# Large Contests

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**Abstract**

We consider contests with many, possibly heterogeneous, players and prizes, and show that the equilibrium outcomes of such contests are approximated by the outcomes of an appropriately defined set of mechanisms.

This makes it possible to easily approximate of the equilibria of contests whose exact equilibrium characterization is complicated, as well as the equilibria of contests for which there is no existing equilibrium characterization. We apply the results to derive the effort-maximizing prize structure given a budget, and to investigate the effect of changing the set of competitors on their equilibrium effort.

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# 1 Introduction

In many settings, agents compete for prizes by expending resources. Some of these settings, such as lobbying, political campaigns, competitions for promotions, and research and development races, typically involve a small number of competitors and prizes; other settings, such as college admissions, grant competitions, competitions by employers for workers on a national or international level (e.g., hospitals for residents), sales competitions in large firms, and certain sports competitions (e.g., marathons) involve many competitors and prizes.[1]

The competitors often differ in their abilities, technologies, access to capital, and prior investments. They may also receive differential treatment in the contest, and their valuations for a given prize may differ. In addition, the prizes may be heterogeneous - colleges differ in their prestige, grants differ in their size, jobs differ in their characteristics, etc. But solving game-theoretic models with asymmetric players and heterogeneous prizes has proven difficult, which has limited our understanding of contests. In particular, contest design questions, such as identifying the effort-maximizing prize distribution given a pool of ex-ante asymmetric competitors, are not well understood.[2]

This paper shows that the equilibrium outcomes of contests with a large (but finite) number of competitors and prizes can be approximated by certain incentive-compatible (IC) and individually rational (IR) mechanisms. The approximation applies even when solving for equilibrium is difficult or impossible, which substantially expands the set of contests that can be studied. This makes it possible to address previously intractable comparative statics and contest design questions by translating them to tractable mechanism design questions. We illustrate this in a flexible asymmetric contest setting by characterizing the effort-maximizing prize distribution and the effect that changing the set of competitors has on the effort they exert.

In our contest framework, $n$ players compete for $n$ prizes (some of which may be worth 0). Each player chooses a non-negative bid, and the player with the highest bid obtains the

[1] In 2012, 4-year colleges in the US received more than 8 mln applications and enrolled approximately 1.5 mln freshmen. In each of the last several years, the National Science Foundation received more than 40,000 grant applications and awarded more than 10,000 grants. In 2012, the National Residency Matching Program, which has been modeled as a contest by Bulow and Levin (2006) (see Section 3), had close to 40,000 doctors who applied to more than 25,000 positions. Cisco, which has more than 15,000 partners in the US, holds several sales competitions among its partners. Between 2010 and 2012, Tokyo, London, New York, Chicago, and Sydney each hosted a marathon with more than 30,000 participants.

[2] For example, Moldovanu and Sela's (2001) important contribution considered only whether one prize or multiple prizes are optimal in an environment with ex-ante symmetric contestants.

highest prize, etc. A player's payoff depends on his bid, his type, and the prize she obtains, and decreases continuously in her bid. This accommodates a wide range of asymmetries among players and heterogeneity in prizes. Players' types are distributed independently, but not necessarily identically, which accommodates both complete-information and incomplete-information asymmetric contests.

We study sequences of contests whose empirical distributions of player types and prizes converge as $n$ grows large. Given the limit type and prize distributions, the approximating IC-IR mechanisms allocate the available prizes to the continuum of agents in a way that is consistent with an "inverse tariff," in which each bid determines a single prize.

The intuition for the approximation is as follows. In any contest, a player can bid 0 and secure the lowest prize. This guarantees IR in the $n$-th contest and implies IR in the limit. As the number of players increases, the mappings from bids to distributions of the bids' percentile rankings (induced by the other players' equilibrium strategies) become similar across players, and coincide in the limit. This is because the rankings of two players who make the same bid differ by at most 1. This "almost" implies that players' mappings from bids to distributions of prizes become similar, and coincide in the limit. Moreover, by a law of large numbers, the common limit mapping from bids to prizes is deterministic, so each bid maps to a single prize. This yields an inverse tariff such that, in the limit, agents choose bids and obtain corresponding prizes, as in a mechanism-design setting. In particular, the mechanism defined by the inverse tariff is IC.

This intuition is incomplete, however, since it assumes that the equilibrium outcomes converge. In addition, for some bids the distributions of prizes may not be similar across players even when the percentile rankings are. This is what happens in the setting of Section 3.2, in which there are half as many identical prizes as players and the limit percentile ranking of a bid $t$ is $1/2$. By bidding slightly above $t$ some players obtain a prize with relatively high probability and other players obtain a prize with relatively low probability, even when the number of players is large. Finally, for some contest specifications the notion of approximation implied by the intuition is substantially weaker than the one we are able to obtain.

Our most general, but weakest, result establishes convergence in weak*-topology of the equilibrium distributions over player types, prizes, and bids as $n$ grows large to a distribution over agent types, prizes, and bids that corresponds to a mechanism. This convergence approximates the average equilibrium distribution over prize-bid pairs of the player types that are close to a given agent type, but this may not be a good approximation of the equilibrium strategy of any single player.

Under a strict single crossing condition on the players' payoff we establish a much stronger form of convergence, which delivers a simultaneous approximation of all players' equilibrium strategies. In this case, as $n$ grows large players' equilibrium strategies become almost deterministic, and the resulting allocation of prizes approaches the assortative one. When payoffs are also quasi-linear with respect to bids, the allocation is efficient, and the unique mechanism that implements it is characterized by applying standard mechanism-design techniques. We are then able to say (approximately, but with an arbitrary degree of precision as $n$ increases and uniformly across all equilibria) how each player will bid, and what prize she will obtain by making any given bid. This result applies to many existing contest models, which are surveyed in Section 1.1, as well as to contest specifications for which no equilibrium characterization exists.

We use this result to derive the prize distribution that maximizes the aggregate bids in a large contest. When the prize budget is sufficiently large, it is optimal to award a limited number of the largest possible prizes. This continues to be true for any prize budget when players' marginal prize utility is weakly increasing. But when players' marginal prize utility is decreasing, the optimal prize distribution with a limited budget includes a range of prizes, and may also include a number of the largest possible prizes. We also show that for any prize distribution, a first-order stochastic dominance shift of the players' type distribution increases the aggregate bids.

The rest of the paper is organized as follows. Section 1.1 surveys the related literature. Section 2 introduces the basic terminology and notation. Section 3 presents some examples that illustrate our results in some settings that have been studied in the literature. Section 4 contains our main results and some discussion of their application. Section 5 develops the contest design application. Section 6 contains the proofs of our main results when the limit set of prizes has full support. Section 7 concludes. Appendix A contains the proofs of intermediate results from Section 6. Appendix B contains the proofs of our main results when the limit set of prizes may not have full support.

## 1.1 Related Literature

Our model includes many variants of the multi-prize all-pay auction with complete and incomplete information, in which each player chooses a bid and pays the associated (and possibly idiosyncratic) cost.[3] Closed-form equilibrium characterizations exist for complete-information contests with two participating players (Hillman and Samet (1987), Hillman and

[3]Other models of competition postulate a probabilistic relation between competitors' efforts and prize allocation. See Tullock (1980) and Lazear and Rosen (1981). For a comprehensive treatment of the literature on competitions with sunk investments, see Nitzan (1994) and Konrad (2007).

Riley (1989), Che and Gale (1998, 2006), Kaplan and Wettstein (2006), Siegel (2010)), with identical prizes and costs (Baye, Kovenock, and de Vries (1993, 1996), González-Díaz (2012), Clark and Riis (1998)), and with identical players (Barut and Kovenock (1998)). Algorithmic equilibrium characterizations exist for some contests with identical prizes and heterogeneous costs (Siegel 2010, 2013b),[4] and with heterogeneous prizes and identical costs (Bulow and Levin (2006), González-Díaz and Siegel (2012), Xiao (2013)). The heterogeneity in prizes, however, is limited to very specific functional forms. Moreover, algorithmic characterizations make further analysis, such as comparative statics and contest design, difficult or impossible. Incomplete-information contests have been solved when there are two players (for example, Amann and Leininger (1996), Siegel (2013a)), and when players are ex-ante identical (for example, Krishna and Morgan (1997), Moldovanu and Sela (2001, 2006)).[5] In contrast, our model accommodates ex-ante asymmetric players, heterogeneous prizes, and complete and incomplete information.

Our paper also contributes to the literature on large games. This literature typically makes continuity assumptions that exclude auction-like games (for example, Kalai (2004) and Carmona and Podczeck (2009, 2010)). A notable exception is Bodoh-Creed (2012), who explicitly considers uniform-price auctions with incomplete information, but assumes enough uncertainty about the set of prizes to exclude the possibility of a small change in the rank order of a bid having a large effect on the prize obtained. Moreover, the analysis in this literature often focuses on $\varepsilon$-equilibria of large games, which may not approximate Nash equilibria well. In contrast, our approach deals with the discontinuities that arise naturally in contests, approximates Nash equilibria, and uncovers a novel connection to mechanism design.

A more closely related paper is Hickman's (2009) theoretical analysis of affirmative action in college admissions. He considers a quasi-linear contest model with incomplete information that satisfies strict single crossing, and approximates the outcome for a large number of applicants by a continuum model in which the limit set of prizes has full support (so a small change in the rank order of a bid cannot have a large effect on the prize obtained). Our paper differs from Hickman's (2009) work in three main ways. First, our approach does not require quasi-linearity, strict single crossing, or full support of the limit prize set (although we obtain stronger results under these conditions), and is therefore applicable to a wider

[4] Siegel (2009, 2013c) gives a closed-form expression for players' equilibrium payoffs, but does not solve for equilibrium.

[5] Parreiras and Rubinchik's (2010) characterize some equilibrium properties in a setting with ex-ante asymmetric players and one prize.

range of settings. Second, we relate the outcomes of large contests to mechanisms, which allows us, under certain conditions, to derive the approximation in closed form. Third, our model accommodates both complete information and incomplete information, and therefore includes many existing contest models as special cases.

Finally, there is a literature on matching and search that employs continuum models to approximate matching models with many participants. Azevedo and Leshno (2012) show that stable matchings are easy to find and are often unique in a two-sided matching model with a continuum of agents on one side. Che and Kojima (2010) show that the random priority mechanism and the probabilistic serial mechanism converge to each other as the number of copies of a given set of object types grows large. Hoppe, Moldovanu, and Sela (2009) consider assortative and random matching in a two-sided model with costly signaling and a continuum of agents on each side. Peters (2010) analyzes a model of directed search with a continuum of workers and firms, which is more tractable than the discrete version of the model.

# 2 Terminology and notation

## 2.1 Agents and prizes

An agent is characterized by a type $x \in X = [0,1]$. We will use the terms "player" for discrete contests and "agent" for the limit case. A prize is characterized by a number $y \in Y = [0,1]$. Prize 0 is "no prize."

Agents' utilities are given by a continuous function $U(x, y, t)$, where $x$ is the agent type, $y$ is the single prize he obtains, and $t \geq 0$ is his bid. The utility of obtaining no prize by bidding 0 is normalized to 0, i.e., $U(x, 0, 0) = 0$ for all $x$. Higher prizes are better and higher bids are more costly, so $U(x, y, t)$ strictly increases in $y$ for every $x > 0$ and $t \geq 0$, and strictly decreases in $t$ for every $x \geq 0$ and $y \geq 0$. The utility is *quasi-linear* in bid if it can be written as

$$U(x, y, t) = v(x, y) - t.$$

We assume that sufficiently high bids are prohibitively costly, so $U(x, 1, b_{\max}) < 0$ for some $b_{\max}$ and all $x$. We therefore restrict the range of bids that agents can make to $B = [0, b_{\max}]$. To simplify our proofs, we choose $b_{\max}$ to be rational.

We say that *weak single crossing* holds if for any $x_1 < x_2$, $t_1 < t_2$, and $y_1 < y_2$ we have that $U(x_1, y_2, t_2) \geq U(x_1, y_1, t_1)$ implies $U(x_2, y_2, t_2) \geq U(x_2, y_1, t_1)$. That is, if some type prefers to obtain a higher prize at a higher bid, then so does any higher type. If the higher type's preference is strict, i.e., the second inequality is strict, then we say that *strict single crossing* holds.

An example that we will use throughout the paper is the quasi-linear utility

$$U(x, y, t) = xh(y) - t, \tag{1}$$

where $h(0) = 0$ and $h$ is continuous and strictly increasing, and therefore satisfies strict single crossing. This functional form generalizes many of the ones used in existing contest models, including those described in Section 3.1 and Section 3.2.

## 2.2 Contests

For every $n$, we define "the $n$-th contest," in which $n$ players compete for $n$ known prizes $y_1^n \leq y_2^n \leq ... \leq y_n^n$ (some of which may be no prize). Player $i$'s privately-known type $x_i^n$ is distributed according to a CDF $F_i^n$, and these distributions are commonly known and independent across players.[6] In the special case of complete information, each CDF corresponds to a Dirac measure. In the contest, each player chooses a bid in $B$, the player with the highest bid obtains the highest prize, the player with the second-highest bid obtains the second-highest prize, and so on. Ties are resolved by a fair lottery. The utility of player $i$ from bidding $t$ and obtaining prize $y_j^n$ is $U\left(x_i^n, y_j^n, t\right)$. A slight adaptation of the proof of Corollary 1 in Siegel (2009) shows that when each player's set of possible types is finite the contest has at least one mixed-strategy equilibrium. For general distributions $F_i^n$, equilibrium existence follows from Corollary 5.2 in Reny (1999).[7]

We let $F^n = \left(\sum_{i=1}^n F_i^n\right)/n$, so $F^n(x)$ approximates the expected percentile ranking of type $x$ given the vector of players' types. We denote by $G^n$ the empirical distribution of prizes, which assigns a mass of $1/n$ to each $y_j^n$ (recall that the prize $y_j^n$ is known). We assume that $F^n$ converges in weak*-topology to a continuous and strictly increasing distribution $F$, and $G^n$ converges to some distribution $G$. We elaborate on this assumption in the next subsection. If $G$ strictly increases on $Y$, we say that $G$ *has full support*, or that *prizes have full support*. Note that $G$ may have full support even if there are masses of identical prizes, i.e., $G$ is discontinuous, or, equivalently, $G$ has atoms.

[6] All probability measures are defined on the $\sigma$-algebra of Borel sets.

[7] Better-reply security of the mixed extension is immediate for strategy profiles that do not involve ties for which winning the tie gives a better prize. For any other strategy profile and a corresponding point in the closure of the graph of vector payoffs, any player who is not winning the limit ties with probability 1 along a sequence of profiles and payoffs converging to the point in the closure can secure a payoff strictly higher than in the closure by increasing her tying bids slightly.

Given an equilibrium of the $n$-th contest, we denote by $D_i^n$ the distribution on $X \times Y \times B$ that describes player $i$'s type, the prize she obtains, and her bids. We refer to the distribution $D^n = \left(\sum_{i=1}^n D_i^n\right)/n$ as the *equilibrium outcome*, and later relate the sequence of distributions $D^1, D^2, \ldots$ to probability distributions $D$ that describe the outcomes of some mechanisms.

## 2.3 Convergence of type and prize distributions

The convergence of $F^n$ and $G^n$ to limit distributions $F$ and $G$ can be interpreted in several ways. First, a modeler studying large contests may specify the limit distributions directly and consider a sequence of discrete contests with distributions converging to the limit distributions. Examples include contests with complete-information in which player $i$'s type is $x_i^n = F^{-1}(i/n)$ and prize $j$ is characterized by $y_j^n = G^{-1}(j/n) = \inf\{z : G(z) \geq j/n\}$, and contests with the same prize structure and IID type distributions $F_i^n = F$. The former specification appears in the examples of Section 3.

Alternatively, the modeler may specify a sequence of contests using analytical formulas that depend on the number of players and prizes, and take the limit of the associated sequence of distributions $F^n$ and $G^n$ as the number grows large. Even if the limit of the entire sequence does not exist, every sequence contains a converging subsequence, and our methods can be applied to every converging subsequence for which the limit of $F^n$ is continuous and strictly increasing.

Finally, given a single discrete contest, a researcher can postulate limit distributions $F$ and $G$ to which the empirical type and prize distributions would converge if the number of players and prizes grew large; if the number of players and prizes in the given contest is sufficiently large, our approximation results can be applied. Moreover, in many settings (such as those in Corollaries 3 and 4 below) the empirical type and prize distributions of the given contest can be used as the limit distributions, since a small change in the limit distribution has a small effect on the approximation.[8]

## 2.4 Limit mechanism-design setting

A (direct) *mechanism* $M$ prescribes for each reported type $x$ a joint probability distribution $Q_x(y,t)$ over prizes and bids. A mechanism is *incentive compatible* (IC) if the expected

[8] In the settings of Corollaries 3 and 4, a small change in $F$ and $G$ leads to a small change in the prize-bid pair that the approximating mechanism specifies for each type (the pair is given by the "assortative allocation," defined in Section 4, and by (5)).

utility of each agent is maximized by reporting truthfully, i.e.,

$$\int_{y\in Y}\int_{t\in B} U(x,y,t)dQ_z(y,t)$$

is maximized at $z=x$.

