1 Spulber's duopoly

The aim of this example is to show that in some specific, but important models, players can attain efficiency by means of simpler penalty-card strategies, without using communication or public randomization. More specifically, we consider a repeated version of Spulber duopoly model, in which two firms meet in periods $t = 1, 2...$ Firm 1's cost type is i.i.d. and takes value $c = \underline{c}$ or $\overline{c}$, each with probability $1/2$. Firm 2's cost type follows a first-order Markov process with support $\{\underline{c}, \overline{c}\}$. If the cost in a certain period is $\underline{c}$, then it will be $\underline{c}$ with probability $p \in (1/2, 1)$ and $\overline{c}$ with the remaining probability in the following period. Similarly, if the cost in a certain period is $\overline{c}$, then it will be $\overline{c}$ with probability p and $\underline{c}$ with the remaining probability in the following period.

In every period t of the dynamic game, firms simultaneously select prices. A single consumer is willing to pay up to r dollars for one unit of the good, and buys from the firm that offers the lower price, and from each firm with the fifty-fifty chance if the two prices are equal. We will assume that r is higher than the higher cost.

Firms are expected profit maximizers and discount future payoffs by a common discount factor $\delta < 1$. The payoffs are normalized by the factor $\varepsilon = 1 - \delta$. Suppose that initially, in period 0, the probability distribution over the cost of each firm is fifty-fifty. Then, the efficient, or most collusive, total payoff of I firms is

$$v = r - \frac{3}{4}\underline{c} - \frac{1}{4}\overline{c}.$$

We obtain the following result:

Theorem 1 *For every $\lambda > 0$, there is a $\underline{\delta}$ such that for every $\delta > \underline{\delta}$, there is an equilibrium of the repeated games in which firms' discount factor is δ such that the ex ante payoff of each firm in this equilibrium exceeds $v/2 - \lambda$.*

2 Description of efficient strategies

In short, the idea behind our strategies on equilibrium path can be described as follows: If in some period, one firm charges a lower price than the other, then it serves the entire market, but obtains a penalty card (or a penalty card of the other firm is annulled). And if one of the firms reaches a limit number of penalty cards, it gives the entire market to the other firm for one period, and one of its penalty cards is annulled. Off equilibrium path, firms play a bad repeated-game equilibrium.

- **The description of strategies on the equilibrium path**:

The strategy profile has $4n + 2$ states. Each state is described by a value of $k \in \{-n, ..., -1, 0, 1...n\}$ and the cost of firm 2 in the previous period. The type of firm 2 will be denoted by L if the cost in the previous period was $\underline{c}$, and it will be denoted by H if the cost was $\overline{c}$.

In all states except when $k = -n$ or n, if the cost of a firm is $\overline{c}$, the firm is supposed to select price r. If the cost is $\underline{c}$, the firm is supposed to select a price which is slightly lower than r, say, $r - \rho$, where ρ is a (small) positive number.

If both firms select the same price, the value of k does not change. If firm 1 selects $r - \rho$, and firm 2 selects r, then k is replaced with $k + 1$ in the next-period state; and if firm 1 selects r, and firm 2 selects $r - \rho$, then k is replaced with $k - 1$ in the next-period state.

If the current state has $k = -n$, then firm 1 is supposed to charge $r - \rho$, and firm 2 is supposed to charge r, independent of its cost; and $k = -n$ is replaced with $k = -n + 1$ at the end of the current period. If the current state has $k = n$, then firm 1 is supposed to charge r, and firm 2 is supposed to charge $r - \rho$, independent of their costs; and $k = n$ is replaced with $k = n - 1$.

For the sake of simplicity, we will sometimes disregard ρ, assuming that this is an infinitesimal number.

- **The description of strategies off the equilibrium path**:

We will assume that when a firm charges a price other than r or $r - \rho$, or does not charge the prescribed price in states in which $k = -n$ or n, the firms switch to playing a "bad" equilibrium, in which both firms obtain relatively low payoffs. The bad equilibrium used in this section can be, for example, the worst carrot and stick equilibrium from Athey and Bagwell (2008).

