A A Non-Constructible Equilibrium[1]

The example depicts a separable contest with three players and one prize of common value 1 (so $v_i(\cdot) = 1 - c_i(\cdot)$). I construct an equilibrium (C, G, G) of the contest, in which player 1's best-response set is the Cantor set. Let $c_1(x) = x$. Let $F(x) = x$ and modify $F(x)$ by mimicking the construction of the Cantor set on $[0, 1]$ in the following way. At every removed interval (a, b) and for every $x \in (a, b)$, let $F(x) = a + (x - a)^2 / (b - a)$. Denote the resulting function by $\tilde{F}$. Then $\tilde{F}(0) = 0, \tilde{F}(1) = 1$, and $\tilde{F}$ is continuous, strictly increasing, equals x precisely on the Cantor set, and is strictly lower than x on its complement. In particular, if player 1's probability of winning when choosing x is $\tilde{F}(x)$, then his best-response set is the Cantor set. To achieve this, let $G(x)$ satisfy $\tilde{F}(x) = 1 - (1 - G(x))(1 - G(x))$ for all $x \in [0, 1]$. That is,

$$G(x) = 1 - \sqrt{1 - \tilde{F}(x)}.$$

Then $G(x)$ is continuous and strictly increasing, with $G(0) = 0$ and $G(1) = 1$. To define player 2 and 3's cost functions, let C be the Cantor function, and recall that it is continuous and weakly increasing, with $C(0) = 0$ and $C(1) = 1$. Let

$$c_2(x) = c_3(x) = 1 - (1 - G(x))(1 - C(x)) = 1 - \left(\sqrt{1 - \tilde{F}(x)}\right)(1 - C(x)).$$

Then c_2 and c_3 are continuous and strictly increasing, with $c_2(0) = c_3(0) = 0$ and $c_2(1) = c_3(1) = 1$. It is straightforward to verify that (C, G, G) is an equilibrium of the contest, in which player 1's best-response set is the Cantor set.

B The Example of Figure 2

Cost functions are $c_2(x) = \frac{3x}{4}$,

$$c_1(x) = \begin{cases} \frac{x}{100} & \text{if } 0 \leq x \leq 0.31948 \\ \frac{x}{100} + 1.0581(x - (0.31948))^2 & \text{if } 0.31948 < x \leq 1 \\ 0.5 + 1.45(x - 1) & \text{if } 1 < x \end{cases}$$

[1] I thank George Mailath for encouraging me to provide this example.

and

$$c_3(x)=\begin{cases}\frac{x}{12} & \text{if} \quad 0\le x\le 0.31948\\ \frac{x}{12}+1.9794\left(x-(0.31948)\right)^2 & \text{if} \quad 0.31948<x\le 0.7259\\ 0.387\,44+1.6923\,(x-0.7259)+25\,(x-0.7259)^2 & \text{if} \quad 0.7259<x\le 0.85\\ 0.98247+\frac{(1-0.98247)}{0.15}\,(x-0.85) & \text{if} \quad 0.85<x\end{cases}$$

These cost functions give powers of $0, \frac{1}{4}$ and $\frac{1}{2}$. Perturbing the cost functions slightly does not change the qualitative aspects of the equilibrium.

C Proofs of the Results of Section 3

C.1 Proof of Theorem 8

That players $m+2,\ldots,n$ do not participate was shown in Corollary 4. Denote by s_i^l the infimum of the scores at which player i is active. Because $m, m+1\in\mathcal{CP}(0)$, we have $s_m^l=s_{m+1}^l=0$. Now, suppose that for $i,j\le m$, $s_i^l\le s_j^l$. Since i and j have positive powers and are not active below s_i^l, $G_i\left(s_i^l\right)=G_j\left(s_i^l\right)=0$. Thus, $P_i\left(s_i^l\right)=P_j\left(s_i^l\right)$.[2] Therefore,

$$\begin{aligned}\frac{u_i\left(s_i^l\right)}{V_i}-\frac{u_j\left(s_i^l\right)}{V_j}&=P_i\left(s_i^l\right)(1-\alpha)\,c\left(s_i^l\right)(a_j-a_i)+\alpha c\left(s_i^l\right)(a_j-a_i)\\&=(a_j-a_i)\,c\left(s_i^l\right)\left(\alpha+P_i\left(s_i^l\right)(1-\alpha)\right).\end{aligned}$$

