. *In particular, the marginal player's power is* 0.

Assumptions B1 and B3 imply that only the marginal player has power 0. In an all-pay auction, for example, a player's power is equal to his valuation for a prize less that of the marginal player.

Theorem 1 *In any equilibrium of a contest, the expected payoff of every player equals the maximum of his power and* 0.

In addition to giving a closed-form formula for players' equilibrium payoffs, Theorem 1 shows that players $1, \ldots, m$ have strictly positive expected payoffs, and players $m+1, \ldots, n$ have expected payoffs of 0. It also shows that a player's expected payoff does not depend

[8] Instead of Assumptions B1-B3, Siegel (2009) makes three assumptions (A1-A3), and specifies two Generic Conditions (Cost Condition and Power Condition) on players' valuations for winning and costs of losing. Assumptions B1 and B2 imply his Assumptions A1-A3 and Cost Condition, and Assumptions B1 and B3 imply his Power Condition. In particular, a contest here is a generic contest of Siegel (2009).

[9] The results follow, respectively, from Theorem 1, Theorem 2, Corollary 1, and Theorem 1, Lemma 1, and footnote 19 of Siegel (2009).

on his cost of losing. Section 3 uses Theorem 1 to derive players' payoffs in a specific class of contests.

A player *participates* in an equilibrium of a contest if with strictly positive probability he chooses strictly positive scores. The second result provides a sufficient condition for players $m+2,\ldots,n$ not to participate in any equilibrium.

Theorem 2 *If the normalized costs of losing and valuations for winning for the marginal player are, respectively, strictly lower and weakly higher than those of player $i > m+1$, that is*

$$\frac{c_{m+1}(x)}{v_{m+1}(0)} < \frac{c_i(x)}{v_i(0)} \text{ for all } x > 0$$

and

$$\frac{v_{m+1}(x)}{v_{m+1}(0)} \geq \frac{v_i(x)}{v_i(0)} \text{ for all } x \geq 0,$$

then player i does not participate in any equilibrium. In particular, if these conditions hold for all players $m+2,\ldots,n$, then in any equilibrium only the $m+1$ players $1,\ldots,m+1$ may participate.

The third result states that an equilibrium always exists.

Lemma 1 *Every contest has a Nash equilibrium.*

The fourth result enumerates four properties of any equilibrium.

Lemma 2 *In any equilibrium of a contest, (1) no score in $(0,T)$ is chosen with strictly positive probability by any player, (2) every score in $(0,T)$ is a best response for at least two players, (3) no score higher than T is a best response for any player, and (4) players $1,\ldots,m+1$ participate.*

2 Solving for Equilibrium

A player's (mixed) strategy is a probability distribution over $[0,\infty)$, and an equilibrium is a profile of strategies, one for each player, such that each player's strategy assigns probability 1 to the player's set of best-responses given the other players' strategies. I describe a strategy of player i by a cumulative distribution function (CDF) G_i, which for every $x \geq 0$ specifies the probability $G_i(x)$ that player i chooses a score lower or equal to x. I denote by $\mathbf{G} = (G_1, \ldots, G_n)$ a vector of players' strategies.

Consider first the relatively simple case of two players and one prize. Part (2) of Lemma 2 shows that the players must behave in a way that makes every positive score up to the threshold a best response for both of them.[10] That both players have interval best-response sets pins down the unique equilibrium even without knowing players' payoffs.

Theorem 3 *In a two-player, single-prize contest the unique equilibrium is given by*

$$\mathbf{G}(x) = \left(\frac{c_2(x)}{v_2(x) + c_2(x)}, \frac{w_1 + c_1(x)}{v_1(x) + c_1(x)} \right)$$

on $[0, T]$.[11]

Proof. It is straightforward to verify that $\mathbf{G}$ is an equilibrium. For uniqueness, choose an equilibrium of the contest. By part (2) of Lemma 2, both players are indifferent among all scores in $(0, T)$. Moreover, no player can choose T with strictly positive probability, otherwise the other player would not have best responses slightly below T, contradicting part (2) of Lemma 2. Therefore, the equilibrium has the form

$$\left(\frac{d_2 + c_2(x)}{v_2(x) + c_2(x)}, \frac{d_1 + c_1(x)}{v_1(x) + c_1(x)} \right)$$

on $(0, T]$ for some d_1 and d_2, and neither CDF can reach 1 before T. By part (3) of Lemma 2, the CDFs of both players must reach 1 at exactly T. Consequently, $d_1 = v_1(T) = w_1$ and $d_2 = v_2(T) = 0$, so the equilibrium is $\mathbf{G}$. ■

Section 3.1 below provides an application of this result to certain non-separable contests. Applied to separable contests, Theorem 3 extends the results of Kaplan and Wettstein

[10] For the purpose of equilibrium construction it is more convenient to work with best responses than with supports.

[11] When there are more than two players, since players 1 and 2 participate in any equilibrium (part (4) of Lemma 2), a single-prize contest has at most one equilibrium in which two players participate. Example 3 in Siegel (2009) shows that a single-prize contest may have multiple equilibria when more than two players participate.

(2006) and Che and Gale (2006), who considered two-player separable contests with ordered cost functions.

I now turn to multiprize contests. Because more than two players participate in any equilibrium (part (4) of Lemma 2), different players may compete on different sets of scores. This makes solving for equilibrium considerably more difficult than in the two-player case. I restrict attention to *regular contests*, which are contests that meet the following regularity condition.

Condition R *The valuations for winning and costs of losing of players* $1, \ldots, m+1$ *are piecewise analytic on* $[0, T]$*.*[12]

Consider the three-player, two-prize, separable regular contest depicted in Figure 2.[13] The contest may be used to model a lobbying setting in which three firms attempt to influence a policy maker through costly "favors," whose cost is incurred regardless of the outcome. The two firms that offer the best favors obtain a manufacturing license whose value is normalized to 1. Each firm's score is the quality of favors it offers. The firms differ in their technologies for producing favors, which leads to differences in their cost functions. Firm 1's costs are everywhere below those of the other firms. Firm 2's costs are linear and are higher than firm 3's costs for low and intermediate scores, but are lower than firm 3's costs for high scores. The contest may also be used to model a setting in which three firms compete in a market that can profitably support only two products, so the firm with the lowest-quality product is excluded from the market. Each firm's score is the quality of its product, and firms' cost functions are derived from their production technologies.

The unique equilibrium of the contest is depicted in Figure 3, drawn as cumulative probability distributions. The best-response set of player 2 is $(0, x_1] \cup [x_2, 1]$, and that of

[12] A function f is piecewise analytic on $[0, T]$ if $[0, T]$ can be partitioned into a finite number of closed intervals such that the restriction of f to each interval is analytic. Analytic functions include polynomials, the exponent function, trigonometric functions, and power functions. Sums, products, compositions, reciprocals, and derivatives of analytic functions are analytic (see, for example, Chapman (2002)).

[13] Players' cost functions are given in the Online Appendix.

player 3 is $[0, x_3] \cup [x_4, 1]$.[14] Each player is defined as being "active" on his best-response set. The points $0, x_1, x_2, x_3, x_4$ are "switching points" above which the set of active players (denoted in curly brackets) changes. As Figure 3 shows, different players compete on different sets of scores and, moreover, players 2 and 3 have non-interval best-response sets. This happens because players 2 and 3 have relative cost advantages at different scores. Non-interval best-response sets are new in the complete-information contest literature.[15]

The equilibrium depicted in Figure 3 is constructed using the algorithm described in Section 2.1. Section 2.2 shows that the equilibrium is unique and explains to what extent the construction applies to n-player contests. Section 2.3 discusses how the equilibrium construction results can help us think about real-world competitions.

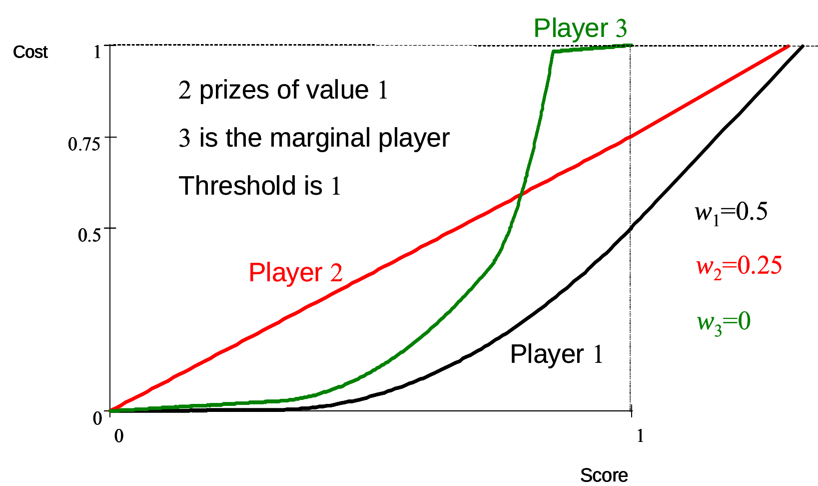

Figure 2: Player's costs, reaches, and powers

[14]When the contest is not regular, players' strategies can be quite "pathological." The Online Appendix depicts an equilibrium in which a player's best-response set is the Cantor set.

[15]In the equilibria of Baye, Kovenock, and de Vries (1993), who considered a single-prize all-pay auction that violates Assumption B3, a player's best-response set may be the union of 0 and a single interval whose lower endpoint is strictly positive. All such equilibria disappear when players' valuations are perturbed slightly to produce unique players with the first- and second-highest valuations (so that Assumption B3 holds). This leaves a single equilibrium, in which the best-response set of each player is an interval (or the singleton 0). A similar perturbation produces a single equilibrium, in which the best-response set of each player is an interval, in González-Díaz (2009). In contrast, the non-interval property that arises here is "fundamental" in nature: it is robust to perturbations in the contest's specification, and, moreover, a player's best-response set may consist of several disjoint intervals of positive length.

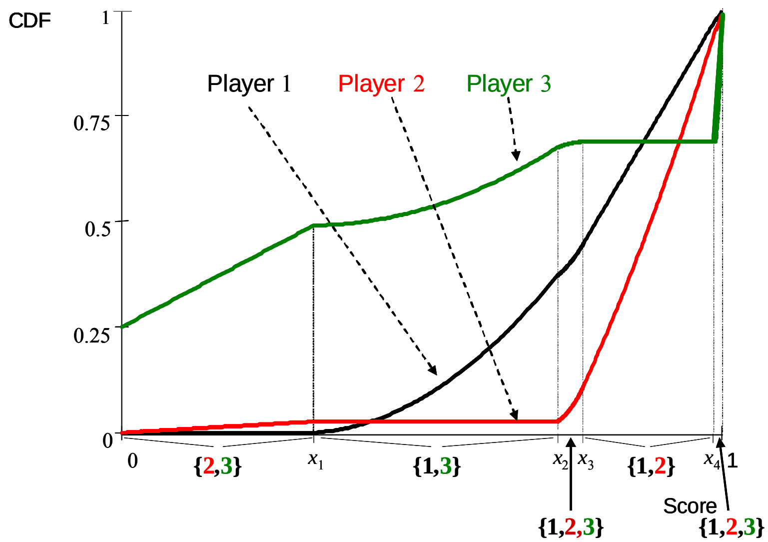

Figure 3: The unique equilibrium

2.1 The Algorithm

The algorithm constructs an equilibrium $\mathbf{G}$ of an $(m+1)$-player regular contest for m prizes. The construction proceeds from 0 to T in several steps. The first step determines $\mathbf{G}(0)$. The second step determines, for any switching point $x < T$ that has been reached, the set $\mathcal{A}^+(x)$ of all the players that will be active immediately above x. This step is divided into two parts. Part one uses $\mathbf{G}(x)$ to identify a set of candidate players $\mathcal{CP}(x)$, which is a superset of $\mathcal{A}^+(x)$. Part two uses $\mathcal{CP}(x)$ to define a supply function S_x with a unique fixed point $H(x)$, and uses $H(x)$ to determine the set $\mathcal{A}^+(x)$ from $\mathcal{CP}(x)$. The third step uses $\mathcal{A}^+(x)$ to extend $\mathbf{G}$ from x up to the next switching point $\bar{x}$ above x. The fourth step identifies $\bar{x}$. Steps two to four are then repeated until T is reached. I show that T is reached in a finite number of steps and that the resulting $\mathbf{G}$ is indeed an equilibrium.