In Athey and Bagwell's carrot and stick equilibria, there are two states. In the war state, all firms choose a price γ lower than r; and in the reward state, all firms charge price r. Firms begin in the war state. In the war state, if all firms choose price $\gamma < r$, the firms switch to the reward state with a probability μ, and return to the war state with the remaining probability. In the reward state, if all firms choose price r, the firms remain in the reward state with probability 1. In each period, if any firm charges a price other than the prescribed price, the firms switch to the war state with probability 1.

Athey and Bagwell study the case in which both firms are Markovian. However, the equilibrium conditions are also satisfied when one of the firms is i.i.d. The off-equilibrium payoff of each firm, when the discount factor converges to 1, is bounded by $r/2 - \underline{c}/2 - \overline{c}2$, independent of the current type of Markovian firm. In turn, we will show that the on-equilibrium payoff of each firm in each state converges to the larger efficient payoff, so firms have incentives to stay on the equilibrium path.

- **Continuation payoffs**:

Assuming that both firms play the prescribed strategies, denote the state-dependent continuation payoff of firm 2 by $V_{k,L}$ and $V_{k,H}$, respectively. These continuation payoffs are computed before firm 2 learns about its cost type in the current period. For $k \in \{-n+1, ..., n-1\}$, we have:

$$\begin{aligned} V_{k,L} =& \frac{p}{2}\left\{(1-\delta)[r-\underline{c}] + \delta V_{k-1,L}\right\} + \frac{p}{2}\left\{(1-\delta)\left[\frac{r-\underline{c}}{2}\right] + \delta V_{k,L}\right\} + \\ &+ \frac{1-p}{2}\left\{(1-\delta)\left[\frac{r-\overline{c}}{2}\right] + \delta V_{k,H}\right\} + \frac{1-p}{2}\delta V_{k+1,H}. \end{aligned}$$

Indeed, the first component of the right-hand side represents the payoff contingent on the type of firm 2 being H and the type of firm 1 being L; the remaining components represent type profiles (H,H), (L,L), and (L,H), respectively.

We will set $\varepsilon = 1-\delta$, and rewrite the formula for $V_{k,L}$ as:

$$\begin{aligned} V_{k,L} =& \varepsilon\left\{\frac{1+2p}{4}r - \frac{3p}{4}\underline{c} - \frac{1-p}{4}\overline{c}\right\} + \\ &+ (1-\varepsilon)\left\{\frac{p}{2}V_{k-1,L} + \frac{p}{2}V_{k,L} + \frac{1-p}{2}V_{k,H} + \frac{1-p}{2}V_{k+1,H}\right\}. \end{aligned}$$

By substituting $1-p$ for p, we obtain:

$$\begin{aligned} V_{k,H} =& \varepsilon\left\{\frac{3-2p}{4}r-\frac{3-3p}{4}\underline{c}-\frac{p}{4}\overline{c}\right\}+ \\ & +(1-\varepsilon)\left\{\frac{1-p}{2}V_{k-1,L}+\frac{1-p}{2}V_{k,L}+\frac{p}{2}V_{k,H}+\frac{p}{2}V_{k+1,H}\right\}. \end{aligned}$$

Letting

$$\overline{b}=\frac{1+2p}{4}r-\frac{3p}{4}\underline{c}-\frac{1-p}{4}\overline{c} \text{ and } \underline{b}=\frac{3-2p}{4}r-\frac{3-3p}{4}\underline{c}-\frac{p}{4}\overline{c},$$

we have

$$V_{k,L}=\varepsilon\overline{b}+(1-\varepsilon)\left\{\frac{p}{2}V_{k-1,L}+\frac{p}{2}V_{k,L}+\frac{1-p}{2}V_{k,H}+\frac{1-p}{2}V_{k+1,H}\right\} \tag{1}$$

and

$$V_{k,H}=\varepsilon\underline{b}+(1-\varepsilon)\left\{\frac{1-p}{2}V_{k-1,L}+\frac{1-p}{2}V_{k,L}+\frac{p}{2}V_{k,H}+\frac{p}{2}V_{k+1,H}\right\}. \tag{2}$$

