Matching and Price Competition: Comment

Julio González-Díaz and Ron Siegel

Research Group in Economic Analysis, University of Vigo

and

Department of Economics, Northwestern University

October 2009

Abstract

We relax the assumption of symmetric linear costs in B&L's (2006) "Matching and Price Competition" and compare the pricing equilibrium that results to the firm-optimal competitive equilibrium. With linear and asymmetric costs, competition may not be localized in the pricing equilibrium, but all other qualitative comparisons of B&L (2006) hold. With non-linear and symmetric costs workers' average utility in the pricing equilibrium may be higher than in the firm-optimal competitive equilibrium. With asymmetric and non-linear costs, firms need not choose scores from an interval in a pricing equilibrium, which may make competition even less localized.

1 Introduction

Jeremy Bulow & Jonathan Levin (2006), henceforth B&L, investigated a matching model that captures certain features of the National Residents Matching Program (NRMP), the central clearinghouse that assigns medical residents to hospitals. In B&L's model, hospitals of known and varying quality compete for residents of known and varying quality by offering wages. The hospital offering the highest wage obtains the best resident, the hospital offering the second-highest wage obtains the second-best resident, etc. Because hospitals commit to wages before the matching is determined, the pricing equilibrium that arises has some of the features of an all-pay auction with complete information. In particular, hospitals employ mixed strategies, so the matching is inefficient with some probability. B&L compare this pricing equilibrium to the one resulting from the hospital-optimal competitive equilibrium, which is the competitive equilibrium with the lowest wages. All competitive equilibria are efficient, matching hospitals and workers assortatively. B&L find that compared to the hospital-optimal competitive equilibrium the pricing equilibrium has lower, more compressed wages and higher hospital profits. They also find that in the pricing equilibrium competition is "localized," in the sense that hospitals compete against hospitals of similar quality.

In this note we investigate the importance of the assumption that hospitals compete by offering wages. As B&L note (pages 653 and 657, and footnote 16), hospitals in fact offer a set of job features, which include wages, reputation, responsibility, work hours, training, quality of facilities, number of senior physicians, etc. This means that the cost of increasing a hospital's attractiveness to residents is not necessarily linear. In addition, different hospitals may differ in their costs of providing these different features. For example, a hospital with many senior physicians may be in a better position to attract additional senior physicians; a reputable hospital usually has an advantage over less-known hospitals; a university hospital may find it less costly to provide its residents with research time or training; a rural hospital may incur lower costs than an urban hospital when increasing wages, because of a lower tax rate. The non-linearity and asymmetry in hospitals' costs suggest that B&L's assumption that residents rank hospitals according to expenditures deserves closer inspection.

This assumption can be separated into two parts. The first is that all hospitals have the same cost of achieving any given level of attractiveness to residents. A hospital's level of attractiveness to residents, which we call "score," is determined by the combination of features the hospital provides. One unit of score has a dollar value of one for residents. B&L assume that all hospitals have the same cost of achieving any given score. The second part of B&L's assumption is that hospitals' technologies of producing score exhibit constant returns to scale, i.e., they are linear and, moreover, the cost of producing score x is x. This means that one dollar of investment by a hospital creates one dollar of value for its residents.

We relax each part of B&L's assumption in turn, and then both parts simultaneously. We find that relaxing symmetry and maintaining linearity does not change B&L's qualitative conclusions regarding residents' salaries, but can lead to hospital profits that are equal to those obtained in the hospital-optimal competitive equilibrium. It can also make competition less localized than in B&L's pricing equilibrium. Relaxing linearity changes B&L's qualitative conclusions regarding residents' utilities, even when symmetry is maintained. With non-linear costs, residents may be better off on average compared to the firm-optimal competitive equilibrium. When both symmetry and linearity are relaxed, hospitals may use non-interval bidding strategies.[1] This too implies that competition may be less localized than in B&L's setting.

