Large Contests

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1 Introduction

This paper shows that the equilibrium outcomes of contests with a large (but finite) number of competitors and prizes can be approximated by the outcome of a certain mechanism in an environment with a single agent that has a continuum of possible types. The approximation applies even when solving for equilibrium is difficult or impossible, which substantially expands the set of contests that can be studied relative to the existing literature.

In our contest framework, n players compete for n prizes (some of which may be worth 0). Each player chooses a non-negative bid, and the player with the highest bid obtains the highest prize, etc. A player's payoff depends on her bid, her type, and the prize she obtains, and satisfies strict single crossing in type and prize-bid pairs. This accommodates a wide range of asymmetries among players and heterogeneity in prizes. Players' types are distributed independently, but not necessarily identically, which accommodates both complete-information and incomplete-information asymmetric contests.

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We study sequences of contests whose empirical distributions of player types and prizes converge as n grows large. Given the limit type and prize distributions, the approximating mechanism implements the assortative allocation of prizes to types, in which the location of a prize in the prize distribution is equal to the location of the type in the type distribution to which the prize is allocated. This mechanism corresponds to an "inverse tariff," in which each bid determines a single prize.

When payoffs are quasi-linear with respect to bids, the approximating mechanism is characterized by applying standard mechanism-design techniques. We are then able to say (approximately, but with an arbitrary degree of precision as n increases, and uniformly across all equilibria) how almost each player will bid, and what prize she will obtain by making any given bid. This result applies to many existing contest models, which are surveyed in Section 1.1, as well as to contest specifications for which no equilibrium characterization exists.

As an illustration, we derive simple approximations for some settings that have been studied in the literature, in which the equilibria were complicated, derived by algorithms, or could be solved only for specific functional forms. Our methods also facilitate further analysis of contests, including contest design, welfare analysis, and comparative statics, by making it possible to recast such questions as questions in mechanism design, which are relatively well understood. This is left for future research.

The rest of the paper is organized as follows. Section 1.1 surveys the related literature. Section 2 introduces the basic terminology and notation. Section 3 presents examples that illustrate our results in some settings that have been studied in the literature. Section 4 contains our main results and some discussion of their application. Section 5 contains the intuition for our results, and discusses the contribution to the methods for studying large games. Section 6 provides rates of convergence for a class of contests with quasi-linear payoffs. The Online Appendix contains the proofs of our main result when the limit set of prizes has full support and when the limit set of prizes may not have full support. Both proofs have a similar structure. The latter, which builds on the former, emphasizes arguments for dealing with the discontinuities that arise when the limit set of prizes does not have full support.

1.1 Related Literature

Our model includes many variants of the multi-prize all-pay auction with complete and incomplete information, in which each player chooses a bid and pays the associated (and possibly idiosyncratic) cost.[1] Closed-form equilibrium characterizations exist for complete-

[1] Other models postulate a probabilistic relation between competitors' efforts and prize allocation. See Tullock (1980) and Lazear and Rosen (1981). For a comprehensive treatment of the literature on competitions

information contests with two participating players (Hillman and Samet (1987), Hillman and Riley (1989), Che and Gale (1998, 2006), Kaplan and Wettstein (2006), Siegel (2010)), with identical prizes and costs (Baye, Kovenock, and de Vries (1993, 1996), González-Díaz (2012), Clark and Riis (1998)), and with identical players (Barut and Kovenock (1998)). Algorithmic equilibrium characterizations exist for some complete-information contests with identical prizes and heterogeneous costs (Siegel 2010, 2014b),[2] and with heterogeneous prizes and identical costs (Bulow and Levin (2006), González-Díaz and Siegel (2012), Xiao (2013)). The heterogeneity in prizes, however, is limited to very specific functional forms. Moreover, algorithmic characterizations make further analysis difficult or impossible. Incomplete-information contests have been solved when there are two players (for example, Amann and Leininger (1996), Siegel (2014a)), and when players are ex-ante identical (for example, Krishna and Morgan (1997), Moldovanu and Sela (2001, 2006)). In contrast, our model accommodates ex-ante asymmetric players, heterogeneous prizes, and complete and incomplete information.

Our paper also contributes to the literature on large games, which typically makes continuity assumptions that exclude auction-like games (for example, Kalai (2004)). An exception is Bodoh-Creed (2013), who explicitly considers uniform-price auctions with incomplete information, but assumes enough uncertainty about the set of prizes to exclude the possibility of a small change in the rank order of a bid having a large effect on the prize obtained. Moreover, the analysis in this literature often focuses on ε-equilibria of large games, which may not approximate Nash equilibria well. In contrast, our approach deals with the discontinuities that arise naturally in contests, approximates Nash equilibria, and uncovers a novel connection to mechanism design.

A more closely related paper is Bodoh-Creed and Hickman's (2015) theoretical analysis of affirmative action in college admissions. They considers an additively separable contest model with incomplete information that satisfies strict single crossing, and approximate the outcome for a large number of applicants by a continuum model in which the limit set of prizes has full support (so a small change in the rank order of a bid cannot have a large effect on the prize obtained). Our paper differs from this work in three main ways. First, our approach does not require additive separability or full support of the limit prize set (although we obtain stronger results under these conditions), and is therefore applicable to a wider range of settings; more importantly, we allow for a general limit prize set, which facilitates the study

with sunk investments, see Nitzan (1994) and Konrad (2007).

[2] Siegel (2009, 2014c) gives a closed-form expression for players' equilibrium payoffs, but does not solve for equilibrium strategies.

of optimal contest design. Second, we relate the outcomes of large contests to mechanisms, which allows us, under certain conditions, to derive the approximation in closed form. Third, our model accommodates both complete information and incomplete information.

2 Terminology and notation

2.1 Agents and prizes

An agent is characterized by a type $x \in X = [0,1]$. We will use the terms "player" for discrete contests and "agent" for the limit case. A prize is characterized by a number $y \in Y = [0,1]$. Prize 0 is "no prize."