For $k=-n$ and n, we have:

$$V_{-n,L}=(1-\varepsilon)\left\{pV_{-n+1,L}+(1-p)V_{-n+1,H}\right\},$$

$$V_{n,H}=\varepsilon\{r-(1-p)\underline{c}-p\overline{c}\}+(1-\varepsilon)\left\{(1-p)V_{n-1,L}+pV_{n-1,H}\right\}. \tag{3}$$

The formulas for $V_{-n,L}$ and $V_{n,H}$ are the same as the formulas for $V_{-n,H}$ and $V_{n,L}$, except for different probabilities of $\overline{c}$ and $\underline{c}$ (that is, p must be replaced with $1-p$ and $1-p$ with p in the formulas for $V_{-n,H}$ and $V_{n,L}$).

2.1 Payoff Efficiency of Prescribed Strategies

- **We will first recursively derive the relation between $V_{k,L}$ and $V_{k+1,H}$.**

We will disregard the expressions of an order lower than ε. That is, our formulas will be approximate. For example, we will replace expression $(1-\varepsilon)^2$ with expression $1-2\varepsilon$. This will make the analysis tractable. We begin with $k=-n+1$.

By plugging the formula for $V_{-n,L}$ into the formula for $V_{-n+1,L}$, we obtain:

$$\begin{aligned} V_{-n+1,L} =& \varepsilon\overline{b}+(1-\varepsilon)\left\{\frac{p}{2}(1-\varepsilon)[pV_{-n+1,L}+(1-p)V_{-n+1,H}]\right\}+ \\ & +(1-\varepsilon)\left\{\frac{p}{2}V_{-n+1,L}+\frac{1-p}{2}V_{-n+1,H}+\frac{1-p}{2}V_{-n+2,H}\right\} \end{aligned}$$

$$= \varepsilon \overline{b} + V_{-n+1,L}\left[\frac{p^2}{2} + \frac{p}{2} - \varepsilon\left(p^2 + \frac{p}{2}\right)\right] +$$

$$+V_{-n+1,H}\left[\left(\frac{1-p}{2}\right)(1+p-\varepsilon(1+2p))\right] + V_{-n+2,L}\left(\frac{1-p}{2}\right)(1-\varepsilon).$$

Similarly, by plugging the formula for $V_{-n,L}$ into the formula for $V_{-n+1,H}$, we obtain:

$$V_{-n+1,H} = \varepsilon \underline{b} + V_{-n+1,L}\left[\frac{1-p^2}{2} - \varepsilon(1-p)\left(p + \frac{1}{2}\right)\right] +$$

$$+V_{-n+1,H}\left[\frac{1-p+p^2}{2} - \varepsilon\left(1 - \frac{3}{2}p + p^2\right)\right] + V_{-n+2,H}\frac{p}{2}(1-\varepsilon).$$

This formula enables us to compute $V_{-n+1,H}$ as a function of $V_{-n+1,L}$ and $V_{-n+2,H}$, as follows:

$$V_{-n+1,H} = \frac{2\varepsilon\underline{b}}{1+p-p^2} + V_{-n+1,L}\frac{1-p^2}{1+p-p^2}\left(1 - \varepsilon\frac{3+2p}{(1+p)(1+p-p^2)}\right) +$$

$$+V_{-n+2,H}\frac{p}{1+p-p^2}\left(1 - \varepsilon\frac{3-2p+p^2}{1+p-p^2}\right).$$

Plugging this formula into the formula for $V_{-n+1,L}$, we obtain the following relation between $V_{-n+1,L}$ and $V_{-n+2,H}$:

$$V_{-n+1,L} = 2\frac{\varepsilon\overline{b}(1+p-p^2) + \varepsilon\underline{b}(1-p^2)}{(1-p)(1+2p)} + V_{-n+2,H}\left(1 - \varepsilon\frac{5+2p-4p^2}{(1-p)(1+2p)}\right).$$