2 The Model of B&L

There are N firms and N workers. The surplus of firm n from employing worker m is $\Delta_n \cdot m$, where $\Delta_n > 0$ is non-decreasing in n. Firms compete by each offering a non-negative wage. Worker m is matched with the firm that offered the $(N - m + 1)^{\text{th}}$ highest wage. B&L show that in equilibrium each firm chooses the wage it offers by randomizing on an interval of wages, and use this observation and the fact that firms have constant marginal costs to provide an algorithm that constructs the equilibrium. The algorithm proceeds from the top, and for every wage identifies every firm's density of offering that wage. The set of "active" firms whose density of offering that wage is positive is the maximal set of firms that have not yet exhausted their probability, such that no equilibrium conditions are violated. Thus, the equilibrium is characterized by a probability distribution G_n for each firm n, where $G_n(x)$ is the probability that firm n offers a wage less than or equal to x.

3 Firms with Asymmetric Linear Costs

Suppose that instead of offering a wage, each firm offers a "score," which is the monetary value that workers assigns to the bundle of attributes associated with working for the firm.

[1] Interval bidding strategies are a key feature in B&L's analysis.

As discussed in the introduction, the cost of offering a given score may vary across firms. Suppose that firm n's cost of offering a score x is given by $c_n(x) = \gamma_n x$ (in B&L, $\gamma_n = 1$ for every firm n). To facilitate the comparison to B&L, we restrict attention to costs such that $\frac{\Delta_n}{\gamma_n}$ are non-decreasing in n. This guarantees that competitive equilibria are efficient (lead to assortative matching).[2]

In a competitive equilibrium in which worker n's price is p_n, firm $n-1$ prefers hiring worker $n-1$ to hiring worker n, so $\gamma_{n-1}p_n - \gamma_{n-1}p_{n-1} \geq \Delta_{n-1}$, or $p_n - p_{n-1} \geq \frac{\Delta_{n-1}}{\gamma_{n-1}}$. This shows that the firm-optimal competitive equilibrium has salaries $p_n = \sum_{k<n} \frac{\Delta_k}{\gamma_k}$.[3] To solve for the pricing equilibrium, we divide every firm n's cost and surplus by γ_n and note that this new market is strategically equivalent to the original one. In this market, every firm's cost is as in B&L, so we apply the algorithm of B&L with $\tilde{\Delta}_n = \frac{\Delta_n}{\gamma_n}$. No other change is needed. B&L's analysis of the market with $\left(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_N\right)$ immediately implies the following.

Claim 1 *When firms' costs are linear with coefficients $(\gamma_1, \ldots, \gamma_N)$ such that $\frac{\Delta_n}{\gamma_n}$ is non-decreasing in n, B&L's qualitative comparisons regarding worker's utilities and firms' profits between the firm-optimal competitive equilibrium and the pricing equilibrium hold.*

Proof. The firm-optimal competitive equilibrium and the pricing equilibrium are identical to those in B&L's setting with $\left(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_N\right)$, so the comparison regarding workers' utilities remains unchanged. The comparison regarding firms' profits also remains unchanged, because each firm n's utility is simply multiplied by γ_n in both equilibria relative to B&L's setting with $\left(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_N\right)$. ■

Asymmetric linear costs can, however, cause the difference between agents' utilities between the competitive equilibrium and the pricing equilibrium to be less pronounced. This is most easily seen by considering the extreme case in which $\frac{\Delta_1}{\gamma_1} = \ldots = \frac{\Delta_N}{\gamma_N}$.

Claim 2 *When the coefficients $(\gamma_1, \ldots \gamma_N)$ are such that $\frac{\Delta_1}{\gamma_1} = \ldots = \frac{\Delta_N}{\gamma_N}$, each firm's profit and workers' average utilities in the pricing equilibrium equal those in the firm-optimal competitive equilibrium.*

[2] This is an increasing differences condition. Firm n's payoff from hiring worker m at price p_m is $\Delta_n \cdot m - \gamma_n p_m$. When firm n faces a price p_m for each worker m, it maximizes $\Delta_n \cdot m - \gamma_n p_m$ or, equivalently, $\frac{\Delta_n \cdot m}{\gamma_n} - p_m$. Therefore, to guarantee that in a competitive equilibrium stronger firms hire better workers (who are more expensive), $\frac{\Delta_n \cdot m}{\gamma_n}$ must have increasing differences. This implies that $\frac{\Delta_n}{\gamma_n}$ is non-decreasing in n.