Also, $w_i/V_i=1-a_ic(T)$ and $w_j/V_j=1-a_jc(T)$, so $w_i/V_i-w_j/V_j=(a_j-a_i)\,c(T)$. Since $w_i/V_i=u_i\left(s_i^l\right)/V_i$ and $w_j/V_j\ge u_j\left(s_i^l\right)/V_j$, we have

(1)
$$0\ge\left(\frac{w_i}{V_i}-\frac{w_j}{V_j}\right)-\left(\frac{u_i\left(s_i^l\right)}{V_i}-\frac{u_j\left(s_i^l\right)}{V_j}\right)=(a_j-a_i)\left(c(T)-c\left(s_i^l\right)\left(\alpha+P_i\left(s_i^l\right)(1-\alpha)\right)\right).$$

Because $\alpha>0$ and $P_i\left(s_i^l\right)\le 1$, we have $\left(\alpha+P_i\left(s_i^l\right)(1-\alpha)\right)>0$, so (1) holds if and only if

$$(a_j-a_i)\left(c(T)-c\left(s_i^l\right)\right)\le 0.$$

[2] If $s_i^l=0$, consider the limit of the probabilities of winning as the score approaches 0 from above, and similarly for $u_i\left(s_i^l\right)$ and $u_j\left(s_i^l\right)$.

Since $c\left(s_i^l\right) < c(T)$, this implies that $a_j \leq a_i$.

It remains to show that if a player becomes active at some score, then he remains active until the threshold. To do this, let us derive some properties of players' semi-elasticities. First normalize players' payoffs so that the prize value is 1 for all players. To this end, note that the contest is strategically equivalent to a contest in which all valuations equal 1 and in which player i's cost is $a_i c$ instead of $\gamma_i c$. We then have

$$q_i(x) = \frac{v_i(x) - v_i(T)}{v_i(x) + c_i(x)} = \frac{a_i\left(\frac{1}{a_{m+1}} - c(x)\right)}{1 - (1-\alpha)\, a_i c(x)}$$

and

$$\varepsilon_i(x) = -\frac{q_i'(x)}{q_i(x)} = \frac{c'(x)\left(a_i(\alpha-1) + a_{m+1}\right)}{\left(1 - c(x)\, a_{m+1}\right)\left(a_i c(x)(\alpha-1) + 1\right)}.$$

Viewed as a function of a_i, we obtain

$$\frac{\partial \varepsilon_i(x)}{\partial a_i} = -\frac{c'(x)(1-\alpha)}{\left(a_i c(x)(\alpha-1)+1\right)^2} \leq 0,$$

so at every score players with a higher reach have higher semi-elasticities. This means that when a new player becomes active all existing active players remain active, and that whether an active player remains active depends only on his semi-elasticity and those of players with lower reaches. In particular, players m and $m+1$ are always active, since their semi-elasticities are always the lowest. To show that players $1, \ldots, m-1$ are active on an interval, observe that the ratio of semi-elasticities of players $j > i$ is non-decreasing in score,

$$\left(\frac{\varepsilon_j(x)}{\varepsilon_i(x)}\right)' = \left(\frac{\left(a_j(\alpha-1) + a_{m+1}\right)\left(a_i c(x)(\alpha-1)+1\right)}{\left(a_i(\alpha-1) + a_{m+1}\right)\left(a_j c(x)(\alpha-1)+1\right)}\right)'$$

$$= \frac{a_{m+1} - (1-\alpha)\, a_j}{a_{m+1} - (1-\alpha)\, a_i} \frac{(1-\alpha)(a_j - a_i)\, c'(x)}{\left(a_j c(x)(\alpha-1)+1\right)^2} \geq 0,$$