Notice that this expression has the form:

$$V_{-n+1,L} = A\varepsilon + V_{-n+2,H}\left(1 - \varepsilon B\right),$$

where expressions

$$A = 2\frac{\overline{b}(1+p-p^2) + \underline{b}(1-p^2)}{(1-p)(1+2p)} \text{ and } B = \frac{5+2p-4p^2}{(1-p)(1+2p)}$$

are independent of ε.

We will show by induction that

$$V_{-n+k,L} = A\varepsilon + (k-1)[2\varepsilon\overline{b} + 2\varepsilon\underline{b}] + V_{-n+k+1,H}\{1 - \varepsilon[4(k-1) + B]\}. \tag{4}$$

The inductive step follows the same lines as the derivation of $V_{-n+1,L}$. We replace $V_{-n+k-1,L}$ in formula (1) for $V_{-n+k,L}$ with the expression (4). This yields a relation between

$V_{-n+k,L}$, $V_{-n+k,H}$, and $V_{-n+k+1,H}$. Similarly, by replacing $V_{-n+k-1,L}$ in formula formula (2) for $V_{-n+k,H}$ with expression (4), we express $V_{-n+k,H}$ as a function of $V_{-n+k,L}$, and $V_{-n+k+1,H}$.

Plugging this last expression into the previously obtained relation between $V_{-n+k,L}$, $V_{-n+k,H}$, and $V_{-n+k+1,H}$, we obtain formula (4) for $V_{-n+k,L}$.

- **We will now compute $V_{n,H}$.**

Letting

$$a = r - (1-p)\underline{c} - p\bar{c},$$

and using the formula (4) for $V_{n-1,L}$ and $V_{n-2,L}$, we obtain from (2) for $k = n-1$:

$$V_{n-1,H} = \varepsilon\underline{b} + (1-\varepsilon)\frac{p}{2}V_{n-1,H} + (1-\varepsilon)\frac{p}{2}V_{n,H} +$$

$$\begin{aligned}
&+(1-\varepsilon)\frac{1-p}{2}\{A\varepsilon + (2n-3)(2\varepsilon\bar{b} + 2\varepsilon\underline{b}) + V_{n-1,L}[1-\varepsilon(8n-12+B)]\} \\
&+(1-\varepsilon)\frac{1-p}{2}\{A\varepsilon + (2n-2)(2\varepsilon\bar{b} + 2\varepsilon\underline{b}) + V_{n,L}[1-\varepsilon(8n-8+B)]\} \\
&\qquad = \varepsilon\underline{b} + (1-p)[A\varepsilon + (4n-5)(\varepsilon\bar{b} + \varepsilon\underline{b})] +
\end{aligned}$$

$$\begin{aligned}
&+\frac{1}{2}V_{n-1,H}\left[1-\varepsilon-\varepsilon(1-p)(8n-12+B)\right] + \\
&+\frac{1}{2}V_{n,H}\left[1-\varepsilon-\varepsilon\,(1-p)(8n-8+B)\right].
\end{aligned}$$

This yields

$$V_{n-1,H} = 2\varepsilon\underline{b} + 2(1-p)[A\varepsilon + (4n-5)(\varepsilon\bar{b} + \varepsilon\underline{b})] + V_{n,H}[1-2\varepsilon-\varepsilon\,(1-p)\,(16n-20+2B)].$$