A mechanism is *individually rational* (IR) if the expected utility of each agent from reporting truthfully is at least as high as the utility from bidding 0 and obtaining the "lowest" available prize, i.e.,

$$\int_{y\in Y}\int_{t\in B} U(x,y,t)dQ_x(y,t) \geq U\left(x, y_{\inf}, 0\right), \tag{2}$$

where $y_{\inf} = \inf\left\{y : G\left(y\right) > 0\right\}$; in addition, we require that the inequality is an equality for at least one type $x$. If prizes have full support, then $y_{\inf} = 0$, so the right-hand side of (2) is $U\left(x,0,0\right) = 0$.

An *inverse tariff* is a non-decreasing upper semi-continuous function that maps bids to prizes; a *tariff mechanism* is an IR mechanism for which there exists an inverse tariff such that for every type $x$ the distribution $Q_x(y,t)$ assigns probability 1 to the set of prize-bid pairs that maximize $U\left(x,y,t\right)$ among the prize-bid pairs in the graph of the inverse tariff. A tariff mechanism is clearly IC.

A *consistent allocation* is a probability distribution $H$ on $X \times Y$ whose marginal on $X$ coincides with $F$ and whose marginal on $Y$ coincides with $G$. With a continuum of agents and prizes distributed according to $F$ and $G$, this condition says that all the prizes are allocated to agents, and each agent obtains exactly one prize (which can be no prize). The conditional distribution $H_x$ is interpreted as the lottery over prizes faced by an agent of type $x$.

With quasi-linear utility, an allocation $H$ is *efficient* if it allocates the prizes in a way that maximizes the non linear part of the agents' utility, i.e., it maximizes

$$\int_{x\in X}\int_{y\in Y} v(x,y)dH(x,y)$$

across all consistent allocations.

A mechanism *implements* an allocation $H$ if the marginal of $Q_x$ on $Y$ coincides with $H_x$ for almost every type $x$. Distributions $H$ and $\{Q_x : x \in X\}$ may not determine a probability distribution on $X \times Y \times B$, because there may be no Borel probability distribution on $X \times Y \times B$ with such conditionals and marginals.[9] When such a distribution $D$ exists, which

[9]For example, take some non-measurable function $f : X \to [0,\infty)$, have $H$ distributed uniformly on, and assigning probability 1 to, the diagonal $\{(x,x) : x \in X\}$, and have $Q_x$ assigning probability 1 to the pair $(x, f\left(x\right))$. That is, type $x$ is prescribed prize $x$ and bid $f\left(x\right)$.

will be the case in all our results, we say that the mechanism that implements $H$ is *regular*, and refer to $D$ as the *outcome* of the mechanism.

Our results will show convergence of $D^n$ to $D$. The notion of convergence, however, will vary across the results. We will introduce the appropriate notions when we formulate our results.

# 3 Examples

We first demonstrate our approximation approach and main results in a few contest settings that appeared in the literature. We focus on complete-information contests, because no equilibrium characterization exists for contests with incomplete information and more than two ex-ante asymmetric players.

## 3.1 Heterogeneous prizes and multiplicative utilities

Consider (1) with $h(y) = y$, let $x_i^n = i/n$ (so $F_i^n$ is a Dirac measure), and let $y_j^n = j/n$. Thus, the limit distributions $F$ and $G$ are uniform. (Note that prizes have full support.) The $n$-th contest is an all-pay auction with $n$ players and $n$ prizes, and the value of prize $j$ to player $i$ is $ij/n^2$. Such contests were studied by Bulow and Levin (2006), hence forth B&L, who considered hospitals that compete for residents by offering identity-independent wages.[10] Hospitals are players, their posted wages are bids, and residents are prizes. The best resident goes to the hospital with the highest wage, and so on.

Consider the efficient allocation in the limit setting, in which an agent of type $x$ obtains prize $x$. The unique IC-IR mechanism that implements this allocation prescribes for every type $x$ bid $x^2/2$. This mechanism is a tariff mechanism with a continuous inverse tariff that maps every bid $t \in [0,1]$ to prize $\sqrt{2t}$.[11] Corollary 1 below shows that this mechanism approximates the equilibrium outcome for large $n$, in that every player $i$ obtains a prize close to $i/n$ and bids close to $(i/n)^2/2$.

This simple approximation contrasts with the elegant, but rather complicated, algorithm developed by B&L to derive the unique mixed-strategy equilibrium of the contest.

[10] In the notation of B&L, $\Delta_i = i/n^2$, as in their Proposition 5.

[11] It is easy to see that this mechanism, and those in later examples, are regular.

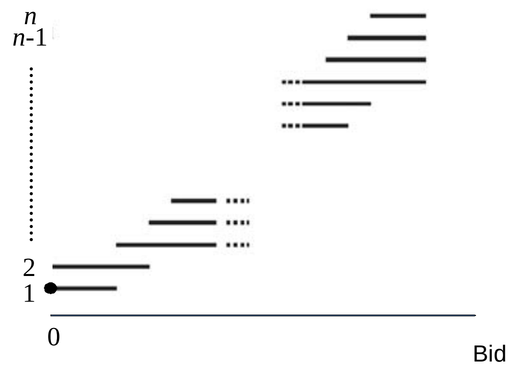


Figure 1: The support of players' strategies in the unique equilibrium

B&L show that, in equilibrium, each player chooses a bid from an interval. The intervals are staggered, so a higher player has an interval with (weakly) higher lower and upper bounds. (The intervals are depicted in Figure 1.) In particular, if a bid $t$ is contained in some player's bidding interval, then it is contained in the bidding intervals of consecutive players $l$, $l+1$,..., $m$, where $l$ is the lowest player whose interval contains $t$, and $m$ is the highest player whose interval contains $t$.

B&L show that

$$l = \arg\min_q \left\{ \frac{1}{m-q} \sum_{k=q}^{m} \frac{n^2}{k} - \frac{n^2}{q} > 0 \right\}, \tag{3}$$

and the density of the strategy of player $q$, $l \le q \le m$, at bid $t$ that belongs to her bidding interval is

$$\frac{1}{m-l} \sum_{k=l}^{m} \frac{n^2}{k} - \frac{n^2}{l}. \tag{4}$$

By iteratively applying (3) and (4), B&L compute the endpoints of players' bidding intervals and the densities of their bidding strategies.

For the rate of convergence of the approximation, because $m-l$ is of order $\sqrt{2l}$ (Lemma 3 in B&L), any player $i$ is outbid with certainty by every bidder $j > i$, except for a number of players $j$ that is of order $\sqrt{n}$. Thus, player $i$ obtains a prize that differs from $i/n$ by at most an expression of order $1/\sqrt{n}$. A similar, but slightly more involved, argument shows that the bidding interval of player $i$ shrinks quickly, so that any bid in the interval differs from $(i/n)^2/2$ by at most an expression of order $1/\sqrt{n}$.

### 3.2 Identical prizes

Consider (1) with $h(y) = y$, let $x_i^n = i/n$ (so $F_i^n$ is a Dirac measure), and let $y_j^n = 0$ if $j/n \leq 1/2$ and $y_j^n = 1$ if $j/n > 1/2$. Thus, the limit distribution $F$ is uniform, and the limit distribution $G$ has $G(y) = 1/2$ for all $y \in [0, 1)$ and $G(1) = 1$. (Note that $G$ does not have full support.) The $n$-th contest is an all-pay auction with $n$ players and $m \equiv \ulcorner n/2 \urcorner$ identical (non-zero) prizes,[12] and the value of a prize to player $i$ is $i/n$. Such contests were studied by Clark and Riis (1998), who considered competitions for promotions, rent seeking, and rationing by waiting in line (see also Siegel (2010)).

Consider the efficient allocation in the limit setting, which assigns a prize to agents with types higher than 1/2. The unique IC-IR mechanism that implements this allocation prescribes for every type $x \leq 1/2$ bid 0 and for every type $x > 1/2$ bid 1/2. This mechanism is a tariff mechanism with a discontinuous (but upper semi-continuous) inverse tariff that maps bids $t \in [0, 1/2)$ to prize 0 and bids $t \geq 1/2$ to prize 1. Corollary 2 below shows that this mechanism approximates the equilibrium outcome for large $n$, in that a large fraction of players $i$ with $i/n > 1/2$ obtain a prize and bid close to 1/2 with high probability, and a large fraction of players $i$ with $i/n < 1/2$ obtain no prize and bid close to 0 with high probability. The approximation applies to a large fraction of players, and not to all players as in Section 3.1, because $G$ does not have full support. This, however, makes little difference on the aggregate level (for welfare, total expected bids, etc.).

This simple approximation contrasts with the complete, but less than straightforward, closed-form equilibrium characterization derived by Clark and Riis (1998).

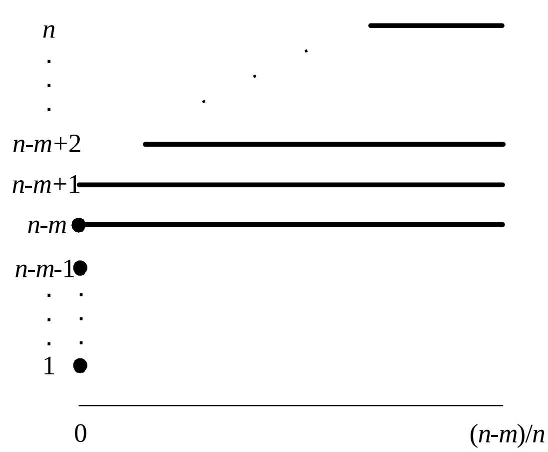


Figure 2: The support of players' strategies (dots represent atoms) in the unique equilibrium

[12]The analysis that follows applies to any fixed limit ratio $p \in (0, 1)$ of prizes to players.

As depicted in Figure 2, in the equilibrium of the $n$-th contest the $n-m-1$ players with the lowest valuations bid 0, and each of the $m+1$ players with the highest valuations bids on an interval, so $m$ of them obtain a prize. The common upper bound of the intervals is $(n-m)/n = 1 - m/n \in \{1/2, 1/2 + 1/2n\}$, and the lower bound of the interval of player $i > n - m$ is

$$\left(1 - \frac{m}{n}\right)\left(1 - \prod_{k=n-m+1}^{i} \frac{k}{i}\right),$$

which increases in $i$.

Thus, for every $\varepsilon > 0$, as $n$ grows large the number of players with valuations greater than 1/2 who bid on $[\varepsilon, 1 - m/n]$ grows large. This may appear to contradict the equilibrium approximation for large $n$. The apparent discrepancy is overcome by noting that the lower bound of the bidding interval of a player with valuation approximately $(1/2) + \varepsilon$ is for large $n$ approximately

$$\frac{1}{2}\left(1 - \frac{\frac{1}{2}\cdot\left(\frac{1}{2}+\frac{1}{n}\right)\cdot \ldots \cdot\left(\frac{1}{2}+\frac{\varepsilon n}{n}\right)}{\left(\frac{1}{2}+\varepsilon\right)^{\varepsilon n}}\right) = \frac{1}{2}\left(1 - \frac{\frac{1}{2}\cdot \ldots \cdot\left(\frac{1}{2}+\frac{\varepsilon n}{2n}\right)}{\left(\frac{1}{2}+\varepsilon\right)^{\frac{\varepsilon n}{2}}} \cdot \frac{\left(\frac{1}{2}+\frac{\varepsilon n}{2n}+\frac{1}{n}\right)\cdot \ldots \cdot\left(\frac{1}{2}+\frac{\varepsilon n}{n}\right)}{\left(\frac{1}{2}+\varepsilon\right)^{\frac{\varepsilon n}{2}}}\right).$$

The first fraction is bounded above by $\left((1/2 + \varepsilon/2)/(1/2 + \varepsilon)\right)^{\varepsilon n/2}$, and the second fraction is bounded above by 1, so as $n$ increases the lower bound of the bidding interval approaches 1/2 as fast as $1 - b^n$ approaches 1, where $b < 1$. Therefore, for any $\varepsilon > 0$, for sufficiently large $n$ at most a fraction $\varepsilon$ of the players bid more than $\varepsilon$ away from what the mechanism prescribes for the types that correspond to them.

## 3.3 Heterogeneous prizes and identical players

Consider $U(x, y, t) = y - t$ (so strict single crossing fails), let $x_i^n = i/n$ (so $F_i^n$ is a Dirac measure), and let $y_j^n = j/n$. The limit distributions $F$ and $G$ are uniform. The $n$-th contest is an all-pay auction with $n$ players and $n$ prizes, and the value of prize $j$ to all players is $j/n$. Such contests were studied by Barut and Kovenock (1998), who considered grading, promotions, procurement settings, and political competitions.

Consider the uniform allocation, whose density is $h(x, y) = 1$ for all values of $x$ and $y$. The unique IC-IR mechanism that implements this allocation has $Q_x(y, t)$ distributed uniformly on the diagonal $y = t$. This is a tariff mechanism with a continuous inverse tariff that maps every bid $t \in [0, 1]$ to prize $t$. Theorem 3 below shows that the mechanism approximates the equilibrium outcome for large $n$ in an aggregate sense. This is weaker than the approximation in the previous subsections, because strict single crossing fails. Indeed, Barut and Kovenock (1998) showed that the $n$-th contest has a unique equilibrium, in which all players randomize uniformly across all bids $t \in [0,1]$.[13]

# 4 Main results

## 4.1 Results with strict single crossing

We show that under strict single crossing the equilibria of large contests are approximated by tariff mechanisms that implement the *assortative allocation*, in which type $x$ obtains prize $y = G^{-1}(F(x))$, where $G^{-1}(z) = \inf\{y : G(y) \geq z\}$ for $z > 0$ and $G^{-1}(0) = y_{\inf} = \inf\{y : G(y) > 0\}$. We first consider settings in which prizes have full support, so $G^{-1}$ is continuous. This guarantees that when the number of players and prizes is large, it is enough to know the approximate rank-order of a player's bid to know the approximate prize she obtains.

**Theorem 1** *Suppose that strict single crossing holds and prizes have full support. Then, for any $\varepsilon > 0$, there is an $N$ such that for all $n \geq N$ in any equilibrium of the $n$-th contest,*

*(a) every player $i$ obtains with probability at least $1-\varepsilon$ a prize that differs by at most $\varepsilon$ from $G^{-1}(F(x_i^n))$;*

*(b) there is a regular tariff mechanism with a continuous inverse tariff that implements the assortative allocation, such that the bid of every player $i$ differs with probability $1$ by at most $\varepsilon$ from the bid that the mechanism prescribes for type $x_i^n$.*

There may not be a unique mechanism that implements the assortative allocation, but when the mechanism is unique it coincides with the one in part (b) of Theorem 1. For example, if $U$ is quasi-linear and satisfies the conditions of Milgrom and Segal's (2002) envelope theorem,[14] then their Corollary 1 shows that the unique IC-IR mechanism that implements the assortative allocation prescribes for type $x$ bid

$$br(x) = v\left(x, G^{-1}(F(x))\right) - \int_0^x v_x\left(z, G^{-1}(F(z))\right) dz - v(0, y_{\inf}). \tag{5}$$

In this case, (5) provides an explicit formula for the tariff mechanism in part (b) of Theorem 1.[15] We therefore obtain the following corollary of Theorem 1, which applies to the setting

[13] The notion of approximation implied by Theorem 3 is in general weaker than in this example.

[14] The conditions are that $v$ is differentiable and absolutely continuous in $x$, and $\sup_{y \in Y} |v_x(x,y)|$ is integrable on $X$.

[15] Other sufficient conditions for (5) are described in Krishna and Maenner's (2001) Proposition 1.

of Section 3.1.

**Corollary 1** *Suppose that strict single crossing holds, prizes have full support, and $U$ is quasi-linear and satisfies the conditions of the envelope theorem. Then, for any $\varepsilon > 0$ there is an $N$ such that for all $n \geq N$, in any equilibrium of the $n$-th contest every player $i$ obtains with probability at least $1 - \varepsilon$ a prize that differs by at most $\varepsilon$ from $G^{-1}\left(F\left(x_i^n\right)\right)$, and bids with probability $1$ within $\varepsilon$ of $br\left(x_i^n\right)$ given by (5).*

Note that quasi-linearity also guarantees that the assortative allocation is efficient, because $v\left(x_2, y_2\right) + v\left(x_1, y_1\right) > v\left(x_2, y_1\right) + v\left(x_1, y_2\right)$ for any $x_1 < x_2$ and $y_1 < y_2$.[16] In addition, for complete-information contests, such as those in Section 3.1, the proof of Theorem 1 shows that the $1 - \varepsilon$ in Theorem 1 and Corollary 1 can be replaced with 1.