Agents' utilities are given by a continuous function $U(x,y,t)$, where x is the agent type, y is the single prize she obtains, and $t \geq 0$ is her bid. The utility of obtaining no prize by bidding 0 is normalized to 0, i.e., $U(x,0,0) = 0$ for all x. Higher prizes are better and higher bids are more costly, so $U(x,y,t)$ strictly increases in y for every $x > 0$ and $t \geq 0$, and strictly decreases in t for every $x \geq 0$ and $y \geq 0$. The utility satisfies strict single crossing, i.e., if an agent of some type prefers to obtain a higher prize at a higher bid, then an agent of any higher type strictly prefers to obtain the higher prize at the higher bid.[3] Sufficiently high bids are prohibitively costly, so $U(x,1,b_{\max}) < 0$ for some $b_{\max}$ and all x. We therefore restrict the range of bids that agents can make to $B = [0, b_{\max}]$.

The utility is *quasi-linear* in bid if it can be written as

$$U(x,y,t) = v(x,y) - t.$$

An example that we will use throughout the paper is the quasi-linear utility

$$U(x,y,t) = xh(y) - t, \tag{1}$$

where $h(0) = 0$ and h is continuous and strictly increasing. This functional form generalizes many of the ones used in existing contest models, including those described in Section 3.

2.2 Contests

For every n, we define "the n-th contest," in which n players compete for n known prizes $y_1^n \leq y_2^n \leq ... \leq y_n^n$ (some of which may be no prize). Player i's privately-known type

[3] That is, for any $x_1 < x_2$, $t_1 < t_2$, and $y_1 < y_2$ we have that $U(x_1,y_2,t_2) \geq U(x_1,y_1,t_1)$ implies $U(x_2,y_2,t_2) > U(x_2,y_1,t_1)$.

x_i^n is distributed according to a CDF F_i^n, and these distributions are commonly known and independent across players.[4] In the special case of complete information, each CDF corresponds to a Dirac measure. Each player, after learning her type, chooses a bid in B, the player with the highest bid obtains the highest prize, the player with the second-highest bid obtains the second-highest prize, and so on. Ties are resolved by a fair lottery. The utility of player i from bidding t and obtaining prize y_j^n is $U\left(x_i^n, y_j^n, t\right)$. A slight adaptation of the proof of Corollary 1 in Siegel (2009) shows that when each player's set of possible types is finite the contest has at least one mixed-strategy Bayesian Nash equilibrium. For general distributions F_i^n, equilibrium existence follows from Corollary 5.2 in Reny (1999).

We let $F^n = \left(\sum_{i=1}^n F_i^n\right)/n$, so $F^n(x)$ is the expected percentile ranking of type x given the vector of players' types. We denote by G^n the empirical distribution of prizes, which assigns a mass of $1/n$ to each y_j^n (recall that each prize y_j^n is known). We assume that F^n converges pointwise to a continuous and strictly increasing distribution F, and G^n converges pointwise to some distribution G.[5] We elaborate on this assumption in the next subsection. If G strictly increases on Y, we say that G *has full support*, or that *prizes have full support*. Note that G may have full support even if there are masses of identical prizes, i.e., G is discontinuous, or, equivalently, G has atoms.

Our results are stronger and their proofs are simpler when prizes have full support. We wish to emphasize, however, the importance of the results for general distributions G. One reason is that without such general results our methods would be of limited use in studying optimal contest design.[6]

2.3 Convergence of type and prize distributions

The convergence of F^n and G^n to limit distributions F and G can be interpreted in several ways. First, a modeler studying large contests may specify the limit distributions directly and consider a sequence of discrete contests with distributions that converge to the limit distributions. Examples include contests with complete information in which player i's type is $x_i^n = F^{-1}(i/n)$ and prize j is $y_j^n = G^{-1}(j/n) = \inf\{y : G(y) \geq j/n\}$, as well as contests with incomplete information in which players have IID type distributions $F_i^n = F$ and prize j is $y_j^n = G^{-1}(j/n)$. The former specification appears in the examples of Section 3.

[4] All probability measures are defined on the σ-algebra of Borel sets.

[5] Our proofs only require pointwise convergence of G^n to G at points of continuity of G, which is equivalent to convergence in weak*-topology.

[6] For example, it is plausible that in some settings the optimal prize structure is a mass of identical prizes.

Alternatively, the modeler may specify a sequence of contests using analytical formulas that depend on the number of players and prizes, and take the limit of the associated sequence of distributions F^n and G^n as the number grows large.

Finally, given a single discrete contest, a researcher can postulate limit distributions F and G to which the empirical type and prize distributions would converge if the number of players and prizes grew large; if the number of players and prizes in the given contest is sufficiently large, our approximation results can be applied. Moreover, in many settings (such as those of Corollaries 1 and 2 below) the empirical type and prize distributions of the given contest can be used as the limit distributions, since a small change in the limit distribution has a small effect on the approximation.[7]

2.4 Limit mechanism-design setting

An *inverse tariff* is a non-decreasing, upper semi-continuous function that maps bids to prizes. Given an inverse tariff, a *tariff mechanism* prescribes for each type x a prize-bid pair (y,t) that maximizes $U(x,y,t)$ among the prize-bid pairs in the graph of the inverse tariff. In addition, we require that $U(x,y,t) \geq U(x,y_{\text{inf}},0)$ for each type x and its prescribed prize-bid pair (y,t), with an equality for at least one type x, where $y_{\text{inf}} = \inf\{y : G(y) > 0\}$. Note that if G has full support, then $y_{\text{inf}} = 0$ and $U(x,y_{\text{inf}},0) = 0$. Every tariff mechanism is therefore incentive compatible and individually rational.

The *assortative allocation* maps to each type x prize $y = G^{-1}(F(x))$, where $G^{-1}(z) = \inf\{y : G(y) \geq z\}$ for $z > 0$ and $G^{-1}(0) = y_{\text{inf}}$. A tariff mechanism *implements* the assortative allocation if it prescribes for each type x prize $G^{-1}(F(x))$.

