Does the New Keynesian Model Have a Uniqueness Problem?*

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Abstract

This paper addresses whether non-uniqueness of equilibrium is a substantive problem for the analysis of fiscal policy in New-Keynesian (NK) models at the zero lower bound (ZLB). There would be a substantive problem if there were no compelling way to select among different equilibria that give different answers to critical policy questions. We argue that learnability provides such a criterion. We study a fully non-linear NK model with Calvo pricing frictions. Our main finding is that the model we analyze has a unique E-stable rational expectations equilibrium at the ZLB. That equilibrium is also stable-under-learning and inherits all of the key properties of linearized NK models for fiscal policy.

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1 Introduction

This paper addresses the question: is the non-uniqueness of equilibria a *substantive* problem for the analysis of fiscal policy in New Keynesian (NK) models? There would be a substantive problem if there were no compelling way to select among different equilibria that give different answers to critical policy questions. To be concrete we focus our analysis on the impact of changes in government consumption when the economy is at the (zero lower bound) ZLB.[1]

Eggertsson and Woodford (2003)(EW) and Eggertsson (2004) develop an elegant and transparent framework for studying fiscal policy in the NK model at the ZLB. A key result from this literature is that the output multiplier associated with government consumption is larger when the ZLB binds than when it does not. The more binding is the ZLB constraint, the larger is the multiplier.[2]

These results are based on linearized versions of the NK model which have a unique bounded minimum-state solution. In fact, non-linear versions of these models can have multiple rational expectations equilibria. This multiplicity occurs even if one restricts attention to bounded minimum-state-variable solutions. The efficacy of fiscal policy can vary a great deal across those equilibria (see Mertens and Ravn (2014)). At some ZLB equilibria, the government consumption multiplier (or 'multiplier' for short) is small or even negative. In others, it is very large. So, in principle, non-uniqueness of equilibria poses an enormous challenge for the analysis of fiscal policy in NK models.

In the spirit of the literature summarized by Evans and Honkapohja (2001), we adopt E-stability as an equilibrium selection criterion in our analysis. The concept of E-stability is closely linked to least-squares learnability in linear models.[3] We view E-stability as a necessary condition for an equilibrium and the associated policy implications to be empirically plausible.[4]

We apply the E-stability criterion to a standard, fully non-linear NK model with Calvo-pricing frictions. Working with this model poses an interesting challenge. Unlike linearized NK models of the type considered by EW, rational expectations equilibria cannot be characterized by one set of

[1]The key constraint is that there is some lower bound (not necessarily zero) on the nominal interest rate.

[2]See, for example, EW, Eggertsson (2011), Christiano et al. (2011)(CER), and Woodford (2011).

[3]See Evans and Honkapohja (2001) and McCallum (2007).

[4]Our notion of stability contrasts sharply with the one used in Benhabib et al. (2001) (BSGU). They study an endowment economy populated by a representative household with preferences defined over consumption and real balances. There is also a monetary authority that sets the nominal interest rate using a Taylor rule subject to a ZLB constraint. BSGU show that there are two steady states corresponding to a nonnegative inflation rate and a deflation rate equal to the household's gross discount rate. They consider the scenario in which actual inflation π_t is close to but not exactly equal to either of the steady state values. BSGU say that a steady state is stable if the economy converges back to it in a rational expectations equilibrium. BSGU show that the deflation steady state equilibrium is the unique stable steady state in their sense of the term.

numbers when the ZLB binds. Because there is an endogenous state variable (price dispersion), finding an equilibrium amounts to finding a set of functions. To this end, we conduct our analysis using non-linear functions that approximate the equilibrium mapping from the state of the economy to the vector of time-t endogenous variables.

Some authors have analyzed fiscal policy in a non-linear version of the NK model with Rotemberg pricing frictions (see Boneva et al. (2016), and Evans et al. (2008)). An advantage of the Rotemberg model is that it eliminates the endogenous state variable of price dispersion. However, others have argued that the reduced-form adjustment cost specification in that model can have a large influence on its implications for fiscal policy (see Ngo and Miao (2015)). To avoid this issue, we work directly with the Calvo model. Our appendix shows that for reasonable adjustment cost specifications, a Rotemberg model delivers qualitatively similar results to the fully non-linear Calvo model.

Using numerical methods, we show that there exists a unique interior, minimum-state equilibrium of our non-linear Calvo model that is E-stable. The predictions of the linearized NK model about fiscal policy at the ZLB hold in the unique interior E-stable rational expectations equilibrium of the non-linear model.

We also study fiscal policy at the ZLB when agents are learning about the structure of the economy. This part of the analysis is in the spirit of Lucas (1978) and Evans and Honkapohja (2001). In particular, Lucas (1978) explored the robustness of his results under rational expectations to a situation where agents are learning about their environment.[5]

The conceptual difficulty that we face in modeling learning is that, in our model, agents are learning about non-linear functions with a state variable. We call an equilibrium stable-under-learning if the outcomes in a learning equilibrium converge to the outcomes under rational expectations. Our key findings are: 1) the unique E-stable equilibrium is stable under learning, 2) the other minimum state variable equilibria are not stable under learning.

In contrast to Mertens and Ravn (2014), we find that the ZLB fiscal multiplier is large in *all* of our learning equilibria. The basic reason is as follows. In calculating the multiplier, Mertens and Ravn (2014) assume that an increase in government spending is associated with a one-time adjustment of agents' expectations.[6] Specifically, agents' expectations about inflation and consumption

[5] In motivating his learning analysis, Lucas (1978) writes: "... the model described above 'assumes' that agents know a great deal about the structure of the economy and perform some non-routine computations. It is in order to ask, then: will an economy with agents armed with 'sensible' rules-of-thumb, revising these rules from time to time so as to claim observed rents, tend as time passes to behave as described[...].".

[6] Brendon et al. (2015) have also considered ZLB episodes because of confidence shocks.

fall discretely so that they are close to the values that they would have in the post-shock rational expectations equilibrium. In contrast, we do not have assume a one-time 'jump' in agents' expectations when government consumption initially changes. While we think our experiment is closer in spirit to learning literature, further research on how actual agents learn under these circumstances is certainly warranted.

Finally, our paper assess the accuracy of linear approximations of the NK model at the ZLB. We show that the standard log-linear approximation delivers predictions about the effects of government spending that are qualitatively similar to those of the non-linear E-stable equilibrium. In addition, we propose a new log-linear approximation to the ZLB equilibrium which is substantially more accurate than the linearization used in much of the literature.

Viewed overall, our results support the view that the standard NK model does not have a substantive uniqueness problem, at least with regards to the effects of fiscal policy at the ZLB. The unique E-stable minimum state variable equilibrium of the non-linear NK model has properties that are qualitatively and quantitatively similar to the linear NK model.

The remainder of this paper is organized as follows. In section 2 we analyze rational expectations equilibria at the ZLB in a nonlinear Calvo model. Section 3 contains our main results regarding E-stability. In section 4 we discuss learning equilibria. Concluding remarks are contained in section 5. Finally, the Appendix shows that our main results hold in a model with Rotemberg pricing frictions.

2 Fiscal Policy in the ZLB

In this section we derive the implications of the NK model for the effects of changes in government purchases when the ZLB in binding. We conduct our analysis in a non-linear version of the NK model in which firms face Calvo price-setting frictions.

2.1 Model Economy

A representative household maximizes

$$E_0 \sum_{t=0}^{\infty} d_t \left[\log\left(C_t\right) - \frac{\chi}{2} N_t^2 \right] \tag{1}$$

where C_t is consumption, N_t are hours worked, and

$$d_0 = 1,\ d_t = \prod_{j=1}^{t} \left(\frac{1}{1 + r_{j-1}} \right),\ t \geq 1. \tag{2}$$

Here, d_t is the household's rate of time discounting. As in EW, the rate of time discounting varies over time as r_t changes. EW refer to r_t as the natural rate of interest. We assume that r_t can take on two values: r and r^ℓ, where $r^\ell < 0$.[7] The stochastic process for r_t is given by

$$\Pr\left[r_{t+1} = r^\ell | r_t = r^\ell\right] = p,\ \ \Pr\left[r_{t+1} = r | r_t = r^\ell\right] = 1 - p,\ \ \Pr\left[r_{t+1} = r^\ell | r_t = r\right] = 0. \tag{3}$$

We assume that r_t is known at time t. The household faces the flow budget constraint

$$P_t C_t + B_t \leq (1 + R_{t-1}) B_{t-1} + W_t N_t + T_t. \tag{4}$$

Here, P_t is the price of the consumption good, B_t is the quantity of risk-free nominal bonds held by the household, R_{t-1} is the gross nominal interest rate paid on bonds held from period $t-1$ to period t, W_t is the nominal wage, and T_t represents lump-sum profits net of lump-sum government taxes and transfers.

A final homogeneous good, Y_t, is produced by competitive and identical firms using the technology

$$Y_t = \left[\int_0^1 (Y_{j,t})^{\frac{\varepsilon}{\varepsilon - 1}} \, dj \right]^{\frac{\varepsilon - 1}{\varepsilon}}, \tag{5}$$

where $\varepsilon > 1$. The representative firm chooses inputs, $Y_{j,t}$, to maximize profits

$$P_t Y_t - \int_0^1 P_{j,t} Y_{j,t} dj, \tag{6}$$

subject to (5). The firm's first order condition for the j^{th} input is

$$Y_{j,t} = (P_t / P_{j,t})^{\varepsilon}\, Y_t. \tag{7}$$

The j^{th} input in (5) is produced by firm j who is a monopolist in the product market and is

[7] We use the letter ℓ to indicate that the natural rate of interest is 'low'.

competitive in factor markets. Monopolist j has the production function

$$Y_{j,t} = N_{j,t}. \tag{8}$$

Here, $N_{j,t}$ is the quantity of labor used by the j^{th} monopolist. The monopolist maximizes

$$E_t \sum_{k=0}^{\infty} d_{t+k} \lambda_{t+k} \left((1+\upsilon) P_{j,t+k} - P_{t+k} s_{t+k} \right) Y_{j,t+k}, \tag{9}$$

where s_t is real marginal cost and λ_t is the utility value of a dollar to the household. The variable υ is a subsidy that is designed to remove monopoly power distortions in a deterministic steady state when $r_t = r$. The j^{th} monopolist sets its price, $P_{j,t}$, subject to the demand curve, (7), and the following Calvo sticky price friction (10)

$$P_{j,t} = \begin{cases} P_{j,t-1} & \text{with probability } \theta \\ \tilde{P}_{j,t} & \text{with probability } 1-\theta \end{cases}. \tag{10}$$

Here $\tilde{P}_{j,t}$ is the price chosen by monopolist j in the event that it can re-optimize its price. The monopolist satisfies whatever demand occurs at its posted price.

The aggregate resource constraint is given by

$$C_t + G_t \leq Y_t. \tag{11}$$

In equilibrium, this constraint is satisfied as an equality because households and the government go to the boundary of their budget constraints. We consider two processes for G_t. In the first, the level of government purchases is a constant, G, independent of r_t for all t. In the second

$$G_t = \begin{cases} G & r_t = r \\ G^{\ell} & r_t = r^{\ell} \end{cases}.$$

The government runs a balanced budget in each period with lump-sum taxation and monetary policy is conducted according to an interest rate rule

$$R_t = \max\left\{1, 1 + r + \alpha\left(\pi_t - 1\right)\right\}. \tag{12}$$

The max operator reflects the ZLB constraint on nominal interest rates and α is assumed to be larger than $1+r$. As in BSGU, the latter assumption guarantees the existence of two steady states.