When prizes do not have full support $G^{-1}$ is discontinuous, so the approximate rank-order of a player's bid may be insufficient to determine the approximate prize she obtains. Consequently, some players' bids may be significantly different from what the limit mechanism specifies, even when the contest is large. For example, in the setting of Section 3.2, when the percentile rank-order of a player's bid is slightly above $1/2$ she obtains $G^{-1}$ of the rank-order, which is 1, and when it is slightly below $1/2$ she obtains $G^{-1}$ of the rank-order, which is 0. And for large $n$ there are many players with valuations greater than $1/2$ who bid substantially less than $1/2$. Thus, Theorem 1 does not hold.

Nevertheless, even when prizes do not have full support, the approximation of Theorem 1 holds for all but a small fraction of players.

**Theorem 2** *Suppose that strict single crossing holds (but prizes may not have full support). Then, for any $\varepsilon > 0$, there is an $N$ such that for all $n \geq N$ in any equilibrium of the $n$-th contest,*

*(a) a fraction of at least $1 - \varepsilon$ of the players $i$ obtain with probability at least $1 - \varepsilon$ a prize that differs by at most $\varepsilon$ from $G^{-1}\left(F\left(x_i^n\right)\right)$;*

*(b) there is a regular tariff mechanism that implements the assortative allocation, such that the bid of each of a fraction of at least $1 - \varepsilon$ of the players $i$ differs with probability at least $1 - \varepsilon$ by at most $\varepsilon$ from the bid that the mechanism prescribes for type $x_i^n$.*

[16] To see why, apply strict single crossing with $t_1 = v(x_1, y_1)$ and $t_2 = v\left(x_1, y_2\right)$. Since type $x_1$ is indifferent between obtaining $y_1$ and paying $t_1$ and obtaining $y_2$ and paying $t_2$, type $x_2$ must strictly prefer the latter option, which yields

$$v\left(x_2, y_2\right) - v\left(x_1, y_2\right) > v(x_2, y_1) - v(x_1, y_1).$$

We also have an analogue of Corollary 1, which applies to the setting of Section 3.2.

**Corollary 2** *Suppose that strict single crossing holds and $U$ is quasi-linear and satisfies the conditions of the envelope theorem. Then, for any $\varepsilon > 0$ there is an $N$ such that for all $n \geq N$, in any equilibrium of the $n$-th contest each of a fraction of at least $1-\varepsilon$ of the players $i$ obtains with probability at least $1-\varepsilon$ a prize that differs by at most $\varepsilon$ from $G^{-1}(F(x_i^n))$, and bids with probability at least $1-\varepsilon$ within $\varepsilon$ of $br(x_i^n)$ given by (5).*

The approximation results apply to many contests for which there is no existing equilibrium characterization. For example, consider (**1**) with $F$ and $G$ uniform. The assortative allocation assigns prize $x$ to type $x$, and (5) shows that $br(x) = xh(x) - \int_0^x h(y)\,dy$. By Corollary 1, for $x_i^n = i/n$ (so $F_i^n$ is a Dirac measure) and $y_j^n = j/n$, when $n$ is large a player with type $x$ bids close to $x - \int_0^x h(y)\,dy$ and obtains a prize close to $x$. While $h(y) = y$ corresponds to the setting of Section 3.1 and $h(y) = y^2$ and $h(y) = e^y$ correspond to Xiao's (2013) quadratic and geometric prize sequences, for which he provides an equilibrium characterization,[17] no equilibrium characterization exists for other, non-trivial functions $h$ (including $h(y) = y^m$ for $m > 2$). The same implication of Corollary 1 holds for contests with ex-ante asymmetric players and incomplete information for which $F^n$ converges in weak*-topology to the uniform distribution, although no equilibrium characterization exists for such contests.

Another example is contests that combine heterogeneous and identical (non-zero) prizes, which have not been studied in the literature. Consider $y_j^n = 2j/n$ for $j/n \leq 1/2$ and $y_j^n = 1$ for $j/n > 1/2$, so $G(y) = y/2$ for $y < 1$ and $G(1) = 1$, with (**1**) and $F$ uniform. The assortative allocation assigns prize $2x$ to type $x < 1/2$, and prize 1 to type $x \geq 1/2$. If $F^n$ converges in weak*-topology to the uniform distribution, then Corollary 1 and (5) show that for large $n$ a player with type $x$ bids close to $\min\{x^2, 1/4\}$ and with high probability obtains a prize close to $\min\{2x, 1\}$.

In addition, the approximation results hold for quasi-linear utilities in which $v(x, y)$ is not multiplicatively separable, and for utilities that are not quasi-linear.

## 4.2 Results without strict single crossing

To formulate the results without strict single crossing, we recall that the weak*-topology is defined on the set of probability distributions $\Delta(\Omega)$ over a compact space $\Omega$, and consists of

[17] Xiao's (2013) characterization is considerably more complicated than B&L's, because in his setting equilibria involve bidding strategies with non-interval support.

all unions of finite intersections of sets of the form

$$\{Q \in \Delta(\Omega) : \left| \int h dP - \int h dQ \right| < \varepsilon\},$$

where $P \in \Delta(\Omega)$, $\varepsilon > 0$, and $h$ is a real-valued and continuous function on $\Omega$ (see Chapter 3 of Rudin (1973)).

**Theorem 3** *Regardless of single crossing (or prizes having full support), for any $\varepsilon > 0$ and any metrization of the weak*-topology there is an $N$ such that for all $n \geq N$, for any equilibrium of the $n$-th contest there is a regular tariff mechanism that implements a consistent allocation whose outcome is $\varepsilon$-close (in the metrization) to the outcome of the equilibrium. If prizes have full support, then the inverse tariff is continuous. If weak single crossing holds and the utility is quasi-linear, then the consistent allocation is also efficient.*

The approximating mechanism may implement a consistent allocation other than the assortative one, as the setting of Section 3.3 shows. In addition, Theorem 3 only allows us to approximate the joint equilibrium distribution over prize-bid pairs of players with types close to a given type, which is weaker than the approximation in the previous results (in the sense of being implied). If, however, for every type $x$ there is a unique prize-bid pair $(y(x), t(x))$ that maximizes $U(x, y, t)$ among the prize-bid pairs from the graph of the inverse tariff, then convergence in weak*-topology implies convergence in a sense similar to that of Theorem 2. Namely, for any $\varepsilon > 0$ and sufficiently large $n$, for a fraction $1 - \varepsilon$ of the players, with probability $1 - \varepsilon$ the prize that player $i$ obtains differs from $y(x_i^n)$ by at most $\varepsilon$ and the bid of player $i$ differs from $t(x_i^n)$ by at most $\varepsilon$.

Another limitation of Theorem 3 is that the set of tariff mechanisms that implement a consistent allocation may be quite large even if every contest has a unique equilibrium. For example, in the setting of Section 3.3 there is a continuum of efficient allocations and tariff mechanisms that implement them, all associated with the same inverse tariff that maps each bid $t \in [0, 1]$ to prize $t$. We conjecture, however, that Theorem 3 is the strongest general convergence result that one can obtain, because some contests have many equilibria, and different sequences of equilibria may be approximated by different mechanisms.

Finally, note that while Theorem 3 and the results that precede it are in the spirit of upper hemi-continuity, they are not implied by standard upper hemi-continuity arguments, because discrete contests and the limit mechanisms belong to different spaces (for example, one has a finite number of players and the other a continuum of agents).

# 5 Contest Design

As an application of our results, we investigate the prize distribution that maximizes players' aggregate bids (effort) in a large contest. This has not been possible so far for discrete contests, because no equilibrium characterization exists for discrete contests with ex-ante asymmetric players and general prize distributions. Section 4 indicates that we can instead consider the approximating limit setting with a continuum of agents and prizes, which makes such an investigation possible.

Consider (1) with $h$ continuously differentiable,[18] and suppose that $F$ has a continuous, strictly positive density $f$. Suppose also that there is an upper bound on the highest possible prize, which corresponds to $y = 1$. (We will comment on the effect of increasing this upper bound.) We first analyze the optimal prize distribution when the prize budget is unrestricted, and then impose a budget constraint. In the latter case, we assume that $h$ is defined so that prize $y$ costs $y$.

We begin by considering a fixed prize distribution $G$. Corollary 2 shows that the approximating limit prize allocation is assortative, i.e., type $x > 0$ obtains prize $y = G^{-1}(F(x))$, where $G^{-1}(z) = \inf\{y : G(y) \geq z\}$. Using integration by parts, (5) shows that the resulting aggregate bids are approximated by

$$\int_0^1 h\left(G^{-1}(F(x))\right)\left(x - \frac{1 - F(x)}{f(x)}\right) f(x)\, dx. \tag{6}$$

This expression coincides with the expected revenue from a bidder in a single-object independent private-value auction if we replace $h(G^{-1}(F(x)))$ with the probability that the bidder wins the object when his type is $x$ (Myerson (1981)). Increasing the probability that type $x$ obtains the object along with the price he is charged allows the auctioneer to capture the entire increase in surplus, but requires a decrease in the price that higher types are charged (to maintain incentive compatibility). This net increase in revenue, or "virtual utility," also coincides with a monopolist's marginal revenue (Bulow and Roberts (1989)). In our contest setting, increasing the prize that type $x$ obtains increases competition with slightly lower types, which "competes away" the additional prize value, but decreases competition by higher types for their prizes, since the prize of type $x$ becomes more attractive to them.

Before analyzing the optimal prize distribution, we consider the effect that a change in the agent type distribution has on the aggregate bids for a given prize distribution. For this,

[18] A similar analysis can be conducted for $U(x, y, t) = v(x, y) - t$ with twice differentiable $v(x, y)$ that has positive second-order cross derivatives.

it is convenient to rewrite (6) using a change-of-variable $z = F(x)$ to obtain

$$\int_0^1 h\left(G^{-1}(z)\right)\left(F^{-1}(z) - \frac{1-z}{f\left(F^{-1}(z)\right)}\right)dz = \int_0^1 h\left(G^{-1}(z)\right)k(z)\,dz, \tag{7}$$

where $k(z) = F^{-1}(z) - (1-z)/f\left(F^{-1}(z)\right)$.[19] Integration by parts shows that for differentiable $G^{-1}$ (7) is equal to

$$\int_0^1 \frac{\partial h\left(G^{-1}(z)\right)}{\partial z}\left((1-z)\,F^{-1}(z)\right)dz,$$

so a first-order stochastic dominance (FOSD) shift in the agent type distribution increases aggregate bids. Since for any prize distribution $G$ values arbitrarily close to (7) can be achieved with prize distributions whose inverse is differentiable, we obtain the following observation.

**Claim 1** *For any prize distribution, a FOSD shift in the agent type distribution increases the aggregate bids.*

We now turn to analyzing the optimal prize distribution for a given type distribution $F$. As is standard in the mechanism design literature, we assume that $x - (1 - F(x))/f(x)$ is strictly increasing in $x$, which is implied, for example, by a monotone hazard rate. Thus, $k(z)$ strictly increases in $z$. Denote by $x^* \in (0,1)$ the unique type that satisfies $x^* - (1 - F(x^*))/f(x^*) = 0$, and let $z^* = F(x^*) \in (0,1)$, so $k(z^*) = 0$.

Suppose first that the prize budget is unrestricted. Then, optimizing the integrand in (7) leads to $G^{-1}(z) = 0$ if $z \leq z^*$ and $G^{-1}(z) = 1$ if $z > z^*$. This $G^{-1}$ is left-continuous and monotonic, so $G$ is a prize distribution and is therefore optimal. We thus obtain the following observation.

**Claim 2** *If the prize budget is unrestricted, then for any function $h$ the optimal prize distribution assigns mass $1 - F(x^*) \in (0,1)$ to the highest possible prize and mass $F(x^*)$ to prize 0.*

Claim 2 shows that an all-pay auction with identical prizes (Clark and Riis (1998)) is optimal when the budget is unrestricted. The claim also shows that increasing the highest possible prize does not change the optimal quantity of prizes, but does increase the resulting aggregate bids (because they are denominated in units of the highest possible prize).

Now suppose that the prize budget is restricted. We model this by introducing the budget constraint $\int_0^1 y\,dG(y) \leq C$. The parameter $C$ is the per-competitor budget, denominated in units of the maximal prize (recall that prize $y$ costs $y$). The following observation is an immediate implication of Claim 2.

[19] Even though $G^{-1}$ may be discontinuous, it is monotonic, so the change-of-variable applies.

**Claim 3** *If $C \geq 1 - F(x^*)$, then the optimal prize distribution coincides with the one in the unrestricted budget case.*

Claim 3 shows that when the prize budget is large some of it is optimally left unused. This is analogous to the seller in an optimal auction keeping the item some of the time (by setting a reserve price), even if he does not value the item, and is also analogous to a monopolist limiting the quantity sold.

Now consider a budget $C < 1 - F(x^*)$. Because $G$ is a probability measure on $[0, 1]$, we have $\int_0^1 y dG(y) = \int_0^1 (1 - G(y))\, dy$ (by integrating by parts) and $\int_0^1 G^{-1}(z)\, dz + \int_0^1 G(y)\, dy = 1$ (by looking at the graphs of $G$ and $G^{-1}$ in the square $[0, 1]^2$). Thus, the budget constraint can be rewritten as

$$\int_0^1 G^{-1}(z)\, dz \leq C. \tag{8}$$

This is a substantial simplification, because maximizing (7) subject to (8) is a calculus of variations problem.

To solve this problem, consider an optimal $G^{-1}$. Because it is non-decreasing, left-continuous, and takes values in $[0, 1]$, there are $z_{\min} \leq z_{\max}$ in $[0, 1]$ such that $G^{-1}(z) = 0$ for $z \leq z_{\min}$, $G^{-1}(z) = 1$ for $z > z_{\max}$, and $G^{-1}(z) \in (0, 1)$ for $z \in (z_{\min}, z_{\max})$. Moreover, $z_{\min} \geq z^*$, because increasing $z_{\min} < z^*$ to $z^*$ increases the value of (7) without violating (8). In addition, $C < 1 - F(x^*)$ implies that $z_{\max} > z^*$ and that (8) holds with equality (the budget constraint binds). Because $k(z)$ is continuous and strictly increasing, standard calculus-of-variations arguments show that if $z_{\min} = z_{\max}$, then $h'(0)\, k(z_{\min}) \leq h'(1)\, k(z_{\max})$, and if $z_{\min} < z_{\max}$, then there exists some $\lambda \geq 0$ such that $h'(G^{-1}(z))\, k(z) = \lambda$ for $z \in (z_{\min}, z_{\max})$, with $h'(0)\, k(z_{\min}) \leq \lambda$ if $z_{\min} > z^*$ and $h'(1)\, k(z_{\max}) \geq \lambda$ if $z_{\max} < 1$.

These properties pin down $G^{-1}$ when $h$ is convex or concave. To see this, suppose first that $h$ is weakly convex. Then $z_{\min} = z_{\max}$, because $h'(G^{-1}(z'))\, k(z') < h'(G^{-1}(z''))\, k(z'')$ for any $z' < z''$ in $(z_{\min}, z_{\max})$. Therefore, the binding budget constraint implies the following observation.

**Claim 4** *If $C < 1 - F(x^*)$ and $h$ is weakly convex, then the optimal prize distribution assigns mass $C$ to the highest possible prize and mass $1 - C$ to prize $0$.*

Claim 4 shows that an all-pay auction with identical prizes remains optimal when the prize budget is small, provided that agents' marginal prize utility is non-decreasing. In contrast to the case of a large or unrestricted budget, however, Claim 4 shows that increasing the highest possible prize without changing the budget (effectively decreasing $C$) decreases the the optimal quantity of prizes, and increasing the budget without increasing the highest possible prize increases the optimal quantity of prizes. Both changes increase the resulting aggregate bids.

Things are different when $h$ is weakly concave (but not linear). Then $z_{\min} < z_{\max}$, because $h'(0) > h'(1)$, so the optimal prize distribution includes a variety of prizes.

**Claim 5** *If $C < 1 - F(x^*)$ and $h$ is weakly concave (but not linear), then the optimal prize distribution includes a range (a continuum) of non-zero prizes.*

As we will show below, the prize distribution may or may not assign positive mass to the highest possible prize.