3 Examples

We first demonstrate our approximation approach and results in two contest settings that appeared in the literature, by comparing our approximating mechanism to the contest equilibria. We focus on complete-information contests, because no equilibrium characterization exists for contests with incomplete information and more than two ex-ante asymmetric players.

[7] In the settings of Corollaries 1 and 2, a small change in F and G leads to a small change in the prize-bid pair that the approximating mechanism specifies for each type (the pair is given by the "assortative allocation," defined in Section 2.4, and by (4)).

3.1 Heterogeneous prizes and multiplicative utilities

Consider (1) with $h(y) = y$, let $x_i^n = i/n$ (so F_i^n is a Dirac measure), and let $y_j^n = j/n$. Thus, the limit distributions F and G are uniform. (Note that prizes have full support.) The n-th contest is an all-pay auction with n players and n prizes, and the value of prize j to player i is ij/n^2. These contests were studied by Bulow and Levin (2006), henceforth B&L, who considered hospitals that have a common ranking for residents and compete for them by offering identity-independent wages. Hospitals are players, their posted wages are bids, and residents are prizes. The best resident goes to the hospital with the highest wage, and so on.[8]

Consider the assortative allocation in the limit setting, which assigns prize x to an agent of type x. This allocation is implementable by a tariff mechanism that prescribes for every type x prize x and bid $x^2/2$. The corresponding inverse tariff is continuous, and maps every bid $t \in [0, 1]$ to prize $\sqrt{2t}$. Corollary 1 below shows that for large n this mechanism approximates the equilibrium outcome, in that every player i obtains a prize close to i/n and bids close to $(i/n)^2/2$.

This simple approximation contrasts with the algorithm developed by B&L to derive the unique mixed-strategy equilibrium of the n-th contest.

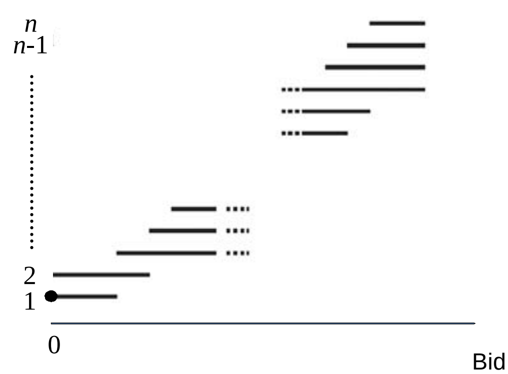

Figure 1: The support of players' strategies (dots represent atoms) in the unique equilibrium

B&L show that, in equilibrium, each player chooses a bid from an interval. The intervals are staggered, so a higher player has an interval with (weakly) higher lower and upper bounds. (The intervals are depicted in Figure 1.) In particular, if a bid t is contained in some player's bidding interval, then it is contained in the bidding intervals of consecutive

[8]The outcome coincides with that of Gale and Shapley's (1962) deferred acceptance algorithm when hospitals rank residents by their ability and residents rank hospitals by their wages.

players l, $l+1$,..., m, where l is the lowest player whose interval contains t, and m is the highest player whose interval contains t.

B&L show that

$$l = \arg\min_q \left\{ \frac{1}{m-q} \sum_{k=q}^{m} \frac{n^2}{k} - \frac{n^2}{q} > 0 \right\}, \tag{2}$$

and the density of the strategy of player q, $l \leq q \leq m$, at bid t that belongs to her bidding interval is

$$\frac{1}{m-l} \sum_{k=l}^{m} \frac{n^2}{k} - \frac{n^2}{l}. \tag{3}$$

By iteratively applying (2) and (3), B&L compute the endpoints of players' bidding intervals and the densities of their bidding strategies.

For the rate of convergence of our approximation, because $m-l$ is of order $\sqrt{2l}$ (Lemma 3 in B&L), any player i is outbid with certainty by every bidder $j > i$, except for a number of players j that is of order $\sqrt{n}$. Thus, player i obtains a prize that differs from i/n by at most an expression of order $1/\sqrt{n}$. A similar but slightly more involved argument shows that the bidding interval of player i shrinks quickly, so that any bid in the interval differs from $(i/n)^2/2$ by at most an expression of order $1/\sqrt{n}$.

3.2 Identical prizes

Consider (1) with $h(y) = y$, let $x_i^n = i/n$ (so F_i^n is a Dirac measure), and let $y_j^n = 0$ if $j/n \leq 1/2$ and $y_j^n = 1$ if $j/n > 1/2$. Thus, the limit distribution F is uniform, and the limit distribution G has $G(y) = 1/2$ for all $y \in [0,1)$ and $G(1) = 1$. (Note that G does not have full support.) The n-th contest is an all-pay auction with n players and $m \equiv \ulcorner n/2 \urcorner$ identical (non-zero) prizes, where $\ulcorner \cdot \urcorner$ is the ceiling function,[9] and the value of a prize to player i is i/n. These contests were studied by Clark and Riis (1998), who considered competitions for promotions, rent seeking, and rationing by waiting in line (see also Siegel (2010)).

Consider the assortative allocation in the limit setting, which assigns a prize to each agent with a type higher than 1/2. This allocation is implementable by the tariff mechanism that prescribes for every type $x \leq 1/2$ prize 0 and bid 0, and for every type $x > 1/2$ prize 1 and bid 1/2. The corresponding inverse tariff is discontinuous (but upper semi-continuous), and maps bids $t \in [0, 1/2)$ to prize 0 and bids $t \geq 1/2$ to prize 1. Corollary 2 below shows that for large n this mechanism approximates the equilibrium outcome, in that all but a small fraction of players i with $i/n > 1/2$ obtain a prize and bid close to 1/2 with high probability, and all but a small fraction of players i with $i/n < 1/2$ obtain no prize and bid close to 0

[9] The analysis of this example applies to any fixed limit ratio $p \in (0,1)$ of prizes to players.