Step 1: To determine $\mathbf{G}(0)$, recall that players $1,\ldots,m$ have strictly positive equilibrium payoffs (because they have strictly positive powers). Therefore, none of these players has an atom at 0 (choose 0 with strictly positive probability), because by choosing 0 a player loses for sure, so his payoff is 0.[16] By part (2) of Lemma 2, every score slightly

[16] Because there are $m+1$ players, it suffices to beat one player to win a prize. This implies that no two players have an atom at 0, otherwise scores slightly above 0 would be profitable deviations. Therefore, a player with an atom at 0 loses for sure when choosing 0.

above 0 is a best response for at least two players. For this to hold, player $m+1$ must have an atom at 0, and the size of the atom must be such that at least one other player obtains his equilibrium payoff if he chooses scores immediately above 0. The atom cannot be larger, otherwise some player will be able to obtain more than his equilibrium payoff by choosing scores immediately above 0. These observations prove the following lemma.

Lemma 3 *In any equilibrium* $\mathbf{G}$ *of an* $(m+1)$*-player regular contest,* $G_i(0) = 0$ *for* $i < m+1$ *and*

$$G_{m+1}(0) = \min_{j \leq m} \frac{w_j}{v_j(0)} < 1.$$

Step 2 part one: Suppose $\mathbf{G}$ has been defined up to a switching point $x < T$. We would like to identify a set of players, the "candidate players at x," who may be active immediately above x when $\mathbf{G}$ is extended above x. Consider a player i who chooses scores immediately above x when other players choose scores according to $\mathbf{G}$. Because CDFs are right-continuous and the player must only beat one other player to win a prize, $1 - \Pi_{j \neq i}(1 - G_j(x)) + \varepsilon$ is the probability that the player wins a prize, for some small $\varepsilon \geq 0$. As player i chooses scores closer to x, ε approaches 0. Therefore, if

$$(1) \qquad (1 - \Pi_{j \neq i}(1 - G_j(x)))\, v_i(x) - (\Pi_{j \neq i}(1 - G_j(x)))\, c_i(x) < w_i,$$

then player i cannot be active immediately above x, because he cannot obtain his equilibrium payoff there. The candidate players at x are the other players, and for each such player i we have

$$(2) \qquad (1 - \Pi_{j \neq i}(1 - G_j(x)))\, v_i(x) - (\Pi_{j \neq i}(1 - G_j(x)))\, c_i(x) = w_i.$$[17]

We therefore denote the set of candidate players at x by

$$\mathcal{CP}(x) = \{\text{players } i \text{ for which (2) holds}\}.$$

The set $\mathcal{CP}(x)$ contains at least two players.[18] If $\mathcal{CP}(x)$ contains only two players, part (2) of Lemma 2 shows that they must both be active immediately above x. The difficulty

[17] The proof of Lemma 5 in the Appendix shows that $>$ does not hold for any player.

[18] This is shown in the proof of Lemma 5 in the Appendix.

is to determine which of the candidate players at x are active immediately above x when $\mathcal{CP}(x)$ contains more than two players.

Step 2 part two: Denote the set of players active immediately above x by $\mathcal{A}^+(x) \subseteq \mathcal{CP}(x)$. To identify $\mathcal{A}^+(x)$ we use the following three observations. First, a player in $\mathcal{A}^+(x)$ obtains his equilibrium payoff when choosing scores immediately above x. For such a player the increase in cost associated with choosing a higher score is exactly offset by the increase in the probability of winning. Second, players who are not in $\mathcal{A}^+(x)$ do not choose scores immediately above x, so their CDFs do not change. Third, no player should obtain more than his equilibrium payoff by choosing scores immediately above x. To express these observations formally, let

$$q_i(x) = 1 - \frac{w_i + c_i(x)}{v_i(x) + c_i(x)}$$

and rewrite (2) as

$$\Pi_{j \neq i}(1 - G_j(x)) = q_i(x). \tag{3}$$

Differentiating (3), taking logs, and recalling that the CDFs of players not in $\mathcal{A}^+(x)$ do not change, we obtain that

$$\text{for every player } i \text{ in } \mathcal{A}^+(x),\ \varepsilon_i(x) = \sum_{j \in \mathcal{A}^+(x) \setminus \{i\}} h_j(x), \tag{4}$$

where $\varepsilon_i(x) = -q_i'(x)/q_i(x) > 0$ denotes player i's *semi-elasticity at* x, $h_j(x) = -(1 - G_j(x))'/(1 - G_j(x))$ denotes player j's *hazard rate at* x, and all derivatives are right-derivatives.[19] Because no player should obtain more than his equilibrium payoff immediately above x, we also have

$$\text{for every player } i \text{ in } \mathcal{CP}(x),\ \varepsilon_i(x) \geq \sum_{j \in \mathcal{A}^+(x) \setminus \{i\}} h_j(x). \tag{5}$$

[19] By Condition R, q_i is right-continuously differentiable. For $x < T$, $q_i'(x)$ is strictly negative, because

$$q_i'(x) = \left(\frac{v_i(x) - v_i(T)}{v_i(x) + c_i(x)}\right)' = \frac{\overbrace{v_i'(x)}^{\text{Negative}}\overbrace{(c_i(x) + v_i(T))}^{\text{Positive}} + \overbrace{c_i'(x)}^{\text{Positive}}\overbrace{(v_i(T) - v_i(x))}^{\text{Negative}}}{(v_i(x) + c_i(x))^2}.$$

Letting $H(x) = \sum_{j \in \mathcal{A}^+(x)} h_j(x)$ and noting that $h_i(x) > 0$ implies that player i in $\mathcal{CP}(x)$ is in $\mathcal{A}^+(x)$, (4) and (5) can be combined to give

$$\text{(6)} \qquad \text{for every player } i \text{ in } \mathcal{CP}(x),\ h_i(x) = \max\{H(x) - \varepsilon_i(x), 0\}.$$

By adding up (6) for players i in $\mathcal{CP}(x)$ (and recalling that the hazard rates at x of players not in $\mathcal{A}^+(x)$ are 0), we have that

$$H(x) = \sum_{i \in \mathcal{CP}(x)} \max\{H(x) - \varepsilon_i(x), 0\}.$$

This means that $H(x)$ is a fixed point of the function

$$S_x(H) = \sum_{i \in \mathcal{CP}(x)} \max\{H - \varepsilon_i(x), 0\}.$$

The function S_x can be thought of as an aggregate supply function. To see this, think of the right-hand side of (6) with H instead of $H(x)$ as player i's "supply curve" of "hazard rate" as a function of "price" H. Then $S_x(H)$ is the aggregate supply of hazard rates at x given H. To determine $H(x)$ from S_x, note that S_x is a piecewise linear function, whose slope increases by 1 every time H exceeds the semi-elasticity of a player in $\mathcal{CP}(x)$. Since all semi-elasticities are positive and $\mathcal{CP}(x)$ includes at least two players, $S'_x(0) = 0$ and $H(x) > 0$. So, S_x is a convex function that starts below the diagonal and reaches a slope of at least 2. Therefore, it intersects the diagonal precisely once above 0, at $H(x)$ (see Figure 4 below). Note that the definition of S_x does not involve $\mathcal{A}^+(x)$. Therefore, $H(x)$ can be deduced from $\mathcal{CP}(x)$ without knowing $\mathcal{A}^+(x)$.

In fact, $H(x)$ determines $\mathcal{A}^+(x)$. If $\varepsilon_i(x) < H(x)$ for a player i in $\mathcal{CP}(x)$, then the player's hazard rate at x is $H(x) - \varepsilon_i(x) > 0$ so the player is in $\mathcal{A}^+(x)$. By definition of $H(x)$ as the fixed point of $S_x(H)$, there are at least two such players (recall that $\mathcal{CP}(x)$ contains at least two players). If $\varepsilon_l(x) > H(x)$ for a player l in $\mathcal{CP}(x)$, then the player cannot obtain his equilibrium payoff immediately above x and is therefore not in $\mathcal{A}^+(x)$. This is depicted in Figure 4: $\mathcal{CP}(x) = \{i, j, l\}$ and $\mathcal{A}^+(x) = \{i, j\}$, because $\varepsilon_i(x) < H(x), \varepsilon_j(x) < H(x)$, and $\varepsilon_l(x) > H(x)$. Also, S'_x does not increase at $\varepsilon_k(x)$, since player k is not a candidate player at x (k is not in $\mathcal{CP}(x)$).

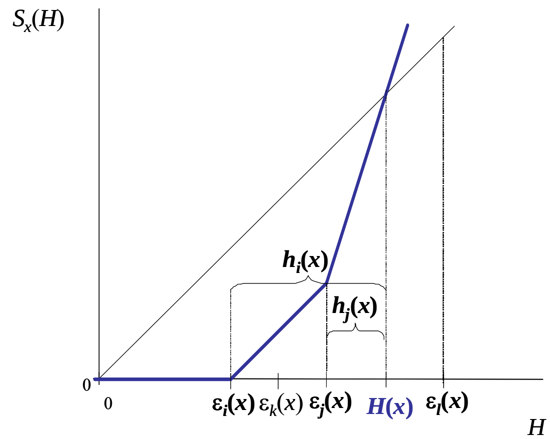

Figure 4: The function S_x and its fixed point $H(x)$

A complication arises if $\varepsilon_i(x) = H(x)$ for a player i in $\mathcal{CP}(x)$. In this case, what determines whether player i is in $\mathcal{A}^+(x)$ is how the player's semi-elasticity and the aggregate hazard rate change immediately above x. For this, consider the unique fixed point $H(x,y)$ of the function

$$S_{x,y}(H) = \sum_{i \in \mathcal{CP}(x)} \max\{H - \varepsilon_i(y), 0\}.$$

for scores y immediately above x. The lowest-order right-derivatives of $\varepsilon_i(\cdot)$ and $H(x,\cdot)$ at x that differ determine whether the player is in $\mathcal{A}^+(x)$ ($<$ means the player is in $\mathcal{A}^+(x)$, $>$ means the player is not in $\mathcal{A}^+(x)$). (This will "generically" stop at the first derivatives.) If all derivatives are equal, then player i is in $\mathcal{A}^+(x)$.[20]

[20] $H(x,y)$ can also be computed as follows. Order the players in $\mathcal{CP}(x)$ in any increasing order of semi-elasticity on some right-neighborhood of x (this can be done since by Condition R semi-elasticities are analytic on some right-neighborhood of x, and an analytic function with an accumulation point of roots is identically 0 in the connected component of the accumulation point). Let $L(x)$ be the highest $l \geq 2$ (in this ordering) such that

$$\frac{1}{l-1} \sum_{j \in \mathcal{CP}(x), j \leq l} \varepsilon_j(y) - \varepsilon_l(y) \geq 0$$

for every score y in some right-neighborhood of x (that $L(x)$ is well defined follows from analyticity as well). Then

$$H(x,y) = \frac{1}{L(x)-1} \sum_{j \in \mathcal{CP}(x), j \leq L(x)} \varepsilon_j(y).$$

This follows from the uniqueness of the positive fixed point of $S_{x,y}$, and shows that $H(x,\cdot)$ is analytic so has derivatives of any order. Note that $H(x,x) = H(x)$.