We now show that when $h$ is strictly concave the optimal distribution is pinned down by the properties described after (8) once it is determined whether the inequalities $z_{\min} \geq z^*$ and $z_{\max} \leq 1$ are strict. There are four cases to consider. Suppose first that $z_{\min} > z^*$ and $z_{\max} < 1$. Then, $h'(G^{-1}(z))\,k(z) = \lambda$ for $z \in (z_{\min}, z_{\max})$. Since $h$ is strictly concave and $k$ is continuous, $h'(0)\,k(z_{\min}) \leq \lambda$ implies that $h'(0)\,k(z_{\min}) = \lambda$ and $h'(1)\,k(z_{\max}) \geq \lambda$ implies that $h'(1)\,k(z_{\max}) = \lambda$. This yields $z_{\min} = k^{-1}(h'(1)\,k(z_{\max})/h'(0))$ and

$$G^{-1}(z) = h'^{-1}\left(\frac{h'(1)\,k(z_{\max})}{k(z)}\right) \text{ for } z \in (z_{\min}, z_{\max}), \tag{9}$$

so $G^{-1}$ is pinned down by $z_{\max}$.

Similarly, if $z_{\min} = z^*$ and $z_{\max} < 1$, then (9) holds; if $z_{\min} > z^*$ and $z_{\max} = 1$, then $G^{-1}(z) = h'^{-1}(h'(0)\,k(z_{\min})/k(z))$ for $z \in (z_{\min}, 1)$; and if $z_{\min} = z^*$ and $z_{\max} = 1$, then for some $\lambda > 0$ we have $h'(G^{-1}(z))\,k(z) = \lambda$ for $z \in (z_{\min}, z_{\max})$, so

$$G^{-1}(z) = h'^{-1}\left(\frac{\lambda}{k(z)}\right). \tag{10}$$

In all four cases, the remaining variable ($z_{\max}$, $z_{\max}$, $z_{\min}$, and $\lambda$) is pinned down by the binding budget constraint.

To illustrate this solution, consider $F$ uniform, $C < 1 - F(x^*) = 1/2$, and $h(y) = \sqrt{y}$. Then $x^* = z^* = 1/2$, $k(z) = 2z - 1$, $h'(0) = \infty$, $h'(1) = 1/2$, and $h'^{-1}(a) = 1/4a^2$. Since $h'(0) = \infty$, $z_{\min} = 1/2$. Suppose that $z_{\max} < 1$. Then (9) applies, so the binding budget constraint implies that $\int_{\frac{1}{2}}^{z_{\max}} ((2z-1)/(2z_{\max}-1))^2\,dz + \int_{z_{\max}}^{1} 1dz = C \Rightarrow z_{\max} = (5-6C)/4$, which implies that $C > 1/6$. We therefore have $G^{-1}(z) = (2(2z-1)/(3(1-2C)))^2$ for $z \in [0, (5-6C)/4]$ and $G^{-1}(z) = 1$ for $z \in [(5-6C)/4, 1]$. This implies that

$$G(y) = \begin{cases} \frac{1}{2} + \frac{3\sqrt{y}(1-2C)}{4} & y \in [0,1) \\ 1 & y = 1 \end{cases},$$

so there is a continuous distribution of positive intermediate prizes and a mass $(6C-1)/4$ of the highest possible prize. The resulting aggregate bids, given by (7), are $(12C(1-C)+1)/16$.

Now suppose that $z_{\max}=1$. Then (10) applies, and we have

$$\int_{\frac{1}{2}}^{1} 4\left(\lambda/\left(2z-1\right)\right)^{-2} dz = C,$$

so $\lambda = 1/\sqrt{24C}$, which implies that $C \leq 1/6$. We therefore have $G^{-1}(z) = 6C\left(2z-1\right)^2$, so

$$G(y) = \begin{cases} \frac{1}{2}+\sqrt{\frac{y}{24C}} & y\in[0,6C] \\ 1 & y=[6C,1] \end{cases}$$

and there is a continuous distribution of positive intermediate prizes with no mass of any positive prize. The resulting aggregate bids are $\sqrt{6C}/6$. The following figure depicts these results.

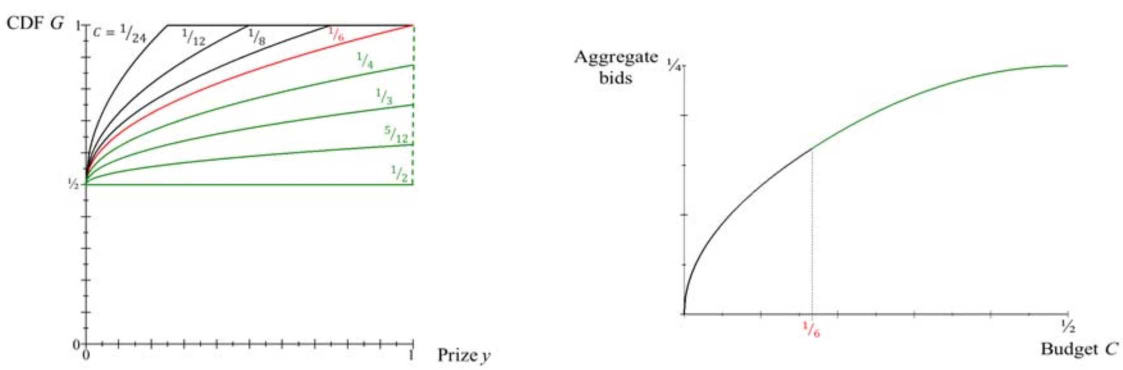


Figure 3: The optimal prize distribution as $C$ increases from 0 to 1/2 (left) and the resulting aggregate bids (right)

To summarize the example, for any budget there is a mass 1/2 of zero prizes. As $C$ increases from 0 to 1/6, the maximal prize allocated increases from the lowest to the highest possible prize, and the prize distribution remains continuous above 0. Once $C$ reaches 1/6, the maximal prize allocated is the highest possible prize, and as $C$ increases from 1/6 to 1/2, the mass of the highest possible prize increases from 0 to 1/2, so the prize distribution is discontinuous at 1. In particular, increasing the highest possible prize without changing the budget (effectively decreasing $C$) changes the optimal prize distribution and increases the aggregate bids if and only if $C > 1/6$.

# 6 Proofs

We choose an equilibrium for each contest, and refer to the sequence in which the $n$-th element is the equilibrium of the $n$-th contest as the sequence of equilibria. For each of the theorems, we will show that every subsequence of this sequence contains a further subsequence that satisfies the statement of the theorem. This suffices, because the following observation can be applied with $Z_n$ being the set of equilibria of contest $n$.

(**Subsequence Property**) Given a sequence of sets $\{Z_n : n = 1, 2, \ldots\}$, suppose that for every sequence $\{z_n : n = 1, 2, \ldots\}$ with $z_n \in Z_n$, every subsequence $\{z_{n_k} : k = 1, 2, \ldots\}$ contains a further subsequence $\{z_{n_{k_l}} : l = 1, 2, ...\}$ such that every element $z_{n_{k_l}}$ has some property. Then there exists an $N$ such that for every $n \geq N$ every element in $Z_n$ has this property.[20]

To simplify notation, we take the subsequence to be the sequence of equilibria (this has no effect on the proofs).

The proof of Theorem 2 is in Appendix B. The structure of the proof is similar to that of Theorem 1 below, but the proof of Theorem 1 relies heavily on the continuity of $G^{-1}$, so to prove Theorem 2 almost every step of the proof must be modified and additional results must be derived.

## 6.1 Proof of Theorem 1

We denote the equilibrium of the $n$-th contest by $\sigma^n = (\sigma_1^n, \ldots, \sigma_n^n)$, where $\sigma_i^n$ is player $i$'s equilibrium strategy; a strategy of player $i$ is a random variable taking values in $X \times B$ whose marginal distribution on $X$ coincides with the distribution of player $i$'s types $F_i^n$. By referring to player $i$ bidding with some probability in a subset $S$ of $B$, we mean the probability of the set $X \times S$, i.e., the probability of $S$ measured by the marginal distribution of player $i$'s strategy on $B$.

We denote by $R_i^n(t)$ the random variable that is the percentile location of player $i$ in the ordinal ranking of the players in the $n$-th contest if she bids slightly above $t$ and the other players employ their equilibrium strategies.[21] That is,

$$R_i^n(t) = \frac{1}{n}\left(1 + \sum_{k \neq i} 1_{\{\sigma_k^n \in X \times [0,t]\}}\right),$$

where $1_{\{\sigma \in X \times [0,t]\}}$ is 1 if $\sigma \in X \times [0, t]$ and 0 otherwise. Let

$$A_i^n(t) = \frac{1}{n}\left(1 + \sum_{k \neq i} \Pr\left(\sigma_k^n \in X \times [0, t]\right)\right)$$

[20] Otherwise, there would be a subsequence of elements without the property.

[21] This is the infimum of her ranking if she bids above $t$, which is equivalent to bidding $t$ and winning any ties there. If ties happen with probability 0, then this is equivalent to bidding $t$.

be the expected percentile ranking of player $i$. Then, by Hoeffding's inequality, for all $t$ in $B$ we have

$$\Pr\left(\left|R_i^n(t) - A_i^n(t)\right| > \delta\right) < 2\exp\left\{-2\delta^2(n-1)\right\}. \tag{11}$$

Finally, let

$$A^n(t) = \frac{1}{n}\sum_{i=1}^{n} A_i^n(t)$$

be the average of the expected percentiles rankings of the players in the $n$-th contest if they bid $t$ and the other players employ their equilibrium strategies.

Let $T^n$ be the mapping from bids to prizes induced by $A^n$. That is, $T^n(t) = (G^n)^{-1}(A^n(t))$, where $(G^n)^{-1}(z) = \inf\{y : G^n(y) \geq z\}$ for $z > 0$, and $(G^n)^{-1}(0) = \inf\{y : G^n(y) > 0\}$. (In words, $(G^n)^{-1}(z)$ is the prize of an agent with percentile ranking $z$ when prizes are distributed according to $G^n$.)

Take an ordering of the rationals in $B$, denoted by $q_1, q_2, \ldots$. Take a converging subsequence of the sequence $T^n(q_1)$, denote it by $T^{n_1}(q_1)$, and denote its limit by $T(q_1)$. Take a converging subsequence of the sequence $T^{n_1}(q_2)$, denote it by $T^{n_2}(q_2)$, and denote its limit by $T(q_2)$. Continue in this fashion to obtain a function $T : \{q_1, q_2, \ldots\} \to [0,1]$, which is weakly increasing (because every $T^n$ is). In addition, define a subsequence of $T^n$ such that its $k$-th element is the $k$-th element in the sequence $T^{n_k}$. For the rest of the proof, denote this new sequence by $T^n$. Note that $T^n$ converges to $T$ on $\{q_1, q_2, \ldots\}$.

The following lemma shows that $T$ can be extended uniquely to a continuous function on the entire interval $B$.

**Lemma 1** *For any $t \in B$ (not necessarily rational) and any two sequences $q^m \uparrow t$ and $r^m \downarrow t$ of rationals in $B$, we have $\lim T(q^m) = \lim T(r^m)$.*

The proof of Lemma 1, and those of other results in this subsection, is in Appendix A. The idea of the proof is that if $T$ is discontinuous at some $t$, then for large $n$ it is better to bid slightly above $t$ than slightly below $t$. But if no player bids slightly below $t$, then players bidding slightly above $t$ would profitably deviate by lowering their bids.

We extend $T$ to the entire interval $B$ by setting $T(t) = \lim T(t^m)$ for some sequence $t^m \to t$ of rationals in $B$. Lemma 1 shows that $T(t)$ is the same regardless of the chosen sequence $t^m$. Lemma 1 also guarantees that this is indeed an extension, and that the extended $T$ is continuous. Continuity and monotonicity of $T$ imply the following result.

**Lemma 2** *$T^n$ converges to $T$ uniformly on $B$.*

We now relate the inverse tariff $T$ to players' behavior in the equilibria that correspond to the sequence $T^n$. Denote by $BR_x$ type $x$'s set of optimal bids given $T$, i.e., the bids $t$ that maximize $U(x, T(t), t)$. Denote by $BR(\varepsilon)$ the $\varepsilon$-neighborhood of the graph of the correspondence that assigns to every type $x$ the set $BR_x$.[22] Denote by $BR_x(\varepsilon)$ the set of bids $t$ such that $(x,t) \in BR(\varepsilon)$.

Note that $BR(\varepsilon)$ is a 2-dimensional open set, while each $BR_x(\varepsilon)$ is a 1-dimensional "slice" of $BR(\varepsilon)$. Note also that $BR_x(\varepsilon)$ may contain bids whose distance from every bid in $BR_x$ is more than $\varepsilon$. Using sets $BR_x(\varepsilon)$, we can characterize players' equilibrium behavior.

**Lemma 3** *For every $\varepsilon > 0$, there is an $N$ such that for every $n \geq N$, in the equilibrium of the $n$-th contest every best response of every type $x_i^n$ of every player $i$ belongs to $BR_{x_i^n}(\varepsilon)$.*

The proof up to this point did not rely on strict single crossing, a fact we will use in the proof of Theorem 3. We now show that under strict single crossing $BR_x$ is a singleton.

**Lemma 4** *If strict single crossing holds, then for all $x$ the set $BR_x$ is a singleton. In addition, the function $br$ that assigns to $x$ the single element of $BR_x$ is continuous and weakly increasing.*

Lemma 4 implies that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $BR_x(\delta) \subseteq [br(x) - \varepsilon, br(x) + \varepsilon]$ for every type $x$. We therefore have the following corollary of Lemmas 3 and 4.

**Corollary 3** *For every $\varepsilon > 0$, there is an $N$ such that for every $n \geq N$, in the equilibrium of the $n$-th contest every best response of every type $x_i^n$ of every player $i$ belongs to $(br(x_i^n) - \varepsilon, br(x_i^n) + \varepsilon)$.*

To prove part (b) of the theorem we need to show that $T \circ br$ is the assortative allocation. This is done by the following lemma.

**Lemma 5** *$G^{-1}(F(x)) = T(br(x))$ for all types $x$.*

Thus, the mechanism that prescribes for type $x$ prize $T(br(x))$ and bid $br(x)$ is a tariff mechanism that implements the assortative allocation. Moreover, IR holds: every type can get at least 0 by bidding 0, and type 0 gets no more than 0 (because $T(br(0)) = 0$).

To complete the proof, it remains to show part (a) of the theorem. This part follows from Corollary 3 and Hoeffding's inequality. We provide the details of the proof in Appendix A.

[22] I.e., $BR(\varepsilon)$ is the union over all types $x$ and bids $t \in BR_x$ of the open balls of radius $\varepsilon$ centered at $(x,t)$.

## 6.2 Proof of Theorem 3

In order not to refer to material we present later, we prove the theorem under the assumption that prizes have full support.[23] Recall that the proofs of Lemmas 1-3 did not rely on strict single crossing. Without strict single crossing, however, best response sets may not be singletons, so we cannot pin down each player's limiting bidding behavior or a unique mechanism induced by $T$.

From the sequence of equilibria that corresponds to the sequence $T^n$ that converges uniformly to $T$, choose a subsequence such that $D^n$ converges to some probability distribution $D$ in weak*-topology. We will now show that $D$ assigns probability 1 to the set $C = \{(x, y, t) : t \in BR_x \text{ and } y = T(t)\} \subset X \times Y \times B$.

By standard arguments the correspondence that assigns $BR_x$ to type $x$ is upper hemicontinuous. Therefore, the set $\{(x, t) : t \in BR_x\} \subset X \times B$ is closed, and by continuity of $T$, $C$ is also closed. Suppose to the contrary that $D$ assigns a positive probability to the complement of $C$. Then, for some $\varepsilon > 0$, $D$ assigns a positive probability to the complement of the $2\varepsilon$-neighborhood $O$ of $C$, that is, to the set $X \times Y \times B - O$. Consider the $\varepsilon$-neighborhood $V$ of $C$ and its closure $\bar{V}$ (which is contained in $O$), and take a continuous function $f : X \times Y \times B \rightarrow [0, 1]$ such that $f(\bar{V}) = 1$ and $f(X \times Y \times B - O) = 0$. Then,

$$\int f dD < 1.$$

But by Lemma 3, for sufficiently large $n$ every best response of every type $x_i^n$ of every player $i$ is in $BR_{x_i^n}(\varepsilon)$. And by (11), when a player bids $t$ her percentile location in the ordinal ranking $R_i^n(t)$ is with high probability close to $A_i^n(t)$, so she obtains a prize close to $T^n(t)$. By uniform convergence of $T^n$ to $T$, the prize is close to $T(t)$. Thus, for sufficiently large $n$ every type $x_i^n$ of player $i$ bids $t$ and obtains with arbitrarily high probability a prize $y$ such that $(x_i^n, y, t) \in V$. This yields

$$\int f dD^n \rightarrow 1, \tag{12}$$

a contradiction.