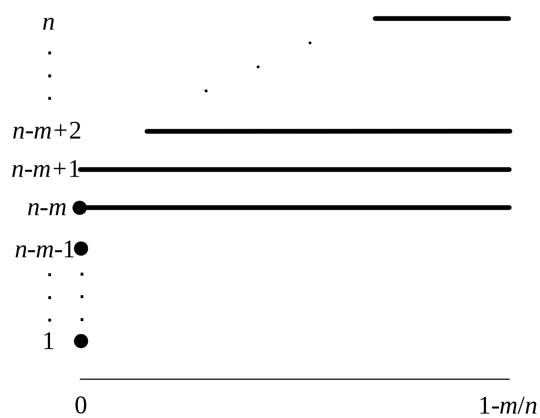

Figure 2: The support of players' strategies (dots represent atoms) in the unique equilibrium

with high probability. The approximation does not apply to all players as in Section 3.1, because G does not have full support, but the fraction of players it applies to converges to 1 as n grows large.

This simple approximation contrasts with the closed-form equilibrium characterization derived by Clark and Riis (1998).

As depicted in Figure 2, in the equilibrium of the n-th contest the $n-m-1$ players with the lowest valuations bid 0, and each of the $m+1$ players with the highest valuations bids on an interval, so m of them obtain a prize. (Recall that $m = \lceil n/2 \rceil$.) The common upper bound of the intervals is $1 - m/n \in \{1/2, 1/2 + 1/2n\}$, and the lower bound of the interval of player $i > n - m$ is

$$\left(1 - \frac{m}{n}\right)\left(1 - \prod_{k=n-m+1}^{i} \frac{k}{i}\right),$$

which increases in i.

Thus, for every $\varepsilon > 0$, as n grows large the number of players with valuations greater than 1/2 who bid on $[\varepsilon, 1 - m/n]$ grows large. This may appear to contradict the equilibrium approximation for large n. The apparent discrepancy is overcome by noting that the lower bound of the bidding interval of a player with valuation approximately $1/2 + \varepsilon$ is for large n approximately

$$\frac{1}{2}\left(1 - \frac{\frac{1}{2}\cdot\left(\frac{1}{2}+\frac{1}{n}\right)\cdot\ldots\cdot\left(\frac{1}{2}+\frac{\varepsilon n}{n}\right)}{\left(\frac{1}{2}+\varepsilon\right)^{\varepsilon n}}\right) = \frac{1}{2}\left(1 - \frac{\frac{1}{2}\cdot\ldots\cdot\left(\frac{1}{2}+\frac{\varepsilon n}{2n}\right)}{\left(\frac{1}{2}+\varepsilon\right)^{\frac{\varepsilon n}{2}}} \cdot \frac{\left(\frac{1}{2}+\frac{\varepsilon n}{2n}+\frac{1}{n}\right)\cdot\ldots\cdot\left(\frac{1}{2}+\frac{\varepsilon n}{n}\right)}{\left(\frac{1}{2}+\varepsilon\right)^{\frac{\varepsilon n}{2}}}\right).$$

The first fraction in parenthesis on the right-hand side is bounded above by $((1/2+\varepsilon/2)/(1/2+\varepsilon))^{\varepsilon n/2}$, and the second fraction is bounded above by 1, so as n increases the lower bound of the bidding interval approaches $1/2$ as fast as $1-b^n$ approaches 1, where $b<1$. Therefore, for any $\varepsilon>0$, for sufficiently large n at most a fraction ε of the players bid more than ε away from what the mechanism prescribes for the types that correspond to them.

4 Results

We show that the equilibria of large contests are approximated by tariff mechanisms that implement the assortative allocation. We first consider settings in which prizes have full support, so G^{-1} is continuous. This guarantees that when the number of players and prizes is large, it is enough to know the approximate rank-order of a player's bid to know the approximate prize she obtains.

Theorem 1 *Suppose that prizes have full support. Then, for any $\varepsilon>0$, there is an N such that for all $n\geq N$ in any equilibrium of the n-th contest,*

(a) every player i obtains with probability at least $1-\varepsilon$ a prize that differs by at most ε from $G^{-1}(F(x_i^n))$;

(b) there is a tariff mechanism with a continuous inverse tariff that implements the assortative allocation, such that the bid of every player i differs with probability 1 by at most ε from the bid that the mechanism prescribes for type x_i^n.

When there is a unique mechanism that implements the assortative allocation (and satisfies the requirement that $U(x,y,t)\geq U(x,y_{\inf},0)$ for each type x and its prescribed prize-bid pair (y,t), with an equality for at least one type x), this mechanism coincides with the tariff mechanism in part (b) of Theorem 1. For example, if U is quasi-linear and satisfies the conditions of Milgrom and Segal's (2002) envelope theorem,[10] then their Corollary 1 shows that the unique mechanism that implements the assortative allocation prescribes for type x bid

$$br(x)=v\left(x,G^{-1}(F(x))\right)-\int_0^x v_x\left(z,G^{-1}(F(z))\right)dz-v(0,y_{\inf}). \tag{4}$$

In this case, (4) provides an explicit formula for the tariff mechanism in part (b) of Theorem 1.[11] We therefore obtain the following corollary of Theorem 1, which applies to the setting of Section 3.1.

[10] The conditions are that v is differentiable and absolutely continuous in x, and $\sup_{y\in Y}|v_x(x,y)|$ is integrable on X.

[11] Other sufficient conditions for (4) are described in Krishna and Maenner's (2001) Proposition 1.

Corollary 1 *Suppose that prizes have full support and U is quasi-linear and satisfies the conditions of the envelope theorem. Then, for any $\varepsilon > 0$ there is an N such that for all $n \geq N$, in any equilibrium of the n-th contest every player i obtains with probability at least $1-\varepsilon$ a prize that differs by at most ε from $G^{-1}(F(x_i^n))$, and bids with probability 1 within ε of $br(x_i^n)$ given by (4).*

For complete-information contests, such as those in Section 3.1, the proof of Theorem 1 shows that the $1-\varepsilon$ in Theorem 1 and Corollary 1 can be replaced with 1.