[3] Similarly, the worst competitive equilibrium from the firms' perspective has $p_n = \sum_{k \leq n} \frac{\Delta_k}{\gamma_k}$.

Proof. Let $\alpha = \frac{\Delta_1}{\gamma_1} = \ldots = \frac{\Delta_N}{\gamma_N}$. Each worker n's utility in the competitive equilibrium is $\alpha (n-1)$, so firm n's profit is $n\Delta_n - \gamma_n (n-1) \frac{\Delta_N}{\gamma_N} = \Delta_n$. To obtain the pricing equilibrium, consider B&L's algorithm. Because $\frac{\Delta_1}{\gamma_1} = \ldots = \frac{\Delta_N}{\gamma_N}$, firms have identical densities and exhaust their probabilities simultaneously. Therefore, there is only one interval of competition. On this interval, each firm's density is $\frac{1}{\alpha(N-1)}$. Firms compete by mixing uniformly on $[0, \alpha (N-1)]$. Because there is no atom at 0, each firm obtains the worst worker by offering 0, so every firm n's payoff is Δ_n, just like in the firm-optimal competitive equilibrium. The average of workers' utilities is the middle of the interval, $\frac{\alpha(N-1)}{2}$. This is equal to the average of workers' utilities in the firm-optimal competitive equilibrium, $\frac{\sum_{n=1}^{N} \alpha(n-1)}{N}$. ■

This extreme example shows why workers' utilities are compressed: in the firm-optimal competitive equilibrium workers' utilities are spaced evenly on the interval $[0, \alpha (N-1)]$, whereas in the pricing equilibrium expected utilities are spaced evenly on the interval $\left[\frac{\alpha(N-1)}{N+1}, \frac{\alpha(N-1)N}{N+1}\right]$, whose length is $\frac{\alpha(N-1)^2}{N+1} < \frac{\alpha(N+1)(N-1)}{N+1} = \alpha (N-1)$.

Asymmetric linear costs can also make competition less localized than in the setting of B&L when $\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n$ are close to each other. As the proof of Proposition 2 shows, even when $\Delta_1, \ldots, \Delta_n$ differ, so that with symmetric linear costs competition is localized, if $\tilde{\Delta}_1 = \ldots = \tilde{\Delta}_n$ then *all* firms compete on the same interval.

4 Firms with Symmetric Non-Linear Costs

A key feature of B&L's pricing equilibrium is that workers' average utilities decrease in the pricing equilibrium relative to the firm-optimal competitive equilibrium. With non-linear costs this is not always the case, and even with identical non-linear costs average utilities may increase. This is shown in Example 1 below.

When firms' costs are not linear, firms' marginal costs change with their chosen score. This may seem to imply that B&L's algorithm cannot be used to solve for equilibrium. An application of the following result, however, shows that the algorithm of B&L can still be used.

Proposition 1 *Consider a market in which every firm i's cost function $c_i : [0, \infty) \to [0, \infty)$ is continuous and strictly increasing. Let $f \colon [0, \infty) \to [0, \infty)$ be a strictly increasing bijection and, for each $n \leq N$, let $\hat{c}_n = c_n \circ f$. Let $\mathbf{G} = (G_1, \ldots, G_n)$ and $\hat{\mathbf{G}} = \left(\hat{G}_1, \ldots, \hat{G}_n\right)$ be two strategy profiles such that, for each $n \leq N$ and each $x \in [0, \infty)$, $\hat{G}_n(x) = G_n(f(x))$. Then, $\mathbf{G}$ is an equilibrium of the market with costs $\mathbf{C} = (c_1, \ldots, c_N)$ if and only if $\hat{\mathbf{G}}$ is* *an equilibrium of the market with costs* $\hat{\mathbf{C}} = (\hat{c}_1, \ldots, \hat{c}_N)$*. Moreover, firms have the same payoffs in both equilibria.*