and also that this ratio is at most 1 (since players with a higher reach have higher semi-elasticities). Suppose in contradiction that there is a player who is not active on an interval, and let $i \leq m-1$ be the player with the highest index among such players. Suppose that player i is active at s_i. Denote by $H^{i+1,\ldots,m+1}(\cdot)$ the positive fixed point of the "supply function" defined using the semi-elasticities of players $i+1, \ldots, m+1$, and consider a score $s_i' \in [s_i, T]$. By definition of player i and because players with higher reaches become active at higher scores, players $m+1, \ldots, i+1$ are active at s_i'. So, because the semi-elasticities of all players $1,\ldots,i-1$ are weakly higher than that of player i, $\varepsilon_i(s_i') \leq H(s_i')$ if and only if $\varepsilon_i(s_i') \leq H^{i+1,\ldots,m+1}(s_i')$. Let $b = \varepsilon_i(s_i')/\varepsilon_i(s_i)$. For all $j > i$, since $\varepsilon_j(s_i')/\varepsilon_i(s_i') \geq \varepsilon_j(s_i)/\varepsilon_i(s_i)$, we have $\varepsilon_i(s_i)/\varepsilon_j(s_i) \geq b$. Therefore,

$$H^{i+1,\ldots,m+1}(s_i') = \frac{1}{m-i}\sum_{j=i+1}^{m+1}\varepsilon_j(s_i') \geq \frac{1}{m-i}\sum_{j=i+1}^{m+1} b\varepsilon_j(s_i) = bH^{i+1,\ldots,m+1}(s_i).$$

Because player i is active at s_i, $\varepsilon_i(s_i) \leq H(s_i)$ so $\varepsilon_i(s_i) \leq H^{i+1,\ldots,m+1}(s_i)$. Therefore, $\varepsilon_i(s_i') = b\varepsilon_i(s_i) \leq bH^{i+1,\ldots,m+1}(s_i) \leq H^{i+1,\ldots,m+1}(s_i')$ so $\varepsilon_i(s_i') \leq H(s_i')$. This shows that once a player becomes active he remains active until the threshold.

C.2 Proof of Corollary 5

Because $s_j^l \leq s_i^l$, we have $G_i(x) \leq G_j(x)$ for any $x \leq s_i^l$. By Theorem 8, both players are active on on $\left[s_i^l, T\right]$. Also, $\varepsilon_i \geq \varepsilon_j$, so because both players are active on $\left[s_i^l, T\right]$ the algorithm shows us that $h_i \leq h_j$ on $\left[s_i^l, T\right]$. So i starts being active later and has a lower hazard rate than j, which means that G_i FOSD G_j. To see this, recall that $h_i(x) = -(1-G_i(x))'/(1-G_i(x))$, so $h_i \leq h_j$ implies that $(1-G_i)'/(1-G_i) \geq (1-G_j)'/(1-G_j)$. This implies that for $y \in \left[s_i^l, T\right]$,

$$0 \leq \int_{s_i^l}^{y}\left(\frac{(1-G_i(x))'}{1-G_i(x)} - \frac{(1-G_j(x))'}{1-G_j(x)}\right)dx = \ln\left(\frac{1-G_i(x)}{1-G_j(x)}\right)\Big|_{s_i^l}^{y}$$

$$= \ln\left(\frac{1-G_i(y)}{1-G_j(y)}\right) - \ln\left(\frac{1-G_i\left(s_i^l\right)}{1-G_j\left(s_i^l\right)}\right).$$

Because $G_i\left(s_i^l\right) \leq G_j\left(s_i^l\right)$, we have $\left(1-G_i\left(s_i^l\right)\right)/\left(1-G_j\left(s_i^l\right)\right) \geq 1$, so by taking exponents the previous inequality implies $\frac{1-G_i(y)}{1-G_j(y)} \geq 1$, or $G_j(y) \geq G_i(y)$, as required.

This FOSD implies that probability of winning is higher than that of player j for any given score, and hence also in expectation. To see this, note that by choosing $x > 0$ i beats j with probability $G_j(x)$, whereas by choosing x j beats i with probability $G_i(x)$. Therefore, because $G_j(x) \geq G_i(x)$, for any given score i wins with at least as high a probability as j does, i.e., $P_i(x) \geq P_j(x)$. Therefore, $P_i = \int P_i(x)\,dG_i \geq \int P_j(x)\,dG_i$, and because $P_j(\cdot)$ is non-decreasing, by FOSD $\int P_j(x)\,dG_i \geq \int P_j(x)\,dG_j = P_j$.