Step 3: Use $\mathbf{G}(x)$ and $\mathcal{A}^+(x)$ in the following way to extend $\mathbf{G}$ from x to $\bar{x}$, where $\bar{x}$ is the first switching point above x (the first score above which the set of active players changes). For a player j not in $\mathcal{A}^+(x)$, set $G_j(y) = G_j(x)$ for every score y in this interval, and let $D = \Pi_{j \notin \mathcal{A}^+(x)} (1 - G_j(x))$ (if $\mathcal{A}^+(x) = \mathcal{N}$, then $D = 1$). For a player i in $\mathcal{A}^+(x)$ and a score y in $(x, \bar{x})$, rewrite (3) with y instead of x as

$$\sum_{j \in \mathcal{A}^+(x) \setminus \{i\}} \ln (1 - G_j(y)) = \ln q_i(y) - \ln D \tag{7}$$

by taking natural logs.[21] The system of $|\mathcal{A}^+(x)|$ linear equations (where $|\mathcal{S}|$ denotes the cardinality of a set $\mathcal{S}$) in $|\mathcal{A}^+(x)|$ unknowns $(1 - G_j(y))$, j in $\mathcal{A}^+(x)$, given by (7) for every player i in $\mathcal{A}^+(x)$ has a unique solution.[22] In this solution, for every player i in $\mathcal{A}^+(x)$ we have

$$G_i(y) = 1 - \frac{\Pi_{j \in \mathcal{A}^+(x)} q_j(y)^{\frac{1}{|\mathcal{A}^+(x)|-1}}}{q_i(y) D^{\frac{1}{|\mathcal{A}^+(x)|-1}}}. \tag{8}$$

Therefore, for every player i in $\mathcal{A}^+(x)$ and every score y in $(x, \bar{x})$ set $G_i(y)$ according to (8); if $\bar{x} < T$, set $G_i(\bar{x})$ according to (8). Set $G_i(T) = 1$ for every player i.

Step 4: The switching point $\bar{x}$ is the first score above x for which $\mathcal{A}^+(x) \neq \mathcal{A}^+(\bar{x})$. This can happen for one of two reasons. The first reason is that a new player becomes active above $\bar{x}$, so $\mathcal{A}^+(\bar{x})$ includes a player not in $\mathcal{A}^+(x)$. Because $\mathcal{A}^+(\bar{x}) \subseteq \mathcal{CP}(\bar{x})$, this is a player i who obtains his equilibrium payoff at $\bar{x}$ ((3) holds with $\bar{x}$ instead of x). The second reason is that an active player becomes inactive above $\bar{x}$, so a player in $\mathcal{A}^+(x)$ is not in $\mathcal{A}^+(\bar{x})$. This means that the player's hazard rate at $\bar{x}$ is 0. Therefore, to identify $\bar{x}$ extend $\mathbf{G}$ above x according to (8) and consider the first score $y > x$ such that one of

[21] $q_i(x) > 0$ because $x < T$. $D > 0$ because $G_i(x) < 1$ for every player i. This follows from the first statement shown in the proof of Lemma 5 in the Appendix, because if $G_k(x) \geq 1$ for any player k at $x < T$, then neither (2) nor (1) hold for any other player i.

[22] To see this, write the system of equations in matrix form, and note that the matrix $\mathbf{1}_{|\mathcal{A}^+(x)| \times |\mathcal{A}^+(x)|} - \mathbf{I}_{|\mathcal{A}^+(x)| \times |\mathcal{A}^+(x)|}$, where $\mathbf{1}_{M \times M}$ and $\mathbf{I}_{M \times M}$ are a matrix of ones and the identity matrix of dimensions $M \times M$, is invertible:

$$\left(\mathbf{1}_{|\mathcal{A}^+(x)| \times |\mathcal{A}^+(x)|} - \mathbf{I}_{|\mathcal{A}^+(x)| \times |\mathcal{A}^+(x)|}\right)^{-1} = \left(\frac{1}{|\mathcal{A}^+(x)| - 1} \cdot \mathbf{1}_{|\mathcal{A}^+(x)| \times |\mathcal{A}^+(x)|} - \mathbf{I}_{|\mathcal{A}^+(x)| \times |\mathcal{A}^+(x)|}\right).$$

the following happens. Either (3) with y instead of x holds for a player i not in $\mathcal{A}^+(x)$, or $h_j(y) = 0$ for a player j in $\mathcal{A}^+(x)$, or y is a concatenation point of the cost function of a player in $\mathcal{A}^+(x)$ (recall that costs are piecewise-defined functions), or $y = T$.[23,24] If $y \neq T$, determine $\mathcal{CP}(y)$ from $\mathbf{G}(y)$ and use $H(y)$ to determine $\mathcal{A}^+(y)$ from $\mathcal{CP}(y)$ as described in Step 2 above. If $\mathcal{A}^+(y) \neq \mathcal{A}^+(x)$, then y is the switching point $\bar{x}$. If $\mathcal{A}^+(y) = \mathcal{A}^+(x)$, then y is not a true switching point, and the search continues above y for the next candidate switching point. This can only repeat a finite number of times before $\bar{x}$ is identified.[25]

The following lemma shows that the algorithm reaches T via a finite number of switching points. Its proof and those of Theorems 4 and 5 below are in the Appendix.

Lemma 4 *The algorithm does not stop before reaching T, and the number of switching points in $[0, T]$ identified by the algorithm is finite. In addition, all players are active sufficiently close to T.*

Theorem 4 summarizes the construction.

Theorem 4 *For any $(m+1)$-player regular contest, the algorithm constructs an equilibrium $\mathbf{G}$. The equilibrium is characterized by a partition into a finite number of intervals of positive length, on the interior of which the set of active players remains constant. The value of $\mathbf{G}$ on each interval is given by (8).*

The supply function shows that when a player obtains his equilibrium payoff and his semi-elasticity is low relative to those of the other players, he remains active. When his semi-elasticity is high relative to those of the other players, he becomes inactive. This

[23] Ignore any player i for whom $\varepsilon_i(x) = H(x)$ and all order derivatives of $\varepsilon_i(\cdot)$ and $H(x, \cdot)$ at x are equal. Such a player's hazard rate will not drop below 0, because, by analyticity, his semi-elasticity will equal the aggregate hazard rate as long as G is extended according to (8).

[24] That y is strictly greater than x follows from the definition of $\mathcal{A}^+(x)$. Indeed, when G is extended above x according to (8), the hazard rate at scores y immediately above x of every player i is $\max\{H(x, y) - \varepsilon_i(y), 0\}$. Therefore, immediately above x the hazard rates of players in $\mathcal{A}^+(x)$ are non-negative, and for the other players (1) holds.

[25] This follows from analyticity.

means that relative cost advantages, which may change in different regions of the competition, determine the regions on which different players compete. For an illustration, consider the supply function S_x and its positive fixed point $H(x)$ in the context of Figure 3 above. $\mathcal{CP}(0) = \mathcal{A}^+(0) = \{2, 3\}$. As x increases from 0 to T, the set of active players changes. At the switching point x_1, player 1 becomes active since he obtains his equilibrium payoff there. This changes S_x and $H(x)$ discontinuously. As a result, $H(x_1)$ falls below player 2's semi-elasticity, and he becomes inactive immediately above x_1. At x_2, player 2 rejoins the set of active players, and all three players are active up to x_3. Thus, the addition of an active player may or may not cause another to become inactive. At x_3, player 3's hazard rate reaches 0 (his semi-elasticity equals the aggregate hazard rate) and he becomes inactive immediately above x_3.[26] Player 3 rejoins the set of active players at x_4, and all three players remain active up to the threshold.[27]

In an all-pay auction, players' semi-elasticities are equal, so once a player becomes active, he remains active up to the threshold. This explains why in a multiprize all-pay auction (Clark and Riis (1998)) each player chooses a bid from an interval. Section 3 studies a class of contests that generalize all-pay auctions and have this interval property.

2.2 Equilibrium Uniqueness and n-player Contests

The algorithm constructs an equilibrium $\mathbf{G}$ with the property that every player's best-response set is a finite union of disjoint intervals. The logic underlying the algorithm can be used to show that $\mathbf{G}$ is the only equilibrium with this property. To prove that

[26] This happens because

$$\varepsilon_3'(x_3) > \frac{\partial H(x,y)}{\partial y}|_{(x_3,x_3)}.$$

[27] Jeremy I. Bulow and John Levin (2006) constructed the equilibrium of a game in which players have linear costs and compete for heterogeneous prizes. Their construction proceeds from the top, without first identifying players' equilibrium payoffs. This is possible because each player's best-response set is an interval and players' marginal costs are identical. Such a procedure does not work here, since the set of players active immediately *below* x cannot be uniquely determined from $\mathbf{G}(x)$ and players' payoffs using local conditions.

the equilibrium is unique, however, we must rule out the existence of all other equilibria, including those that do not have this property.[28] This is done by the following theorem.

Theorem 5 *For any $(m+1)$-player regular contest, the algorithm constructs the unique equilibrium of the contest.*

I now turn to regular contests with any number of players.

Theorem 6 *Regular contests have* at most *one equilibrium in which precisely $m+1$ players participate. The candidate for this equilibrium is the unique equilibrium of the reduced contest with players $1, \ldots, m+1$. It is an equilibrium of the original contest if and only if players $m+2, \ldots, n$ cannot obtain strictly positive payoffs by participating. If they can, then in every equilibrium at least $m+2$ players participate.*

Proof. Immediate from Theorem 5 and part (4) of Lemma 2. ■

In some regular contests, players $m+2, \ldots, n$ do not participate in any equilibrium. A sufficient condition for this is given by Theorem 2. For such contests, a much stronger result can be stated.

Theorem 7 *Regular contests in which players $m+2, \ldots, n$ do not participate in any equilibrium have a unique equilibrium. In this equilibrium, players $m+2, \ldots, n$ choose 0 and players $1, \ldots, m+1$ behave as in the unique equilibrium of the reduced contest with players $1, \ldots, m+1$. A sufficient condition for this is that the conditions of Theorem 2 hold for every player $m+2, \ldots, n$.*

Proof. By Theorem 5, the only candidate equilibrium is the one described in the statement of the theorem. This is an equilibrium of the contest, because every contest has an equilibrium (Lemma 1). ■

Note that the conditions of Theorem 2 do not place any restrictions on how the valuations for winning and costs of losing relate among players $\mathcal{N} \setminus \{m+1\}$. Thus, Theorem 7 applies to a wide class of contests.

---

[28] Equilibria for which this property does not hold may exist if the contest is not regular. The Online Appendix depicts an equilibrium of a contest that is not regular in which a player's best-response set is the Cantor set.