When prizes do not have full support G^{-1} is discontinuous, so the approximate rank-order of a player's bid may be insufficient to determine the approximate prize she obtains. Consequently, some players' bids may be significantly different from what the limit mechanism specifies, even when the contest is large. For example, in the setting of Section 3.2, when the percentile rank-order of a player's bid is slightly above 1/2 she obtains G^{-1} of the rank-order, which is 1, and when it is slightly below 1/2 she obtains G^{-1} of the rank-order, which is 0. And for large n there are many players with valuations greater than 1/2 who bid substantially less than 1/2. Thus, Theorem 1 does not hold.

Nevertheless, even when prizes do not have full support, the approximation of Theorem 1 holds for all but a small fraction of players.

Theorem 2 *For any $\varepsilon > 0$, there is an N such that for all $n \geq N$ in any equilibrium of the n-th contest,*

(a) a fraction of at least $1-\varepsilon$ of the players i obtain with probability at least $1-\varepsilon$ a prize that differs by at most ε from $G^{-1}(F(x_i^n))$;

(b) there is a tariff mechanism that implements the assortative allocation, such that the bid of each of a fraction of at least $1-\varepsilon$ of the players i differs with probability at least $1-\varepsilon$ by at most ε from the bid that the mechanism prescribes for type x_i^n.

We also have an analogue of Corollary 1, which applies to the setting of Section 3.2.

Corollary 2 *Suppose that U is quasi-linear and satisfies the conditions of the envelope theorem. Then, for any $\varepsilon > 0$ there is an N such that for all $n \geq N$, in any equilibrium of the n-th contest each of a fraction of at least $1-\varepsilon$ of the players i obtains with probability at least $1-\varepsilon$ a prize that differs by at most ε from $G^{-1}(F(x_i^n))$, and bids with probability at least $1-\varepsilon$ within ε of $br(x_i^n)$ given by (4).*

The approximation results apply to many contests for which there is no existing equilibrium characterization. For example, consider (1) with F and G uniform. The assortative allocation assigns prize x to type x, and (4) shows that $br(x) = xh(x) - \int_0^x h(y)\,dy$. By

Corollary 1, for $x_i^n = i/n$ (so F_i^n is a Dirac measure) and $y_j^n = j/n$, when n is large a player with type x bids close to $x - \int_0^x h(y)\,dy$ and obtains a prize close to x. While $h(y) = y$ corresponds to the setting of Section 3.1 and $h(y) = y^2$ and $h(y) = e^y$ correspond to Xiao's (2013) quadratic and geometric prize sequences, for which he provides an equilibrium characterization,[12] no equilibrium characterization exists for other, non-trivial functions h (including $h(y) = y^m$ for $m > 2$). The same implication of Corollary 1 holds for contests with ex-ante asymmetric players and incomplete information for which F^n converges to the uniform distribution. No equilibrium characterization exists for such contests.

Another example is contests that combine heterogeneous and identical (non-zero) prizes, which have not been studied in the literature. Consider $y_j^n = 2j/n$ for $j/n \leq 1/2$ and $y_j^n = 1$ for $j/n > 1/2$, so $G(y) = y/2$ for $y < 1$ and $G(1) = 1$, with (1), $h(y) = y$, and F uniform. The assortative allocation assigns prize $2x$ to type $x < 1/2$, and prize 1 to type $x \geq 1/2$. Corollary 1 and (4) show that for large n a player with type x bids close to $\min\{x^2, 1/4\}$ and with high probability obtains a prize close to $\min\{2x, 1\}$.

In addition, the approximation results hold for quasi-linear utilities in which $v(x, y)$ is not multiplicatively separable, and for utilities that are not quasi-linear.

5 Intuition for the results and contribution to the analysis of large games

The intuition for our results is as follows. As the number of players increases, the competition they face becomes similar. That is, the mappings between players' bids and the distributions of their percentile rankings (given the other players' equilibrium strategies) become similar for all players, and coincide in the limit. This is because the rankings of two players who make the same bid differ by at most 1. This "almost" implies that also the mappings between bids and the distributions of prizes become similar, and coincide in the limit. Moreover, by the law of large numbers, the common limit mapping is deterministic, so each bid maps to a single prize. This "almost" yields an inverse tariff such that in the limit players choose their bids as in a mechanism-design setting in which a single agent faces the inverse tariff. Since a player can always bid 0 and secure the lowest prize, we obtain a tariff mechanism. Strict single crossing guarantees that higher types choose higher prizes, so the mechanism implements the assortative allocation.

[12] Xiao's (2013) characterization is considerably more complicated than B&L's, because in his setting some players' equilibrium bidding strategies have non-interval supports.

This intuition is incomplete for two reasons. First, it assumes that the equilibrium outcomes converge. To show convergence we apply Helly's (1912) selection theorem for monotone functions, which implies that the average of players' mappings between bids and expected equilibrium prizes has a converging subsequence. This does part of the work, but does not directly imply the strong notion of convergence we are able to obtain for the equilibrium outcomes.[13] Second, for some bids the distributions of prizes may not be similar across players even when the percentile rankings are. That is, while a player's bid is sufficient for determining approximately the player's percentile ranking, it may not be sufficient for determining approximately the player's prize. This is what happens in the setting of Section 3.2, in which there are half as many identical prizes as players and the limit percentile ranking of a bid t is $1/2$. By bidding slightly above t some players obtain a prize with relatively high probability and other players obtain a prize with relatively low probability, even when the number of players is large.

This indeterminacy of the distributions of prizes corresponding to bids is a consequence of the discontinuity in the limit supply of prizes, i.e., the fact that there are no intermediate prizes between $y = 0$ and $y = 1$. Theorem 2 shows that such discontinuities affect the equilibrium strategies of only a small fraction of players, which vanishes as the number of players grows large. Our method of dealing with discontinuities may be useful for studying other auction-like settings, in which such discontinuities naturally appear but are typically assumed away in the existing literature on large games (see Section 1.1).

Another advantage of our approach, compared to the existing literature on large games, is that it approximates the equilibrium outcomes of discrete contests by the outcomes of mechanisms with continuum of agent types, which can be characterized by applying methods from the mechanism-design literature, and not by the equilibrium outcomes of contests (or other games) with a continuum of agents.