It is well known that Calvo pricing gives rise to dispersion in the price of intermediate goods. As shown in Appendix A, price dispersion, p_t^*, evolves according to

$$p_t^* = \left[(1-\theta)\left[\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta}\right]^{\frac{-\varepsilon}{1-\varepsilon}} + \theta\pi_t^{\varepsilon}(p_{t-1}^*)^{-1}\right]^{-1}. \tag{13}$$

Moreover, aggregate output is given by

$$Y_t = p_t^* N_t. \tag{14}$$

We use following baseline parameterization of the model

$$\begin{aligned} \varepsilon &= 7.0,\ \beta = 0.99,\ \alpha = 2.0,\ p = 0.75, \\ r^{\ell} &= -0.02/4,\ \theta = 0.85,\ G = 0.2,\ \chi = 1.25. \end{aligned} \tag{15}$$

The value $\varepsilon = 7$ implies markups of roughly 15 percent. We set $\beta = 0.99$, which implies a steady state real interest rate of 4 percent. The parameter $\alpha = 2$ is set larger than 1 so as to satisfy the Taylor principle. We set $\theta = 0.85$, implying that prices are updated a little less than once per year, on average. The parameter $\chi = 1.25$ normalizes labor supply to be 1 in steady state, and government spending is 20 percent of steady state output. We set $p = 0.75$, implying that the probability of remaining at the ZLB in each period is 75 percent, and we set r^{ℓ} so that the natural rate of interest at the ZLB is -2 percent.

2.2 Equilibrium

The model is similar to others in the literature, and in Appendix A, we show that the equilibrium conditions of the model can be summarized as

$$\left[(1-\theta)\left[\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta}\right]^{\frac{\varepsilon}{\varepsilon-1}}+\frac{\theta\pi_t^{\varepsilon}}{p_{t-1}^*}\right]^{-1}-p_t^*=0 \tag{16}$$

$$\frac{1}{1+r_t}\max\left(1,1+r+\alpha\left(\pi_t-1\right)\right)E_t\frac{1}{Y_{t+1}-G_{t+1}}\frac{1}{\pi_{t+1}}-\frac{1}{Y_t-G_t}=0 \tag{17}$$

$$E_t\sum_{j=0}^{\infty}d_{t+j}\theta^j\left(\frac{\tilde{p}_t}{\prod_{k=0}^{j}\pi_{t+k}}-\chi\frac{Y_{t+j}^2}{p_{t+j}^*}\right)\left(\prod_{k=0}^{j}\pi_{t+k}\right)^{-\varepsilon}\frac{Y_{t+j}}{Y_{t+j}-G_{t+j}}=0. \tag{18}$$

At time t the model has two state variables, lagged price dispersion, p_{t-1}^*, and the current period realization of the discount rate, r_t. We define a rational expectations equilibrium as a set of functions

$$\mathbf{X}\left(p_{t-1}^*,r_t\right)=\left[\begin{array}{ccc}\mathbf{Y}(p_{t-1}^*,r_t) & \pi(p_{t-1}^*,r_t) & \mathbf{p}^*(p_{t-1}^*,r_t)\end{array}\right] \tag{19}$$

that satisfy (16)-(18) for every p_{t-1}^* and r_t. Since the equilibrium functions depend only on p_{t-1}^* and r_t, we have restricted ourselves to minimum-state-variable equilibria. Also since the equations in (16) hold with strict equality, we have restricted ourselves to interior equilibria.

We use the following recursive procedure for computing equilibria. We first solve for $\mathbf{X}\left(p_{t-1}^*,r_t=r\right)$, because $r_t=r$ is an absorbing state for r_t. We then use this solution to find $\mathbf{X}\left(p_{t-1}^*,r_t=r^\ell\right)$. In both cases $\mathbf{X}$ solves a particular fixed point problem. Consider $r_t=r$. Given a function, $\tilde{\mathbf{X}}\left(p_t^*,r\right)$, which determines aggregate prices and quantities after period t, and a value of p_{t-1}^*, we use (16)-(18) to solve for Y_t, π_t, p_t^*. By varying p_{t-1}^* we obtain a new function, $\bar{\mathbf{X}}(p_{t-1}^*,r)$. This procedure defines a mapping on the space of candidate equilibrium functions. An equilibrium, $\mathbf{X}(p_{t-1}^*,r)$, is a fixed point of the mapping. Now consider $r_t=r^\ell$. Conditional on $\mathbf{X}(p_{t-1}^*,r)$, $\mathbf{X}\left(p_{t-1}^*,r^\ell\right)$ is a fixed point of another mapping defined by (16)-(18). Interiority of the equilibrium requires that consumption and gross inflation are positive. In practice, we approximate $\mathbf{X}(p_{t-1}^*,r)$ and $\mathbf{X}\left(p_{t-1}^*,r^\ell\right)$ by functions that are piecewise linear and continuous in p_{t-1}^* (see the Appendix C for details).

Significantly, we find two functions $\mathbf{X}(p_{t-1}^*,r)$ that satisfy (16)-(18). That is, we find equilibria, each of which is associated with a different steady state. Corresponding to each function $\mathbf{X}(p_{t-1}^*,r)$, we find two functions, $\mathbf{X}\left(p_{t-1}^*,r^\ell\right)$. That is, for each $\mathbf{X}(p_{t-1}^*,r)$, we find two equilibria when the zero lower bound is binding. So, in total we find four equilibria.

It is useful to discuss how the two equilibrium functions $\mathbf{X}\left(p_{t-1}^{*}, r\right)$ relate to the two steady states discussed in BSGU. A steady-state is the 3-tuple

$$Y \equiv \mathbf{Y}(p^{*}, r),\ \pi \equiv \pi(p^{*}, r),\ p^{*} = \mathbf{p}^{*}(p^{*}, r).$$

As in BSGU, there are two steady-state equilibria corresponding to the two solutions of equation (17) evaluated in steady state

$$\frac{1}{1+r} \max\left(1, 1+r+\alpha\left(\pi-1\right)\right) \frac{1}{\pi} = 1$$

The solutions are $\pi = 1$ and $\pi = 1/\left(1+r\right)$. The corresponding value of Y is computed using equation (17).

Corresponding to each steady state, there is an equilibrium function, $\mathbf{X}\left(p_{t-1}^{*}, r\right)$, that satisfies (16)-(18).[8] We call the function $\mathbf{X}$ associated with steady-state inflation $\pi = 1$ the *target-inflation* equilibrium because it is associated with the target-inflation steady-state. We call the function $\mathbf{X}$ associated with steady-state inflation of $\pi = \beta$ the *deflation* equilibrium because it is associated with the deflation steady state.

We now discuss our procedure for finding the two equilibrium functions, $\mathbf{X}\left(p_{t-1}^{*}, r^{\ell}\right)$, conditional on a given $\mathbf{X}(p_{t-1}^{*}, r)$. In order to construction the functions $\mathbf{X}\left(p_{t-1}^{*}, r^{\ell}\right)$, it is useful to consider how the economy would evolve if $r_t = r^{\ell}$ for a very long time. As pointed out by Mertens and Ravn (2014), in such an equilibrium, p_t^{*} converges to a constant $p^{*\ell}$, where

$$p^{*\ell} = \mathbf{p}^{*}(p^{*\ell}, r^{\ell}).$$

The other equilibrium quantities converge to the constants

$$Y^{\ell} \equiv \mathbf{Y}(p^{*\ell}, r^{\ell}),\ \pi^{\ell} \equiv \pi(p^{*\ell}, r^{\ell}).$$

When the system has converged, (16)-(18) consists of three equations in Y^{ℓ}, π^{ℓ}, and $p^{*\ell}$. These

[8]These two functions are distinct in the sense that there is no transition path from one steady state to the other.

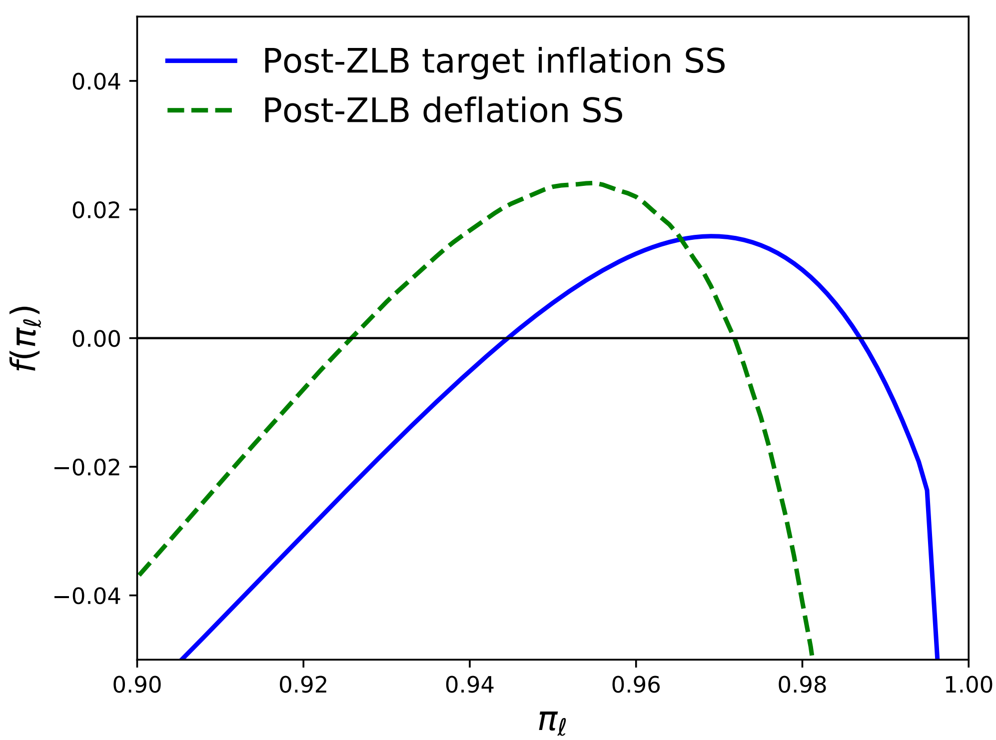
equations can be collapsed into one equation in one unknown, π^{ℓ} :[9]

$$f(\pi^{\ell}) = 0. \tag{20}$$

For every case that we consider, there are two values of π^{ℓ} which solve (20). It follows that there are always two points to which the economy could converge if it remained at the ZLB for a very long period of time. For each of these points of convergence, there is a corresponding function, $\mathbf{X}\left(p_{t-1}^{*}, r^{\ell}\right)$, which we construct.

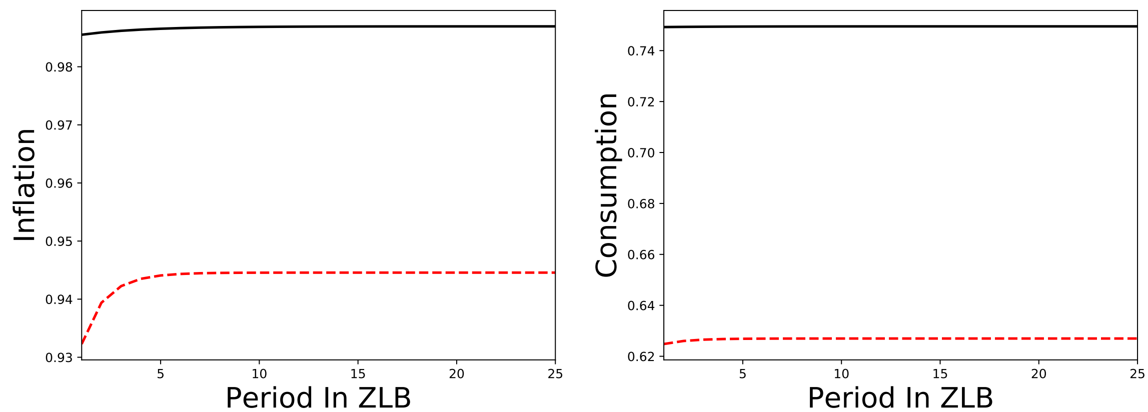
The two curves in Figure 1 display the function $f(\pi^{\ell})$ for the two steady states. In each case $f(\pi^{\ell})$ has an inverted 'U' shape and there are two points to which inflation could converge if $r_t = r^{\ell}$ for many periods.