2.3 Discussion

Theorems 3 and 7 and the equilibrium construction algorithm provide strong predictions when there are $m+1$ players or the valuations for winning and costs of losing of players $m+2, \ldots, n$ are ranked relative to those of the marginal player. These predictions, along with the flexibility of the contest framework, can improve our understanding of competitions with an all-pay feature, and help us think about policy and design questions. Consider, for example, a researcher studying a lobbying setting in which several firms make outlays to influence a policy maker. Suppose the researcher has data on the value of a favorable outcome to each firm and on the determinants of firms' lobbying technologies, such as their costs of capital, reputation, and political connections. If the researcher also has some idea of how each firm's investment translates into influence over the policy maker, then he can formulate firms' valuations for winning and costs of losing to obtain a contest, as in Figure 2. He can then use the equilibrium results to predict firms' payoffs, the probability that each firm obtains a favorable outcome, and firms' outlays, which may be unobservable. Policy measures, such as imposing lobbying caps, restricting the type of lobbying activities in which firms can engage, or otherwise changing firms' lobbying technologies, translate directly to changes in firms' valuations for winning and costs of losing. Equilibrium analysis of the resulting contest can help predict the effects of such policy measures and whether they are desirable according to certain criteria, such as efficiency and equality. The effects of restricting or enlarging the set of lobbying firms (by limiting or encouraging entry) can be similarly predicted.

The framework can also guide a contest designer in using the set of instruments available to him. Consider, for example, an R&D race intended to foster innovation. Depending on the context, the designer may have different goals in mind: maximizing the quality of the best innovation, the quality of the best two innovations, or perhaps the average quality of contestants' innovations. The designer may have the flexibility to determine the value and number of prizes, the scoring rule used, which may treat different contestants differently, and the degree of functionality of the innovation that must be demonstrated before the winner or winners are selected. Each specification of these factors, together with contestants' innovation technologies, can be translated to a contest. The tools developed above can then be used to predict the value of the outcome variables in which the designer is interested. By optimizing over the set of feasible contests, the designer can get a sense of how to best use the instruments at his disposal to achieve his goals.

One may also be interested in studying competitions that are not fully specified. An example is a competition in which all contestants employ technologies with quadratic costs, whose coefficients may not be known or have not yet been determined. Because the corresponding contests are not fully specified, the equilibrium construction results, even though they apply in principal, may not deliver formulae describing players' strategies when there are multiple prizes. Nevertheless, our understanding of how the equilibrium is constructed and, in particular, how the supply function relates the set of active players in each region to players' semi-elasticities, may be used to derive certain qualitative features of the outcome. When the class of contests is parameterized by one or more variables, we may be able to conclude how these qualitative features change with the parameters. Section 3 conducts such an exercise to study how the size of the all-pay component affects the outcome of a competition.

3 Simple Contests

This section studies a class of contests that allows for a combination of unconditional and conditional investments, while accommodating a limited degree of asymmetry among players. I show that contests in this class have a unique equilibrium, in which every player's best response set is an interval. I rank players' strategies in terms of first-order stochastic dominance (FOSD), and explore the connection between the unconditional component of the investment and how efficiently prizes are allocated. The results I obtain are new, since complete-information contests with partly-conditional investments have not been studied previously. Complete-information all-pay auctions are a special case in which all investments are unconditional.

Suppose that all players share a common underlying technology, captured by a strictly increasing function $c(\cdot)$ with $c(0) = 0$, but may differ in how well they employ this technology. This difference is captured by every player i's idiosyncratic *cost coefficient* $\gamma_i > 0$. For example, firms competing in an R&D race may have access to similar technologies, but may differ in the skill and innovativeness of their workers. A positive fraction $\alpha \leq 1$ of the cost is sunk. The remaining fraction of the cost, $1 - \alpha$, is borne only if the player wins a prize. As explained in the introduction, the fraction of sunk cost may correspond to the degree to which the various features of a firm's product must be developed before the winners are determined. For every player i, we therefore have $v_i(s_i) = V_i - \gamma_i c(s_i)$ and $c_i(s_i) = \alpha\gamma_i c(s_i)$, where $V_i > 0$ is player i's valuation for a prize. I call a contest in this family a *simple contest* (see Figure 5 below).[29] When $\alpha = 1$, all investments are sunk.[30] If, in addition, $c(x) = x$ and $\gamma_i = 1$ for every player i, we have an all-pay auction.

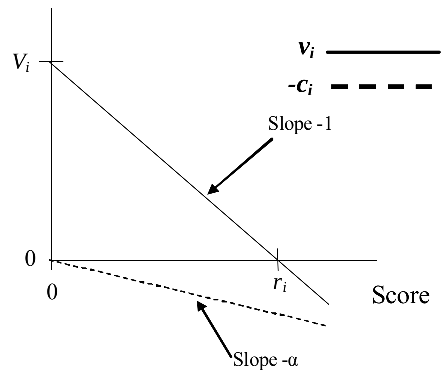

Figure 5: The valuation for winning and cost of losing for player i with $\gamma_i = 1$ in a simple contest with $c(x) = x$

The reach r_i of player i satisfies $v_i(r_i) = 0$, so $r_i = c^{-1}(V_i/\gamma_i)$. Since c is strictly increasing and players are ordered in decreasing order of reach, $(V_1/\gamma_1) \geq ... \geq (V_n/\gamma_n)$. The contest's threshold is $T = r_{m+1} = c^{-1}\left(V_{m+1}/\gamma_{m+1}\right)$, so the range of scores over which

[29]The contest framework can also accommodate player- and score-specific fractions $\alpha_i(s_i)$ and valuations $V_i(s_i)$. Of course, the more general the class of contests considered, the weaker the conclusions regarding players' equilibrium behavior.

[30]The contest is then similar to the one in Moldovanu and Sela (2001). The informational assumptions, however, are different. In their model, all players are ex-ante symmetric. The individual coefficients γ_i are privately known and drawn iid from a commonly known distribution. Moldovanu and Sela (2001) solve for the symmetric equilibrium, and do not characterize all equilibria. In contrast, the model here is of complete information and has a unique equilibrium.

players compete, $[0, T]$, is independent of α. For Assumption B3 to hold, assume that V_{m+1}/γ_{m+1} is different from V_i/γ_i for all $i \neq m+1$. Theorem 1 then shows that the equilibrium payoff of player $i < m+1$ is

$$w_i = v_i(T) = V_i - \gamma_i c \left(c^{-1} \left(\frac{V_{m+1}}{\gamma_{m+1}} \right) \right) = V_i - \gamma_i \frac{V_{m+1}}{\gamma_{m+1}}.$$

We therefore have the following corollary of Theorem 1.

Corollary 1 *In a simple contest, the payoff of every player $i < m+1$ is $V_i - (\gamma_i V_{m+1})/\gamma_{m+1}$. The payoffs of players $m+1, \ldots, n$ are 0. Payoffs are independent of α and c.*

Corollary 1 shows that the payoff of every player $i < m+1$ increases in his valuation for a prize and in the marginal player's cost coefficient, and decreases in the player's cost coefficient and in the marginal player's valuation for a prize. In particular, the payoff of player i is not affected by the characteristics of any player other than players i and $m+1$.

Aggregate expenditures equal the allocation value of the prizes less players' utilities. We therefore have another corollary of Theorem 1.

Corollary 2 *In a simple contest in which $V_1 = \ldots = V_n = V$, aggregate expenditures are*

$$mV - \sum_{i=1}^{m} \left(V - \gamma_i \frac{V}{\gamma_{m+1}} \right) = V \sum_{i=1}^{m} \left(\frac{\gamma_i}{\gamma_{m+1}} \right),$$

and are independent of α and c.

Corollary 2 shows that when players' valuations for a prize are equal, as is the case when prizes are monetary, aggregate expenditures increase in valuations and in the cost coefficients of players $1, \ldots, m$, and decrease in the marginal player's cost coefficient.

3.1 Simple Contests with a Single Prize

Theorems 2 and 3 show that a simple contest with a single prize has a unique equilibrium, which is described in the following corollary.

Corollary 3 *A simple contest with a single prize has a unique equilibrium. In this equilibrium, players* $3, \ldots, n$ *choose* 0 *and the CDFs of players* 1 *and* 2 *are*

$$\mathbf{G}^{\alpha}(x) = \left(\frac{\alpha \gamma_2 c(x)}{V_2 - (1-\alpha)\gamma_2 c(x)}, \frac{V_1 - \gamma_1 \frac{V_2}{\gamma_2} + \alpha \gamma_1 c(x)}{V_1 - (1-\alpha)\gamma_1 c(x)} \right)$$

on $[0, c^{-1}(V_2/\gamma_2)]$.

The corollary shows that the unique equilibrium is not independent of α and c (although equilibrium payoffs are independent of α and c). It is straightforward to verify that for any α and c player 1's CDF first-order stochastically dominates (FOSD) that of player 2. Therefore, player 1 wins the prize with a higher probability than player 2 (see Corollary 5 below). Moreover, $\partial G_1^{\alpha}(x)/\partial\alpha > 0$ and $\partial G_2^{\alpha}(x)/\partial\alpha > 0$ for all x in $(0,T)$, so the equilibria for lower values of α FOSD those of higher values of α: players tend to choose higher scores as the unconditional component of the investment decreases. It can also be shown that player 1's probability of winning increases (and that of player 2 decreases) as α decreases.

In the limit, as α approaches 0, player 1 chooses T and wins the prize with certainty, and player 2 chooses scores lower or equal to $x \leq T$ with probability

$$\frac{V_1 - \gamma_1 \frac{V_2}{\gamma_2}}{V_1 - \gamma_1 c(x)}.$$

This is an equilibrium of the limit game in which every player i's payoff when choosing s_i is 0 if he loses and $V_i - \gamma_i c(s_i)$ if he wins (see Section 3.2.1 below). A special case is the complete-information first-price auction.

3.2 Simple Contests with Multiple Prizes

For the remainder of the section assume that $c(\cdot)$ is piecewise analytic. The following result is an immediate corollary of Theorem 7, since players $m+2, \ldots, n$ do not participate in any equilibrium (the conditions of Theorem 2 hold).

Corollary 4 *A simple contest with multiple prizes has a unique equilibrium. In the unique equilibrium, the strategies of players* $1, \ldots, m+1$ *are given by the algorithm, and players* $m+2, \ldots, n$ *choose* 0.

Let $a_i = \gamma_i / V_i$, and note that a_i is increasing in i. The following result shows that in the unique equilibrium, the best response set of every player $i \leq m+1$ is a single interval whose upper bound is T and whose lower bound increases in the player's reach (or, equivalently, decreases in a_i).

Theorem 8 *In the unique equilibrium of a simple contest, every player $i \leq m+1$ is active on the interval $\left[s_i^l, T\right]$ for some $s_i^l \geq 0$, with $s_m^l = s_{m+1}^l = 0$.*[31] *For players $i, j \leq m$, $s_i^l \leq s_j^l$ if and only if $a_j \leq a_i$. Players $m+2, \ldots, n$ choose 0.*

The proof of Theorem 8 and those of Corollary 5, Theorem 9, and Corollary 6 below are in the Online Appendix. The outline of the proof is as follows. First, the higher a player's reach, the higher the score at which he becomes active for the first time. Second, semi-elasticities at every score increase in players' reaches, so when a player becomes active his semi-elasticity is higher than those of the other active players. Thus, no active players become inactive as a result of a new player becoming active.[32] Third, the semi-elasticity of an active player never increases above the aggregate hazard rate.[33] This shows that once a player becomes active he stays active until the threshold. Figure 6 depicts players' atoms and the regions in which players are active in the unique equilibrium.