Proof. Denote by $u_n(x)$ firm n's expected payoff in the market with costs $\mathbf{C}$ when the firm chooses score x and all other firms choose scores according to $\mathbf{G}$. Similarly, denote by $\hat{u}_n(y)$ firm n's expected payoff in the market with costs $\hat{\mathbf{C}}$ when the firm chooses score y and all other firms choose scores according to $\hat{\mathbf{G}}$. Then, by definition of $\hat{\mathbf{G}}$ and $\hat{\mathbf{C}}$, for any score $x \geq 0$ at which no player has an atom, i.e., no player chooses x with positive probability, we have

$$\hat{u}_n(x) = \left(1 + \sum_{k \neq n} \hat{G}_k(x)\right) \Delta_n - \hat{c}_n(x) = \left(1 + \sum_{k \neq n} G_k(f(x))\right) \Delta_n - c_n(f(x)) = u_n(f(x)).$$

Hence, a score $x \geq 0$ is a profitable deviation from $\hat{G}_n$ for firm n in the market with costs $\hat{\mathbf{C}}$ when all other firms choose scores according to $\hat{\mathbf{G}}$ if and only if $f(x)$ is a profitable deviation from G_n for firm n in the market with costs $\mathbf{C}$ when all other firms choose scores according to $\mathbf{G}$.[4] This implies the first part of the proposition. The coincidence of firms' payoffs in the two markets follows from the equality

$$\int_0^\infty \hat{u}_n(x)\, d\hat{G}_n(x) = \int_0^\infty u_n(f(x))\, dG_n(f(x)) \overset{f \text{ is a bijection}}{=} \int_0^\infty u_n(x)\, dG_n(x).$$

■

The following corollary shows how to obtain the equilibrium when firms have the same non-linear costs.[5]

Corollary 1 *Let c be each firm's continuous and strictly increasing cost function. Then, the following statements hold:*

1. *The market has a unique pricing equilibrium.*
2. *The pricing equilibrium can be constructed in two steps. First, construct B&L's pricing equilibrium in the market with linear costs, i.e., with $c_n(x) = x$. Second, transform the pricing equilibrium from the first step as described in Proposition 1.*

[4] If x is a profitable deviation, then so are all scores slightly higher than x, by continuity of firms' costs. Therefore, if a profitable deviation exists we can find a score that is a profitable deviation and at which no player has an atom.

[5] Proposition 1 can also be used to obtain the equilibrium when firms' non-linear costs differ by a multiplicative constant.

3. Every firm chooses scores from an interval.

Proof. Part 1 follows from Proposition 1 and the uniqueness of the equilibrium for the contest with costs $c_n(x) = x$. Part 2 follows directly from Proposition 1. Part 3 follows from Proposition 1 and the fact that in the equilibrium of the contest with costs $c_n(x) = x$ Every firm chooses scores from an interval. ■

Although Proposition 1 states that firms' equilibrium payoffs remain unchanged after an identical transformation is applied to all firms' cost functions, workers' average utilities may change. This is shown in the following example.

Example 1 *There are three workers and three firms, with* $\Delta_1 = 4, \Delta_2 = 5,$ *and* $\Delta_3 = 20$*. The firms' common cost function is depicted on the left-hand side of Figure 1. Worker's utilities (scores) in the firm-optimal competitive equilibrium are* $0, 4.9,$ *and* 5.87*, with an average of* 3.59*. Workers' expected utilities (scores) in the pricing equilibrium, which is depicted on the right-hand side of Figure 1, are* $2.26, 4.48,$ *and* 5.09*, with an average of* 3.94*. Thus, workers' average utility increases in the pricing equilibrium relative to the firm-optimal competitive equilibrium.*[6]