Figure 1: Graph of $f\left(\pi_{\ell}\right)$



[9]To define f, solve the first equation in (16) for $p^{*\ell} = p_{t-1}^{*} = p_{t}^{*}$ conditional on a given value π^{ℓ}. Then, use $\mathbf{X}(p^{*\ell}, r)$ and the second equation in (16) to solve for Y^{ℓ}. The third equation in (16) can then be used to solve for F^{ℓ}. The value of $f(\pi^{\ell})$ is the expression on the left of the equality in the fourth equation of (16).

Figure 2: RE Equilibrium Paths In ZLB



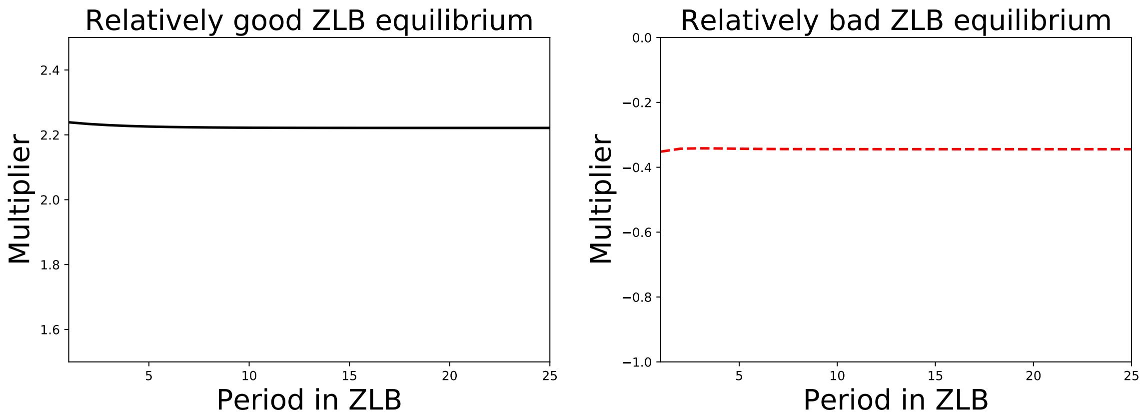
2.3 Baseline Fiscal Multiplier Results

In this sub-section, we consider the following experiment. At time $t = 0$ the economy is in a nonstochastic steady state for $r_0 = r$ and $\pi_0 = p_0^* = 1$. Consistent with (3), agents assign zero probability to a change in r_t. In period $t = 1$, $r_1 = r^\ell$. Afterward, r_t evolves according to (3). For now, we assume that the economy will return to the target inflation steady state after $r_t = r$. The solid and dashed lines in each panel of Figure 2 display the dynamic path of inflation and consumption in the two rational expectations equilibria when $r_t = r^\ell$. Given our assumptions, the ZLB is binding.

A number of features of these equilibria are worth noting. First, in one equilibrium, quarterly inflation and consumption drop in the impact period of the shock by roughly 1.5 and 5.1 percentage points, respectively. After about 5 quarters these declines stabilize at 1.3 and 5 percentage points, respectively. We refer to this equilibrium as the *relatively-good* equilibrium. Second, in the other equilibrium, quarterly inflation and consumption drop in the impact period of the shock by 6.8 and 17.5 percentage points, respectively. After about 5 quarters these declines stabilize at 5.5 and 17.3 percentage points, respectively. We refer to this equilibrium as the *relatively-bad* equilibrium. Third, the dynamics induced by the evolving state variable p_t^* are larger, the larger is the drop in inflation. In both equilibria, p_t^*, C_t, and π_t effectively converge to constants after roughly a year.

To derive values for the multiplier we assume that $G^\ell = 1.05 \times G$, i.e. when $r_t = r^\ell$, G rises by

Figure 3: RE Multiplier at the ZLB



1 percent of steady state output. We define the multiplier to be

$$\frac{\Delta C_t + 0.05G}{0.05G}.$$

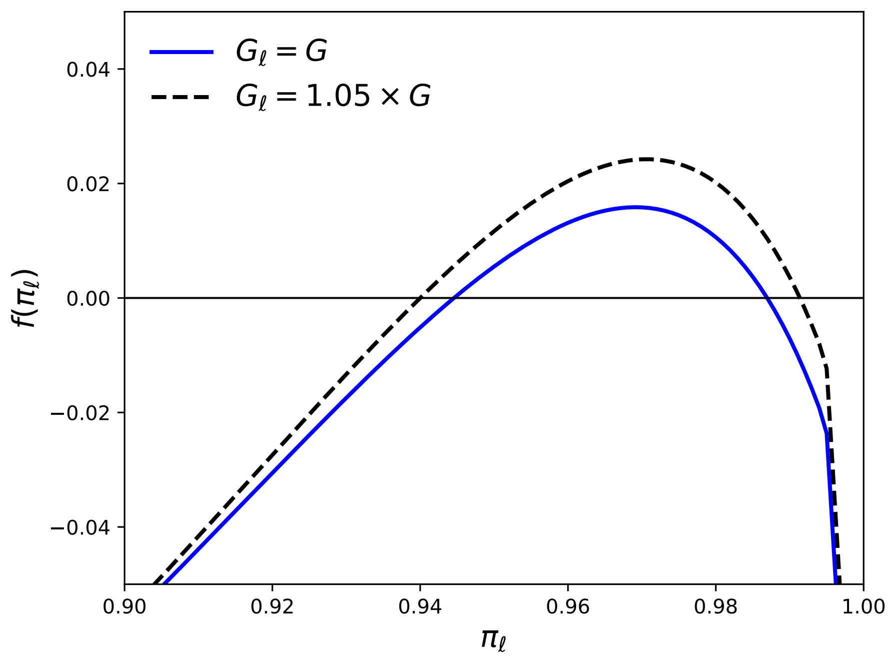
Here, the Δ refers to the difference between outcomes when government purchases are G^{ℓ} and the ZLB and when government purchases are G at the ZLB. We compute the multiplier assuming that if the economy is in the relatively good equilibrium then it is in the relatively good equilibrium when government purchases are higher. We make the symmetric assumption for the other equilibrium.[10]

The two panels of Figure 3 display the multiplier in the two equilibria at the ZLB as a function of time. Notice that the multiplier in the relatively good ZLB equilibrium is large, exceeding two over the time period displayed. In contrast, the multiplier is actually *negative* in the relatively-bad ZLB equilibrium. To understand why the multiplier depends on the equilibrium, note that an increase in G^{ℓ} shifts f upwards (see Figure 4). This shift implies that the effect of an increase in G depends on which equilibrium we consider. At the relatively-bad equilibrium (the crossing to the left), inflation falls. That decline induces a rise the real interest rate and a fall in consumption. At the relatively-good equilibrium (the crossing to the right), inflation rises. That rise induces a decline in the real interest rate and an increase in consumption.

In the analysis above, the ZLB becomes binding because of a shock to the household's discount rate. In Appendix B we suppose, as in Mertens and Ravn (2015), that the ZLB binds because of a

[10]This assumption is non-trivial because one can easily construct examples in which G serves as a sunspot inducing a switch from one equilibrium to the other. As in Mertens and Ravn (2014), we abstract from this issue.

Figure 4: Effects of an Increase in G on $f\left(\pi^{\ell}\right)$



non-fundamental sunspot shock. We find qualitatively similar results to those discussed above.

3 E-Stability at the ZLB

In this section, we investigate whether or not rational expectations equilibria at the ZLB are E-stable. This section is organized as follows. In the subsection 3.1, we restate the model in terms of general beliefs, rather than imposing rational expectations. In the subsection 3.2, we define E-stability for a non-linear model and show that only the relatively good ZLB equilibrium that returns to the target-inflation steady state is E-stable. In subsection 3.3, we analyze the relationship between our E-stability results and those that obtain in a log-linear version of the NK model. We propose a new log-linearization that turns out to be more accurate than the traditional one used in the literature. Finally, in section 3.4 we investigate the magnitude of the fiscal multiplier in the non-linear model and in linearized versions of the model.

3.1 Beliefs

To solve agents problems, we must attribute to them beliefs about all relevant aggregate variables.[11] We assume that households and firms enter the period with the same beliefs.

Denote beliefs about time-t aggregate consumption and inflation by $C^{e,t}\left(p^*_{t-1}, r_t\right)$ and $\pi^{e,t}\left(p^*_{t-1}, r_t\right)$. The first-order condition for firms that can pick $\tilde{p}_t \equiv \tilde{P}_t/P_{t-1}$ is given by

$$E_t \sum_{j=0}^{\infty} d_{t+j}\theta^j \left(\frac{\tilde{p}_t}{\prod_{k=0}^{j} \pi^{e,t}_{t+k}} - \chi \frac{(C^{e,t}_{t+j} + G_{t+j})^2}{p^{*e,t}_{t+j}} \right) \left(\prod_{k=0}^{j} \pi^{e,t}_{t+k} \right)^{-\varepsilon} \frac{C^{e,t}_{t+j} + G_{t+j}}{C^{e,t}_{t+j}} = 0. \tag{21}$$

Here we have used the aggregate resource constraint and the intratemporal Euler equation. Consistent with (13), firms form beliefs about p^*_t according to

$$p^{*,e,t}_t = \left[(1-\theta) \left[\frac{1 - \theta \left[\pi^{e,t}\left(p^*_{t-1}, r_t\right)\right]^{\varepsilon - 1}}{1-\theta} \right]^{\frac{-\varepsilon}{1-\varepsilon}} + \theta \left[\pi^{e,t}\left(p^*_{t-1}, r_t\right)\right]^{\varepsilon} \left(p^*_{t-1}\right)^{-1} \right]^{-1}. \tag{22}$$

Equations (21)-(22) correspond to the standard optimality condition for $\tilde{p}_t$ in the Calvo model and the law of motion for p^*_{t-1}, except that we have not imposed rational expectations.

Note that firms choose $\tilde{p}_t$ based on beliefs about time-t aggregate prices and quantities. The current values of these variables are a function of firms' collective time-t decisions. So firms cannot actually observe these variables when they choose $\tilde{p}_t$. The standard assumption is that firms form a "belief" about these variables, based on state variables and contemporaneous shocks, when they make their decisions. In a rational expectations equilibrium that belief is correct. In a world where firms do not necessarily have rational expectations it is not natural to assume that firms see the current values of aggregate variables when they choose prices.

Households observe the current-period wage when they supply labor and know the current-period inflation rate when they purchase consumption goods. In addition, they know their own discount rate, r_t. They form time-t beliefs about aggregate inflation in the next period, $\pi^{e,t+1}\left(p^*_t, r_{t+1}\right)$. This notation reflects that, in contrast to firms, households know π_t and p^*_t when forming expectations

[11] Under rational expectations, intermediate good firms choose their price level, $\tilde{P}_t$, based on the current and expected values of aggregate output, consumption, and inflation. The current values of these variables are a function firms' collective time t decisions. So firms cannot actually observe these variables when they choose $\tilde{P}_t$. The standard assumption is that when firms make their time-t decisions, they form 'beliefs' about these variables, based on the values of the time-t state variables and realizations of the shocks. In a rational expectations equilibrium those beliefs are correct. If firms do not have rational expectations it is not natural to assume that they see the current values of aggregate variables when they choose their prices.

about inflation at time $t+1$. The household intertemporal Euler equation is given by

$$\frac{1}{C\left(p_{t-1}^{*}, r_t\right)} = \frac{1}{1+r_t} \max\left(1, 1+r+\alpha\left(\pi_t - 1\right)\right) E_t \frac{1}{C^{e,t}\left(p_t^{*}, r_{t+1}\right)} \frac{1}{\pi^{e,t+1}\left(p_t^{*}, r_{t+1}\right)}. \tag{23}$$

Here, $C^{e,t}\left(p_t^{*}, r_{t+1}\right)$ denotes households' future consumption plan. When $r_t = r^{\ell}$, (23) can be written as

$$\begin{aligned} \frac{1}{C\left(p_{t-1}^{*}, r_t = r^{\ell}\right)} = \frac{1}{1+r_t} \Big[& \frac{p}{C^{e,t}\left(p_t^{*}, r_{t+1} = r^{\ell}\right)} \frac{1}{\pi^{e,t+1}\left(p_t^{*}, r_{t+1} = r^{\ell}\right)} \\ & + \frac{1-p}{C^{e,t}\left(p_t^{*}, r_{t+1} = r\right)} \frac{1}{\pi^{e,t+1}\left(p_t^{*}, r_{t+1} = r\right)} \Big]. \end{aligned} \tag{24}$$

In their analysis of learning in an infinite-horizon model, studies like Preston (2005) use the household's lifetime budget constraint in conjunction with the household's first-order conditions to expression consumption as a function of current and future aggregate variables. This strategy is feasible with log-linear versions of the model. But, as far as we know, those strategies do not work in our non-linear setting.[12]

3.2 E-stability in the non-linear model

E-stability, as defined by Evans and Honkapohja (2001), is a relationship between the parameters characterizing beliefs and the parameters characterizing outcomes. Loosely speaking, E-stability implies that small deviations from parameters that characterize rational expectations beliefs induce smaller deviations in the parameters that characterize equilibrium outcomes. In non-linear models with state variables, rational expectations equilibrium beliefs cannot be exactly characterized by a finite number of parameters. However, these beliefs can be well-approximated with a finite number of parameters (see Judd (1998)). We use this fact to when assessing the E-stability of equilibria in our non-linear model.