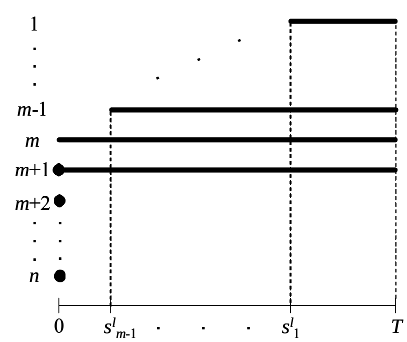

Figure 6: The unique equilibrium of a simple contest

[31] The score 0 is not a best response for players $i < m+1$.

[32] Contrast this with what happens in Figure 3 at the switching point x_1: player 1 becomes active, and because his semi-elasticity is sufficiently lower than that of player 2, player 2 becomes inactive immediately above x_1.

[33] Contrast this with what happens in Figure 3 at the switching point x_3: the hazard rate of player 3 drops to 0 and he becomes inactive immediately above x_3.

A corollary of Theorem 8 and the fact that players' semi-elasticities are increasing in reach is that players' equilibrium CDFs can be ranked in terms of FOSD.

Corollary 5 *For every positive $\alpha \leq 1$ and every $i < j \leq m+1$, the CDF of player i FOSD that of player j. This implies that player i chooses higher scores than player j, on average, and also that player i wins a prize with higher probability than player j.*[34]

Using Theorem 8, I now derive an expression for the equilibrium strategies of players $1, \ldots, m+1$. Recall that for any $y < T$ the formula for player i's CDF at y is given by (8). Since each player is active on an interval, D equals 1 in (8), and the switching points are the scores s_i^l at which players become active. Because players with higher reaches become active at higher scores, for every $y < T$ there is a unique $j = 1, \ldots, m$ such that y is in $\left[s_j^l, s_{j-1}^l\right)$ (where $s_0^l = T$) and the set of players active immediately above s_j^l is $j, \ldots, m+1$. Thus, for a player $i < j$ we have $G_j(y) = 0$, and for a player $i \geq j$, by substituting the expression for q_i into (8) and simplifying, we have

$$(9) \qquad G_i(y) = 1 - \left(\frac{1}{a_{m+1}} - c(y)\right)^{\frac{1}{m+1-j}} \frac{\Pi_{k=j}^{m+1}\left[\frac{a_k}{1-(1-\alpha)a_k c(y)}\right]^{\frac{1}{m+1-j}}}{\frac{a_i}{1-(1-\alpha)a_i c(y)}}.$$

We still have to identify the scores s_i^l at which players become active. For simplicity assume that the a_is are distinct.[35] Recall that $s_m = s_{m+1} = 0$. The score s_i^l at which player $i < m$ becomes active is the lowest score x at which he obtains his power, so that (2) holds with s_i^l instead of x. Substituting (9) (with active players $i+1, \ldots, m+1$) into (2), it can be shown that s_i^l is the lowest score x that satisfies

$$(10) \qquad \frac{\Pi_{k=i+1}^{m} a_k}{a_i^{m-i}} = \frac{1}{(1 - a_{m+1}c(x))} \frac{\Pi_{k=i+1}^{m+1}(1 - a_k c(x)(1-\alpha))}{(1 - a_i c(x)(1-\alpha))^{m-i}}.$$

[34] In contrast to the two-player case, a player's CDF for low values of α does not always FOSD his CDF for higher values of α.

[35] If they are not distinct but $a_{m-1} \neq a_m$, all of the results hold without change, but the proof of Theorem 9 and the notation in (10) require slight modifications. If $a_{m-1} = a_m$, then $s_i^l = 0$ for every positive $\alpha \leq 1$ and every player $i < m$ for which $a_i = a_m$. Theorem 9 holds for every other player $i < m$. Corollary 6 holds, but requires a modified proof.

This equation characterizes s_i^l implicitly, and provides a closed-form expression for s_i^l when $\alpha = 1$ (see Section 3.2.2 below). It can also be used to show the following result.

Theorem 9 *For every positive $\alpha < 1$ and every player $i < m$, s_i^l decreases in α. As α approaches 0, s_i^l approaches T.*

Theorem 9 shows that as investments become more conditional, for every player $i < m$ the lower endpoint of the interval from which the player chooses scores approaches the threshold. This has the following implication regarding the allocation of prizes. Call a simple contest *β-efficient*, for some positive $\beta < 1$, if each of the players $1, \ldots, m$ (players with positive power) obtains a prize with probability at least β in the unique equilibrium of the contest.

Corollary 6 *Choose a family of simple contests parameterized by α. For any positive $\beta < 1$, every simple contest in the family with a small enough $\alpha > 0$ is β-efficient.*

In particular, the corollary shows that when players differ only in their valuations for a prize, so that players $1, \ldots, m$ are those with the highest valuations for a prize, allocative efficiency of the prizes can be approached arbitrarily closely by reducing the unconditional investment component α. The limiting equilibrium corresponding to $\alpha = 0$ is efficient (see Section 3.2.1 below). Since players' payoffs remain the same for all $\alpha > 0$ (Corollary 1), this immediately implies that as α approaches 0 expenditures approach their maximal value,

$$\sum_{i=1}^{m} V_i - \sum_{i=1}^{m} \left(V_i - \gamma_i \frac{V_{m+1}}{\gamma_{m+1}} \right) = V_{m+1} \sum_{i=1}^{m} \frac{\gamma_i}{\gamma_{m+1}}.$$

Although players' individual expenditures and probabilities of winning, as well as aggregate expenditures, change with α, a simple change-of-variable argument can be used to show that they are independent of c.

3.2.1 Fully Conditional Investments ($\alpha = 0$)

The game with $\alpha = 0$ is not a contest, since Assumption B3 is violated. Instead, it is a complete-information, discriminatory-price multiprize auction in which player i's cost of bidding x is $\gamma_i c(x)$ if he wins and 0 if he loses. This game has many equilibria, some of which involve players playing weakly-dominated strategies, and some of which rely on specific tie-breaking rules. Different equilibria lead to different payoffs.[36] Considering the limit of the equilibria of simple contests as α approaches 0, we obtain the following equilibrium of the game with $\alpha = 0$. Players $1, \ldots, m-1$ bid T. Players $m+2, \ldots, n$ bid 0. Players m and $m+1$ bid as specified by the limit of the equilibria of two-player simple contests as α approaches 0. As shown in Section 3.1, this means that player m bids T and player $m+1$ bids according to the CDF

$$G_{m+1}(x) = \frac{V_m - \gamma_m \frac{V_{m+1}}{\gamma_{m+1}}}{V_m - \gamma_m c(x)}.$$

In particular, only player $m+1$ employs a mixed strategy. Players $1, \ldots, m$ win with certainty and players $m+1, \ldots, n$ lose with certainty. Players' payoffs are given by Corollary 1. Thus, taking the fraction of the all-pay component to 0 can serve as a selection criterion that delivers a unique equilibrium of the limit game. This equilibrium is robust to the tie-breaking rule, and is "close" to the equilibria of "nearby" contests with a small all-pay component. A special case is the complete-information, pay-your-bid multiprize auction ($\gamma_i = 1$ and $c(x) = x$).

3.2.2 Fully Unconditional Investments ($\alpha = 1$)

When $\alpha = 1$ the contest is separable and all investments are unconditional. We can then simplify (9) to obtain

$$G_i(y) = 1 - \left(\frac{1}{a_{m+1}} - c(y)\right)^{\frac{1}{m+1-j}} \frac{\Pi_{k=j}^{m+1} a_k^{\frac{1}{m+1-j}}}{a_i}. \tag{11}$$

The following closed-form expression for s_i^l, $i < m$ (recall that $s_m^l = s_{m+1}^l = 0$) is provided by (10):

$$\frac{\Pi_{k=i+1}^{m} a_k}{a_i^{m-i}} = \frac{1}{\left(1 - a_{m+1} c\left(s_i^l\right)\right)} \Rightarrow s_i^l = c^{-1}\left(\frac{1}{a_{m+1}} - \frac{a_i^{m-i}}{\Pi_{k=i+1}^{m+1} a_k}\right). \tag{12}$$

[36] For example, a weak player could submit a bid whose cost exceeds his valuation for a prize, thereby forcing a stronger player to submit a high bid. As long as the weak player loses for sure, he does not incur any costs. This cannot happen under Assumption B3.

The special case of $c(x) = x$ and $\gamma_i = 1$ is the multiprize all-pay auction, which was first analyzed by Clark and Riis (1998). Setting $c(x) = x$ and $\gamma_i = 1$ in (11) and (12) delivers the equilibrium described in their Proposition 1.[37] The analysis on page 279 of Clark and Riis (1998) applies to simple contests with $\alpha = 1$, and provides a recursive closed-form formula for each player's expenditures and probability of winning, which are independent of c.

4 Conclusion

This paper has investigated equilibrium behavior in a single-prize and multiprize contest model featuring asymmetric contestants and a combination of unconditional and conditional investments. These features, which are common to many real-world competitions, are not well accommodated by existing models and may lead to complicated equilibrium behavior. In particular, the support of a player's equilibrium strategy may consist of several disjoint intervals. I have solved for the unique equilibrium in two-player, single-prize contests, and provided an algorithm that constructs the unique equilibrium for a large class of multiprize contests. What matters for equilibrium uniqueness in a contest for m prizes is that only the strongest $m + 1$ players participate, which is implied when weak players are everywhere disadvantaged relative to the marginal player. Many existing models of competition, including multiprize all-pay auctions, satisfy this condition.

The key to the equilibrium construction is a dynamic supply function, which uses players' equilibrium payoffs and semi-elasticities to determine how the set of active players changes at different scores. A similar function can be used to solve the matching model of Bulow and Levin (2006).[38] An interesting question is whether similar functions can be

[37] Theorem 8 applied to multiprize all-pay auctions corrects two imprecisions in Clark and Riis (1998). The first is that they claimed uniqueness of equilibrium but provided an incorrect proof of this claim, as discussed in footnote 6 above. The second is that their footnote 6 claims that multiple equilibria arise when two or more players have the same valuation. Theorem 8 shows that the equilibrium is unique even if several players have the same valuation for a prize, as long as the valuation of the marginal player is different from those of the other players.

[38] Modify the supply function S_x by replacing ε_i with $1/\Delta_i$. Proceed from the top, and replace $\mathcal{CP}(x)$

used to solve other complete-information games with a continuum of pure strategies in which players compete for multiple prizes, such as wars of attrition.

As an application of the algorithm and the equilibrium uniqueness result, I have investigated the class of simple contests. Simple contests accommodate a limited degree of asymmetry among players and both unconditional and conditional investments. In the unique equilibrium of a simple contest, every player chooses a score from an interval. This explains players' behavior in multiprize all-pay auctions, which are a simple contests in which all investments are unconditional and all costs are linear. When players differ only in their valuations for a prize, higher allocative efficiency can be achieved by reducing the unconditional component of the investment. Doing so, however, increases aggregate expenditures, since players' payoffs are independent of the unconditional component. When investments are wasteful, this implies a trade-off between increasing allocative efficiency, which requires decreasing the unconditional component, and decreasing aggregate expenditures, which requires increasing the unconditional component.[39] Understanding this trade-off may also help us think about R&D competitions in which hold up can arise after the winners are chosen. A larger conditional component leads to higher efficiency, but increases the risk of hold up by the winners.