Thus, the limit mechanism determined by $D$ prescribes for each of a measure 1 of types $x$ bids $t \in BR_x$ and corresponding prizes $T(t)$ with probability 1. This implies that $D$

[23] Relaxing this assumption is straightforward given the proof of Theorem 2 in Appendix B. In the notation of that proof, the relaxation involves replacing $T$ with $T^*$ (this still implies that the set $C$ defined below is closed), and using Lemma 12 from Appendix B to show (12). The rest of the proof stays unchanged.

determines a tariff mechanism. It is regular, because each $D^n$ is regular, and it implements a consistent allocation, because each $D^n$ implements a consistent allocations.

For the last part of the theorem, suppose that for some $x' < x''$ and $t' > t''$ type $x'$ is prescribed $t'$ and type $x''$ is prescribed $t''$. Then $x'$ weakly prefers $t'$ to $t''$ and $x''$ weakly prefers $t''$ to $t'$. By weak single crossing, $x''$ weakly prefers $t'$ to $t''$, and therefore is indifferent between the two bids. Thus, if a lower type is prescribed a higher prize, this is without loss of efficiency.

# 7 Conclusion

In large contests, players' strategies and the resulting distribution of prizes are approximated by the outcomes of certain mechanisms. Under strict single crossing and quasi-linearity of players' utilities, the approximating mechanism is unique. Thus, the outcomes of large contests can be approximated by using standard techniques from the mechanism design literature, even when solving for equilibrium is difficult or impossible. An immediate implication is that the outcome of large contests that satisfy strict single crossing is approximately assortative, regardless of the details of players' utilities, their private information, and the distribution of prizes. In particular, the distinction between complete and incomplete information disappears for large contests. Another implication is that contest design and certain comparative statics exercises can be conducted easily for large contests, as Section 5 demonstrates.

The approach and results can also be used to provide foundations for contest models with a continuum of competitors, in the spirit of Morgan, Sisak, and Várdy (2013). It may also be possible to derive additional results that are weaker than Theorems 1 and 2, but stronger than Theorem 3, by identifying suitable conditions weaker than strict single crossing.

The approximation approach may prove useful for studying other contest specifications. One example is contests in which some players have initial advantages that must be overcome by other players.[24] More generally, the approach may be useful in analyzing other large discontinuous games, such as large auctions and double auctions.

[24] One such contest to which our approximation approach applies is all-pay auctions with head starts and identical valuations (Siegel (2013b)). Obtaining a general result for contests with head starts is not straightforward, however, because they require allowing the agent's utility function to weakly decrease in his bid, whereas many of our arguments rely on the assumption that a higher bid is inferior if it does not increase the probability of obtaining better prizes.

# A Proofs of results from Section 6.1

We first describe some properties of inverse tariff $T$ defined in Section 6, to which we will refer in several proofs:

(1) $T$ is (weakly) increasing, because every $T^n$ is (weakly) increasing.

(2) $T(0) = 0$, otherwise players bidding 0 would have profitable deviations.[25]

(3) $T(b_{\max}) = 1$, because $A^n(b_{\max}) = 1$ and therefore $T^n(b_{\max}) = 1$.

In addition, we will often refer to the following property of discrete contest equilibria:

(No-Gap Property) In any equilibrium, there is no interval $(a, b) \in B$ of positive length in which all players bid with probability 0 and some player bids in $[b, b_{\max}]$ with positive probability.

Proof: Suppose the contrary, and consider such a maximal interval $(a, b)$. A player would only bid $b$ or slightly higher than $b$ if some other player bids $b$ with positive probability. But the player who bids $b$ with positive probability would be better off either by slightly increasing her bid (if another player bids $b$ and winning the tie leads to a higher prize) or by decreasing her bid (in the complementary case).

Finally, we will refer to the following properties, which are implied by convergence in weak*-topology of $F^n$ to $F$ and $G^n$ to $G$:

(a) $F^n$ converges to $F$ pointwise.

(b) $(G^n)^{-1}$ converges to $G^{-1}$ pointwise.

Proof: Convergence in weak*-topology implies pointwise convergence at every point at which the limit is continuous (see Billingsley (1995), Theorem 25.8). Since we assumed that $F$ is continuous, this yields (a).

To show (b), suppose that for some $r \in [0, 1]$ and $\delta > 0$ we have that $(G^n)^{-1}(r) \leq G^{-1}(r) - \delta$ for arbitrarily large $n$. (An analogous argument applies to the case $(G^n)^{-1}(r) \geq G^{-1}(r) + \delta$.) With no loss of generality, assume that the inequality holds for all $n$, and that $(G^n)^{-1}(r)$ converges to some $y \leq G^{-1}(r) - \delta$. There exists a prize $z$ such that $y < z < G^{-1}(r)$ and $G$ is continuous at $z$. By full support, $G$ strictly increases, so $G(z) < r$. Since $G^n(z)$ converges to $G(z)$, we have that $G^n(z) < r$ for large enough $n$. This yields $z \leq (G^n)^{-1}(r)$, contradicting the assumption that $(G^n)^{-1}(r)$ converges to $y < z$.

[25] Indeed, suppose to the contrary that $T(0) > 0$. This means that for some $\delta > 0$ and large enough $n$, $A^n(0) > G^n(0) + \delta$. Thus, a fraction of at least $G^n(0) + \delta$ players bid 0 in the $n$-th contest with positive probability. Any one of them would be better off bidding slightly above 0, and winning against all other players who bid 0, than bidding 0 and with positive probability losing to all other players who bid 0.

## A.1 Proof of Lemma 1

Suppose the lemma is false for some $t \in (0, b_{\max})$, $q^m \uparrow t$, and $r^m \downarrow t$. Let $y' = \lim T(q^m)$ and $y'' = \lim T(r^m)$ (the limits exist by the monotonicity of $T$), and let $\gamma = (y'' - y')/2$.

Suppose first that $U(0, y, t)$ strictly increases in $y$. (Recall that we assumed $U(x, y, t)$ strictly increases in $y$ only for $x > 0$.) Then, by uniform continuity of $U$, there exist $\delta, \Delta > 0$ such that every type $x$ gains at least $\Delta$ from obtaining a prize higher by $\gamma$ at a bid higher by $\delta$. More precisely,

$$U(x, y + \gamma, t + \delta) - U(x, y, t) \geq \Delta \tag{13}$$

for all $x, y$, and $t$ such that $y + \gamma$ and $t + \delta$ belong to the domain of $U$. This implies, as $U$ is bounded, that every type $x$ strictly prefers bidding $t + \delta$ and obtaining with sufficiently high probability a prize sufficiently close to $y + \gamma$ to bidding $t$ and obtaining with sufficiently high probability a prize sufficiently close to $y$, independent of the prizes obtained with the remaining probability.

Choose an element $t' > 0$ of the sequence $q^m$ and an element $t'' < b_{\max}$ of the sequence $r^m$ such that $t'' - t' < \delta$. Next, choose $n$ large enough so that $|T^n(t'') - T(t'')| < \gamma/2$ and $|T^n(t') - T(t')| < \gamma/2$. This implies that $T^n(t'') - T^n(s) > \gamma$ for any bid $s \leq t'$.

By choosing $n$ large enough, we guarantee (see (11), which applies uniformly to all bids) that $R_i^n(s)$, the percentile ranking of player $i$ who bids $s$ in the $n$-th contest, is close to $A^n(s)$ with high probability, and $R_i^n(t'')$ is close to $A^n(t'')$ with high probability. Thus, every type $x$ obtains a prize sufficiently close to $T^n(t'')$ with a sufficiently high probability by bidding (slightly above) $t''$, and obtains a prize that is with a sufficiently high probability at most slightly higher than $T^n(t')$ by bidding (slightly above) any $s \leq t'$. Therefore, because $t'' - t' < \delta$**,** no player bids any $s \in (t'' - \delta, t']$ with positive probability, so by the No-Gap Property $T^n(t') = 1$. But $T^n(t') \to T(t') \leq y' < y'' \leq 1$, a contradiction.

When $U(0, y, t)$ only weakly increases in $y$, the argument above shows that for any $\varepsilon > 0$ there exist $\delta, \Delta > 0$ for which (13) holds for every type $x \in [\varepsilon, 1]$. There also exist $t'$ and $t''$ in the sequences $q^m$ and $r^m$ such that $t'' - t' < \delta$ and $T^n(t'') - T^n(t') > 3\gamma/2$ for large enough $n$. Letting $t''' = \inf\{s : T^n(s) \geq T^n(t'') - \gamma\} \in (t', t'']$ we see that only players with types lower than $\varepsilon$ can bid in $[t', t''')$. Thus, for small enough $\varepsilon$ (by continuity of $G^{-1}$ and convergence of $(G^n)^{-1}$ to $G^{-1}$), in order to increase $T^n(t')$ to $T^n(t') + \gamma/2$ multiple players with types $\varepsilon$ or higher must bid $t'''$ with positive probability and therefore tie there. But then any one of these players could profitably deviate to bidding slightly above $t'''$.[26]

[26]**B**y doing so the player would obtain with high probability a prize of at least $T^n(t') + \gamma/2$ instead of

For the case $t = b_{\max}$, set $t'' = b_{\max}$ and repeat the argument above.[27] Finally, suppose that $t = 0$. Then the above proof with $t' = 0$ shows that for large $n$ no player bids $t' = 0$ with positive probability. This means, in turn, that sufficiently small bids give lower payoffs than bid $t''$. Thus, no player bids close to $t' = 0$ with positive probability, which contradicts the No-Gap Property.

## A.2 Proof of Lemma 2

Suppose the contrary. Then, there is some $\delta > 0$ and a sequence of integers $n_1, n_2, \ldots$ such that for every $n_k$ there is some bid $t_k$ with $|T^{n_k}(t_k) - T(t_k)| > \delta$. Passing to a subsequence if necessary, we assume that the sequence $t_k \to t$.

Consider rationals $q'$ and $q''$ such that $q' < t < q''$ and $T(q'') - T(q') < \delta/2$; such numbers exist because $T$ is continuous.[28] For large enough values of $k$, we have that $|T^{n_k}(q') - T(q')| < \delta/2$ and $|T^{n_k}(q'') - T(q'')| < \delta/2$.

For any $t' \in [q', q'']$, either (a) $T^{n_k}(t') \geq T(t')$, or (b) $T^{n_k}(t') \leq T(t')$.

By the monotonicity of $T$ and $T^{n_k}$, we have

$$T^{n_k}(t') - T(t') \leq T^{n_k}(q'') - T(q') \leq |T^{n_k}(q'') - T(q'')| + |T(q'') - T(q')| < \delta$$

in case (a), and

$$T(t') - T^{n_k}(t') \leq T(q'') - T^{n_k}(q') \leq |T(q'') - T(q')| + |T(q') - T^{n_k}(q')| < \delta$$

in case (b).

Since $t_k \in [q', q'']$ for large enough values of $k$, we obtain a contradiction to the assumption that $|T^{n_k}(t_k) - T(t_k)| > \delta$ for all such $k$.

## A.3 Proof of Lemma 3

Suppose to the contrary that for arbitrarily large $n$, in the equilibrium of the $n$-th contest some type $x_i^n$ of some player $i$ has a best response that belongs to the complement of $BR_{x_i^n}(\varepsilon)$. Passing to a convergent subsequence if necessary, we assume that $x_i^n \to x^*$.

losing the tie with positive probability and then obtaining with high probability a prize of at most $T^n(t') + \varepsilon$.

[27] The only difference is that bidding "slightly above $b_{\max}$" is impossible. But by bidding $b_{\max}$ a player wins with probability 1, because $b_{\max}$ is strictly dominated by 0 for all players.

[28] If $t = 0$ set $q' = 0$, and if $t = b_{\max}$ set $q'' = b_{\max}$.

Notice that for every $x$ there is a $\delta_x > 0$ such that (under the inverse tariff) any bid from the complement of $BR_x(\varepsilon)$ gives type $x$ a payoff lower than any element of $BR_x$ does by at least $\delta_x$. Let $\delta = \delta_{x^*}$.

We have that:

1. The maximal payoff of type $x$, attained at any bid from $BR_x$, is continuous in $x$.

This follows from Berge's Theorem.

2. For every $\rho > 0$, for sufficiently large $n$ the highest payoff that type $x_i^n$ can obtain by bidding in the complement of $BR_{x_i^n}(\varepsilon)$ cannot exceed by $\rho$ the highest payoff that type $x^*$ can obtain by bidding in the complement of $BR_{x^*}(\varepsilon)$.

Indeed, suppose that for a sequence $n_k$ diverging to $\infty$ type $x_i^{n_k}$ obtains by bidding some $t_k$ in the complement of $BR_{x_i^{n_k}}(\varepsilon)$ a payoff at least $\rho$ higher than the highest payoff that type $x^*$ can obtain by bidding in the complement of $BR_{x^*}(\varepsilon)$. Passing to a convergent subsequence if necessary, we assume that $t_k \to t$. Since every $(x_i^{n_k}, t_k)$ belongs to the complement of $BR(\varepsilon)$, so does $(x^*, t)$; thus, $(x^*, t)$ belongs to the complement of $BR_{x^*}(\varepsilon)$. However, by continuity of the payoff functions, bidding $t$ gives type $x^*$ a payoff by at least $\rho$ higher than the highest payoff that type $x^*$ can obtain by bidding in the complement of $BR_{x^*}(\varepsilon)$, a contradiction.

By 1 and 2, for sufficiently large $n$, any bid in the complement of $BR_{x_i^n}(\varepsilon)$ gives type $x_i^n$ a payoff lower by at least $\delta/2$ than any bid in $BR_{x_i^n}$. Indeed, by 2 applied to $\rho = \delta/4$, any bid in the complement of $BR_{x_i^n}(\varepsilon)$ gives type $x_i^n$ a payoff at most $\delta/4$ higher than the highest payoff that type $x^*$ can obtain by bidding in the complement of $BR_{x^*}(\varepsilon)$. This last payoff is in turn lower than the payoff that type $x^*$ obtains by bidding in $BR_{x^*}$ by at least $\delta$. And by 1, the payoff that type $x_i^n$ obtains by bidding in $BR_{x_i^n}$ cannot be lower by more than $\delta/4$ than the payoff that type $x^*$ obtains by bidding in $BR_{x^*}$.

By uniform convergence of $T^n$ to $T$, the analogous statement, with $\delta/2$ replaced with some smaller positive number and $T$ replaced with $T^n$, is also true. This means, however, that for sufficiently large $n$, player $i$ would be strictly better off bidding slightly above any bid in $BR_{x_i^n}$ when his type is $x_i^n$ than bidding in the complement of $BR_{x_i^n}(\varepsilon)$. This is because (11) implies that for sufficiently large $n$, by bidding slightly above $t$ the player obtains a prize arbitrarily close to $T^n(t)$ with probability arbitrarily close to 1.

## A.4 Proof of Lemma 4

Observe that for any $x' < x''$, strict single crossing implies that if $t' \in BR_{x'}$ and $t'' \in BR_{x''}$, then $t' \leq t''$. Suppose that $BR_{x'}$ contained two bids, $t_1 < t_2$, for some type $x$. The first observation and Lemma 3 imply that for any $0 < \varepsilon < (t_2 - t_1)/4$, for sufficiently large

$n$ only players with types in $I = [\max\{x' - \varepsilon, 0\}, \min\{x' + \varepsilon, 1\}]$ may bid in the interval $[t_1 + (t_2 - t_1)/4, t_2 - (t_2 - t_1)/4]$.

Consider obtaining a prize that is $\Delta$ higher in the limit prize distribution[29] by increasing the bid from $t_1 + (t_2 - t_1)/4$ to $t_2 - (t_2 - t_1)/2$. If $\Delta$ is sufficiently small, then by continuity of $G^{-1}$ the increase in the prize is small as well, so the associated increment in utility is negative for all types, and uniformly bounded away from 0.

Therefore, taking $\Delta/2 = F(\min\{x' + \varepsilon, 1\}) - F(\max\{x' - \varepsilon, 0\})$, if $\varepsilon > 0$ is sufficiently small, then for sufficiently large $n$ every type of every player is better off bidding $t_1+(t_2-t_1)/4$ than bidding $t_2-(t_2-t_1)/2$. This is so, because with high probability the higher bid leads to a prize that is approximately only $\Delta/2$ higher in the prize distribution $G^n$.[30] By convergence of $(G^n)^{-1}$ to $(G)^{-1}$, for sufficiently large $n$ this prize is not much more than $\Delta/2$ higher in the limit prize distribution.