Recall that the aggregate state variables in the economy are p_{t-1}^{*} and r_t. We assume that, conditional on r_t, firms' beliefs are formed using a finite-elements approximation on a fine grid over p_{t-1}^{*}. This finite-elements approximation is on the same grid that we use to solve for the rational expectations equilibria.

We define a rational expectations equilibrium as a set of functions $Y_t = \mathbf{Y}(p_{t-1}^{*}, r_t; \Psi)$ and $\pi_t =$

[12] Of course, in our analysis the household's per-period budget constraint holds at every date and for every state.

$\pi(p^*_{t-1}, r_t; \Psi)$ that solve (3.2) and (3.3) at all of the grid points in our finite-elements approximation. With N grid points, the procedure yields an N-vector of parameters, Ψ, that characterize the rational expectations equilibrium.

Denote the vector of parameters characterizing agents' beliefs about the perceived equilibrium laws of motion (PLM) by Ψ^{PLM}. Conditional on those beliefs, the actual law of motion (ALM) is defined by the resulting outcomes for aggregate variables, as determined by equations (21)-(23). Denote by Ψ^{ALM} the vector of parameters characterizing the finite-elements approximation to the outcomes for aggregate variables. The model defines a mapping from Ψ^{PLM} to Ψ^{ALM}, given by

$$\Psi^{ALM} = T(\Psi^{PLM}). \tag{25}$$

Evans and Honkapohja (2001)define an equilibrium to be E-stable if the real parts of the eigenvalues of $\frac{\partial T(\Psi)}{\partial \Psi}$ are negative. The idea is that small perturbations of beliefs from an E-stable equilibrium induce dynamics that push the system back to the initial equilibrium.

Based on this definition, we investigate the E-stability of the four rational expectations equilibria in our model. We compute $\frac{\partial T(\Psi)}{\partial \Psi}$ for the parameters of our finite-elements approximation. Our results can be characterized as follows. First, as in McCallum (2009), the negative-inflation steady state equilibrium is not E-stable. So, the two ZLB equilibria that converge to the negative inflation steady state after $r_t = r$ are not E-stable. Second, the relatively bad ZLB equilibrium that converges to the intended steady after $r_t = r$ is not E-stable. Third, the relatively good ZLB equilibrium that converges to the intended steady after $r_t = r$ is E-stable. So, there is a unique (interior minimum state variable) rational expectations equilibrium.

Recall, above we established that the qualitative properties of fiscal policy in the E-stable equilibrium are the same as those stressed in literature based on the log-linearized NK model (see, for example, Christiano et al. (2011)). Taken together, this result and our E-stability results make concrete our claim that non-uniqueness in the NK model is not a substantive problem at least as regards fiscal policy.

3.3 Analyzing E-stability using a linearized version of the model

To provide further insight into the results of the results of section 3.2, it is useful to focus on the limit points of a ZLB equilibrium, assuming $r_t = r^\ell$ for a long time. The advantage of doing so is that we can work with linear approximations to the model, where the point of approximation is taken around the limit point.[13] Note that $p_t^* \neq 1$ at the limit point in the ZLB. So this linearization maintains the dependence of aggregate variables on $\hat{p}_t^*$. Approximating around that point seems reasonable given the relative speed with which the system converges in the ZLB. In fact, equilibrium quantities and prices converge to this limit point after about one year (see Figure 2). When $r_t = r$, we linearize the model around the target-inflation steady state.

We denote log-deviations from steady-state values by $\hat{x}_t$. The equilibrium of the linearized model when $r_t = r$ is characterized by a set of coefficients γ_x, so that

$$\hat{x}_t = \gamma_x \hat{p}_{t-1}^* \tag{26}$$

for each of the endogenous variables. Normally the dependence of $\hat{x}_t$ on $\hat{p}_{t-1}^*$ is ignored in the linearized Calvo model. The rationale is that $\hat{p}_{t-1}^*$ depends only on its own lagged value, so that if the system starts in steady state, $\hat{p}_{t-1}^*$ is identically equal to zero. This assumption does not make sense in our context because $\hat{p}_{t-1}^*$ will not be equal to zero once the system has been in the ZLB. So it makes sense to maintain the dependence of $\hat{x}_t$ on $\hat{p}_{t-1}^*$ when analyzing how the system returns to steady state after the ZLB episode is over.

When $r_t = r^\ell$, we linearize the equilibrium conditions around a limit point in the ZLB. We denote log-deviations of the endogenous variables from their value at that limit point by $\hat{x}_t^\ell$. At the limit point, p_{t-1}^* is a constant that is less than 1. So, when r_t reverts to r, $\hat{p}_{t-1}^*$ has a non-zero value. This makes clear why, in (26), $\hat{x}_t$ is a function of $\hat{p}_t^*$.

When the equilibrium conditions are linearized around the relatively good ZLB equilibrium, the linearized system has a unique bounded rational expectations equilibrium. It is also E-stable. It follows from Evans and Honkapohja (2001) that the equilibrium therefore stable under adaptive and least squares learning. When the equilibrium conditions are linearized around the relatively bad ZLB equilibrium, the linearized system is indeterminate in the sense of Blanchard and Khan and not E-stable. In this way, the E-stability of the non-linear equilibria are closely related to the determinacy of the linearized equilibria.

[13]Linear approximations have the added advantage that they do not require numerical approximation to the equilibrium laws of motion for the linearized equilibrium conditions.

Table 1: Comparing the Linear and Non-linear Models

Model	Multiplier
Standard Linearization	1.63
Our Linearization, rel. good	2.44
Nonlinear, rel. good	2.24
Nonlinear, rel. bad	-0.35

3.4 Multipliers in the linear and non-linear model

In this sub-section, we compare the size of the fiscal multiplier in the non-linear and linear models. Table 1 reports the multiplier in the non-linear model the standard log-linearized model and our log-linearized model. In the latter case, the multiplier is reported for the linearization around the point for which the equilibrium is determinate.

Two key results are worth noting. First, the multipliers in the linearized models are similar to the multiplier in the relatively good equilibrium of the non-linear model. Indeed, for our preferred linearization, the multipliers are nearly identical. Second, in stark contrast, the multiplier at the relatively bad ZLB equilibrium of the nonlinear model is very different than the multipliers in the linear models. These results make concrete our claim that multiplicity in the NK model is not a substantive problem, at least regarding fiscal policy, because the results from the linearized models are qualitatively similar to the results at the relatively good ZLB equilibrium, which is E-stable.

4 Stability Under Learning at the ZLB

In this section we investigate stability under learning of equilibria at the ZLB in the non-linear model. In addition we assess whether the qualitative results of the standard linear NK model continue to hold in the non-linear learning model.

4.1 The Learning Algorithm

An important question in a nonlinear environment is: how do agents learn about nonlinear equilibrium functions mapping the state of the economy to aggregate prices and quantities? We follow Chen and White (1998) and assume that households and firms learn about equilibrium functions using a recursive non-parametric learning scheme. A virtue of this scheme is that agents do not have to know the precise functional forms of the equilibrium mappings.

Specifically, we assume that when the ZLB is binding, households and firms update their beliefs about consumption and inflation according to

$$\pi_{r^\ell}^{e,t+1}(x) = \pi_{r^\ell}^{e,t}(x) + \mathcal{K}\left(\frac{x - p_{t-1}^*}{b_t}\right)\left[\frac{\pi_t - \pi_{r^\ell}^{e,t}(x)}{(t+1)b_t}\right], \tag{27}$$

$$C_{r^\ell}^{e,t+1}(x) = C_{r^\ell}^{e,t}(x) + \mathcal{K}\left(\frac{x - p_{t-1}^*}{b_t}\right)\left[\frac{C_t - C_{r^\ell}^{e,t}(x)}{(t+1)b_t}\right]. \tag{28}$$

Here, the kernel $\mathcal{K}$ is the normal density function and the bandwidth b_t is a sequence of positive numbers that decreases to zero as $t \to \infty$. As in Chen and White (1998), we assume that b_t is of order $O\left((t+1)^{-\delta}\right)$ for some $0 < \delta < \frac{1}{4}$. Chen and White (1998) make this assumption to ensure that beliefs that are updated using their non-parametric learning algorithm they propose converges to rational expectations beliefs in a stochastic environment.[14]

According to equations (27) and (28), agents update their beliefs using what amounts to a non-parametric estimator of the equilibrium mapping. A key feature of the learning algorithm is that when agents see a new value of consumption and inflation, they disproportionately update their beliefs about the equilibrium functions in an area around p_{t-1}^*. The kernel density and bandwidth must meet certain regularity condition so that beliefs converge over time. The presence of the $(t+1)^{-1}$ term implies that over time agents give less weight to new observations. The b_t term controls the rate at which future observations get less weight. The term $[(t+1)b_t]^{-1}$ goes to zero as $t \to \infty$, although less quickly than $(t+1)^{-1}$. Notice in the kernel, $x - p_{t-1}^*$ is divided by b_t. So, as the bandwidth goes to zero, agents are less willing to update their beliefs about the equilibrium mappings for values of p_{t-1}^* that are far away from the current realization.

Unlike Chen and White (1998), we are not working with a model that is stationary and ergodic. To see this, note that the state in which the ZLB is no longer binding is an absorbing state. So agents only observe one sequence of consumption and inflation corresponding to a particular sequence of p_{t-1}^*, at the ZLB.[15] It follows that beliefs in a learning equilibrium can't converge to the rational expectations beliefs for all values of p_{t-1}^*. For this reason, we adopt a more-limited notion of stability under learning that is more-applicable to EW equilibria. Specifically, we say that a rational expectations equilibrium is stable under learning if the values of inflation and consumption

[14] In practice, we assume $b_t = \frac{t+100}{t+1}(t+1)^{-0.24}$.

[15] For example, we assume that a particular value of p_{t-1}^* at the beginning of the ZLB episode, so agents are only going to learn about the properties of the ZLB starting from that point.

in the learning equilibrium when the ZLB binds converge to their values in the limit point of the corresponding rational expectations equilibrium if the ZLB binds for a very long time.

4.2 Results

In what follows, we imagine that the economy is initially in the target-inflation steady state. Then, r_t drops to r^ℓ and obeys the law of motion given by (3). Agents have to learn about consumption and inflation evolve when $r_t = r^\ell$. We establish four key results. First, the relatively good ZLB equilibrium is stable under learning in the sense defined above. Second, the key results from the standard log-linear NK model carry over to the learning environment. Specifically, the multiplier is much bigger when the ZLB constraint is binding compared to when it is not. Third, the relatively bad ZLB equilibrium is not stable under learning. Fourth, the multiplier in the ZLB is large even when agents' beliefs are begin near the relatively bad ZLB equilibrium.