6 Rates of Convergence

The proofs of Theorems 1 and 2 do not address the issue of how quickly the contest equilibria outcomes converge to the outcome of the approximating mechanism. We now present a conceptually simpler proof of our approximation results, which provides rates of convergence but applies to a more restricted environment. This environment satisfies (1), and F, G^{-1}, and h are differentiable with derivatives that are bounded and bounded away from 0, i.e., $0 < \underline{f} \leq F'(x) \leq \overline{f}$, $0 < \underline{g} \leq (G^{-1\prime})(r) \leq \overline{g}$, and $0 < \underline{h} \leq h'(y) \leq \overline{h}$ for all x, y, and

[13] A sketch of the proof of Theorem 1 appears at the beginning of the Online Appendix.

$r \in [0,1]$.[14] For simplicity, we consider complete-information contests in which player i's type is $x_i^n = F^{-1}(i/n)$ and prize j is $y_j^n = G^{-1}(j/n)$.[15]

The idea underlying the simpler proof is that for any type x, the expected fraction of players with types lower than $x - \varepsilon$ who outbid the player with type x falls below ε as the number of players grows large. Indeed, suppose this fraction were bounded below by ε. Then, the highest-bidding player with a type lower than $x-\varepsilon$ outbids this fraction with her highest bid t, which she weakly prefers to an equilibrium bid of the player with type x, which is outbid by this fraction. The improvement of order ε in the prize lottery associated with the bid t would increase the utility of the player with type x at least by an expression of order ε^2 more than it increases the utility of the player with type lower than $x - \varepsilon$. The lotteries that the two players obtain by placing the same bid may differ, but since the corresponding rankings of the two players differ by at most 1, this difference vanishes when the contests become sufficiently large. Moreover, it is of order $1/n$. So, the player with type x would strictly prefer bid t to her equilibrium bid.

By an analogous argument, the expected fraction of players with types higher than $x+\varepsilon$ who lose to the player with type x falls below ε as the number of players grows large. This proves that for large n the expected equilibrium prize allocation is approximately assortative. In fact, if $1/n$ is smaller than an expression of order ε^2, then the expected prize of a player with type x will differ from the corresponding prize in the assortative allocation by at most ε. Together with Hoeffding's inequality, this implies that the actual prize of a player with type x will differ from the corresponding prize in the assortative allocation by at most ε, with probability P such that $1 - P$ is bounded by an expression of order ε if $1/n$ is smaller than an expression of order $-\varepsilon^2/\ln \varepsilon$.

Once this is established, the fact that no player prefers to mimic the bidding behavior of "adjacent" players with higher or lower types and obtain their prize lottery shows that the equilibrium bids satisfy an "approximate envelope formula," and are therefore approximated by (4). We show that the difference between equilibrium bids and (4) is with probability 1 bounded by an expression of order ε if $1/n$ is smaller than an expression of order $-\varepsilon^2/\ln \varepsilon$.

Formally, let

$$P^n(\varepsilon) = 1 - 4\exp\{-\varepsilon^2(n-1)/2\overline{g}^2(\overline{f}+1)^2\} \text{ and } R^n(\varepsilon) = 2\overline{h}[\varepsilon P^n(\varepsilon) + 1 - P^n(\varepsilon)] + 2\overline{h}\overline{g}/n,$$

[14] It will follow from the proof that there is no conceptual difficulty in extending the arguments to an arbitrary quasi-linear utility $v(x,y) - t$ whose derivatives $\partial^2 v(x,y)/\partial x \partial y = \partial^2 v(x,y)/\partial y \partial x$ are bounded and bounded away from 0. The assumption that all derivatives are bounded and bounded away from 0 is essential for the proof.

[15] Analogous results can be obtained for contests with incomplete information.

and notice that $1-P^n(\varepsilon)$ and $R^n(\varepsilon)$ are bounded by expressions of order ε if $1/n$ is smaller than an expression of order $-\varepsilon^2/\ln\varepsilon$; in particular, if $1/n$ is smaller than an expression of order ε^3.

Theorem 3 *Let $n > 2\overline{h}\overline{g}/\underline{h}\underline{g}\varepsilon^2$. Then, in any equilibrium of the n-th contest,*

(a) every player i obtains with probability $P^n(\varepsilon)\to_{n\to\infty} 1$ a prize that differs by at most ε from $G^{-1}(F(x_i^n))$;

(b) the bid of every player i differs with probability 1 by at most $R^n(\varepsilon)\to_{n\to\infty} 2\overline{h}\varepsilon$ from $br(x_i^n)$.

Fix any equilibrium of the n-player contest, and any player i. Let br_i^- and br_i^+ denote the lowest and highest equilibrium best responses of player i. Let σ_k^n denote the equilibrium strategy of player k, which is a distribution over bids. Our first lemma says that few players $k<i$ bid more than br_i^-, and few players $k>i$ bid less than br_i^+.

Lemma 1 *For any $\varepsilon'>0$ and $n > 2\overline{h}\overline{g}/\underline{h}\underline{g}(\varepsilon')^2$, we have that*

$$E\left[\frac{1}{n}\sum_{k:x_k^n\le x_i^n-\varepsilon'} 1_{\{br_i^-\le\sigma_k^n\}}\right]\le\frac{\varepsilon'}{2} \text{ and } E\left[\frac{1}{n}\sum_{k:x_k^n> x_i^n+\varepsilon'} 1_{\{\sigma_k^n\le br_i^+\}}\right]\le\frac{\varepsilon'}{2}.$$

Proof. We will prove the first inequality; the proof of the second inequality is analogous. Suppose the contrary that

$$E\left[\frac{1}{n}\sum_{k:x_k^n\le x_i^n-\varepsilon'} 1_{\{t<\sigma_k^n\le t'\}}\right]>\frac{\varepsilon'}{2}, \tag{5}$$

where $t=br_i^-$ and $t'=\max\{br_k^+ : x_k^n\le x_i^n-\varepsilon'\}$ is the maximal best response of players k such that $x_k^n\le x_i^n-\varepsilon'$.[16]