C.3 Proof of Theorem 9

Let

$$f(\alpha, x) = \frac{1}{(1 - a_{m+1} c(x))} \frac{\Pi_{k=i+1}^{m+1} (1 - a_k c(x)(1 - \alpha))}{(1 - a_i c(x)(1 - \alpha))^{m-i}},$$

and note that f is differentiable in $\alpha < 1$ and $x < T$. Denote by $s_i^l(\alpha)$ the lowest x such that $f(\alpha, x) = \left(\Pi_{k=i+1}^{m} a_k\right) / a_i^{m-i}$. Note that $\left(\Pi_{k=i+1}^{m} a_k\right) / a_i^{m-i} > 1$ (because a_k increases in k) and $f(\alpha, 0) = 1$ (since $c(0) = 0$). Suppose that when α increases to α' the value of f at $s_i^l(\alpha)$ increases. Then, because $f\left(\alpha', s_i^l(\alpha)\right) > \left(\Pi_{k=i+1}^{m} a_k\right) / a_i^{m-i}$ and $f(\alpha', 0) < \left(\Pi_{k=i+1}^{m} a_k\right) / a_i^{m-i}$, the intermediate value theorem shows that $s_i^l(\alpha') < s_i^l(\alpha)$. Therefore, to show that s_i^l decreases in α it suffices to show that $\partial f(a, x) / \partial \alpha > 0$ for $x \in (0, T)$. By considering the expression for $\partial f(a, x) / \partial \alpha$, one can verify that it suffices to show that

$$\left(\sum_{j=i+1}^{m+1} a_j c(x) \prod_{k \in \{i+1, \ldots, m+1\} \setminus j} (1 - a_k c(x)(1 - \alpha)) \right) (1 - a_i c(x)(1 - \alpha))$$

$$> \Pi_{k=i+1}^{m+1} (1 - a_k c(x)(1 - \alpha)) ((m - i) a_i c(x)).$$

For this inequality to hold, it suffices that for every $j = i + 1, \ldots m + 1$,

$$a_j c(x) > a_i c(x) \text{ and } 1 - a_i c(x)(1 - \alpha) > 1 - a_j c(x)(1 - \alpha),$$

and this holds because a_k increases in k. Therefore, s_i^l decreases in $\alpha < 1$ for every player $i = 1, \ldots, m - 1$. Now consider what happens to s_i^l as α approaches 0. For $x < T$,

$$f(0, x) = \frac{1}{(1 - a_{m+1} c(x))} \frac{\Pi_{k=i+1}^{m+1} (1 - a_k c(x))}{(1 - a_i c(x))^{m-i}} = \frac{\Pi_{k=i+1}^{m} (1 - a_k c(x))}{(1 - a_i c(x))^{m-i}} \leq 1.$$

Therefore, by uniform continuity of $f(\alpha, x)$ on $[0, \tilde{\alpha}] \times [0, x]$ for any $\tilde{\alpha} \in (0, 1)$, s_i^l approaches T as α approaches 0.

C.4 Proof of Corollary 6

Choose $\beta < 1$. By Theorem 9 there exist $\tilde{x} < T$ and $\tilde{\alpha} > 0$ such that for all $\alpha < \tilde{\alpha}$ and $i < m$, $s_i^l > \tilde{x}$. Choose such $\tilde{x}$ and $\tilde{\alpha}$ that also satisfy

$$\frac{V_m - \gamma_m \frac{V_{m+1}}{\gamma_{m+1}} + \alpha \gamma_m c(\tilde{x})}{V_m - (1 - \alpha) \gamma_m c(\tilde{x})} > \beta \tag{2}$$

and

$$\text{(3)} \qquad \frac{\alpha\gamma_{m+1}c\left(\tilde{x}\right)}{V_{m+1}-\left(1-\alpha\right)\gamma_{m+1}c\left(\tilde{x}\right)}<1-\beta,$$

Consider the unique equilibrium $\mathbf{G}$ of a such a simple contest with $\alpha < \tilde{\alpha}$. Since $G_i\left(\tilde{x}\right) = 0$ for $i = 1, \ldots, m-1$ and $G_i\left(0\right) = 1$ for $i = m+2, \ldots, n$, Corollary 3 shows that the CDFs of players $m+1$ and m on $[0, \tilde{x}]$ are given by (2) and (3). Since $s_i^l > \tilde{x}$ for $i = 1, \ldots, m-1$, each of these $m-1$ players beats player $m+1$, and therefore wins a prize, with probability of at least β. Player m chooses scores higher than $\tilde{x}$ with probability of at least β, and therefore wins a prize with probability of at least β^2.