The equilibrium construction results developed in this paper and the payoff characterization of Siegel (2009) present a unified framework for analyzing a wide range of competitions with sunk investments. By modeling such competitions as contests, we can derive players' payoffs, expenditures, scores, and probability of obtaining a prize, and compute other variables of economic interest, such as the degree of efficiency and rent dissipation. The effects of changing the number of prizes, the set of participants, the scoring rule, and other characteristics of a competition can also be studied using this framework. As dis-

with the set of players who have not yet exhausted their probability mass (this is the set of candidate players). The fixed point H is players' aggregate density, and a candidate player i's density is $\max\{H - 1/\Delta_i, 0\}$. A player becomes inactive when he exhausts his probability mass, at which time the fixed point changes because the set of candidate players changes.

[39] Daniele Condorelli (2009) and Jason Hartline and Timothy Roughgarden (2008) investigate a similar trade off in the context of mechanism design with money burning and private information.

cussed in Section 2.3, this can help us think about policy and design issues and enhance our understanding of real-world competitions.

References

[1] **Barut, Yasar, and Dan Kovenock.** 1998. "The Symmetric Multiple Prize All-Pay Auction with Complete Information." *European Journal of Political Economy*, 14(4): 627-644.

[2] **Baye, Michael R., Dan Kovenock, and Casper de Vries.** 1993. "Rigging the Lobbying Process: An Application of All-Pay Auctions." *American Economic Review*, 83(1): 289-94.

[3] **Baye, Michael R., Dan Kovenock, and Casper de Vries.** 1996. "The All-Pay Auction with complete-information." *Economic Theory*, 8(2): 291-305.

[4] **Bulow, Jeremy I., and Jonathan Levin.** 2006. "Matching and Price Competition." *American Economic Review*, 96(3): 652-68.

[5] **Chapman, Charles P.** 2002. *Real Mathematical Analysis*, New York: Springer-Verlag.

[6] **Che, Yeon-Koo, and Ian Gale.** 1998. "Caps on Political Lobbying." *American Economic Review,* 88(3): 643–51.

[7] **Che, Yeon-Koo, and Ian Gale.** 2003. "Optimal Design of Research Contests." *American Economic Review,* 93(3): 646–71.

[8] **Che, Yeon-Koo, and Ian Gale.** 2006. "Caps on Political Lobbying: Reply." *American Economic Review,* 96(4): 1355-60.

[9] **Clark, Derek J., and Christian Riis.** 1998. "Competition over More than One Prize." *American Economic Review*, 88(1): 276-289.

[10] **Condorelli, Daniele.** 2009. "Allocation Scarce Resources: What Money Can't Buy." http://ssrn.com/abstract=1420899.

[11] **Cornes, Richard, and Roger Hartley.** 2005. "Asymmetric Contests with General Technologies." *Economic Theory*, 26(4): 923-46.

[12] **Dasgupta, Partha.** 1986. "The Theory of Technological Competition." In *New Developments in the Analysis of Market Structure*, ed. Joseph E. Stiglitz and G. F. Mathewson, 519-47. Cambridge: MIT press.

[13] **Ellingsen, Tore.** 1991. "Strategic Buyers and the Social Cost of Monopoly." *American Economic Review*, 81(3): 648-57.

[14] **González-Díaz, Julio.** 2009. "First-Price Winner-Takes-All Contests." Unpublished.

[15] **Hartline, Jason, and Timothy Roughgarden.** 2008. "Optimal Mechanism Design and Money Burning." Annual ACM Symposium on Theory of Computing, Proceedings of the 40th annual ACM symposium on Theory of computing.

[16] **Hillman, Arye L., and John G. Riley.** 1989. "Politically Contestable Rents and Transfers." *Economics and Politics*, 1(1): 17-39.

[17] **Hillman, Arye L., and Dov Samet.** 1987. "Dissipation of Contestable Rents by Small Numbers of Contenders." *Public Choice*, 54(1): 63-82.

[18] **Kaplan, Todd R., Israel Luski, Aner Sela, and David Wettstein.** 2002. "All-Pay Auctions with Variable Rewards." *Journal of Industrial Economics*, 50(4): 417-30.

[19] **Kaplan, Todd R., and David Wettstein.** 2006. "Caps on Political Lobbying: Comment" *American Economic Review*, 96(4): 1351-4.

[20] **Konrad, Kai A.** 2007. "Strategy in Contests - an Introduction." Berlin Wissenschaftszentrum Discussion Paper SP II 2007 – 01.

[21] **Lazear, Edward P.**, and Sherwin Rosen. 1981. "Rank-Order Tournaments as Optimum Labor Contracts." *Journal of Political Economy*, 89(5): pp. 841-64.

[22] **Moldovanu, Benny, and Aner Sela.** 2001. "The Optimal Allocation of Prizes in Contests." *American Economic Review*, 91(3): 542-58.

[23] **Moldovanu, Benny, and Aner Sela.** 2006. "Contest Architecture." *Journal of Economic Theory*, 126(1): 70-96.

[24] **Nitzan, Shmuel.** 1994. "Modeling Rent-Seeking Contests." *European Journal of Political Economy*, 10(1): 41-60.

[25] **Parreiras, Sergio, and Anna Rubinchik.** 2009. "Contests with Many Heterogeneous Agents." *Games and Economic Behavior*, forthcoming.

[26] **Siegel, Ron.** 2009. "All-Pay Contests." *Econometrica*, 77(1): 71-92.

[27] **Tullock, Gordon.** 1980. "Efficient Rent Seeking." In *Toward a theory of the rent seeking society*, ed. James M. Buchanan, Robert D. Tollison, and Gordon Tullock, 269-82. College Station: Texas A&M University Press.

[28] **Varian, Hal.** 1980. "A Model of Sales." *American Economic Review*, 70(4): 651-58.

A Proofs of the Results of Section 2

A.1 Proof of Lemma 4

The proof includes four lemmas. The first lemma shows that the algorithm can proceed as long as T has not been reached.

Lemma 5 *The algorithm can proceed from any switching point lower than T that has been reached.*

Proof. There are two statements to prove. The first is that at a switching point lower than T, for every player i either (2) holds or (1) holds. The second is that the set of candidate players at the switching point contains at least two players. I will show that these statements hold for any score $x < T$ (not necessarily a switching point) that has been reached by the algorithm. It is easy to check that the statements hold at $x = 0$. Suppose that they hold at the switching point $\hat{x}$ that precedes x, and denote by $\bar{x}$ the next switching point (so $x \in [\hat{x}, \bar{x}]$). Because $\mathbf{G}$ is extended from $\hat{x}$ to $\bar{x}$ according to (8) (with $\mathbf{G}(T) = 1$), (2) holds for every player in $\mathcal{A}^+(\hat{x})$ and any $x \in [\hat{x}, \bar{x}]$. By Step 4 of the algorithm, (1) holds for every player not in $\mathcal{A}^+(\hat{x})$ and any $x \in (\hat{x}, \bar{x})$, and at $\bar{x}$, for every player not in $\mathcal{A}^+(\hat{x})$ either (2) holds or (1) holds. Therefore, the first statement holds. For the second statement, if $|\mathcal{CP}(\hat{x})| \geq 2$ then by definition of the supply function $|\mathcal{A}^+(\hat{x})| \geq 2$. For every player $i \in \mathcal{A}^+(\hat{x})$ and any $x \in [\hat{x}, \bar{x}]$, by construction of $\mathbf{G}$ (2) is satisfied, so $\mathcal{A}^+(\hat{x}) \subseteq \mathcal{CP}(x)$ and $|\mathcal{CP}(x)| \geq 2$. ■

The next lemma shows that the number of switching points bounded away from T is finite.

Lemma 6 *For every $\tilde{x} < T$, the number of switching points in $[0, \tilde{x}]$ is finite.*

Proof. I assume analytic valuations for winning and costs of losing (the obvious extension applies to piecewise analyticity). Choose $\tilde{x} < T$ and rank players' semi-elasticities at every score in $[0, \tilde{x}]$. Since semi-elasticities are analytic, this ranking can change only finitely many times. Thus, $[0, \tilde{x}]$ can be divided into a finite number of intervals such that the ranking of players' semi-elasticities is constant on each interval. Fix one such interval $\mathcal{I}$. For every subset $\mathcal{B} \subseteq \mathcal{N}$ of at least two players and every $x \in \mathcal{I}$, denote by $t_{\mathcal{B}}(x)$ the highest semi-elasticity of a player who can join the set of active players $\mathcal{B}$ and maintain a non-negative hazard rate:

$$t_{\mathcal{B}}(x) = \frac{1}{|\mathcal{B}| - 1} \sum_{j \in \mathcal{B}} \varepsilon_j(x)$$

(the aggregate hazard rates of players in $\mathcal{B}$). Since semi-elasticities are analytic, so is $t_{\mathcal{B}}(\cdot)$. Thus, the interval $\mathcal{I}$ can be divided into a finite number of subintervals such on every subinterval each function in $\{\varepsilon_i - t_{\mathcal{B}} : i \in \mathcal{N}, \mathcal{B} \subseteq \mathcal{N}, |\mathcal{B}| \geq 2\}$ is either positive, negative, or 0. Clearly, on any such subinterval $\mathcal{L} \subseteq \mathcal{I}$ an active player can become inactive only if a player with a strictly lower semi-elasticity becomes active (recall that the order of players' semi-elasticities doesn't change on $\mathcal{I}$). Now, suppose in contradiction that the number of switching points in $\mathcal{L}$ is infinite. This implies that some player i becomes inactive and active an infinite number of times, which, by the above, implies that some player j with semi-elasticity strictly lower than that of i becomes inactive and active an infinite number of times. Continuing in this way, we obtain a contradiction since the number of players is finite. ■

The following two lemmas show that there are no switching points in some left-neighborhood of T.

Lemma 7 *For some $\tilde{x} < T$, $\varepsilon_i(x) < H(x)$ for every player i and every $x \in (\tilde{x}, T)$.*

Proof. First, for every i and j we have

$$\lim_{x \to T} \frac{\varepsilon_i(x)}{\varepsilon_j(x)} = \lim_{x \to T} \frac{q_i'(x)}{q_i(x)} \frac{q_j(x)}{q_j'(x)} = \frac{q_i'(T)}{q_j'(T)} \lim_{x \to T} \frac{q_j(x)}{q_i(x)} = \frac{q_i'(T)}{q_j'(T)} \frac{q_j'(x)}{q_i'(x)} = 1,$$

where the penultimate equality follows from l'Hopital's rule.
Let $\varepsilon_{\min}(x) = \min_{i \in \mathcal{N}} \varepsilon_i(x)$ for $x < T$. Then, by the above, $\lim_{x \to T} \varepsilon_i(x) / \varepsilon_{\min}(x) = 1$, so $\varepsilon_i(x) / \varepsilon_{\min}(x) < n/(n-1)$ for all $x > \tilde{x}$ for some $\tilde{x}$ sufficiently close to T. To conclude, it suffices to show that for every $x > \tilde{x}$, we have $H(x) \geq (n/(n-1)) \varepsilon_{\min}(x)$. Let $S_x^{\min}(H) = n \max\{H - \varepsilon_{\min}(x), 0\}$. Then for every H, we have $S_x(H) \leq S_x^{\min}(H)$, and since $(n/(n-1)) \varepsilon_{\min}(x)$ is the unique positive fixed point of $S_x^{\min}$, $H(x) \geq (n/(n-1)) \varepsilon_{\min}(x)$.