Moreover, every type of every player is better off bidding $t_1+(t_2-t_1)/4$ than bidding any bid in interval $(t_2-(t_2-t_1)/2, t_2-(t_2-t_1)/4)$, because such bids are even more costly than $t_2-(t_2-t_1)/2$, and enable a player to obtain a prize that is with high probability not much more than $\Delta/2$ higher in the limit prize distribution, than the prize the player obtains by bidding $t_2 - (t_2 - t_1)/2$. Therefore, no player bids in the interval $((t_2 - (t_2 - t_1)/2, t_2 - (t_2 - t_1)/4))$ with positive probability, so by the No-Gap Property $T^n(t_2 - (t_2 - t_1)/4) = 1$ for sufficiently large $n$. Thus, $T(t_2 - (t_2 - t_1)/4) = 1$, so $t_2$ cannot be in $BR_x$, because bidding slightly above $t_2 - (t_2 - t_1)/4$ gives type $x$ a higher payoff.

Consequently, $BR_x$ is a singleton for any $x$, and by strict single crossing, $br$ is weakly increasing. An argument analogous to the argument used to show that $BR$ is a singleton also shows that $br$ is continuous.[31]

[29] More precisely, given an initial prize $y'$, the prize that is $\Delta$ higher in the limit prize distribution is the prize $y''$ such that $\Delta = G(y'') - G(y')$. The prize that is $\Delta$ higher in the prize distribution $G^n$ is defined similarly.

[30] This follows from the convergence of $F^n$ to $F$ and Hoeffding's inequality applied to random variables

$$Z_i^n = \begin{cases} 1 \text{ if } \min\{x' + \varepsilon, 1\} \le x_i^n \le \max\{x' - \varepsilon, 0\}, \\ \qquad 0 \text{ otherwise,} \end{cases}$$

for $i = 1, ..., n$.

[31] More precisely, suppose that $br$ is discontinuous at some $x$, and apply the argument to $t_1 = br(x_1)$ and $t_2 = br(x_2)$ where $x_1$ and $x_2$ are slightly lower and higher, respectively, than $x$.

## A.5 Proof of Lemma 5

Consider an arbitrary type $x$. Let $x^{\min} = \min\{z : br(z) = br(x)\}$ and $x^{\max} = \max\{z : br(z) = br(x)\}$ ($x^{\min}$ and $x^{\max}$ are well defined because $br$ is continuous).

First, observe that $G^{-1}(F(x^{\min})) = G^{-1}(F(x^{\max}))$. Indeed, by Corollary 3, for sufficiently large $n$ all types in the interval $[x^{\min}, x^{\max}]$ bid in the $n$-th contest close to $br(x)$. Suppose that $G^{-1}(F(x^{\min})) < G^{-1}(F(x^{\max}))$, and consider the players whose type belongs to $[x^{\min}, x^{\max}]$ with positive probability.[32] Among these players, the one whose expected prize is the lowest contingent on having a type in this interval can profitably deviate to bidding slightly above $br(x)$, thereby outbidding the other players with a type in this interval and obtaining a discretely higher prize.

Suppose that $x^{\min} > 0$. By Corollary 3 for any $\delta > 0$, there is an $N$ such that if $n \geq N$, then the equilibrium bids of every player with type lower than $x^{\min} - \delta$ are lower than $br(x^{\min})$, and the equilibrium bids of every player with type higher than $x^{\min}$ are higher than $br(x^{\min} - \delta)$. Therefore, a player who bids $br(x^{\min})$ outbids all players with types lower than $x^{\min} - \delta$, so $T^n(br(x^{\min})) \geq (G^n)^{-1}(F^n(x^{\min} - \delta))$, and a player who bids $br(x^{\min} - \delta)$ is outbid by all players with types higher than $x^{\min}$, so $T^n(br(x^{\min} - \delta)) \leq (G^n)^{-1}(F^n(x^{\min}))$. Since $T^n$ converges to $T$, $T$ and $br$ are continuous, $(G^n)^{-1}$ converges to $(G^{-1})$, $F^n$ converges to $F$, and $F$ and $G^{-1}$ are continuous, we obtain $T(br(x^{\min})) = G^{-1}(F(x^{\min}))$.

Similarly, if $x^{\max} < 1$, we obtain that $T(br(x^{\max})) = G^{-1}(F(x^{\max}))$.

Thus, since $br(x) = br(x^{\min}) = br(x^{\max})$ and $G^{-1}(F(x)) = G^{-1}(F(x^{\min})) = G^{-1}(F(x^{\max}))$, we have that $T(br(x)) = G^{-1}(F(x))$ when $x^{\min} > 0$ or $x^{\max} < 1$. However, it cannot happen that $x^{\min} = 0$ and $x^{\max} = 1$, because $0 = G^{-1}(F(0)) < G^{-1}(F(1)) = 1$.

## A.6 Proof of Part (a) of Theorem 1

Consider a type $x \in X$. Let $t = br(x)$, and let $t'$ and $t''$ be such that $T(t') = T(t) - \varepsilon/3$ and $T(t'') = T(t) + \varepsilon/3$. Finally, let $x'$ and $x''$ be such that $t' = br(x')$ and $t'' = br(x'')$. (Take $x' = 0$ and $t' = 0$ if $T(t) - \varepsilon/3 < 0$, and $x'' = 1$ and $t'' = br(1)$ if $T(t) + \varepsilon/3 > 1$.)

By Lemma 3, for sufficiently large $n$, every player with type no higher than $x'$ bids less than the player with type $x$, and every player with type no lower than $x''$ bids more than the player with type $x$. By Hoeffding's inequality, this implies that the player of type $x$ outbids with high probability at least a fraction of players close to $F^n(x')$.

[32] For large $n$, at least a fraction of players close to $F(x^{\max}) - F(x^{\min})$ have types that belong to $[x^{\min}, x^{\max}]$ with positive probability.

Since $F^n$ converges to $F$, she outbids with high probability at least a fraction of players close to $F(x')$. So, since $(G^n)^{-1}$ converges to $G^{-1}$, she obtains (with high probability) a prize no lower than $G^{-1}(F(x')) - \varepsilon/3 = T(t) - 2\varepsilon/3$. Similarly, for sufficiently large $n$ a player of type $x$ outbids with high probability at most a fraction of players close to $F(x'')$, and so she obtains (with high probability) a prize no higher than $G^{-1}(F(x'')) = T(t) + 2\varepsilon/3$. (These bounds are immediate if $y - \varepsilon/2 < 0$ or if $y + \varepsilon/2 > 1$.) Thus, type $x$ obtains (with high probability) a prize which differs from $G^{-1}(F(x))$ by at most $2\varepsilon/3$.

This proves part (a) for a single $x$, but we must show that there is an $N$ such that for any $n \geq N$ part (a) holds for all $x$ simultaneously. Such an $N$ can be obtained by taking a finite grid of types $x$, and the corresponding grid of bids $br(x)$ such that $|T(t^1) - T(t^2)| < \varepsilon/3$ for any pair of neighboring elements $t^1$, $t^2$ of the grid, and taking the largest $N$ among the $N$'s corresponding to $x$'s from the grid.

# B Proof of Theorem 2

Recall that $G^{-1}(z) = \inf\{y : G(y) \geq z\}$ for $z > 0$ and $G^{-1}(0) = \inf\{y : G(y) > 0\}$, and note that $G^{-1}$ may be discontinuous (but is left-continuous). Discontinuities require modifying almost all the arguments used in the proof of Theorem 1. In order not to obscure the structure of the proof, we relegate to the end of the section the proofs of all intermediate results.

Let $I_0 = (y_0^l, y_0^u)$ be a longest interval in $[0, G^{-1}(1)]$ to which $G$ assigns measure 0; let $I_1 = (y_1^l, y_1^u)$ be a longest such interval disjoint from $I_1$, and so on. Then, every open interval of prizes that has measure zero is contained in one of the intervals $I_0, I_1, \ldots$. And for any $\varepsilon > 0$, there is a $K$ such that the lengths of $I_{K+1}$, $I_{K+2}$, ... sum up to less than $\varepsilon$.

The definitions of $R_i^n$, $A_i^n$, $A^n$, and $T^n$ are as in Section 6.1. The definition of $T$, however, must be changed. We first define a function $A$ on the rationals in $B$. Take an ordering of all rationals in $B$, denoted by $q_1, q_2, \ldots$. Take a converging subsequence of the sequence $A^n(q_1)$, denote it by $A^{n_1}(q_1)$, and denote its limit by $A(q_1)$. Take a converging subsequence of the sequence $A^{n_1}(q_2)$, denote it by $A^{n_2}(q_2)$, and denote its limit by $A(q_2)$. Continue in this fashion to obtain a function $A : \{q_1, q_2, \ldots\} \to [0, 1]$, which is weakly increasing (because each $A^n$ is). In addition, define a subsequence of $A^n$ such that its $k$-th element is the $k$-th element in the sequence $A^{n_k}$. For the rest of the proof, denote this new sequence by $A^n$ (with the corresponding sequence $T^n = (G^n)^{-1} \circ A^n$). Note that $A^n$ converges to $A$ on $\{q_1, q_2, \ldots\}$.

Let $T = G^{-1} \circ A$. Since $G$ may not have full support, we now have that $T(0) = \inf\{z :$

$G(z) > 0\}$ and $T(b_{\max}) = G^{-1}(1)$; in addition, $T$ is still (weakly) increasing (compare to properties (1)-(3) from Section A).

In addition, $F^n$ pointwise converges to $F$, but $(G^n)^{-1}$ may not pointwise converge to $G^{-1}$. However, the proof of property (b) from Section A gives that $\lim_n (G^n)^{-1}(r) = G^{-1}(r)$ unless $r$ is the value of $G$ on an interval $I_k = (y_k^l, y_k^u)$. Moreover, we can directly observe that $\lim_n (G^n)^{-1}(r) \geq G^{-1}(r)$ for every $r$ that is the value of $G$ on an interval $I_k = (y_k^l, y_k^u)$, but it can happen that $\lim_n (G^n)^{-1}(r) = y_k^u$ and $G^{-1}(r) = y_k^l$.

The discontinuities in $G^{-1}$ imply that $T$ defined on the rationals in $B$ may not be continuous, so Lemma 1 does not hold. Points of discontinuity, however, correspond to open intervals of prizes that have measure zero. More precisely, we have the following result.

**Lemma 6** *For any $t > 0$ in $B$ (not necessarily rational) one of the following conditions holds:*

1. *For any two sequences $q^m \uparrow t$ and $r^m \downarrow t$ of rationals in $B$, we have $\lim T(q^m) = \lim T(r^m)$.*

2. *There is some $k = 1, 2, ...$ such that for any two sequences $q^m \uparrow t$ and $r^m \downarrow t$, of rationals in $B$, we have $\lim T(q^m) = y_k^l$ and $\lim T(r^m) = y_k^u$. Moreover, $\lim A(q^m) = G(y_k^l)$ and $\lim A(r^m) = G(y_k^u)$.*

Using Lemma 6, we define a function $T^*$ on the entire interval $B$ by setting $T^*(t) = \lim T(r^m)$ for some sequence $r^m \downarrow t$ of rationals in $B$ (and $T^*(b_{\max}) = G^{-1}(1)$). Monotonicity of $T$ on the rationals guarantees that $T^*(t)$ is well-defined, and is the same regardless of the sequence $r^m$. In addition, it is easy to check that $T^*$ is (weakly) increasing, right-continuous, and continuous at every bid $t$ such that condition 1 from Lemma 6 holds. Note that $T^*$ may not be an extension of $T$, because when $\lim T(r^m) \neq T(t)$ for a rational $t$, we have that $T^*(t) = \lim T(r^m) \neq T(t)$.

Consider now a bid $t > 0$ such that condition 2 from Lemma 6 holds. Denote this bid $t$ by $t_k$, where $k$ is described in condition 2. Then, there is a bid $t' < t_k$ such that $A(t') = A(t) = G(y_k^l)$, so $A$ is constant on an interval below $t_k$. Indeed, if $A(t') < G(y_k^l)$ for all $t' < t_k$, then, as in the proof of Lemma 6, for large $n$ no player would bid any $t'$ slightly below $t_k$. This would be so, because bidding slightly above $t_k$ would almost certainly give a prize no lower than $y_k^u$, whereas bidding $t'$ would almost certainly give a prize no higher than $y_k^l$. Let

$$t_k^l = \inf\{t' : A(t') = G(y_k^l)\} < t_k.$$

It is also true that every maximal interval on which $T^*$ is constant with a value lower than $G^{-1}(1)$ is $\left[t_k^l, t_k\right)$ for some $k$. Indeed, consider a maximal nontrivial interval with lower bound $t^l$ and upper bound $t^u$ on which the value of $T^*$ is $y < G^{-1}(1)$. It suffices to show that $T^*(t^u) > y$, because then condition 2 from Lemma 6 applies to $t^u$, which implies that $t^u = t_k$ for some $k$; and the maximality of $\left[t^l, t^u\right)$ yields $t^l = t_k^l$. Suppose that $T^*(t^u) = y$. Then, for large enough $n$ bidding $t^u$ almost certainly gives a prize at most slightly higher than $y$, whereas bidding slightly above $t^l$ almost certainly gives a prize not much lower than $y$. But then, for large enough $n$, no player bids in some neighborhood of $t^u$, because bidding slightly above $t^l$ leads to a higher payoff. This contradicts the No-Gap Property, because $y < G^{-1}(1)$.

Because $G^{-1}$ may be discontinuous, $T^n$ need not uniformly converge to $T$ or $T^*$, even on the set of points at which they are continuous. In particular, for a rational $t$ in $\left[t_k^l, t_k\right)$ it may be that $T^n(t) = (G^n)^{-1}(A^n(t)) \geq y_k^u$ for arbitrarily large $n$, whereas $T(t) = T^*(t) = y_k^l$. Nevertheless, $T^n$ "converges uniformly" except on some neighborhoods of a finite number of intervals $[t_k^l, t_k]$. More precisely, we say that $T^n$ *converges uniformly to* $T^*$ *up to* $\beta$ on a set $C$ if there exists an $N$ such that for every $n \geq N$ and $t \in C$ we have that

$$|T^n(t) - T^*(t)| < \beta.$$

We then have the following modification of Lemma 2.

**Lemma 7** *For every $\beta > 0$, there exists a number $K$ such that for every $\gamma > 0$, $T^n$ converges uniformly to $T^*$ up to $\beta$ on the complement of*

$$O_\gamma = \bigcup_{k=1}^{K} (t_k^l - \gamma, t_k + \gamma).$$

We now relate players' equilibrium behavior in large contests to the inverse tariff $T^*$. Define $BR_x$, $BR(\varepsilon)$ and $BR_x(\varepsilon)$ as in Section 6.1 with $T^*$ instead of $T$ (the maximal payoff is achieved because $T^*$ is increasing and right-continuous, so is upper semi-continuous). Define *the mass expended (in the $n$-*th contest*) in an interval of bids $I$ by players with type $x \in S$* as $\left(\sum_{i=1}^n \Pr(\sigma_i^n \in S \times I)\right)/n$. We then have the following result, which we use in proving the remaining results.

**Lemma 8** *For all $k$ and any $\varepsilon > 0$ and $L > 0$, there exists $\gamma > 0$ such that for sufficiently large $n$ we have that:*

*(i) The mass expended in $(t_k^l - \gamma, t_k^l + \gamma)$ by players with types $x$ for which $t_k^l \notin BR_x(\varepsilon)$ is less than $\varepsilon/3L$;*

*(ii) The mass expended in $(t_k - \gamma, t_k]$ by players with types $x$ for which $t_k \notin BR_x(\varepsilon)$ is less than $\varepsilon/3L$.*

*In addition, for any $\alpha > 0$, for sufficiently large $n$ we have that:*

*(iii) The mass expended in $\left[t_k^l + \alpha, t_k - \alpha\right]$ by all players is less than $\varepsilon/3L$.*

Lemma 3 must also to be modified.

**Lemma 9** *For every $\varepsilon > 0$, there exist $K$ such that for every $\gamma > 0$, there is an $N$ such that for every $n \geq N$ in the equilibrium of the $n$-th contest every best response of every type $x_i^n$ of every player $i$ belongs to*

$$BR_{x_i^n}(\varepsilon) \cup \bigcup_{k=1}^{K} (t_k^l - \gamma, t_k).$$

Strict single crossing no longer implies that $BR_x$ is a singleton. Instead, we have the following result.

**Lemma 10** *If strict single crossing holds, then for all but a countable number of types the set $BR_x$ is a singleton. For those types for which it is not a singleton, $BR_x$ contains precisely two elements: $t_k^l$ and $t_k$ for some $k$. The correspondence that assigns to type $x$ the set $BR_x$ is weakly increasing (i.e., for any $x' < x''$, if $t' \in BR_{x'}$ and $t'' \in BR_{x''}$, then $t' \leq t''$) and upper hemi-continuous.*

Let $br(x) = \min BR_x$, and note that $br$ is increasing and left continuous, and is not right continuous precisely at types $x$ for which $BR_x$ is not a singleton. We then have the following corollary of Lemmas 8, 9, and 10, which is a modification of Corollary 3.