Plugging this expression and (4) for $k = 2n-1$ into (3), we obtain:

$$\begin{aligned}
V_{n,H} = {}& \varepsilon a + (1-p)(1+2p)A\varepsilon + 2(1-p)[2n-2+p(4n-5)](\varepsilon\bar{b} + \varepsilon\underline{b}) + 2p\varepsilon\underline{b} + \\
&+V_{n,H}\left[1-\varepsilon\left\{(1-p)(8n-7+B) + p[(1-p)(16n-18+2B) + 2p+1]\right\}\right],
\end{aligned}$$

which yields

$$V_{n,H} = \frac{a + (1-p)(1+2p)A + 2(1-p)[2n-2+p(4n-5)](\bar{b} + \underline{b}) + 2p\underline{b}}{(1-p)(8n-7+B) + p[(1-p)(16n-18+2B) + 2p+1]}.$$

This last formula leads us to the following claim:

Claim 1 *The value $V_{n,H}$ is independent of ε, and*

$$\lim_{n\to\infty} V_{n,H} = \frac{1}{2}(\overline{b} + \underline{b}) = \frac{r}{2} - \frac{3}{8}\underline{c} - \frac{1}{8}\overline{c}.$$

In order to compute the limit, divide the numerator and the denominator by n, and remove the expressions of order $1/n$.

By applying (4) and (3), and then (1) and (2) recursively, we obtain:

Claim 2 *For every $k \in \{-n+1, ..., n-1\}$,*

$$\lim_{\varepsilon\to 0} V_{k,H} = \lim_{\varepsilon\to 0} V_{k,L} = V_{n,H},$$

and

$$\lim_{\varepsilon\to 0} V_{-n,L} = V_{n,H}.$$

Thus,

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} V_{0,L} = \lim_{n\to\infty}\lim_{\varepsilon\to 0} V_{0,H} = \frac{1}{2}(\overline{b} + \underline{b}) = \frac{r}{2} - \frac{3}{8}\underline{c} - \frac{1}{8}\overline{c}. \tag{5}$$

The analysis of the state-dependent continuation payoff of firm 1 is analogous. Denoting the payoffs by $V'_{k',L'}$ and $V'_{k',H'}$, where $k' = -k$, $L' = H$, and $H' = L$, we obtain identical formulas, except that a is replaced with

$$a' = r - \frac{1}{2}\underline{c} - \frac{1}{2}\overline{c},$$

and $\underline{b}$ and $\overline{b}$ are replaced with

$$\underline{b'} = \frac{3-2p}{4}r - \frac{2-p}{4}\underline{c} - \frac{1-p}{4}\overline{c} \text{ and } \overline{b'} = \frac{1+2p}{4}r - \frac{1+p}{4}\underline{c} - \frac{p}{4}\overline{c},$$

respectively. All the conclusions regarding the value functions are the same, since

$$\underline{b'} + \overline{b'} = r - \frac{3}{4}\underline{c} - \frac{1}{4}\overline{c}.$$

From this observation, together with (5), it follows that as $n \to \infty$, our strategies approximate the efficient total payoff.

2.2 Incentive Constraints of Markovian Firm

We will now turn to verifying the incentive constraints. It is sufficient to check the constraints for sufficiently large n's. In the extreme states, when $k = -n$ or n, they follow from the fact that

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0} V_{-n,L} = \lim_{n\to\infty}\lim_{\varepsilon\to 0} V_{n,H} = \frac{r}{2} - \frac{3}{8}\underline{c} - \frac{1}{8}\overline{c}.$$

This limit is higher than the payoff in the bad continuation equilibrium, which is going to be played contingent on deviations. Similarly, no firm has an incentive to deviate to an off-equilibrium action for any other value of k.

Consider a state with $-n < k < n$, and the incentive constraints of firm 2. Suppose first that the cost of the firm is $\underline{c}$.