Throughout, we characterize agents' beliefs by a finite elements approximation on the grid for p^*_{t-1} that we use to calculate the rational expectations equilibria. Agents update their expectations according to (27) and (28), where these equations are evaluated at each grid point. In our initial experiment, we assume that agents' initial beliefs are equal to the mappings for the rational expectations equilibrium that converges to target-inflation steady state. Agents also think that, in the post-ZLB period, when $r_t = r$, the equilibrium laws of motion correspond to the rational expectations equilibrium that eventually converges to the target-inflation steady state.[16] These assumptions are reasonable given the E-stability of the target-inflation steady state equilibrium, and the presumption that $r_t = r^\ell$ is a relatively rare event.

The purple solid lines in Figure 5 display the paths of consumption and inflation in the learning equilibrium, assuming no change in government spending. The corresponding rational expectations equilibrium paths are given by the blue dashed lines. Notice that, in the learning equilibrium, consumption and inflation converge from above to their limiting values in the relatively good ZLB equilibrium. Significantly, the ZLB is not binding in the first few periods because inflation is relatively high as a result of agents initially high inflation expectations. In those periods, the Taylor rule calls for a non-negative interest rate. After 3 periods, the ZLB becomes binding. As expectations about inflation and consumption adjust downward in response to low realized outcomes, actual

[16] In Appendix B, we analyze the case in which agents assume that inflation and consumption converge to the negative inflation steady state when $r_t = r$. We find similar results.

inflation and consumption decline.

The top panel of Figure 6 displays the value of the multiplier in the learning equilibrium. We calculate the multiplier assuming that G_t increases by 1 percent of steady state output for as long as long as $r_t = r^\ell$. After the ZLB binds, the multiplier rises above 1 and then converges to its value in the corresponding rational expectations equilibrium.[17]

It is instructive to consider an analogous experiment, but where $r^\ell = r$ and the ZLB does not bind. That is, in period 0, G increases. When G increases, agents also think that it will revert to its steady state value with probability $1 - p$.[18] After G reverts to its steady state value, agent believe that the equilibrium laws of motion correspond to the rational expectations equilibrium that eventually converges to the target-inflation steady state. We assume that agents' initial beliefs about consumption and inflation are equal to the mappings for the rational expectations equilibrium that converges to target-inflation steady state. The bottom panel of Figure 6 displays the value of the multiplier in the learning equilibrium for this case. Notice that the multiplier is lower than 1, and quickly converges to its value in the corresponding rational expectations equilibrium.

We now re-do the ZLB experiment assuming that initial beliefs about the equilibrium mapping for consumption and inflation are slightly higher than their relatively bad ZLB equilibrium values. The top two panels of Figure 7 are the analogue of Figure 5. Notice that consumption and inflation diverge from the limit points in the relatively bad ZLB equilibrium. Indeed, after many periods, they converge to the limit points of the relatively good ZLB equilibrium.[19] The purple solid line in the bottom panel of Figure 7 displays the multiplier in the learning equilibrium. The blue dashed line displays the multiplier in the relatively bad ZLB equilibrium. Note the multiplier is initially about 1. Then, reflecting relatively high inflation rates, the multiplier becomes very large.[20] These properties contrast sharply with the properties of the multiplier in the corresponding rational expectations equilibrium.

We conclude that for all of the different learning equilibria that we consider, the multiplier is large when the ZLB binds. Moreover, the multiplier is relatively small when the ZLB is not binding. These results are qualitatively very similar to the results that we obtain in E-stable rational expectations

[17] The value of the multiplier is initially low because the ZLB isn't binding in the first few periods.

[18] In these equilibria, we assume that beliefs are such that the ZLB does not bind. This is in contrast to the assumption we make in Appendix B.

[19] If we re-do the experiment assuming that the initial beliefs about consumption and inflation are lower than the relatively bad ZLB equilibrium values, the outcomes for inflation and consumption diverge toward zero.

[20] Eventually the multiplier converges from above to its value in the limiting point of the relatively good ZLB equilibrium.

Figure 5: Learning Equilibrium, Starting at Steady State RE Equilibrium

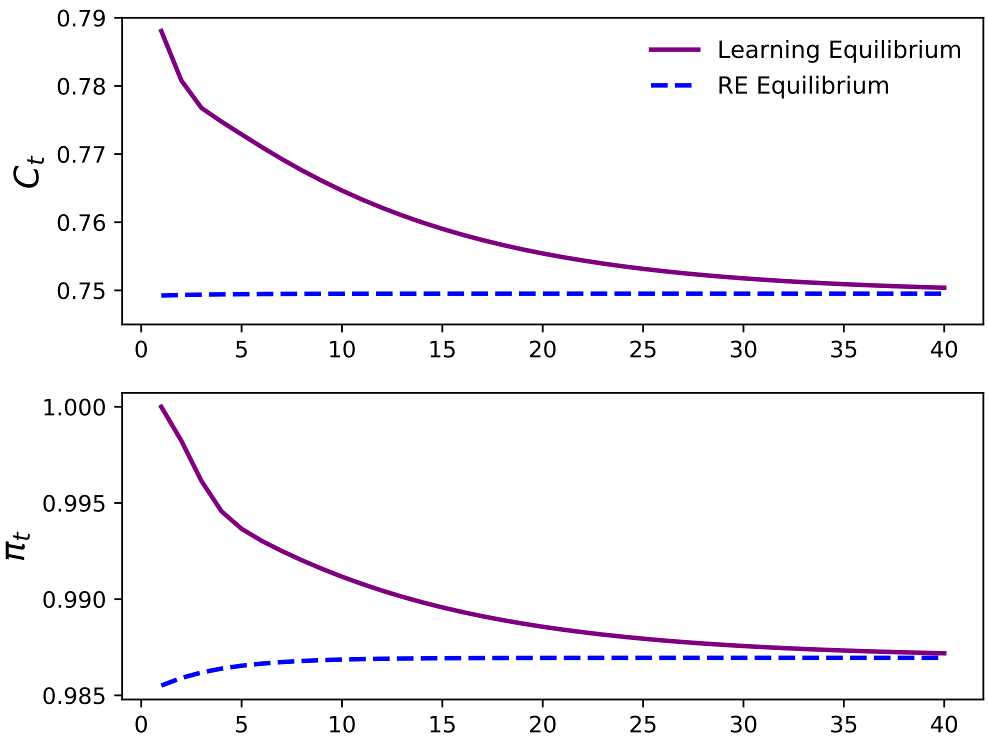


Figure 6: Learning Equilibrium, Starting at Steady State RE Equilibrium

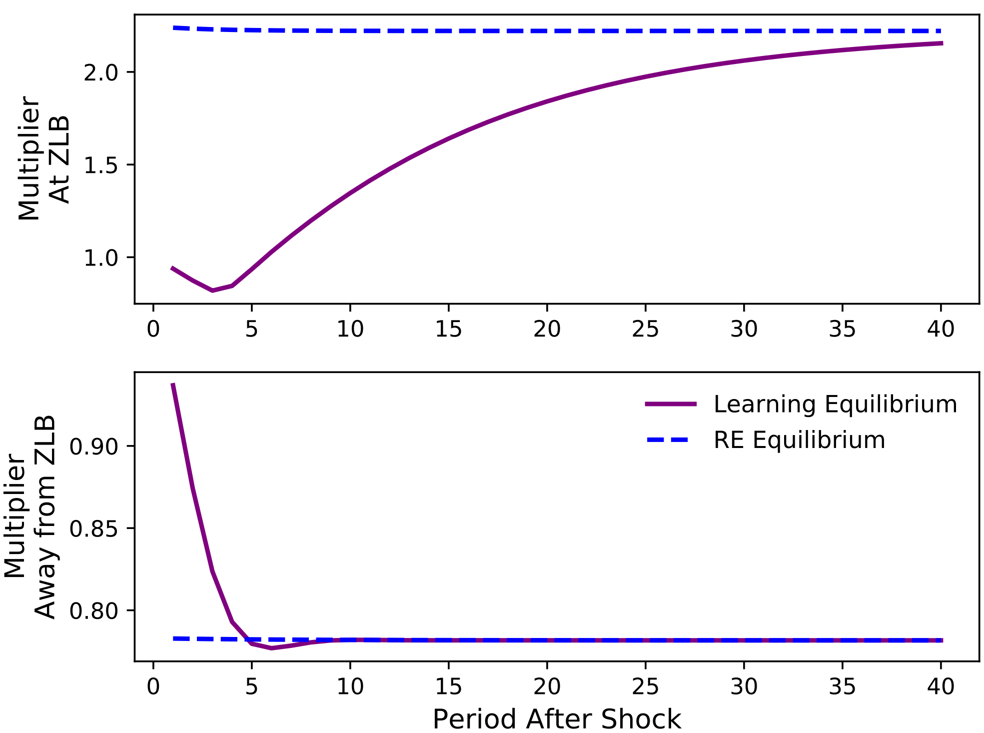
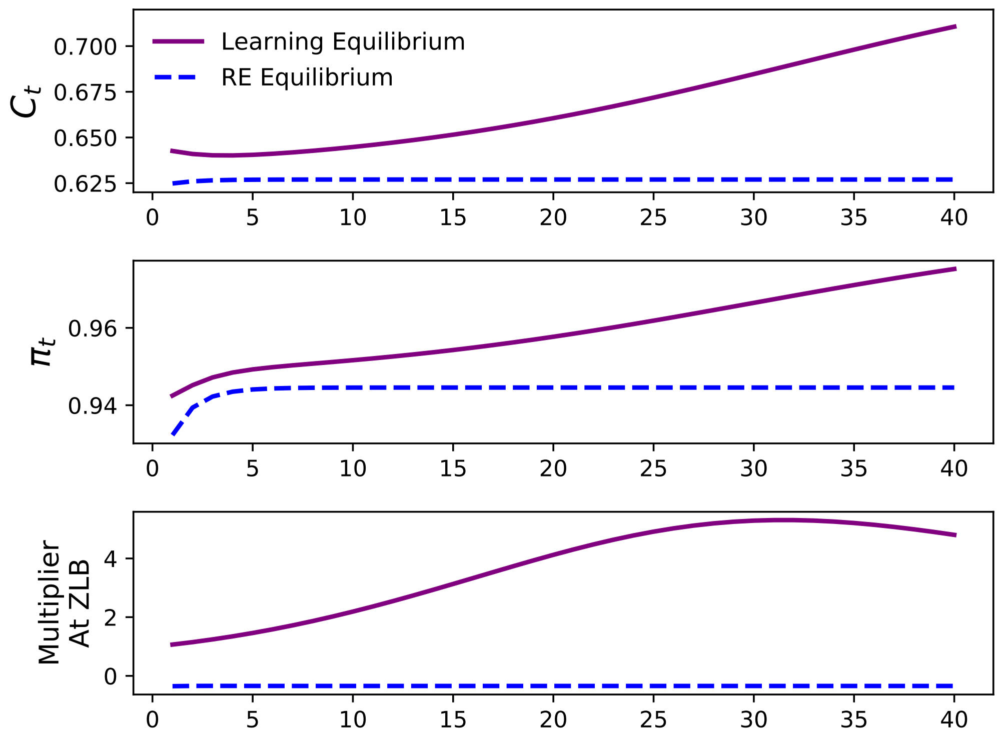


Figure 7: Learning Equilibrium, Starting Near Relatively Bad ZLB Equilibrium



equilibria of the NK model. We infer that the multiplier results do not depend sensitively on the rational expectations hypothesis being literally true.