**Corollary 4** *For every $\varepsilon > 0$, there is an $N$ such that for $n \geq N$ a fraction $1 - \varepsilon$ of of players $i$ bid in $(br(x_i^n) - \varepsilon, br(x_i^n) + \varepsilon)$ with probability at least $1 - \varepsilon$.*

To prove part (b) of the theorem it remains to show that $T^* \circ br$ is the assortative allocation. This is done by the following lemma, which is a modification of Lemma 5 that accommodates the discontinuities in $T^*$ and $br$.

**Lemma 11** $G^{-1}(F(x)) = T^*(br(x))$ *for any type $x > 0$.*

To complete the proof, it remains to show (a) in the statement of the theorem. To do so, we use the following result, which we also need to prove Theorem 3 without assuming full support of prizes.

**Lemma 12** *For every $\varepsilon, \delta > 0$, there is an $N$ such that for $n \geq N$, each type $x$ from a set whose $F^n$-measure is at least $1-\varepsilon$ bids at least with probability $1-\varepsilon$ a $t$, and obtains at least with probability $1-\varepsilon$ a $y$ such that:*

*(1) $|y - T^*(t)| < \delta$, or*

*(2) $|t - r| < \delta$ and $|y - T^*(r)| < \delta$ for some $r$ in $BR_x$.*

*Moreover, if strict single crossing holds, then (2) must hold for every such $(x, y, t)$.*

To see that Lemma 12 implies (a) in the statement of the theorem, choose some $\varepsilon > 0$. Lemma 12 shows that for every $\delta > 0$, there is an $N$ such that for $n \geq N$ and for a fraction $1-\varepsilon$ of players $i$, the $F_i^n$-measure of their types $x_i^n$ that satisfy the condition of Lemma 12 is at least $(1-\varepsilon)$. This means that each such player $i$ obtains with probability at least $1-\varepsilon$ a prize $y$ that differs by at most $\delta$ from the prize $T^*(t)$ for some optimal bid $t$ of the player's type. For types $x_i^n > 0$ such that $br\left(x_i^n\right)$ is a unique optimal bid, this yields (a) by Lemma 11. However, by Lemma 9 and strict single crossing, there is only a countable number of other types $x_i^n$. And the $F$-measure of such types is 0 since $F$ has no atoms, so the $F^n$-measure of such types is arbitrarily small for sufficiently large $n$.

## B.1 Proof of Lemma 6

Let $\lim T\left(q^m\right) = y'$ and $\lim T\left(r^m\right) = y''$. Both limits $y'$ and $y''$ exist and $y' \leq y''$ by monotonicity. Suppose that $y' < y''$. If $G$ assigns a positive measure to $(y', y'')$, then it assigns a positive measure to any interval with endpoints sufficiently close to $y'$ and $y''$. In such a case, we obtain a contradiction by arguments similar to those used in the proof of Lemma 1. Indeed, for sufficiently large $n$ no bidder would bid slightly below $t$, because bidding slightly above $t$ would almost certainly give a better prize.

Thus, $G$ assigns measure zero to $(y', y'')$. This implies that $(y', y'') \subseteq (y_k^l, y_k^u)$ for some $k$. By definition, $T$ takes values in $[0, y_k^l] \cup [y_k^u, 1]$, so $y' = y_k^l$ and $y'' = y_k^u$. Moreover, the monotonicity of $T$ implies that $k$ is the same for any sequences $q^m \uparrow t$ and $r^m \downarrow t$ of rationals in $B$. It remains to show that $\lim A\left(q^m\right) = G(y_k^l)$ and $\lim A\left(r^m\right) = G(y_k^u)$.

For this, note that if $\lim A\left(q^m\right) > G(y_k^l)$, then $\lim T\left(q^m\right) > y_k^l$. Similarly, if $\lim A\left(r^m\right) > G(y_k^u)$, then $\lim T\left(r^m\right) > y_k^u$. The inequalities $\lim A\left(q^m\right) < G(y_k^l)$ and $\lim A\left(r^m\right) < G(y_k^u)$ can be ruled out similarly if $G$ does not have atoms at $y_k^l$ or $y_k^u$. Suppose that $G$ has an atom at $y_k^u$ and $\lim A(r^m) < G(y_k^u)$. Since $\lim T(r^m) = y_k^u$, $A(r^m) > G(y_k^l)$ for sufficiently large $m$. Take two rationals $r^m$ such that $G(y_k^l) < A(r^m) < G(y_k^u)$; denote them by $t' < t''$. Then, for sufficiently large $n$ any player obtains a prize close to $y_k^u$ with arbitrarily high probability by bidding any $t \in [t', t'']$. Thus, for sufficiently large $n$, no player would bid in the interval $[(t' + t'')/2, t'']$ with positive probability. This contradicts the No-Gap Property.

Suppose that $G$ has an atom at $y_k^l$ and $\lim A(q^m) < G(y_k^l)$. Then, for sufficiently large $n$ bidding $q^m$ almost certainly gives a prize at most slightly better than $y_k^l$. In contrast, bidding $r^m$ almost certainly gives a prize at least as good as $y_k^u$. This follows directly from (11) if $G$ has an atom at $y_k^u$. If $G$ does not, then this again follows from (11) for large enough $n$, because $A(r^m) > G(y_k^u)$ for any $m$. For large enough $n$ a contradiction with the No-Gap Property is obtained similarly to the last part of the proof of Lemma 1 that deals with $U(0, y, t)$ strictly increasing in $y$.

## B.2 Proof of Lemma 7

The proof is analogous to the proof of Lemma 2. Take a $K$ such that the lengths of $I_{K+1}$, $I_{K+2}, \ldots$ sum up to less than $\beta/2$. Take any $\gamma > 0$, and suppose to the contrary that there is an increasing sequence of integers $n_1, n_2, \ldots, n_m, \ldots$ such that for every $n_m$ there is some bid $t_m \notin O_\gamma$ with $|T^{n_m}(t_m) - T^*(t_m)| \geq \beta$. Passing to a subsequence if necessary, we assume that the sequence $t_m \to t$. Take rationals $q'$ and $q''$ such that $q' < t < q''$ and $T^*(q'') - T^*(q') < \beta/2$,[33] and

$$[q', q''] \subset B - \bigcup_{k=1}^{K} [t_k^l, t_k].$$

This is possible, since the lengths of $I_{K+1}$, $I_{K+2}, \ldots$ sum up to less than $\beta/2$. In addition, for large enough $k$ we have that $|T^{n_k}(q') - T^*(q')| < \beta/2$ and $|T^{n_k}(q'') - T^*(q'')| < \beta/2$, since the length of each $I_{K+1}$, $I_{K+2}, \ldots$ is less than $\beta/2$. The rest of the proof coincides with the proof of Lemma 2.

## B.3 Proof of Lemma 8

First, observe that the maximal payoff of type $x$, attained at any bid in $BR_x$, is still continuous in $x$. Indeed, upper semi-continuity of $T^*$is all that is needed for the continuity of the maximal payoff. This observation implies that there exists a $\delta > 0$ such that for any type $x$

[33] If $t = 0$, take $q' = 0$.

any bid in the complement of $BR_x(\varepsilon)$ gives type $x$ a payoff lower by at least $\delta$ than any bid in $BR_x$.

For (i), suppose the contrary that for any $\gamma > 0$ there are arbitrarily large $n$ such that the mass expended in $(t_k^l - \gamma, t_k^l + \gamma)$ by players with types $x$ for which $t_k^l \notin BR_x(\varepsilon)$ is at least $\varepsilon/3L$. Take $\gamma$ small enough so that the payoff that such players obtain by bidding slightly more than any bid in $BR_x$ is higher by $\delta/2$ than the payoff that they would obtain by bidding $t_k^l - \gamma$ and getting $y_k^l$.[34]

Suppose first that $t_k^l > 0$. We can assume, that $t_k^l - \gamma$ is a rational. By monotonicity of $A$ and the definition of $t_k^l$, we have that $A\left(t_k^l - \gamma\right) < G\left(y_k^l\right)$. Take a positive $\alpha < \varepsilon/6L$ such that $A\left(t_k^l - \gamma\right) < G\left(y_k^l\right) - \alpha$. For any $t \geq t_k^l - \gamma$ and sufficiently large $n$, if $A^n(t) < G\left(y_k^l\right) - \alpha/2$, then no player of type $x$ such that $t_k^l \notin BR_x(\varepsilon)$ bids $t$, because by bidding $t$ such a player would obtain with high probability a prize no higher than $y_k^l$, and therefore would obtain a higher payoff by bidding slightly more than any bid in $BR_x$.

Let $\gamma_n$ be defined by $t_k^l - \gamma_n = \inf\left\{t : A^n(t) \geq G\left(y_k^l\right) - \alpha\right\}$. Since $A\left(t_k^l - \gamma\right) < G\left(y_k^l\right) - \alpha$ and $A^n(t)$ is right-continuous, we have that $\gamma_n < \gamma$ (for sufficiently large $n$). And since for every $t < t_k^l - \gamma_n$ we have $A^n(t) < G\left(y_k^l\right) - \alpha/2$ (by definition of $\gamma_n$), players with types $x$ for which $t_k^l \notin BR_x(\varepsilon)$ must expend the mass of at least $\varepsilon/3L$ in $[t_k^l - \gamma_n, t_k^l + \gamma)$.

If more than half of this mass is expended in $\left(t_k^l - \gamma_n, t_k^l + \gamma\right)$, then we have that $A^n\left(t_k^l + \gamma\right) > A^n\left(t_k^l - \gamma_n\right) + \varepsilon/6L \geq G\left(y_k^l\right) - \alpha + \varepsilon/6L > G\left(y_k^l\right)$. This cannot happen for sufficiently large $n$, because for $t \in \left[t_k^l, t_k\right)$ is $A(t) = G\left(y_k^l\right)$. Thus, the players with types $x$ for which $t_k^l \notin BR_x(\varepsilon)$ bid precisely $t_k^l - \gamma_n$ with probability at least $\varepsilon/6L$. Since these players tie with each other at $t_k^l - \gamma_n$, by bidding $t_k^l - \gamma_n$ they must obtain a prize of a specific type $y$ with probability 1, even if they lose all ties at $t_k^l - \gamma_n$. (Otherwise, each of them could obtain a higher payoff by bidding slightly above $t_k^l - \gamma + \gamma_n$ and winning the ties at $t_k^l - \gamma_n$.) But a player who loses all ties at $t_k^l - \gamma_n$ has rank order no higher than $G\left(y_k^l\right) - \alpha$, by definition of $\gamma_n$, so $y \leq y_k^l$. Therefore, such a player would obtain a strictly

[34] To see why bidding slightly above any $t \in BR_x$ gives at least a payoff close to $U(x, T^*(t), t)$, consider the following two cases:

(a) $G^{-1}(A(r^m)) > T^*(t)$ for all rationals $r^m > t$; in this case, since $\lim_n (G^n)^{-1}(A(r^m)) \geq G^{-1}(A(r^m))$, for any $r^m > t$, if $n$ is sufficiently large, then $(G^n)^{-1}(A(r^m)) > T^*(t)$. This implies that a player obtains a prize higher than $T^*(t)$ with arbitrarily high probability by bidding $r^m$.

(b) $G^{-1}(A(r^m)) = T^*(t)$ for rationals $r^m > t$ close enough to $t$; in this case, $T^*(r^m) = T^*(t)$ for such rationals $r^m$. This implies that $t = t_{k'}^l$ for some $k'$. The claim now follows from left-continuity of $G^{-1}$ and the fact that $\lim_n (G^n)^{-1}(q) \geq G^{-1}(q)$ for any $q$.

higher payoff by bidding slightly more than any bid in $BR_x$.

Now suppose that $t_k^l = 0$. Then $A(t_k^l) \le G\left(y_k^l\right)$. The case $A(t_k^l) < G\left(y_k^l\right)$ is handled as in the case $t_k^l > 0$ above. Suppose that $A(t_k^l) = G\left(y_k^l\right)$. Then, for any $\gamma > 0$ such that $t_k^l + \gamma < t_k$, for sufficiently large $n$ the mass expended in $\left(t_k^l, t_k^l + \gamma\right)$ by all players is smaller than $\varepsilon/6L$, because $A(t) = G\left(y_k^l\right)$ for any rational $t \in \left(t_k^l + \gamma, t_k\right)$. Thus, if (i) does not hold, for sufficiently large $n$ the mass expended precisely at $t_k^l$ by the players with types $x$ for which $t_k^l \notin BR_x(\varepsilon)$ is at least $\varepsilon/6L$, and so the ranking of a player who ties at $t_k^l$ and loses is at most $G\left(y_k^l\right) - \varepsilon/12L$. But in this case each player of type $x$ for which $t_k^l \notin BR_x(\varepsilon)$ would strictly prefer bidding slightly more than any bid in $BR_x$ to bidding $t_k^l$, a contradiction.

To show (ii), note that if $t_k = t_{k'}^l$ for some $k'$, then (ii) follows from (i). Thus, suppose that $t_k \neq t_{k'}^l$ for any $k'$. Suppose the contrary that for any $\gamma > 0$ there is an arbitrarily large $n$ such that the mass expended in $(t_k - \gamma, t_k]$ by players with types $x$ for which $t_k \notin BR_x(\varepsilon)$ is at least $\varepsilon/3L$. Take $\gamma$ small enough so that the payoff that such players obtain by bidding slightly more than any bid in $BR_x$ is higher by $\delta/2$ than the payoff that they would obtain by bidding $t_k - \gamma$ and getting $y_k^u$. Observe that for sufficiently large $n$, by bidding $t_k$ any player almost certainly obtains a prize at most slightly better than $T^*(t_k) = y_k^u$. This is so, because $t_k \neq t_{k'}^l$ and so $A(t_k) \neq G(y_{k'}^l)$ for any $k'$. Therefore, for large enough $n$ a player with type $x$ for which $t_k \notin BR_x(\varepsilon)$ would be better off bidding slightly above any $t \in BR_x$ than bidding in $(t_k - \gamma, t_k]$.

Part (iii) follows immediately from the fact that the value of $A$ on $\left[t_k^l, t_k\right)$ is $G\left(y_k^l\right)$, by the definition of $t_k^l$.

## B.4 Proof of Lemma 9

Take a $\delta > 0$ such that for any type $x$ any bid in the complement of $BR_x(\varepsilon)$ gives type $x$ a payoff lower by at least $\delta$ than any bid in $BR_x$. Take $\beta > 0$ such that for any type $x$, bid $t$, and prizes $y'$ and $y''$ with $|y' - y''| \le \beta$ we have

$$|U(x, y', t) - U(x, y'', t)| \le \frac{\delta}{3}.$$

Next, take a $K$ guaranteed by Lemma 7 for this $\beta$. In addition, take $K$ large enough so the lengths of $I_{K+1}$, $I_{K+2}, \ldots$ sum up to less than $\beta/2$. Finally, for any $\lambda > 0$ take an $N_\lambda$ that satisfies the definition of uniform convergence up to $\beta$ on the complement of $O_\lambda$. (Note that $K$ is the same for all $\lambda$.)

Suppose to the contrary of the statement of the lemma that there is a $\gamma > 0$ and a subsequence of contests such that a type $x_i^n$ of player $i$ in the $n$-th contest has a best response

$t^n$ to the strategies of the other players that does not belong to $BR_{x_i^n}(\varepsilon) \cup \bigcup_{k=1}^{K} (t_k^l - \gamma, t_k)$. As usual, we assume that the subsequence is the entire sequence; moreover, we assume that $x_i^n \to x^*$ and $t^n \to t^*$.

Consider the following two cases:

A. ($t^* \neq t_k$ for any $k = 1, ..., K$) In this case, for some $\lambda > 0$ there is a neighborhood of $t^*$ that is disjoint from $O_\lambda$. By uniform convergence of $T^n$ to $T^*$ up to $\beta$ on the complement of $O_\lambda$,

$$U\left(x_i^n, T^n\left(t^n\right), t^n\right) - U\left(x_i^n, T^*\left(t^n\right), t^n\right) \leq \frac{\delta}{3}$$

for $n \geq N_\lambda$. And because $t^n \notin BR_{x_i^n}(\varepsilon)$, for any $t \in BR_{x_i^n}$ we have

$$U\left(x_i^n, T^*\left(t\right), t\right) - U\left(x_i^n, T^*\left(t^n\right), t^n\right) \geq \delta.$$

Thus, we obtain

$$U\left(x_i^n, T^*\left(t\right), t\right) - U\left(x_i^n, T^n\left(t^n\right), t^n\right) \geq \frac{2\delta}{3}.$$

Observe that any bid $t'$ higher than $t$ guarantees, for sufficiently large $n$, a prize not much worse than $T^*(t)$ with arbitrarily high probability.[35]

We will now show that by bidding $t^n$, for sufficiently high $n$ type $x_i^n$ obtains with arbitrarily high probability a prize no better than $T^n\left(t^n\right) + \beta$. Indeed, since $t^*$ does not belong to $\left[t_k^l, t_k\right]$ for any $k \leq K$, we have that $A\left(t'\right)$ is bounded away from $G\left(y_1^l\right), \ldots, G\left(y_K^l\right)$ for rationals $t'$ sufficiently close to $t^*$. Therefore, $A^n\left(t^n\right)$ is also bounded away from $G\left(y_1^l\right), \ldots, G\left(y_K^l\right)$ for sufficiently large $n$. And for sufficiently large $n$, bidding $t^n$ gives with arbitrarily high probability a rank order arbitrarily close to $A^n\left(t^n\right)$. Since the lengths of $I_{K+1}$, $I_{K+2}, ...$ sum up to less than $\beta/2$, and for any $r$ other than $G\left(y_1^l\right), \ldots, G\left(y_K^l\right)$ and sufficiently large $n$ the difference between $(G^n)^{-1}(r)$ and $G^{-1}(r)$ is no larger than the length of $I_{K+1}$, by bidding $t^n$ a player obtains with arbitrarily high probability a prize no better than $(G^n)^{-1}\left(A^n\left(t^n\right)\right) + \beta$.