If the firm honestly "reports" the cost, its payoff will be

$$\frac{1}{2}\left[\varepsilon(r-\underline{c}) + (1-\varepsilon)V_{k-1,L}\right] + \frac{1}{2}\left[\varepsilon\frac{(r-\underline{c})}{2} + (1-\varepsilon)V_{k,L}\right].$$

By reporting $\overline{c}$, the firm gets

$$\frac{1}{2}\left[\varepsilon\frac{(r-\underline{c})}{2} + (1-\varepsilon)V_{k,L}\right] + \frac{1}{2}(1-\varepsilon)V_{k+1,L}.$$

Thus, the incentive constraint reduces to

$$\varepsilon(r-\underline{c}) \geq (1-\varepsilon)(V_{k+1,L} - V_{k-1,L}).$$

- **We will express $V_{-n+k+1,L} - V_{-n+k-1,L}$ as a function of $V_{-n+k,L}$.**

By (4),

$$V_{-n+k+1,H} = -A\varepsilon - (k-1)(2\varepsilon\overline{b} + 2\varepsilon\underline{b}) + V_{-n+k,L}\{1+\varepsilon[4(k-1)+B]\}. \tag{6}$$

Remember that we are omitting the expressions of order ε^2 or lower. In particular,

$$\{1-\varepsilon[4(k-1)+B]\}^{-1} = 1+\varepsilon[4(k-1)+B].$$

Similarly,

$$V_{-n+k+2,H} = -A\varepsilon - k(2\varepsilon\overline{b} + 2\varepsilon\underline{b}) + V_{-n+k+1,L}[1+\varepsilon(4k+B)]. \tag{7}$$

By (1),

$$V_{-n+k+1,L} = \varepsilon\overline{b} +$$
$$+(1-\varepsilon)\left(\frac{p}{2}V_{-n+k,L} + \frac{p}{2}V_{-n+k+1,L} + \frac{1-p}{2}V_{-n+k+1,H} + \frac{1-p}{2}V_{-n+k+2,H}\right).$$

Plugging(6) and (7) into this last formula, we obtain:

$$V_{-n+k+1,L} = \varepsilon\overline{b} - \frac{1-p}{2}[A\varepsilon + (k-1)(2\varepsilon\overline{b} + 2\varepsilon\underline{b})] - \frac{1-p}{2}[A\varepsilon + k(2\varepsilon\overline{b} + 2\varepsilon\underline{b})] +$$

$$+V_{-n+k,L}\left\{\frac{1}{2} + \varepsilon\left[-\frac{p}{2} + \frac{1-p}{2}(4k-5+B)\right]\right\} +$$
$$+\frac{p}{2}(1-\varepsilon)V_{-n+k+1,L} + \frac{1-p}{2}V_{-n+k+1,L}[1+\varepsilon(4k-1+B)],$$

which yields

$$V_{-n+k+1,L} = 2\varepsilon\overline{b} - (1-p)[2A\varepsilon + (2k-1)\left(2\varepsilon\overline{b} + 2\varepsilon\underline{b}\right)] +$$
$$+V_{-n+k,L}\{1+\varepsilon\left[-2p + (1-p)(8k-6+2B)\right]\}.$$

Take the analogous expression for $V_{-n+k,L}$, divide it by $(1+\varepsilon[-2p+ (1-p)(8(k-1) -6+2B)])$, or equivalently, multiply it by $(1-\varepsilon[-2p+ (1-p)(8(k-1)- 6+2B)])$. (Recall that we omit expressions of order ε^2 or lower.) This yields

$$V_{-n+k-1,L} = -2\varepsilon\overline{b} + (1-p)[2A\varepsilon + (2k-3)(2\varepsilon\overline{b} + 2\varepsilon\underline{b})] +$$
$$+V_{-n+k,L}\{1-\varepsilon\left[-2p + (1-p)(8k-14+2B)\right]\}.$$

Thus,

$$V_{-n+k+1,L} - V_{-n+k-1,L} =$$

$$= 4\varepsilon\overline{b} - (1-p)[4A\varepsilon + (4k-4)(2\varepsilon\overline{b} + 2\varepsilon\underline{b})] +$$
$$+V_{-n+k,H}\varepsilon\left[-4p + (1-p)(16k-20+4B)\right].$$

This last expression increases or decreases in k when $2V_{-n+k,L} > \overline{b} + \underline{b}$ or $2V_{-n+k,L} < \overline{b} + \underline{b}$, respectively.