4.3 Reconciling with Mertens and Ravn (2014)

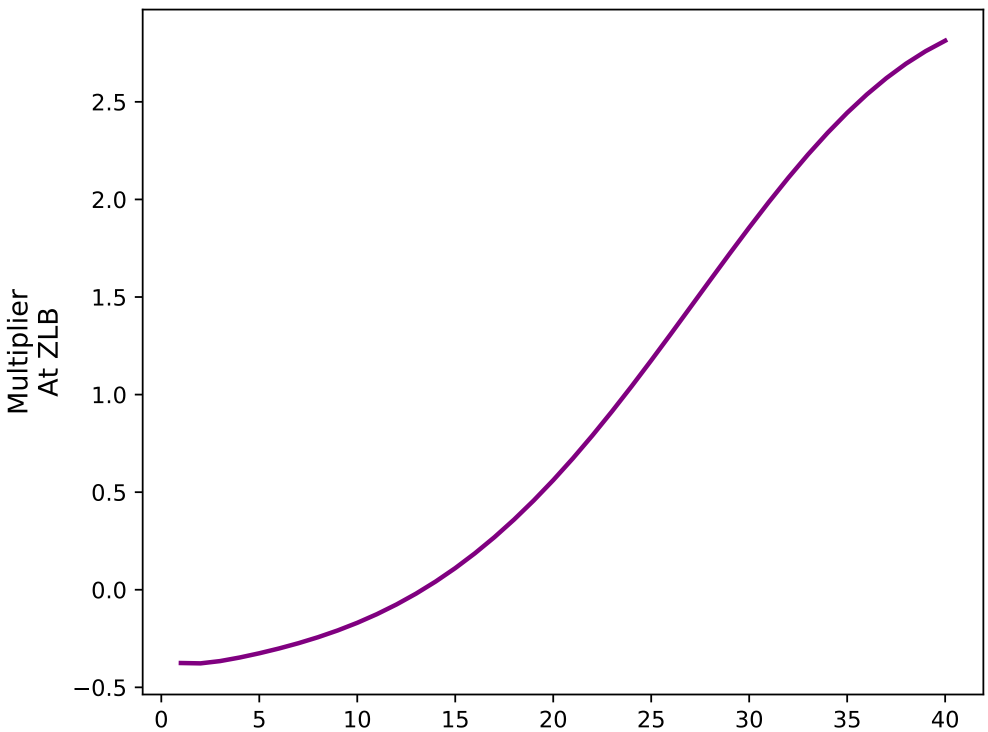
Mertens and Ravn (2014) report that the fiscal multiplier is small in a learning equilibrium. In their experiment, agents' initial beliefs are near a rational expectations equilibrium that is analogous to our relatively bad ZLB equilibrium. Their result contrasts sharply with our finding that the multiplier is large when we begin with beliefs that are near the relatively bad ZLB equilibrium. There are three differences our analyses. First, they work with a linearized Calvo model when they study the learning equilibrium. Second, they assume that when firms choose prices, they see the contemporaneous aggregate price level. Third, the experiment that underlies their multiplier calculation is subtly different than ours. When we calculate the multiplier, we assume agents' initial expectations about consumption and inflation are the same regardless of the value of G. Recall that in the relatively bad ZLB equilibrium, inflation actually falls by ϵ_π when G increases. Similarly, C_t falls by ϵ_C. When Mertens and Ravn (2014) initially raise G in a learning equilibrium, they simultaneously decrease agents' expectations of inflation by ϵ_π and consumption by ϵ_C. Consistent with the analysis above, this fall in inflation expectations would in and of itself reduce output in the ZLB.

It turns out that the first two differences between our analysis and that of Mertens and Ravn (2014) do not have a large impact on the multiplier. In contrast the third difference is very important. Figure 8 displays the multiplier if we adopt the assumption in Mertens and Ravn (2014) about how expectations about inflation change when G increases. Notice that that the multiplier is negative for roughly 3 years. In this example, the initial drop in inflation expectations is quantitatively much more important than the increase in G.

5 Conclusion

In this paper we analyze whether non-uniqueness of equilibria poses a substantive challenge to the key conclusions of NK models about the efficacy of fiscal policy in the ZLB. We argue that it does not. This conclusion rests on our view that if a rational expectations equilibrium is not E-stable or stable under learning, then it is simply too fragile to be a description of the data. We have also

Figure 8: Multiplier with Shock to Expectations



argued that the key properties of linear and non-linear models are very similar. Indeed, under our proposed new way of linearizing the NK model in the ZLB, the quantitative value of the multiplier is very similar in the linear and non-linear model.

To asses stability under learning, we adopted a particular model of learning. It is certainly possible that there exist some models of how agents learn for which our results would not hold. Exploring stability under learning under alternative plausible learning models is an important avenue for future research.

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A Benchmark NK Model

Representative households maximize

$$E_0 \sum_{t=0}^{\infty} d_t \left[\log\left(C_t\right) - \frac{\chi}{2} N_t^2 \right]$$

where C_t is consumption, N_t are hours worked, and

$$d_0 = 1,\ d_t = \prod_{j=1}^{t} \left(\frac{1}{1 + r_{j-1}} \right)$$

The stochastic process for r_t is given by

$$\Pr\left[r_{t+1} = r^{\ell} | r_t = r^{\ell}\right] = p,\ \ \Pr\left[r_{t+1} = r | r_t = r^{\ell}\right] = 1 - p,\ \ \Pr\left[r_{t+1} = r^{\ell} | r_t = r\right] = 0$$

The household face the budget constraint

$$P_t C_t + B_t \leq \left(1 + R_{t-1}\right) B_{t-1} + W_t N_t + T_t$$

Here P_t is the price of the consumption good, B_t is the quantity of risk-free nominal bonds held by the household, R_{t-1} is the gross nominal interest rate paid on bonds held from period $t-1$ to period t, W_t is the nominal wage, and T_t represents lump-sum profits net of lump-sum government taxes. First-order necessary conditions

$$\frac{1}{C_t} = \lambda_t$$

$$\chi N_t = \frac{W_t}{P_t} \lambda_t$$

$$\lambda_t = d_t R_t E_t \frac{\lambda_{t+1}}{\pi_{t+1}}$$

A final homogeneous good, Y_t, is produced by competitive and identical firms using the technology:

$$Y_t = \left[\int_0^1 \left(Y_{j,t}\right)^{\frac{\varepsilon}{\varepsilon - 1}} dj \right]^{\frac{\varepsilon - 1}{\varepsilon}}$$

where $\varepsilon > 1$. The representative firm chooses inputs, $Y_{j,t}$, to maximize profits:

$$P_t Y_t - \int_0^1 P_{j,t} Y_{j,t} dj,$$

The first-order necessary condition is

$$Y_{j,t} = \left(P_t / P_{j,t}\right)^{\varepsilon} Y_t.$$

Monopolists maximize

$$E_t \sum_{k=0}^{\infty} d_{t+k} \frac{\lambda_{t+k}}{P_{t+k}} \left((1+\upsilon)P_{j,t+k} - P_{t+k}s_{t+k}\right) Y_{j,t+k}$$

where s_t is real marginal cost and υ is a subsidy that can be used to offset monopoly distortions in steady state. The j^{th} monopolist sets its price, $P_{j,t}$, subject to the demand curve, and the following Calvo sticky price friction. The probability that a firm updates its price is $1-\theta$. The monopolist maximizes

$$E_t \sum_{k=0}^{\infty} d_{t+k}\lambda_{t+k} \left((1+\upsilon)P_{j,t+k} - P_{t+k}s_{t+k}\right) \left(\frac{P_{j,t+k}}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k}.$$

The first-order conditions is

$$E_t \sum_{k=0}^{\infty} \theta^k d_{t+k}\lambda_{t+k} \left(\frac{\tilde{P}_t}{P_{t+k}} - s_{t+k}\right) \left(\frac{P_t}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k} = 0,$$

which can be written as

$$E_t \sum_{j=0}^{\infty} (\theta)^j \, d_{t+j}\lambda_{t+j} \left(\frac{\tilde{p}_t}{\Pi_{t,t+j}}\pi_t - s_{t+j}\right) \Pi^{\varepsilon}_{t,t+j} Y_{t+j} = 0$$

where $\tilde{p}_t \equiv \tilde{P}_t / P_{t-1}$ and $\Pi_{t,t+j} \equiv P_{t+j}/P_t$. It is convenient to write this as

$$\tilde{p}_t = \pi_t \frac{E_t \sum_{j=0}^{\infty} d_{t+j}\theta^j \chi \Pi^{\varepsilon}_{t,t+j} Y^2_{t+j}/p^*_{t+j}}{E_t \sum_{j=0}^{\infty} d_{t+j}\theta^j \Pi^{\varepsilon-1}_{t,t+j} \frac{Y_{t+j}}{C_{t+j}}}.$$

This expression can be simplified by writing the infinite sums recursively,

$$\tilde{p}_t = \pi_t \frac{K_t}{F_t}$$

where

$$F_t = \lambda_t Y_t + \theta \frac{1}{1+r_t} E_t \pi^{\varepsilon-1}_{t+1} F_{t+1}$$

and

$$K_t = \lambda_t Y_t s_t + \theta \frac{1}{1+r_t} E_t \pi^{\varepsilon}_{t+1} K_{t+1}.$$

The price index evolves so that

$$P_t = \left(\int_0^1 P_{i,t}^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}} = \left((1-\theta) \tilde{P}_t^{1-\varepsilon} + \theta \int_0^1 P_{i,t-1}^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}} = \left((1-\theta) \tilde{P}_t^{1-\varepsilon} + \theta P_{t-1}^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}}.$$

meaning

$$1 = \left((1-\theta) \left(\frac{\tilde{p}_t}{\pi_t} \right)^{1-\varepsilon} + \theta \pi_t^{\varepsilon-1} \right)^{\frac{1}{1-\varepsilon}},$$

market clearing in the labor market implies

$$N_t = \int_0^1 N_{j,t} dj$$

Define

$$p_t^* \equiv \left(\int_0^1 \left(\frac{P_{j,t}}{P_t} \right)^{-\varepsilon} dj \right)^{-1}$$

Then

$$Y_t = p_t^* N_t$$

and

$$p_t^* = \left((1-\theta) \left(\frac{\tilde{p}_t}{\pi_t} \right)^{-\varepsilon} + \theta \pi_t^{\varepsilon} \left(p_{t-1}^* \right)^{-1} \right)^{-1}$$

The aggregate resource constraint is

$$C_t + G_t \leq Y_t$$

In equilibrium, household and firm optimality conditions are satisfied, and the budget and resource constraints hold. In equilibrium we need to determine C_t, h_t, Y_t, w_t, π_t, λ_t, R_t, s_t, F_t, K_t, $\tilde{p}_t$, and p_t^* . To do that, we use the following equations:

$$\frac{1}{C_t} = \lambda_t$$

$$\chi N_t = \frac{W_t}{P_t} \lambda_t$$

$$\lambda_t = \frac{1}{1+r_t} R_t E_t \frac{\lambda_{t+1}}{\pi_{t+1}}$$

$$\tilde{p}_t = \pi_t \frac{K_t}{F_t}$$

$$F_t = \lambda_t Y_t + \theta \frac{1}{1+r_t} E_t \pi_{t+1}^{\varepsilon-1} F_{t+1}$$

$$K_t = \lambda_t Y_t s_t + \theta \frac{1}{1+r_t} E_t \pi_{t+1}^{\varepsilon} K_{t+1}$$

$$s_t = w_t$$

$$1 = \left((1-\theta) \left(\frac{\tilde{p}_t}{\pi_t} \right)^{1-\varepsilon} + \theta \pi_t^{\varepsilon-1} \right)^{\frac{1}{1-\varepsilon}}$$

$$Y_t = p_t^* N_t$$

$$p_t^* = \left((1-\theta) \left(\frac{\tilde{p}_t}{\pi_t} \right)^{-\varepsilon} + \theta \pi_t^{\varepsilon} \left(p_{t-1}^* \right)^{-1} \right)^{-1}$$

$$C_t + G_t = Y_t$$

The monetary policy equation is

$$R_t = \max \left\{ 1, \frac{1}{\beta} + \theta_\pi \left(\pi_t - 1 \right) \right\}.$$

These equations can be simplified to the following 4 equations

$$\frac{1}{C_t} = \frac{1}{1+r_t} R_t E_t \frac{1}{C_{t+1}\pi_{t+1}}$$

$$F_t = \frac{C_t + G_t}{C_t} + \theta \frac{1}{1+r_t} E_t \pi_{t+1}^{\varepsilon-1} F_{t+1}$$

$$\left(\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{1-\varepsilon}} F_t = \chi \frac{\left(C_t + G_t\right)^2}{p_t^*} + \theta \frac{1}{1+r_t} E_t \pi_{t+1}^{\varepsilon} \left(\frac{1-\theta\pi_{t+1}^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{1-\varepsilon}} F_{t+1}$$

$$p_t^* = \left[(1-\theta) \left(\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta} \right)^{-\frac{\varepsilon}{1-\varepsilon}} + \theta \pi_t^{\varepsilon} \left(p_{t-1}^* \right)^{-1} \right]^{-1}$$

Alternatively, the equations can be simplified to (16)-(18).