■

Since active players with semi-elasticities strictly lower than the aggregate hazard rate remain active, in order to complete the proof it suffices to show the following.

Lemma 8 *Every player i has scores x arbitrarily close to T such that $q_i(x) = \Pi_{j \neq i}(1 - G_j(x))$.*

Proof. Suppose, in contradiction, that for every $x \in (\tilde{x}_i, T)$, we have $q_i(x) < \Pi_{j \neq i}(1 - G_j(x))$ for some player i and some $\tilde{x}_i > \tilde{x}$, where $\tilde{x}$ is the score identified in the previous lemma. Then, $f(x) = \sum_{j \neq i} \ln(1 - G_j(x)) - \ln q_i(x) > 0$. Since $i \notin \mathcal{CP}(x)$, for every $x \in (\tilde{x}_i, T)$ we have

$$H(x) = \frac{|\mathcal{A}^+(x)|}{|\mathcal{A}^+(x)| - 1} \sum_{j \in \mathcal{A}^+(x)} \varepsilon_j(x) = \sum_{j \in \mathcal{N}} \frac{G_i'(x)}{(1 - G_i(x))} = \sum_{j \neq i} \frac{G_i'(x)}{(1 - G_i(x))}.$$

Thus,

$$f'(x) = \varepsilon_i(x) - \frac{|\mathcal{A}^+(x)|}{|\mathcal{A}^+(x)| - 1} \sum_{j \in \mathcal{A}^+(x)} \varepsilon_j(x) \leq \varepsilon_i(x) - \frac{n-1}{n-2} \sum_{j \in \mathcal{A}^+(x)} \varepsilon_j(x) =$$

$$-\frac{1}{n-2} \varepsilon_{\min}(x) + o(\varepsilon_{\min}(x))$$

as $x \to T$, by the proof of the previous lemma. Since

$$-\frac{1}{n-2} \int_x^T \varepsilon_{\min}(y)\, dy = \lim_{z \to T} \frac{1}{n-2} (\ln q_{\min}(z) - \ln q_{\min}(x)) = -\infty,$$

f crosses 0 at a score in $(\tilde{x}_i T)$, a contradiction. ■

A.2 Proof of Theorem 4

To prove that $\mathbf{G}$ is an equilibrium, it suffices to show three things: (i) $\mathbf{G}$ is a profile of CDFs (non-decreasing, right-continuous, and $\lim_{x\to T}\mathbf{G}(x)=1$), (ii) no player can obtain a payoff higher than his power by choosing any score when the other players choose scores according to $\mathbf{G}$, and (iii) every player i assigns G_i-measure 1 to scores that give him a payoff equal to his power when the other players choose scores according to $\mathbf{G}$. For (i), consider any two consecutive switching points $x<\bar{x}$ and any player $i\in\mathcal{A}^+(x)$. The hazard rate of G_i at every $y\in[x,\bar{x})$ is, by construction, equal to $H(x,y)-\varepsilon_i(y)\geq 0$. This shows that $\mathbf{G}$ is non-decreasing. That $\mathbf{G}$ is continuous on $(0,T)$ and right-continuous at 0 follows from (8). The next lemma shows that $\lim_{x\to T}\mathbf{G}(x)=1$.

Lemma 9 *For every player i, $\lim_{x\to T}G_i(x)=G_i(T)=1$.*

Proof. By Lemma 4, there is a score $\tilde{x}<T$ such that for every $x\in(\tilde{x},T)$, we have $\mathcal{CP}(x)=\mathcal{N}$. Now (8) implies that for every player i and every $x\in(\tilde{x},T)$,

$$\ln(1-G_i(x))=\frac{1}{n-1}\sum_{j\in\mathcal{N}}\ln q_j(x)-\ln q_i(x).$$

To show that $G_i(x)\underset{x\to T}{\to}1$, it therefore suffices to show that

$$\frac{1}{n-1}\sum_{j\in\mathcal{N}}\ln q_j(x)-\ln q_i(x)\underset{x\to T}{\to}-\infty.$$

Since $\ln q_i(x)\underset{x\to T}{\to}-\infty$, it suffices to show that

$$\frac{1}{n-1}\frac{\sum_{j\in\mathcal{N}}\ln q_j(x)}{\ln q_i(x)}>1+\delta \text{ for some } \delta>0$$

for large enough x. The inequality follows from l'Hopital's rule and the fact that $\lim_{x\to T}\varepsilon_i(x)/\varepsilon_j(x)=1$, which is shown in the proof of Lemma 4. ■

For (ii), Lemma 5 shows that for every $x<T$ and every player i either (2) holds or (1) holds, so no player can obtain more than his power by choosing $x<T$. By definition of power, no player obtains more than his power if he chooses a score higher or equal to T even if he wins a prize with certainty. For (iii), by construction G_i increases only when (2) holds for player i, so scores for which (2) does not hold have G_i-measure 0. Consequently, scores for which (2) does hold have G_i-measure 1.

A.3 Proof of Theorem 5

The proof requires the following definition.

Definition 3 *An equilibrium is* constructible *if for every score $x < T$ there exists some $\bar{x} > x$ such that for each player either every score in $(x, \bar{x})$ is a best response, or no score in $(x, \bar{x})$ is a best response.*

By construction, the output of the algorithm is a constructible equilibrium. Thus, an $(m+1)$-player regular contest has a constructible equilibrium. The following result shows that the existence of a constructible equilibrium guarantees uniqueness. Theorem 5 is an immediate corollary.

Theorem 10 *If an $(m+1)$-player contest, regular or not, has a constructible equilibrium, then that is the unique equilibrium of the contest.*

Suppose that the contest has a constructible equilibrium $\mathbf{G}$. For expositional simplicity, I assume that the number of switching points in $\mathbf{G}$ is finite (this is true for the equilibrium constructed by the algorithm). It is straightforward to accommodate a countably infinite number of switching points by defining the limit of a sequence of switching points to be a switching point and modifying the proof appropriately.

In what follows, x_k denotes switching point k in $\mathbf{G}$. The last switching point is T. $\mathcal{CP}(x)$ and $\mathcal{A}^+(x)$ are defined as in Section 2. Choose any equilibrium $\tilde{\mathbf{G}}$ of the contest, and recall that $\tilde{\mathbf{G}}$ is continuous above 0 (part (1) of Lemma 2). $\widetilde{\mathcal{CP}}(x)$ denotes the set of players active at x under $\tilde{\mathbf{G}}$, i.e., the set of players defined by (3) with $\tilde{\mathbf{G}}$ instead of $\mathbf{G}$. Using $\mathcal{A}^+(x)$, I will show that $\mathcal{CP}(x) = \widetilde{\mathcal{CP}}(x)$ for all $x \in [0, T]$. The following lemma shows that doing so is sufficient.

Lemma 10 *Let $\tilde{x} \in [0, T]$. If $\widetilde{\mathcal{CP}}(x) = \mathcal{CP}(x)$ for every $x \in [0, \tilde{x}]$, then $\tilde{\mathbf{G}}(x) = \mathbf{G}(x)$ for every $x \in [0, \tilde{x}]$.*

Proof. Denote by $\bar{x}$ the infimum of the scores on which $\tilde{\mathbf{G}}$ differs from $\mathbf{G}$. Since $\tilde{\mathbf{G}}(0) = \mathbf{G}(0)$ (Lemma 3 does not rely on analyticity), $\mathbf{G}$ and $\tilde{\mathbf{G}}$ are continuous on $(0, T)$ (part (1) Lemma 2), and $\tilde{\mathbf{G}}(T) = \mathbf{G}(T) = 1$ (part (3) of Lemma 2), we have $\tilde{\mathbf{G}}(\bar{x}) = \mathbf{G}(\bar{x})$. If $\bar{x} < \tilde{x}$, then by constructibility of $\mathbf{G}$ and because $\widetilde{\mathcal{CP}}(x) = \mathcal{CP}(x)$ for every $\in [0, \tilde{x}]$, we have that on a right-neighborhood of $\bar{x}$ both $\tilde{G}_i$ and G_i are given by (8) for $i \in \mathcal{A}^+(\bar{x})$ and remain constant for $i \notin \mathcal{A}^+(x)$. This contradicts the definition of $\bar{x}$, so $\bar{x} = \tilde{x}$. ■

Now, let x_k be the highest switching point such that $\widetilde{\mathcal{CP}}(x) = \mathcal{CP}(x)$ on $[0, x_k]$ (if no such switching point exists, let $x_k = 0$), and suppose in contradiction that $x_k < T$. Choose $x \in (x_k, x_{k+1})$ such that $\widetilde{\mathcal{CP}}(x) \neq \mathcal{CP}(x)$. Since $x_k < T$, such an x exists, otherwise Lemma 10 and continuity would imply that $\widetilde{\mathcal{CP}}(x_{k+1}) = \mathcal{CP}(x_{k+1})$. The following lemmas show that $\widetilde{\mathcal{CP}}(x) \subseteq \mathcal{CP}(x)$ and $\mathcal{CP}(x) \subseteq \widetilde{\mathcal{CP}}(x)$, which completes the proof.

Lemma 11 *$\widetilde{\mathcal{CP}}(x) \subseteq \mathcal{CP}(x)$.*

Proof. Suppose $\widetilde{\mathcal{CP}}(x) \nsubseteq \mathcal{CP}(x)$, and let $i_0 \in \widetilde{\mathcal{CP}}(x) \backslash \mathrm{CP}(x)$. Since $i_0 \notin \mathcal{CP}(x)$, we have

$$\frac{w_{i_0} + c_{i_0}(y)}{v_{i_0}(y) + c_{i_0}(y)} > P_{i_0}(x) = 1 - \Pi_{j \neq i_0}(1 - G_j(x))$$

or, equivalently,

$$\frac{v_{i_0}(y) - w_{i_0}}{v_{i_0}(y) + c_{i_0}(y)} < \Pi_{j \neq i_0} (1 - G_j(x)).$$

Since $i_0 \in \widetilde{\mathcal{CP}}(x)$, we have

$$\frac{v_{i_0}(y) - w_{i_0}}{v_{i_0}(y) + c_{i_0}(y)} = \Pi_{j \neq i_0} \left(1 - \tilde{G}_j(x)\right),$$

so

$$\Pi_{j \neq i_0} \left(1 - \tilde{G}_j(x)\right) < \Pi_{j \neq i_0} (1 - G_j(x)).$$

Let $\mathcal{J}_1 = \mathcal{N} \setminus \{i_0\}$. Then

$$\Pi_{j \in \mathcal{J}_1} \left(1 - \tilde{G}_j(x)\right) < \Pi_{j \in \mathcal{J}_1} (1 - G_j(x)). \tag{13}$$

By the Threshold Lemma of Siegel (2009), the expression on each side of (13) is a product of $n - 1$ strictly positive numbers. Therefore, there exists a player $i_1 \in \mathcal{J}_1$ such that

$$\Pi_{\mathcal{J}_1 \setminus \{i_1\}} \left(1 - \tilde{G}_j(x)\right) < \Pi_{\mathcal{J}_1 \setminus \{i_1\}} (1 - G_j(x)) \tag{14}$$