- **We will determine the sign of $2V_{-n+k,L} - (\overline{b} + \underline{b})$.**

In the limit as $\varepsilon \to 0$,

$$V_{-n+k,L} = \frac{a + (1-p)(1+2p)A + 2(1-p)[2n-2+p(4n-5)](\overline{b}+\underline{b}) + 2p\underline{b}}{(1-p)(8n-7+B) + p[(1-p)(16n-18+2B)+2p+1]},$$

by Claim 2. Notice that the denominator of this expression is positive. Therefore, the sign of $2V_{-n+k,L} - (\overline{b}+\underline{b})$ (in the limit) coincides with

$$\{a + (1-p)(1+2p)A + 2(1-p)[2n-2+p(4n-5)](\overline{b}+\underline{b}) + 2p\underline{b}\} -$$
$$-\{(1-p)(8n-7+B) + p[(1-p)(16n-18+2B)+2p+1]\}(\overline{b}+\underline{b})/2.$$

Simple algebra shows that this is equal to $a - (\overline{b}+\underline{b}) < 0$.

Thus, the value of $V_{-n+k+1,L} - V_{-n+k-1,L}$ decreases with k, and it suffices now to verify the incentive constraint for $k = 1$. By applying the formula for $V_{-n+k+1,L} - V_{-n+k-1,L}$ in the case of $k = 1$, we obtain

$$\frac{V_{-n+2,L} - V_{-n,L}}{\varepsilon} = 4\overline{b} - (1-p)4A + V_{-n+1,L}\left[-4p + (1-p)(-4+4B)\right],$$

and since

$$\lim_{n\to\infty} V_{-n+1,L} = \frac{\overline{b}+\underline{b}}{2},$$

we have (also in the limit as $n \to \infty$)

$$\frac{V_{-n+2,L} - V_{-n,L}}{\varepsilon} = 4\overline{b} - 8\frac{\overline{b}(1+p-p^2) + \underline{b}(1-p^2)}{(1+2p)} +$$

$$+\frac{\overline{b}+\underline{b}}{2}\left[-4p + -4(1-p) + 4\frac{5+2p-4p^2}{(1+2p)}\right] = \frac{4\overline{b}}{(1+2p)}.$$

Finally, $4\overline{b}/(1+2p) < (r - \underline{c})$ is equivalent to $(1-p)\underline{c} < (1-p)\overline{c}$, so the incentive constraint is satisfied.

Suppose now that the cost of firm 2 is $\overline{c}$. Recall that $-n < k < n$.

If the firm honestly "reports" the cost, its payoff will be

$$\frac{1}{2}(1-\varepsilon)V_{k+1,H} + \frac{1}{2}\left[\varepsilon\frac{(r-\overline{c})}{2} + (1-\varepsilon)V_{k,H}\right].$$

By reporting $\underline{c}$, the firm gets

$$\frac{1}{2}\left[\varepsilon\frac{(r-\overline{c})}{2}+(1-\varepsilon)V_{k,H}\right]+\frac{1}{2}[\varepsilon(r-\overline{c})+(1-\varepsilon)V_{k-1,H}].$$

This time the incentive constraint reduces to

$$\varepsilon(r-\overline{c})\leq(1-\varepsilon)(V_{k+1,H}-V_{k-1,H}).$$

By (4), we represent $V_{k+1,H}-V_{k-1,H}$ as

$$V_{k+1,H}-V_{k-1,H}=(V_{k,L}-V_{k-2,L})\{1+\varepsilon[4(n+k-1)+B]\}+8V_{k-2,L}\varepsilon-2(2\varepsilon\overline{b}+2\varepsilon\underline{b}). \quad (8)$$

Since (i) $8V_{k-2,L}\varepsilon-2(2\varepsilon\overline{b}+2\varepsilon\underline{b})$ is close to 0 for large enough values of n, (ii) the sign of $V_{k+1,H}-V_{k-1,H}$ is determined by $V_{k,L}-V_{k-2,L}$, and (iii) this last term is decreasing in k, it suffices to check the incentive constraint just for $k=n-1$.