A.1 Target-inflation steady state

The intended steady state can be found as follows. Normalize $N = 1$ and assume that $G/Y = 0.2$. In steady state,$\pi = 1$ ($R = 1$), so $p^* = 1$, $\tilde{p} = 1$, and $Y = 1$. From the resource constraint, $C = 0.8$. Firm optimality implies $K = F$, meaning that $s = 1$. We can then get χ from the intratemporal

Euler equation.

A.2 Linearization away from ZLB

We linearize the model around the target-inflation steady state. Denote the log deviation of a variable from steady state by $\hat{x}_t$. The log-linear equations are

$$-\hat{C}_t = \hat{\lambda}_t$$

$$\hat{N}_t = \hat{w}_t - \hat{C}_t$$

$$-\hat{C}_t = \hat{R}_t - \beta\widehat{(1+r_t)} - E_t\left(\hat{C}_{t+1} + \hat{\pi}_{t+1}\right)$$

$$\hat{\tilde{p}}_t = \hat{\pi}_t + \hat{K}_t - \hat{F}_t$$

$$\hat{F}_t = (1-\beta\theta)\left(\hat{\lambda}_t + \hat{Y}_t\right) - \theta\beta\widehat{(1+r_t)} + \theta\beta\left(\varepsilon - 1\right)E_t\hat{\pi}_{t+1} + \theta\beta E_t\hat{F}_{t+1}$$

$$\hat{K}_t = (1-\beta\theta)\left(\hat{\lambda}_t + \hat{Y}_t + \hat{s}_t\right) - \theta\beta\widehat{(1+r_t)} + \theta\beta\varepsilon E_t\hat{\pi}_{t+1} + \theta\beta E_t\hat{K}_{t+1}$$

$$\hat{s}_t = \hat{w}_t$$

$$\frac{1}{1-\theta}\hat{\pi}_t = \hat{\tilde{p}}_t$$

$$\hat{Y}_t = \hat{p}^*_t + \hat{N}_t$$

$$\hat{p}^*_t = (1-\theta)\,\varepsilon\left(\hat{\tilde{p}}_t - \hat{\pi}_t\right) - \theta\varepsilon\hat{\pi}_t + \theta\hat{p}^*_{t-1}$$

$$\frac{C}{Y}\hat{C}_t + \frac{Y-C}{Y}\hat{G}_t = \hat{Y}_t$$

After some more algebra

$$-\hat{C}_t = \hat{R}_t - \beta\widehat{(1+r_t)} - E_t\left(\hat{C}_{t+1} + \hat{\pi}_{t+1}\right)$$

$$\hat{\pi}_t = \frac{(1-\beta\theta)\,(1-\theta)}{\theta}\left(\frac{C+Y}{Y}\hat{C}_t + \frac{Y-C}{Y}\hat{G}_t - \hat{p}^*_t\right) + \beta E_t\hat{\pi}_{t+1}$$

$$\hat{p}^*_t = \theta\hat{p}^*_{t-1}$$

Setting $r_t = r$, ignoring the ZLB, and assuming our Taylor rule holds, we have

$$-\hat{C}_t = \beta\theta_\pi \hat{\pi}_t - E_t\left(\hat{C}_{t+1} + \hat{\pi}_{t+1}\right)$$

$$\hat{\pi}_t = \frac{(1-\beta\theta)(1-\theta)}{\theta}\left(\frac{C+Y}{Y}\hat{C}_t + \frac{Y-C}{Y}\hat{G}_t - \hat{p}_t^*\right) + \beta E_t \hat{\pi}_{t+1}$$

$$\hat{p}_t^* = \theta \hat{p}_{t-1}^*$$

We want a solution of the form

$$\hat{C}_t = \gamma_C \hat{p}_{t-1}^*$$

$$\hat{\pi}_t = \gamma_\pi \hat{p}_{t-1}^*$$

So

$$-\gamma_C \hat{p}_{t-1}^* = \beta\theta_\pi \gamma_\pi \hat{p}_{t-1}^* - \left(\gamma_C \theta \hat{p}_{t-1}^* + \gamma_\pi \theta \hat{p}_{t-1}^*\right)$$

$$\gamma_\pi \hat{p}_{t-1}^* = \frac{(1-\beta\theta)(1-\theta)}{\theta}\left(\frac{C+Y}{Y}\gamma_C \hat{p}_{t-1}^* - \theta \hat{p}_{t-1}^*\right) + \beta\gamma_\pi \theta \hat{p}_{t-1}^*$$

For these equations to hold, it must be

$$(\theta - 1)\gamma_C = (\beta\theta_\pi - \theta)\gamma_\pi$$

$$(1-\beta\theta)\gamma_\pi = \frac{(1-\beta\theta)(1-\theta)}{\theta}\left(\frac{C+Y}{Y}\gamma_C - \theta\right)$$

So, after some algebra,

$$\gamma_C = \theta\left(\frac{\theta}{\beta\theta_\pi - \theta} + \frac{C+Y}{Y}\right)^{-1}$$

$$\gamma_\pi = -\frac{1-\theta}{\beta\theta_\pi - \theta}\theta\left(\frac{\theta}{\beta\theta_\pi - \theta} + \frac{C+Y}{Y}\right)^{-1}$$

Note that we also have

$$\hat{F}_t = \gamma_F \hat{p}_{t-1}^*$$

where

$$\gamma_F = \left(1-\theta^2\beta\right)^{-1}\theta^2\beta(\varepsilon - 1)\gamma_\pi - \left(1-\theta^2\beta\right)^{-1}(1-\beta\theta)\frac{G}{Y}\gamma_C$$

A.3 Standard linearization at the ZLB

As is Carlstrom et al. (2014), as well as CER, the linearizaiton at the ZLB starts with the following two equations

$$-\hat{C}_t = \beta\left(R_t - 1 - r_t\right) - E_t\left(\hat{C}_{t+1} + \hat{\pi}_{t+1}\right)$$

$$\hat{\pi}_t = \frac{(1-\beta\theta)(1-\theta)}{\theta}\left(\frac{C+Y}{Y}\hat{C}_t + \frac{Y-C}{Y}\hat{G}_t\right) + \beta E_t\hat{\pi}_{t+1}$$

Note that $\hat{p}_t^*$ is ignored. Also, we have used that $R_t = 1 + r_t = \beta^{-1}$ in steady state. After the ZLB, it is assumed that $\hat{C}_t = \hat{G}_t = \hat{\pi}_t = 0$ because there are no state variables. At the ZLB, these equations become

$$-\hat{C}_t^\ell = -\beta r^\ell - pE_t\left(\hat{C}_{t+1}^\ell + \hat{\pi}_{t+1}^\ell\right)$$

$$\hat{\pi}_t^\ell = \frac{(1-\beta\theta)(1-\theta)}{\theta}\left(\frac{C+Y}{Y}\hat{C}_t^\ell + \frac{Y-C}{Y}\hat{G}_t^\ell\right) + \beta pE_t\hat{\pi}_{t+1}^\ell$$

Combine these equations to get

$$-\frac{Y}{Y+C}\frac{\theta}{(1-\beta\theta)(1-\theta)}\left(\hat{\pi}_t^\ell - \beta pE_t\hat{\pi}_{t+1}^\ell\right) + \frac{Y-C}{Y+C}\hat{G}_t^\ell + \beta r^\ell =$$
$$-p\left(-\frac{Y}{C+Y}\frac{\theta}{(1-\beta\theta)(1-\theta)}\left(E_t\hat{\pi}_{t+1}^\ell - \beta pE_t\hat{\pi}_{t+2}^\ell\right) + \frac{Y-C}{Y+C}E_t\hat{G}_{t+1}^\ell + E_t\hat{\pi}_{t+1}^\ell\right)$$

To simplify, define $\kappa \equiv (1-\beta\theta)(1-\theta)/\theta$, $\eta_G \equiv G/Y$, $\phi \equiv \beta + 1 + (2-\eta_G)\kappa$, and note that $Y = 1$.

$$\hat{\pi}_t^\ell - p\left(\beta + 1 + (2-\eta_G)\kappa\right)E_t\hat{\pi}_{t+1}^\ell + \beta p^2 E_t\hat{\pi}_{t+2}^\ell = \kappa\beta(2-\eta_G)r^\ell + \kappa\eta_G\left(1-p\right)E_t\hat{G}^\ell$$

Similar to Carlestrom, et al. (2014), the full set of solutions is given by

$$\hat{pi}_t = \hat{\pi}^\ell + a_1\lambda_1^t + a_2\lambda_2^t$$

where a_1 and a_2 are arbitrary constants,

$$\hat{\pi}^\ell = \frac{\kappa\beta(2-\eta_G)r^\ell + \kappa\eta_G\left(1-p\right)E_t\hat{G}^\ell}{\Delta}$$

and

$$\Delta = (1-p)(1-\beta p) - p(2-\eta_G)\kappa.$$

The values λ_1 and λ_2 are the solutions to

$$1 - (\beta + 1 + (2 - \eta_G)\kappa)\, p\lambda + \beta p^2 \lambda^2 = 0.$$

Let

$$z(\lambda) \equiv \frac{1}{\lambda} + \beta p^2 \lambda.$$

We seek values of λ so that $z(\lambda) = (\beta + 1 + (2 - \eta_G)\kappa)\, p$. The function f achieves a minimum at

$$\lambda = \sqrt{\frac{1}{\beta p^2}} > 1.$$

Note that

$$z(1) = 1 + \beta p^2 = \Delta + (\beta + 1 + (2 - \eta_G)\kappa)\, p$$

Both values of λ exceed unity if, and only if, $z(1) > (\beta + 1 + (2 - \eta_G)\kappa)\, p$, which is equivalent to

$$\Delta > 0.$$

If both roots exceed unity, then $a_1 = a_2 = 0$ gives the unique bounded solution. We focus on this case. We then have the following proposition.

Proposition 1. *There is a unique non-explosive solution to the linearized system of equations at the ZLB if an only if $\Delta > 0$.*

Note that if $\Delta < 0$, then inflation rises as r^ℓ falls and the system is pushed away from the ZLB.