[6]The common cost function is $c(x) = \frac{5\tan^{-1}(3(x-5))}{\tan^{-1}(15)} + 5$. Firms' equilibrium densities are

$$g_1(x) = \begin{cases} \frac{c'(x)}{\Delta_2} & x_1 > x \\ 0 & \text{otherwise} \end{cases}, \quad g_2(x) = \begin{cases} \frac{c'(x)}{\Delta_1} & x_1 > x \\ \frac{c'(x)}{\Delta_3} & x_1 \le x \le x_2 \\ 0 & \text{otherwise} \end{cases}, \quad g_3(x) = \begin{cases} \frac{c'(x)}{\Delta_2} & x_1 \le x \le x_2 \\ 0 & \text{otherwise} \end{cases},$$

where $x_1 = 5 - \frac{1}{3}\tan\left(\frac{2\tan^{-1}(15)}{5}\right)$ and $x_2 = 5 + \frac{1}{3}\tan\left(\frac{2\tan^{-1}(15)}{5}\right)$.

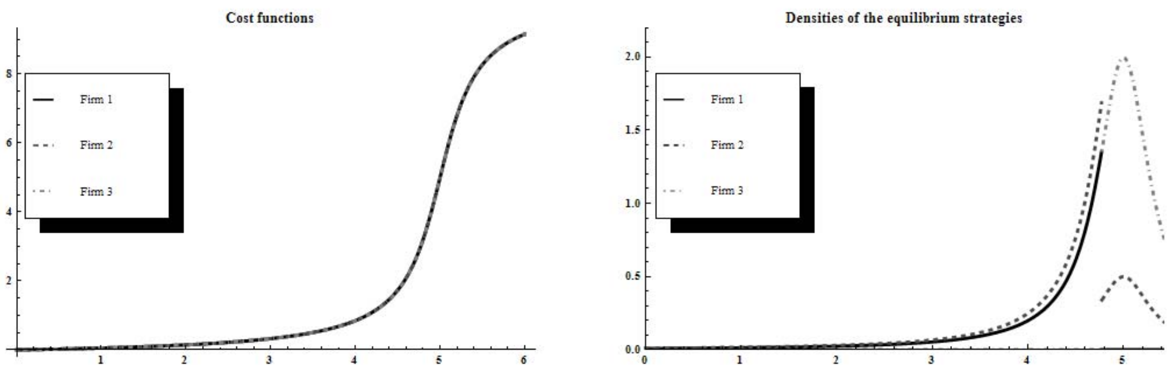

Figure 1: The common cost function (left) and the densities of firms' strategies in the pricing equilibrium (right) of Example 1.

The reason that workers' average utilities increase in the pricing equilibrium relative to the firm-optimal competitive equilibrium is that firms' high marginal costs at high scores result in the two strongest firms concentrating their competition on high scores, which pushes up workers' utilities.[7]

5 Firms with Asymmetric Non-Linear Costs

With asymmetric, non-linear costs, B&L's algorithm cannot be applied to solve for an equilibrium. This is because firms' marginal costs are not constant and there is no simple transformation that changes the market to one for which B&L's algorithm applies. An alternative approach is to apply a version of the algorithm developed in Ron Siegel (2009). This algorithm can be used to solve for a pricing equilibrium by proceeding from the bottom, as long as firms' equilibrium payoffs are known. Although we have been unable to find a general method of identifying firms' payoffs in a pricing equilibrium, we were able

[7] Although worker 2 and 3's utilities are still lower in the pricing equilibrium than in the firm-optimal competitive equilibrium, the reduction is much smaller than with linear costs.

to identify payoffs in some examples.[8]

Example 2 below shows that with asymmetric and non-linear payoffs the utility of a worker who is not the worst worker may increase in the pricing equilibrium relative to the firm-optimal competitive equilibrium, and workers' average utilities may increase as well. It also shows that firms may no longer choose scores from an interval (the density of firms' strategies are depicted on the right-hand side of Figure 2). That firms may employ non-interval strategies suggests that asymmetric, non-linear costs may lead to non-localized competition.