The value of $V_{n,H}-V_{n-2,H}$ can be directly computed, as we have already computed $V_{n-1,L}-V_{n-3,L}$. In the limit as $n\to\infty$, we obtain

$$\frac{V_{n,H}-V_{n-2,H}}{\varepsilon}=\frac{-4\underline{b}+4a}{(1+2p)},$$

and since $(-4\underline{b}+4a)/(1+2p)>\varepsilon(r-\overline{c})$ is equivalent to $(1-p)\underline{c}<(1-p)\overline{c}$, the incentive constraint is again satisfied.

2.3 Incentive Constraints of i.i.d. Firm

The Markovian firm's cost in the previous period does not affect the Markovian firm's incentive constraint, since it knows its current cost before deciding which price to charge, and it faces an i.i.d. opponent with equal probability of having a high or low cost. Thus, for any $k\neq n,-n$, there are only two incentive constraints that must be satisfied for the Markovian firm. The Markovian firm's cost in the previous period does, however, affects the incentive constraints of the i.i.d. firm, since the i.i.d. firm's belief about the Markovian firm's cost depends on the prior state. Thus, for any $k\neq n,-n$, there are four incentive constraints that must be satisfied by the i.i.d. firm.

Consider a state with $k\neq n$ or n, and suppose that the Markovian firm's cost in the previous period was $\underline{c}$. Suppose also that the current cost of the i.i.d. firm is also $\underline{c}$.

If the i.i.d. firm honestly "reports" its cost, its payoff will be

$$(1-p)[\varepsilon(r-\underline{c})+(1-\varepsilon)V'_{k+1,H}]+p\left[\varepsilon\frac{(r-\underline{c})}{2}+(1-\varepsilon)V'_{k+1,L}\right].$$

If the i.i.d. firms lies about its cost, it gets

$$(1-p)\left[\varepsilon\frac{(r-\underline{c})}{2}+(1-\varepsilon)V'_{k,H}\right]+p(1-\varepsilon)V'_{k-1,L}.$$

Letting $L'=H$ and $H'=L$, and reversing the chip count (i.e., a positive k represents a surplus of chips for firm 1), the incentive constraint reduces to

$$\frac{1}{2}\varepsilon(r-\underline{c})\geq(1-\varepsilon)[p(V'_{k+1,H'}-V'_{k,H'})+(1-p)(V'_{k,L'}-V'_{k-1,L'})].$$

Similarly, the incentive constraint for the i.i.d. firm with low cost and prior state H is

$$\frac{1}{2}\varepsilon(r-\underline{c})\geq(1-\varepsilon)[(1-p)(V'_{k+1,H'}-V'_{k,H'})+p(V'_{k,L'}-V'_{k-1,L'})];$$

the incentive constraint for the i.i.d. firm with high cost and prior state L is

$$\frac{1}{2}\varepsilon(r-\underline{c})\leq(1-\varepsilon)[p(V'_{k+1,H'}-V'_{k,H'})+(1-p)(V'_{k,L'}-V'_{k-1,L'})];$$

and the incentive constraint for the i.i.d. firm with high cost and prior state H is

$$\frac{1}{2}\varepsilon(r-\underline{c})\leq(1-\varepsilon)[(1-p)(V'_{k+1,H'}-V'_{k,H'})+p(V'_{k,L'}-V'_{k-1,L'})].$$

Notice that each incentive constraint contains the two expressions $V'_{k+1,H'}-V'_{k,H'}$ and $V'_{k,L'}-V'_{k-1,L'}$. With the change of variables, we can use earlier results to show (in a similar manner to the "Markovian" case) that both expressions decrease with k. This implies that each incentive constraint must be verified only for $k=-n+1$ or $n-1$ (depending on which of the four constraints we are verifying). We can do this directly, in a manner similar to the Markovian case.