(otherwise multiplying the products of all subsets of size $n-2$ for $\mathbf{G}$ and for $\tilde{\mathbf{G}}$ would lead to a contradiction).
Now, note that for every player $i \in \mathcal{N}$, we have $\tilde{G}_i(x_k) = G_i(x_k)$, by Lemma 10, and since $\tilde{G}_i$ is non-decreasing

$$\text{for every player } i \notin \mathcal{A}^+(x), (1 - \tilde{G}_i(x) \leq (1 - G_i(x)). \tag{15}$$

Let $\mathcal{K}_1 = \mathcal{N} \setminus \mathcal{J}_1 = \{i_0\}$. Since $\mathcal{A}^+(x) \subseteq \mathcal{CP}(x)$ and $i_0 \notin \mathcal{CP}(x)$, by (15) we have

$$\left(1 - \tilde{G}_{j \in \mathcal{K}_1}(x)\right) \leq (1 - G_{j \in \mathcal{K}_1}(x)). \tag{16}$$

By (14) and (16),

$$\Pi_{j \in \mathcal{J}_1 \cup \mathcal{K}_1 \setminus \{i_1\}} \left(1 - \tilde{G}_j(x)\right) < \Pi_{j \in \mathcal{J}_1 \cup \mathcal{K}_1 \setminus \{i_1\}} (1 - G_j(x)). \tag{17}$$

Since $\mathcal{N} = \mathcal{J}_1 \cup \mathcal{K}_1$, (17) shows that $i_1 \notin \mathcal{A}^+(x_k)$, otherwise i_1 would obtain under $\tilde{\mathbf{G}}$ more than his power by choosing x.
Now repeat the process above, letting $\mathcal{J}_{r+1} = \mathcal{J}_r \setminus \{i_r\}, \mathcal{K}_{r+1} = \mathcal{K}_r \cup \{i_r\}$. By induction on r, (13),(14),(16), and (17) hold with $\mathcal{J}_r$ instead of $\mathcal{J}_1$, $\mathcal{K}_r$ instead of $\mathcal{K}_1$, and i_k instead of i_1, so $\mathcal{K}_r \cap \mathcal{A}^+(x_k) = \phi$. A contradiction is reached at stage $n - 1$, since $|\mathcal{K}_{n-1}| = n - 1$ but $|\mathcal{A}^+(x_k)| \geq 2$. ■

Corollary 7 *For every player $j \notin \mathcal{CP}(x)$ and every $y \in (x_k, x_{k+1})$, we have $\tilde{G}_j(y) = G_j(y) = G_j(x_k)$.*

Proof. Immediate from $\widetilde{\mathcal{CP}}(y) \subseteq \mathcal{CP}(y)$ applied to all points $y \in (x_k, x_{k+1})$. ■

The next two lemmas establish that $\mathcal{CP}(x) \subseteq \widetilde{\mathcal{CP}}(x)$.

Lemma 12 *If $\mathcal{CP}(x) \not\subseteq \widetilde{\mathcal{CP}}(x)$, then $\tilde{G}_i(x) > G_i(x)$ for some $i \in \mathcal{CP}(x) \backslash \widetilde{\mathcal{CP}}(x)$.*

Proof. Let $\mathcal{B} = \mathcal{CP}(x) \backslash \widetilde{\mathcal{CP}}(x)$, and suppose that for every player $j \in \mathcal{B}$ we have $\tilde{G}_j(x) \leq G_j(x)$. This implies that there is a player $j \in \mathcal{B}$ for which $\tilde{G}_j(x) < G_j(x)$. Indeed, by Corollary 7 and (8) with $\widetilde{\mathcal{CP}}(x)$ instead of $\mathcal{A}^+(x)$ and x instead of y, once $\mathbf{G}$ and $\tilde{\mathbf{G}}$ agree on $(\mathcal{N}/\mathcal{CP}(x)) \cup \mathcal{B} = \mathcal{N}/\widetilde{\mathcal{CP}}(x)$ we obtain $\tilde{\mathbf{G}}(x) = \mathbf{G}(x)$ and therefore $\mathcal{CP}(x) = \widetilde{\mathcal{CP}}(x)$. To show that there is a player $i \in \mathcal{B}$ for which $\tilde{G}_i(x) > G_i(x)$, the following observation is required. Fix some values $\bar{G}_j(x)$ for $j \notin \mathcal{CP}(x)$ and use (8) to solve for the values $\bar{G}_l(x), l \in \mathcal{CP}(x)$. Maintaining the value $\bar{G}_l(x)$ for player $l \in \mathcal{CP}(x)$ and solving for $\mathcal{CP}(x) \setminus \{l\}$ using (8) gives the same solutions. If we now lower $\bar{G}_l(x)$ and solve for $\mathcal{CP}(x) \setminus \{l\}$, then the values $\bar{G}_j(x)$ of all players $j \in \mathcal{CP}(x) \setminus \{l\}$ strictly increase (this is easily seen from (8), since D increases). Observe also that setting $\bar{G}_j(x) = \tilde{G}_j(x)$ for $j \notin \widetilde{\mathcal{CP}}(x)$ and solving for the values $\bar{G}_i(x), i \in \widetilde{\mathcal{CP}}(x)$ using (8) with $\widetilde{\mathcal{CP}}(x)$ instead of $\mathcal{CP}(x)$ gives $\bar{\mathbf{G}}(x) = \tilde{\mathbf{G}}(x)$.
Now, consider the following process by which $\tilde{\mathbf{G}}(x)$ is "reached" from $\mathbf{G}(x)$. Set $\bar{G}_l(x)$ equal to $G_l(x) = \tilde{G}_l(x)$ for $l \notin \mathcal{CP}(x)$. Take a player $j \in \mathcal{B}$ for whom $\tilde{G}_j(x) < G_j(x)$. Then, solving for $\mathcal{CP}(x) \setminus \{j\}$ using $\bar{G}_j(x) = G_j(x)$ as described above and then lowering $\bar{G}_j(x)$ to $\tilde{G}_j(x)$ and solving again for $\mathcal{CP}(x) \setminus \{j\}$, raises the solutions above $G_l(x)$ for all $l \in \mathcal{CP}(x) \setminus \{j\}$. Thus, if $\mathcal{B} = \{j\}$ then $\mathcal{CP}(x) \setminus \{j\} = \widetilde{\mathcal{CP}}(x)$ and $\bar{G}_l(x) = \tilde{G}_l(x), l \notin \widetilde{\mathcal{CP}}(x)$, so the solutions for $\bar{G}_i, i \in \widetilde{\mathcal{CP}}(x)$ coincide with $\tilde{\mathbf{G}}$ and player j obtains under $\tilde{\mathbf{G}}$ at x more than his power (since under $\mathbf{G}$ at x player $j \in \mathcal{CP}(x)$ obtains precisely his power). If $\mathcal{B} \neq \{j\}$, continue this process: take a player $l \in \mathcal{B} \backslash \{j\}$ and lower $\bar{G}_l(x)$ obtained in the previous step to $\tilde{G}_l(x)$;[40] solve for $\mathcal{CP}(x) \setminus \{j, l\}$, and remember that after lowering $\bar{G}_j(x)$ from $G_j(x)$ to $\tilde{G}_j(x)$ all players in $\mathcal{CP}(x) \setminus \{j\}$ obtained precisely their power, and $l \in \mathcal{CP}(x) \setminus \{j\}$. Since $\bar{G}_l(x)$ decreases to $\tilde{G}_l(x)$, the solutions for all players in $\mathcal{CP}(x) \setminus \{j, l\}$ strictly increase and $\tilde{G}_j(x)$ does not change, so player l now obtains more than his payoff. Continuing in this way and recalling that $\bar{\mathbf{G}} = \tilde{\mathbf{G}}$ once $\bar{\mathbf{G}}$ and $\tilde{\mathbf{G}}$ agree on $\mathcal{N}/\widetilde{\mathcal{CP}}(x)$, we see that the last player in $\mathcal{B}$ obtains more than his power under $\tilde{\mathbf{G}}$ at x, a contradiction. Therefore, $\tilde{G}_i(x) > G_i(x)$ for some player $i \in \mathcal{B}$. ■

Lemma 13 *$\mathcal{CP}(x) \subseteq \widetilde{\mathcal{CP}}(x)$.*

Proof. Suppose $\mathcal{CP}(x) \not\subseteq \widetilde{\mathcal{CP}}(x)$. By the previous lemma, $\tilde{G}_i(x) > G_i(x)$ for some $i \in \mathcal{CP}(x) \backslash \widetilde{\mathcal{CP}}(x)$. Since $\tilde{G}_i(x_k) = G_i(x_k)$, $\tilde{G}_i(y) > G_i(y)$ for some $y \in (x_k, x)$ such that $i \in \widetilde{\mathcal{CP}}(y)$.[41] This means that $\widetilde{\mathcal{CP}}(y) \neq \mathcal{CP}(y)$ (otherwise Corollary 7 and (8) would

---

[40] Since $\tilde{G}_l(x) \leq G_l(x)$ and lowering $\bar{G}_j(x)$ from $G_j(x)$ to $\tilde{G}_j(x)$ raised the solutions $\bar{G}_l(x)$ for all $l \in CP(x) \setminus \{j\}$, we have $\tilde{G}_l(x) < \bar{G}_l(x)$.

[41] Let $\bar{z} = \sup_{z \in [x_k, x)} \left\{ \tilde{G}_i(z) = G_i(z) \right\}$. By continuity of $\tilde{G}_i$ and G_i, $\bar{z} < x$. If for all $y \in (\bar{z}, x)$ we had $i \notin \tilde{A}(y)$ then $\tilde{G}_i$ would not increase on $(\bar{z}, x)$ so we would have $\tilde{G}_i(x) \leq G_i(x)$.

imply $\tilde{G}_i(y) = G_i(y)$). Let $\hat{\mathcal{B}} = \mathcal{CP}(y) \setminus \widetilde{\mathcal{CP}}(y)$.
Now perform a procedure similar to the one described in the previous lemma, reaching $\tilde{\mathbf{G}}(y)$ from $\mathbf{G}(y)$. Begin with players $l \in \hat{\mathcal{B}}$ for whom $\tilde{G}_l(y) > G_l(y)$. Raising $\bar{G}_l(y)$ from $G_l(y)$ to $\tilde{G}_l(y)$ decreases the solutions for all other players, so the order of raising does not matter - the solutions must be raised for all players $l \in \hat{\mathcal{B}}$ for whom $\tilde{G}_l(y) > G_l(y)$. If the solutions of any remaining players in $\hat{\mathcal{B}}$ now need to be raised to reach their level in $\tilde{\mathbf{G}}$, continue the raising process until no more players in $\hat{\mathcal{B}}$ need their solutions raised. It cannot be that $\hat{\mathcal{B}}$ is exhausted, since $\tilde{G}_i(y) > G_i(y)$ and so far the solutions of all players in $\widetilde{\mathcal{CP}}(y)$ have been repeatedly decreased, starting from their level in $\mathbf{G}$. Thus, there remains a non-empty set $\bar{\mathcal{B}} \subseteq \hat{\mathcal{B}}$ of players whose solutions must now be decreased to reach their level in $\tilde{\mathbf{G}}$. Decreasing these solutions increases the solutions for all other players. By the argument used in the previous lemma, the last player whose solution is decreased receives too high a payoff under $\tilde{\mathbf{G}}$ at y. ■