Example 2 *There are three workers and three firms, with* $\Delta_1 = 4, \Delta_2 = 5,$ *and* $\Delta_3 = 20$. *Firms' cost functions are depicted on the left-hand side of Figure 2. Worker's utilities (scores) in the firm-optimal competitive equilibrium are* $0, 20,$ *and* 40*, with an average of* 20*. Workers' expected utilities (scores) in the pricing equilibrium, which is depicted on the right-hand side of Figure 2, are* $11.06, 23.25,$ *and* 30.77*, with an average of* 21.7*. Both worker 1 and 2's utility and workers' average utility increases in the pricing equilibrium relative to the firm-optimal competitive equilibrium. In addition, the support of firm* 3*'s strategy is not an interval.*[9]

[8] We did this by guessing firms' payoffs, running the algorithm of Siegel (2009) assuming these payoffs, and verifying that the resulting strategies form an equilibrium.

[9] Firms' cost functions are $c_1(x) = \frac{x}{4}$, $c_2(x) = \frac{x}{5}$, and $c_3(x) = 20 - \frac{20\tan^{-1}(20-x)}{\tan^{-1}(20)}$. Firms' equilibrium densities are

$$g_1(x) = g_2(x) = \begin{cases} \frac{c3'(x)}{2\Delta_3} & 0 \le x < x_1 \text{ or } x_2 < x \le x_3 \\ \frac{1}{\Delta_3} & x_1 \le x \le x_2 \\ 0 & \text{otherwise} \end{cases}, \quad g_3(x) = \begin{cases} \frac{1}{\Delta_3} - \frac{c3'(x)}{2\Delta_3} & 0 \le x < x_1 \text{ or } x_2 < x \le x_3 \\ 0 & \text{otherwise} \end{cases},$$

where $x_1 = 20 - \sqrt{-1 + \frac{10}{\tan^{-1}(20)}}$, $x_2 = 35.2305$, and $x_3 = 40$.

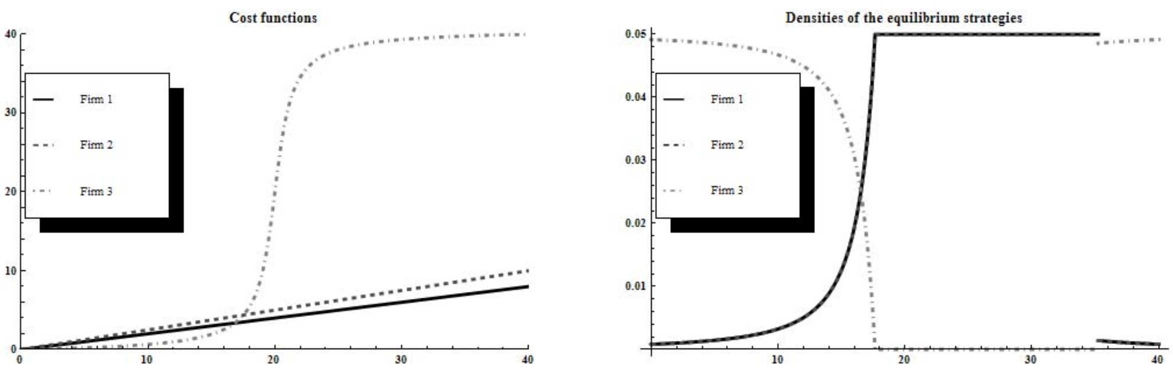

Figure 2: Firms' cost functions (left) and the densities of firms' strategies in the pricing equilibrium (right) of Example 2.

6 Conclusion

Our analysis shows that B&L's simplifying assumption that hospitals are ranked according to expenditures is not without loss of generality. Because non-linearities and asymmetries are common in the real world, our findings suggest that B&L's conclusions should be examined with care.

References

[1] **Bulow, Jeremy, and Jonathan Levin.** 2006. "Matching and Price Competition." *American Economic Review*, 96(3): 652-68.

[2] **Siegel, Ron.** 2009. "Asymmetric Contests with Conditional Investments." *American Economic Review*, forthcoming.