Therefore, by definition of $\beta$, we have that by bidding $t^n$ type $x_i^n$ obtains a payoff that is higher than $U\left(x_i^n, T^n\left(t^n\right), t^n\right)$ by at most slightly more than $\delta/3$. Consequently, for sufficiently large $n$ player $i$ would obtain by bidding some $t' > t$ a payoff strictly higher than by bidding $t^n$, a contradiction.

B. ($t^* = t_k$ for some $k = 1, ..., K$) Then, consider a $t^{**}$ slightly higher than $t^*$, such that $t^{**}$ does not belong to $[t_k^l, t_k]$ for $k = 1, ..., K$, and such that: (i) for sufficiently large $n$ the payoff (of any player) in the $n$-th contest of bidding $t^{**}$ is not much lower than the payoff of bidding $t^n$; (ii) for sufficiently large $n$, we have that the difference between $U(x_i^n, T^*(t), t)$

[35] To see why, see footnote 34.

for any $t$ in $BR_{x_i^n}$ and $U(x_i^n, T^*(t^{**}), t^{**})$ is not much lower than $\delta$. This latter condition is possible because, by definition, $(x^*, t^*) \notin BR(\varepsilon)$, and by right continuity of $T^*$ at $t^*$. Now, using (ii), apply an argument analogous to that from case A with $t^{**}$ playing the role of $t^n$, with a contradiction obtained by referring to (i).

## B.5 Proof of Lemma 10

Monotonicity of the correspondence follows from strict single crossing, and upper hemi-continuity follows from standard arguments.[36]

Suppose that $BR_x$ contains a pair of bids $t_1 < t_2$. Below we will show that for any $\varepsilon > 0$ and any interval $[a, b]$ such that $t_1 < a$ and $b < t_2$, for sufficiently large $n$ the mass expended in $[a, b]$ by all players is at most $\varepsilon$. This implies that the function $A$, and therefore $T^*$, is constant on every such interval $[a, b]$, and therefore on $(t_1, t_2)$. But $T^*(t_2) > T^*(t_1)$ because $t_1 < t_2$ are in $BR_x$, so by definition of the discontinuity points $t_k$ of $T^*$ we must have $(t_1, t_2) \subseteq \left(t_k^l, t_k\right)$ for some $k$. And because $BR_x \subseteq B \backslash \cup \bigcup_{k=1}^{\infty} (t_k^l, t_k)$, we have that $t_1 = t_k^l$ and $t_2 = t_k$.

It remains to show that for any $\varepsilon > 0$, for sufficiently large $n$ the mass expended in $[a, b]$ by all players is at most $\varepsilon$. We will show this for $\varepsilon/2$ and players of types lower than $x$ (a similar argument applies to types higher than $x$). Choose $x' < x$ such that $F(x) - F(x') < \varepsilon/3$. For sufficiently small $\lambda > 0$, $\sup \cup_{z \leq x'} BR_z(\lambda) < a$. (This is because $x' < x$ and $t_1 \in BR_x$, so every bid in $BR_z$ is at most $t_1 < a$.)

Therefore, by Lemma 9, there is some $K$ such that for every $\gamma > 0$ and sufficiently large $n$ any bid in $[a, b]$ made by a player of type $z \leq x'$ in the $n$-th contest is in $\bigcup_{k=1}^{K} (t_k^l - \gamma, t_k)$. Consider one of these $K$ intervals for which $(t_k^l - \gamma, t_k) \cap [a, b] \neq \varnothing$. Since $\sup \cup_{z \leq x'} BR_z(\lambda) < a \leq t_k$, $t_k$ is not in $BR_z(\lambda)$ for any $z \leq x'$. If $t_k^l > \sup \cup_{z \leq x'} BR_z(\lambda)$, then by (i) of Lemma 8 there exists a $\gamma$ such that for sufficiently large $n$ the mass expended in $\left(t_k^l - \gamma, t_k\right)$ by players of type $z \leq x'$ is less than $\varepsilon/6K$. If $t_k^l \leq \sup \cup_{z \leq x'} BR_z(\lambda)$, then by (ii) and (iii) of Lemma 8, for sufficiently large $n$ the mass expended in $[a, t_k)$ by players of type $z \leq x'$ is less than $\varepsilon/6K$.

Therefore, for large enough $n$ the mass expended in $[a, b]$ by players of type $z \leq x'$ is smaller than $\varepsilon/6$, and because $F(x) - F(x') < \varepsilon/3$, the mass expended in $[a, b]$ by players of type $z \leq x$ is smaller than $\varepsilon/2$.

[36] More precisely, this follows from the fact that $BR_x$ is the set of all $t$ such that $(t, T^*(t))$ maximizes type $x$'s utility over the closure of the graph of $T^*$, which is a compact set.

## B.6 Proof of Corollary 4

Choose $\varepsilon > 0$. Lemma 10 implies that there is a finite number of intervals of types with total $F$-mass $\varepsilon/2$, such that for every type $x$ not in one of these intervals, $BR_x \subseteq (br(x) - \varepsilon, br(x) + \varepsilon)$.[37] Consider the $F$-mass $1 - \varepsilon/2$ of types $x$ with the last property, and let $K$ be the one in the statement of Lemma 9. Then, by Lemma 9 and Lemma 8 for $L = K$, for sufficiently large $n$, at most an $F$-mass $\varepsilon/2$ of those types bid outside of $(br(x) - \varepsilon, br(x) + \varepsilon)$.

## B.7 Proof of Lemma 11

The proof is analogous to that of Lemma 5. Consider an arbitrary type $x$. Define $x^{\min} = \min\{z : br(x) \in BR_z\}$ and $x^{\max} = \max\{z : br(x) \in BR_z\}$. By strict single crossing, $BR_z$ has only one element $br(z) = br(x)$ for all $z \in (x^{\min}, x^{\max})$; it may have two elements for $z = x^{\min}$ or $x^{\max}$, in which cases $br(x)$ is the higher one and the lower one of the two, respectively.

The claim that $G^{-1}(F(x^{\min})) = G^{-1}(F(x^{\max}))$ is obtained by the same argument as in the proof of Lemma 5. The rest of the proof requires the following minor changes when $BR_{x^{\min}}$ has two elements (and analogous changes when $BR_{x^{\max}}$ has two elements):

1. Instead of $x^{\min}$, we consider $\underline{x}^{\min} = \min\{z : br(x^{\min}) \in BR_z\}$, and compare the equilibrium bids of every player with type lower than $\underline{x}^{\min} - \delta$ to $br(x^{\min})$, and the equilibrium bids of every player with type higher than $x^{\min}$ to $br(\underline{x}^{\min} - \delta)$. This change does not affect the arguments, since $G^{-1}(F(\underline{x}^{\min})) = G^{-1}(F(x^{\min}))$.

2. It may not be true that the equilibrium bids of every player with type lower than $\underline{x}^{\min} - \delta$ are lower than $br(x^{\min})$, or the equilibrium bids of every player with type higher than $x^{\min}$ are higher than $br(\underline{x}^{\min} - \delta)$, because players may bid in $\bigcup_{k=1}^{K}(t_k^l - \gamma, t_k) - BR(\varepsilon)$ (see Lemma 9). However, this happens only with vanishing probability as $n$ grows large, so the arguments are again not affected.

[37] There is a $K > 0$ such that $\sum_{k>K}(t_k - t_k^l) < \varepsilon$. For each $k \leq K$ such that $BR_{x_k} = \{t_k^l, t_k\}$ for some type $x_k$, consider the interval of types $[x_k - \lambda, x_k + \lambda] \cap [0,1]$, where $\lambda$ is such that (continuous) $F$ increases by no more than $\varepsilon/2K$ on any interval no larger than $2\lambda$. The sum of the $F$-mass of these intervals is no larger than $\varepsilon/2$, and the sum of the "jumps" of $br$ on the complement of these intervals is smaller than $\varepsilon$.

## B.8 Proof of Lemma 12

Take any $\lambda > 0$. By Lemma 9, there is a large $K$ such that for any $\gamma > 0$, if $n$ is sufficiently large, the equilibrium bid of every player $i$ in the $n$-th contest belongs with probability 1 to

$$BR_{x_i^n}(\lambda) \cup \bigcup_{k=1}^{K} (t_k^l - \gamma, t_k).$$

Assume that $K$ is, in addition, large enough so that the lengths of $I_{K+1}$, $I_{K+2}, \ldots$ sum up to less than $\delta/2$.

We first claim that for any $t \notin (t_k^l - \gamma, t_k)$ for all $k = 1, \ldots, K$, there exists an $N_t$ such that for every $n \geq N_t$, a player who bids $t$ in the $n$-th contest obtains (with high probability) a prize $y$ such that $|y - T^*(t)| < \delta/2$. We will also show that there exists an $N = N_t$ that is common for all such bids $t$.

Suppose first that $t \neq t_k$ for any $k = 1, \ldots, K$. Since $A(t)$ differs from $G(y_k^l)$ and $G(y_k^u)$ for any $k = 1, \ldots, K$, any rank order close to $A(t)$ also differs from $G(y_k^l)$ and $G(y_k^u)$. By (11), for sufficiently large $n$, a player who bids $t$ has (with high probability) a rank order close to $A(t)$; in particular, this rank order differs from $G(y_k^l)$ and $G(y_k^u)$. By the assumption that the lengths of $I_{K+1}$, $I_{K+2}, \ldots$ sum up to less than $\delta/2$, this implies that the difference between $T^*(t)$ and the prize obtained by a player who bids $t$ is lower than $\delta/2$ (with high probability).

Suppose that $t = t_k$ for some $k = 1, \ldots, K$. By an argument analogous to the one used in the previous case, the prize obtained by a player who bids $t$ cannot, as $n$ increases, exceed $T^*(t)$ by $\delta/2$ with a probability that is bounded away from 0. And $T^*(t)$ cannot exceed this prize by $\delta/2$ with a probability that is bounded away from 0 as $n$ increases, because the player would profitably deviate by bidding slightly above $t$, which would guarantee a prize no worse than $T^*(t)$ with arbitrarily high probability.

Now, note that the number $N_t$ that was chosen for any bid $t$ has the required property also for all bids close enough to $t$; in the case of $t = t_k$ for some $k = 1, \ldots, K$, we mean bids close enough and higher than $t$. That is, for every $t$ there is a neighborhood $W_t$ of that $t$ with $N_t$ that is common for all bids from this neighborhood. The family of sets $W_t$ is an open covering of the compact set of bids $t$ that satisfy $t \notin (t_k^l - \gamma, t_k)$ for $k = 1, \ldots, K$. Thus, it contains a finite subcovering, and any number $N$ that exceeds numbers $N_t$ for all elements of this finite subcovering has the required property.

This yields part (1) of the lemma for bids $t \notin (t_k^l - \gamma, t_k)$ for all $k = 1, \ldots, K$. If strict single crossing holds, and $\lambda$ is sufficiently small, then part (2) of the lemma also holds for such bids. To see why, notice that $BR_x$ is a singleton, and so $br(x)$ is its only element, for all except a countable number of types $x$. Since $F$ has no atoms, the set of such types has $F$-measure 1. And for such types $x$, equilibrium bids $t \notin (t_k^l - \gamma, t_k)$ for all $k = 1, ..., K$ belong to $(br(x) - \lambda, br(x) + \lambda)$. If $\lambda$ is sufficiently small, and $x$ is bounded away from 0, then $|t - r| < \delta$ for $r = br(x)$, and $T^*(t) - T^*(r) \leq \delta/2$.[38] And if $\lambda$ is sufficiently small, then also $T^*(r) - T^*(t) \leq \delta/2$, because $t \notin (t_k^l - \gamma, t_k)$ for all $k = 1, ..., K$ and the lengths of $I_{K+1}$, $I_{K+2}, ...$ sum up to less than $\delta/2$. Finally, by our first claim, the prize $y$ obtained by bidding $t$ must satisfy $|y - T^*(t)| < \delta/2$, so $|y - T^*(r)| < \delta$.

Now consider bids $t$ such that $t$ is in $(t_k^l - \gamma, t_k)$ for some $k = 1, ..., K$. By (iii) of Lemma 8, we can disregard bids $t$ in $[t_k^l + \gamma, t_k - \gamma]$. Suppose that $t$ is in $(t_k - \gamma, t_k)$ and $t_k \neq t_{k'}^l$ for all other $k' = 1, ..., K$. By (ii) of Lemma 8, one can assume that $t_k \in BR_{x_i^n}$.[39] We will show that for sufficiently small $\gamma$ and for sufficiently large $n$, player $i$ obtains by bidding $t$ (with arbitrarily high probability) a prize $y$ in $(T^*(t_k) - \delta, T^*(t_k) + \delta)$. First, notice that player $i$ cannot obtain by bidding $t$ a prize lower than $T^*(t_k) - \delta$ (with probability bounded away from 0), because for small enough $\gamma$ it would be profitable to deviate to bidding slightly above $t_k$, and obtain a prize not much lower than $T^*(t_k)$ with high probability. Player $i$ cannot obtain by bidding $t$ a prize higher than $T^*(t_k) + \delta$ (with probability bounded away from 0), because by (11), for any rational $r^m > t_k$ and sufficiently large $n$ the rank order of player $i$ is with arbitrarily high probability bounded above by $A(r^m)$ . Thus, the upper bound on the prize follows from the assumption that $t_k \neq t_{k'}^l$ for all other $k' = 1, ..., K$, and the lengths of $I_{K+1}, I_{K+2}, ...$ sum up to less than $\delta/2$.

Finally, suppose that $t$ is in $\left(t_k^l - \gamma, t_k^l + \gamma\right)$ for some $k = 1, ..., K$. By (i) of Lemma 8, one can assume that $t_k^l \in BR_{x_i^n}$. We will show that for sufficiently small $\gamma$ and for sufficiently large $n$, equilibrium bidding in $\left(t_k^l - \gamma, t_k^l + \gamma\right)$ leads (with arbitrarily high probability) to a prize $y \in (T^*(t_k^l) - \delta, T^*(t_k^l) + \delta)$, except a small probability event. Indeed, by an argument similar to that from the previous case, such a bid cannot lead to a prize lower than $T^*(t_k^l) - \delta$ (with probability bounded away from 0). To obtain a prize higher than $T^*(t_k^l) + \delta$ with a nonvanishing probability, a player's expected rank order when bidding $t$ cannot be lower than $G\left(y_k^l\right)$ by a nonvanishing constant. But, if a nonvanishing fraction of players win a

[38] Indeed, for types bounded away from 0, and for sufficiently small $\lambda$, we have that $U(x, y, t) > U(x, y', t')$ whenever $y - y' > \delta/2$ and $t - t' < \lambda$. (The assumption that types are bounded away fro 0 is essential, because we did not assume that $U(0, y, t)$ strictly increases in $y$.) However, since $r = br(x)$, we cannot have $U(x, T^*(t), t) > U(x, T^*(r), r)$.

[39] The lemma says only that the mass expended in $(t_k - \gamma, t_k]$ by types $x$ for which $t_k \notin BR_x(\lambda)$ for some small $\lambda > 0$ is small. However, if $\lambda > 0$ is sufficiently small, then the mass of types $x$ such that $t_k \notin BR_x$ but $t_k \in BR_x(\lambda)$ is small.

prize higher than $T^{*}(t_k^l)+\delta$ with a nonvanishing probability by bidding in $\left(t_k^l-\gamma, t_k^l+\gamma\right)$, then the increase in expected rank order on the interval $\left(t_k^l-\gamma, t_k^l+\gamma\right)$ is bounded away from 0 for all $n$, which contradicts the fact that $A^n(t_k^l+\gamma)$ approaches $G(y_k^l)$ as $n$ increases.

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