Since some player $k<i$ weakly prefers bidding t' to bidding t, we have that

$$x_k^n\left\{E\left[h(y)\mid k,t'\right]-E[h(y)\mid k,t]\right\}\ge t'-t, \tag{6}$$

where the expected values refer to the lottery over prizes faced by player k who bids t' or t against the equilibrium strategies of the other players. We will now show that player i strictly

[16]Notice that we replaced $t\le\sigma_k^n$ from the statement of the lemma with $t<\sigma_k^n$. This is without loss of generality, because no player k's strategy can have an atom at br_i^-. If it had an atom, then player i would strictly prefer to bid slightly more than br_i^-. Thus, the probability that $\sigma_k^n=br_i^-$ for some k is zero.

prefers bidding a t'' slightly higher than t' to bidding t, which contradicts the definition of t as a best response.

By bidding t'' player i outbids player k with probability 1, and for all realizations of equilibrium bids of players other than i and k, player i outbids no fewer of those other players than player k does by bidding t'. Thus, $E[h(y) \mid i, t''] \geq E[h(y) \mid k, t']$; and since for all realizations of equilibrium bids of players other than i and k, by bidding t player i outbids at most one more player than player k does by bidding t (namely player k), we have that $E[h(y) \mid i, t] \leq E[h(y + \overline{g}/n) \mid k, t]$. These two inequalities and $x_k^n \leq x_i^n - \varepsilon'$ yield

$$\begin{aligned} & x_i^n \{E[h(y) \mid i, t''] - E[h(y) \mid i, t]\} \\ & \geq x_k^n \{E[h(y) \mid i, t''] - E[h(y) \mid i, t]\} + \varepsilon' \{E[h(y) \mid i, t''] - E[h(y) \mid i, t]\} \\ & \geq x_k^n \{E[h(y) \mid k, t'] - E[h(y + \overline{g}/n) \mid k, t]\} + \varepsilon' \{E[h(y) \mid i, t''] - E[h(y) \mid i, t]\}. \end{aligned}$$

By (5), this is at least

$$\begin{aligned} & x_k^n \{E[h(y) \mid k, t'] - E[h(y + \overline{g}/n) \mid k, t]\} + (\varepsilon')^2 \underline{h}\underline{g}/2 \\ & \geq x_k^n \left\{E[h(y) \mid k, t'] - E[h(y) \mid k, t] - \overline{h}\overline{g}/n\right\} + (\varepsilon')^2 \underline{h}\underline{g}/2, \end{aligned}$$

and by (6) and $x_k^n \leq 1$, this is at least

$$(t' - t) - \overline{h}\overline{g}/n + (\varepsilon')^2 \underline{h}\underline{g}/2 > t'' - t,$$

where the final inequality follows from $n > 2\overline{h}\overline{g}/\underline{h}\underline{g}(\varepsilon')^2$ for t'' converging to t'. ■

We can now derive part (a) of our result. Since

$$E\left[\frac{1}{n} \sum_{k: x_k^n \leq x_i^n - \varepsilon'} 1_{\{\sigma_k^n \leq br_i^-\}}\right] = F(x_i^n - \varepsilon') - E\left[\frac{1}{n} \sum_{k: x_k^n \leq x_i^n - \varepsilon'} 1_{\{br_i^- < \sigma_k^n\}}\right] \geq F(x_i^n - \varepsilon') - \varepsilon'/2,$$

by Hoeffding's inequality,

$$\Pr\left\{\frac{1}{n} \sum_{k: x_k^n \leq x_i^n - \varepsilon'} 1_{\{\sigma_k^n \leq br_i^-\}} > F(x_i^n - \varepsilon') - \varepsilon'\right\} \geq 1 - 2\exp\{-(\varepsilon')^2(n-1)/2\}.$$

That is, player i obtains at least with probability $1 - 2\exp\{-(\varepsilon')^2(n-1)/2\}$ a prize no lower than $G^{-1}(F(x_i^n - \varepsilon') - \varepsilon') \geq G^{-1}(F(x_i^n)) - \overline{g}(\overline{f} + 1)\varepsilon'$.

Similarly, player i obtains at least with probability $1 - 2\exp\{-(\varepsilon')^2(n-1)/2\}$ a prize no higher than $G^{-1}(F(x_i^n + \varepsilon') + \varepsilon') \leq G^{-1}(F(x_i^n)) + \overline{g}(\overline{f} + 1)\varepsilon'$. Noticing that $[1 - 2\exp\{-(\varepsilon')^2(n-1)/2\}]^2 \geq 1 - 4\exp\{-(\varepsilon')^2(n-1)/2\}$, and replacing ε' with $\varepsilon/\overline{g}(\overline{f} + 1)$, we obtain part (a) of Theorem 3.

Our next lemma shows that the equilibrium utility of player i is close to that from obtaining prize $G^{-1}(F(x_i^n))$ by bidding $br(x_i^n)$.

Lemma 2 *Player i's equilibrium utility U_i^n in n-th contest satisfies*

$$\left| U_i^n - \int_0^{x_i^n} h(G^{-1}(F(z)))dz \right| \leq \overline{h}[\varepsilon P^n(\varepsilon) + 1 - P^n(\varepsilon)] + 2\overline{h}\overline{g}/n.$$

Proof. To establish this result we will show that

$$U_i^n \geq \int_0^{x_i^n} h(G^{-1}(F(x)))dx - \overline{h}[\varepsilon P^n(\varepsilon) + 1 - P^n(\varepsilon)] - \overline{h}\overline{g}/n, \tag{7}$$

and that

$$U_i^n \leq \int_0^{x_i^n} h(G^{-1}(F(x)))dx + \overline{h}[\varepsilon P^n(\varepsilon) + 1 - P^n(\varepsilon)] + 2\overline{h}\overline{g}/n. \tag{8}$$

It will be convenient to let $t_k = br_k^+$ for $k = 1, ..., n$. To show (7), notice first that since player i can bid any $t' > t_{i-1}$, we have that

$$U_i^n \geq x_i^n Eh(y \mid i, t') - t' \geq x_i^n Eh(y \mid i-1, t_{i-1}) - t'$$

$$= x_{i-1}^n Eh(y \mid i-1, t_{i-1}) - t' + (x_i^n - x_{i-1}^n)Eh(y \mid i-1, t_{i-1})$$

$$= U_{i-1}^n - (t' - t_{i-1}) + (x_i^n - x_{i-1}^n)Eh(y \mid i-1, t_{i-1}) > U_{i-1}^n + (x_i^n - x_{i-1}^n)Eh(y \mid i-1, t_{i-1}),$$

where the second inequality holds because by bidding $t\prime$ player i outbids player $i-1$ with probability 1, and for all realizations of equilibrium bids of players other than i and $i-1$, outbids more of these players than player $i-1$ does by bidding t_{i-1}. The last expression is equal to

$$U_{i-1}^n + (x_i^n - x_{i-1}^n)h(G^{-1}(F(x_{i-1}^n))) + (x_i^n - x_{i-1}^n)E\{h(y \mid i-1, t_{i-1}) - h(G^{-1}(F(x_{i-1}^n)))\}$$

$$\geq U_{i-1}^n + (x_i^n - x_{i-1}^n)h(G^{-1}(F(x_{i-1}^n))) - (x_i^n - x_{i-1}^n)\overline{h}[\varepsilon P^n(\varepsilon) + 1 - P^n(\varepsilon)],$$

where the last inequality follows from part (a) of the theorem.

By repeating this argument, we obtain that

$$U_i^n \geq \sum_{k=1}^{i}(x_k^n - x_{k-1}^n)h(G^{-1}(F(x_{k-1}^n))) - \sum_{k=1}^{i}(x_k^n - x_{k-1}^n)\overline{h}[\varepsilon P^n(\varepsilon) + 1 - P^n(\varepsilon)]$$

$$\geq \sum_{k=1}^{i}(x_k^n - x_{k-1}^n)h(G^{-1}(F(x_{k-1}^n))) - \overline{h}[\varepsilon P^n(\varepsilon) + 1 - P^n(\varepsilon)].$$

The first part of the last expression is a familiar Riemann sum of $h(G^{-1}(F(x)))$ with respect to the tagged partition $0 = x_0^n, x_1^n, \ldots, x_i^n$. This sum differs from the integral of $h(G^{-1}(F(x)))$ over $[0, x_i^n]$ by at most

$$\sum_{k=1}^{i}(x_k^n - x_{k-1}^n)[h(G^{-1}(F(x_k^n))) - h(G^{-1}(F(x_{k-1}^n)))] \leq \sum_{k=1}^{i}(x_k^n - x_{k-1}^n)\overline{h}\overline{g}/n \leq \overline{h}\overline{g}/n.$$

Thus,

$$\sum_{k=1}^{i}(x_k^n - x_{k-1}^n)h(G^{-1}(F(x_{k-1}^n))) \geq \int_0^{x_i^n} h(G^{-1}(F(x)))dx - \overline{h}\overline{g}/n,$$

which gives (7).

To show (8), notice that since player $i-1$ can bid any $t' > t_i$, we have that

$$U_{i-1}^n \geq x_{i-1}^n Eh(y \mid i-1, t') - t' = x_i^n Eh(y \mid i-1, t') + (x_{i-1}^n - x_i^n)Eh(y \mid i-1, t') - t'.$$

By bidding t' player $i-1$ outbids player i with probability 1, and for all realizations of equilibrium bids of players other than i and $i-1$, outbids more of these players than player i does by bidding t_i. Thus, $Eh(y \mid i-1, t') \geq Eh(y \mid i, t_i)$. Now, by bidding any t' player $i-1$ outbids at most one more player (player i) than player i does by bidding t'. Thus, $Eh(y \mid i-1, t') \leq Eh(y + \overline{g}/n \mid i, t')$. These two inequalities yield

$$U_{i-1}^n \geq x_i^n Eh(y \mid i, t_i) + (x_{i-1}^n - x_i^n)Eh(y + \overline{g}/n \mid i, t') - t'.$$

Since this inequality holds for all $t' > t_i$, we obtain

$$\begin{aligned} U_{i-1}^n &\geq x_i^n Eh(y \mid i, t_i) + (x_{i-1}^n - x_i^n)Eh(y + \overline{g}/n \mid i, t_i) - t_i \\ &\geq U_i^n + (x_{i-1}^n - x_i^n)\left\{Eh(y \mid i, t_i) + \overline{h}\overline{g}/n\right\}. \end{aligned}$$

(Recall that no strategy of player other than i can have an atom at t_i. Therefore, $Eh(y+\overline{g}/n \mid i, t')$ converges to $Eh(y + \overline{g}/n \mid i, t_i)$.) Thus,

$$U_i^n \leq U_{i-1}^n + (x_i^n - x_{i-1}^n)\left\{Eh(y \mid i, t_i) + \overline{h}\overline{g}/n\right\},$$

and we can proceed similarly to the proof of (7) to obtain (8). ■

We can now prove part (b) of our result. Let t denote any best response of player i to the equilibrium strategies of the other players. Then,

$$\begin{aligned} |t - br\,(x_i^n)| &\leq \left|U_i^n - \int_0^{x_i^n} h(G^{-1}(F(x)))dx\right| + \left|x_i^n Eh(y \mid i, t) - x_i^n h(G^{-1}(F(x_i^n)))\right| \\ &\leq \overline{h}[\varepsilon P^n(\varepsilon) + 1 - P^n(\varepsilon)] + 2\overline{h}\overline{g}/n + \overline{h}[\varepsilon P^n(\varepsilon) + 1 - P^n(\varepsilon)], \end{aligned}$$

where the first inequality follows from (4) and the definition of U_i^n and the second inequality follows from Lemma 2, part (a) of Theorem 3, and $x_k^n, y \leq 1$.

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