The linearization predicts that at the ZLB, the drop in output is given by

$$\hat{Y}^\ell = \frac{1 - \eta_G}{2 - \eta_G} \frac{1}{\kappa} \left[(1 - \beta p)\hat{\pi}^\ell \kappa \frac{\eta_G}{1 + \eta_G} \hat{G}^\ell \right].$$

The government purchases multiplier is then

$$\frac{dY^\ell}{dG^\ell} = \frac{Y}{G} \frac{d\hat{Y}^\ell}{d\hat{G}^\ell} = \frac{1 - \eta_G}{2 - \eta_G} \left[\frac{(1 - \beta p)(1 - p)}{\Delta} + \frac{1}{1 - \eta_G} \right].$$

A.4 Our linearization at ZLB

Start by writing the non-linear system of equations at the ZLB explicitly, where we use a superscript ℓ for the value at the lower bound.

$$\frac{1}{C_t^\ell} = \frac{1}{1+r^\ell} E_t \left[\frac{p}{C_{t+1}^\ell \pi_{t+1}^\ell} + \frac{1-p}{C_{t+1}\pi_{t+1}} \right]$$

$$F_t^\ell = \frac{C_t^\ell + G_t^\ell}{C_t^\ell} + \theta \frac{1}{1+r^\ell} E_t \left[p \left(\pi_{t+1}^\ell\right)^{\varepsilon-1} F_{t+1}^\ell + (1-p)\left(\pi_{t+1}\right)^{\varepsilon-1} F_{t+1} \right]$$

$$\left(\frac{1-\left(\pi_t^\ell\right)^{\varepsilon-1}\theta}{1-\theta}\right)^{\frac{1}{1-\varepsilon}} F_t^\ell = \chi \frac{\left(C_t^\ell + G_t^\ell\right)^2}{p_t^{\ell*}} + \theta \frac{1}{1+r^\ell} E_t \Bigg[p \left(\pi_{t+1}^\ell\right)^{\varepsilon} \left(\frac{1-\left(\pi_{t+1}^\ell\right)^{\varepsilon-1}\theta}{1-\theta}\right)^{\frac{1}{1-\varepsilon}} F_{t+1}^\ell + \\ (1-p)\,\pi_{t+1}^{\varepsilon} \left(\frac{1-\pi_{t+1}^{\varepsilon-1}\theta}{1-\theta}\right)^{\frac{1}{1-\varepsilon}} F_{t+1} \Bigg]$$

$$p_t^{\ell*} = \left[(1-\theta) \left(\frac{1-\theta\left(\pi_t^\ell\right)^{\varepsilon-1}}{1-\theta}\right)^{-\frac{\varepsilon}{1-\varepsilon}} + \theta \left(\pi_t^\ell\right)^{\varepsilon} \left(p_{t-1}^{\ell*}\right)^{-1} \right]^{-1}$$

Now, note that for the purposes of our linearization, we are going to linearize around the limit point at the ZLB when the ZLB is binding and around the point to which the system will jump when the ZLB is not binding, both from the nonlinear model. It is useful to note that

$$\hat{p}_t^* \approx \log\left(p_t^*\right)$$

and

$$\hat{p}_t^{\ell*} \approx \log\left(\frac{p_t^*}{p^{\ell*}}\right) \approx \hat{p}_t^* - \log\left(p^{\ell*}\right)$$

Denote the points to which the system jumps with $\overrightarrow{X}_{t+1}$. Note that

$$\hat{\overrightarrow{X}}_{t+1} \approx \log\left(\frac{X_{t+1}}{\overrightarrow{X}}\right) \approx \log\left(\frac{X_{t+1}}{X}\right) + \log\left(\frac{X}{\overrightarrow{X}}\right) \approx \hat{X}_{t+1} + \log\left(\frac{X}{\overrightarrow{X}}\right)$$

We know that we can solve for

$$\hat{X}_{t+1} \approx \gamma_X \hat{p}_t^*$$

So

$$\hat{\overrightarrow{X}}_{t+1} \approx \gamma_X \hat{p}_t^* + \log\left(\frac{X}{\overrightarrow{X}}\right) \approx \gamma_X \hat{p}_t^{\ell *}$$

This means that we can write the $t+1$ variables if $r_t = r$ as a function of $\hat{p}_t^{\ell *}$. Also, we can write,

$$X_{t+1} \approx \overrightarrow{X} \exp\left(\hat{\overrightarrow{X}}_{t+1}\right) \approx \overrightarrow{X} \exp\left(\gamma_X \hat{p}_t^{\ell *}\right)$$

Thus, we can write the system as

$$\frac{1}{C_t^\ell} = \frac{1}{1+r^\ell} E_t\left[\frac{p}{C_{t+1}^\ell \pi_{t+1}^\ell} + \frac{1-p}{\overrightarrow{C}\exp\left(\gamma_C \hat{p}_t^{\ell *}\right)\overrightarrow{\pi}\exp\left(\gamma_\pi \hat{p}_t^{\ell *}\right)}\right]$$

$$F_t^\ell \left(\pi_t^\ell\right)^{1-\varepsilon} = \frac{C_t^\ell + G_t^\ell}{C_t^\ell} + \theta \frac{1}{1+r^\ell} E_t\left[p F_{t+1}^\ell + (1-p)\overrightarrow{F}\exp\left(\gamma_F \hat{p}_t^{\ell *}\right)\right]$$

$$\begin{aligned}\left(\frac{\left(\pi_t^\ell\right)^{1-\varepsilon} - \theta}{1-\theta}\right)^{\frac{1}{1-\varepsilon}} F_t^\ell \left(\pi_t^\ell\right)^{-\varepsilon} = \chi \frac{\left(C_t^\ell + G_t^\ell\right)^2}{p_t^{\ell *}} + \theta \frac{1}{1+r^\ell} E_t\Bigg[p\left(\frac{\left(\pi_{t+1}^\ell\right)^{1-\varepsilon} - \theta}{1-\theta}\right)^{\frac{1}{1-\varepsilon}} F_{t+1}^\ell + \\ (1-p)\left(\frac{\left(\overrightarrow{\pi}\exp\left(\gamma_\pi \hat{p}_t^{\ell *}\right)\right)^{1-\varepsilon} - \theta}{1-\theta}\right)^{\frac{1}{1-\varepsilon}} \overrightarrow{F}\exp\left(\gamma_F \hat{p}_t^{\ell *}\right)\Bigg]\end{aligned}$$

$$p_t^{\ell *} = \left[(1-\theta)\left(\frac{1-\theta\left(\pi_t^\ell\right)^{\varepsilon-1}}{1-\theta}\right)^{-\frac{\varepsilon}{1-\varepsilon}} + \theta\left(\pi_t^\ell\right)^\varepsilon \left(p_{t-1}^{\ell *}\right)^{-1}\right]^{-1}$$

We log-linearize these equations around the limit point at the ZLB to obtain

$$-\frac{1}{C^\ell}\hat{C}_t^\ell = \frac{1}{1+r^\ell} E_t\left[-\frac{p}{C^\ell \pi^\ell}\left(\hat{C}_{t+1}^\ell + \hat{\pi}_{t+1}^\ell\right) - \frac{1-p}{\overrightarrow{C}\overrightarrow{\pi}}\left(\gamma_C + \gamma_\pi\right)\hat{p}_t^{\ell *}\right]$$

$$F^\ell \left(\pi^\ell\right)^{1-\varepsilon}\left(\hat{F}_t^\ell + (1-\varepsilon)\hat{\pi}_t^\ell\right) = \frac{G^\ell}{C^\ell}\left(\hat{G}_t^\ell - \hat{C}_t^\ell\right) + \theta \frac{1}{1+r^\ell} E_t\left[p F^\ell \hat{F}_{t+1}^\ell + (1-p)\overrightarrow{F}\gamma_F \hat{p}_t^{\ell *}\right]$$

$$
\begin{aligned}
0 = &- \left(\frac{\left(\pi^{\ell}\right)^{1-\varepsilon}-\theta}{1-\theta}\right)^{\frac{1}{1-\varepsilon}} F^{\ell}\left(\pi^{\ell}\right)^{-\varepsilon}\left(\left[\left(\frac{\left(\pi^{\ell}\right)^{1-\varepsilon}-\theta}{1-\theta}\right)^{-1}\frac{\left(\pi^{\ell}\right)^{1-\varepsilon}}{1-\theta}-\varepsilon\right]\hat{\pi}_t^{\ell}+\hat{F}_t^{\ell}\right) \\
&+\chi 2\frac{\left(C^{\ell}+G_t^{\ell}\right)}{p^{\ell *}}C^{\ell}\hat{C}_t^{\ell}+\chi 2\frac{\left(C^{\ell}+G^{\ell}\right)}{p^{\ell *}}G^{\ell}\hat{G}_t^{\ell}-\chi\frac{\left(C^{\ell}+G^{\ell}\right)^2}{p^{\ell *}}\hat{p}_t^{\ell *} \\
&+\theta\frac{p}{1+r^{\ell}}\left(\left(\frac{\left(\pi^{\ell}\right)^{1-\varepsilon}-\theta}{1-\theta}\right)^{\frac{1}{1-\varepsilon}}F^{\ell}\hat{F}_{t+1}^{\ell}+\left(\frac{\left(\pi^{\ell}\right)^{1-\varepsilon}-\theta}{1-\theta}\right)^{\frac{1}{1-\varepsilon}-1}F^{\ell}\frac{1}{1-\theta}\left(\pi^{\ell}\right)^{1-\varepsilon}\hat{\pi}_{t+1}^{\ell}\right) \\
&+\theta\frac{1-p}{1+r^{\ell}}\left(\left(\frac{\left(\overrightarrow{\pi}\right)^{1-\varepsilon}-\theta}{1-\theta}\right)^{\frac{1}{1-\varepsilon}}\overrightarrow{F}\gamma_F+\left(\frac{\left(\overrightarrow{\pi}\right)^{1-\varepsilon}-\theta}{1-\theta}\right)^{\frac{1}{1-\varepsilon}-1}\overrightarrow{F}\frac{1}{1-\theta}\left(\overrightarrow{\pi}\right)^{1-\varepsilon}\gamma_{\pi}\right)\hat{p}_t^{\ell *}
\end{aligned}
$$

$$
-\left(p^{\ell *}\right)^{-1}\hat{p}_t^{\ell *}=\varepsilon\theta\left(\left(\pi^{\ell}\right)^{\varepsilon}\left(p^{\ell *}\right)^{-1}-\left(\frac{1-\theta\left(\pi^{\ell}\right)^{\varepsilon-1}}{1-\theta}\right)^{-\frac{\varepsilon}{1-\varepsilon}-1}\left(\pi^{\ell}\right)^{\varepsilon-1}\right)\hat{\pi}_t^{\ell}-\theta\left(\pi^{\ell}\right)^{\varepsilon}\left(p^{\ell *}\right)^{-1}\hat{p}_{t-1}^{\ell *}
$$

An alternative approach to finding the point around which to linearize would be to calculate the limit point using the linear approximation that obtains after the ZLB no longer binds. Note that

$$
\hat{p}_t^{*} \approx \hat{p}_t^{\ell *}+\log\left(p^{\ell *}\right)
$$

$$
\overrightarrow{X} \approx X\exp\left(\gamma_X\log\left(p^{\ell *}\right)\right)
$$

Then we can write

$$
\frac{1}{C^{\ell}}=\frac{1}{1+r^{\ell}}\left[\frac{p}{C^{\ell}\pi^{\ell}}+\frac{1-p}{C\exp\left(\gamma_C\log\left(p^{\ell *}\right)\right)\pi\exp\left(\gamma_{\pi}\log\left(p^{\ell *}\right)\right)}\right]
$$

$$
F^{\ell}\left(\pi^{\ell}\right)^{1-\varepsilon}=\frac{C^{\ell}+G^{\ell}}{C^{\ell}}+\theta\frac{1}{1+r^{\ell}}\left[pF^{\ell}+(1-p)F\exp\left(\gamma_F\log\left(p^{\ell *}\right)\right)\right]
$$

$$
\begin{aligned}
\left(\frac{\left(\pi^{\ell}\right)^{1-\varepsilon}-\theta}{1-\theta}\right)^{\frac{1}{1-\varepsilon}}F^{\ell}\left(\pi^{\ell}\right)^{-\varepsilon}=&\chi\frac{\left(C^{\ell}+G^{\ell}\right)^2}{p^{*\ell}}+\theta\frac{1}{1+r^{\ell}}E_t\Bigg[p\left(\frac{\left(\pi^{\ell}\right)^{1-\varepsilon}-\theta}{1-\theta}\right)^{\frac{1}{1-\varepsilon}}F^{\ell} \\
&+(1-p)\left(\frac{\left(\pi\exp\left(\gamma_{\pi}\log\left(p^{\ell *}\right)\right)\right)^{1-\varepsilon}-\theta}{1-\theta}\right)^{\frac{1}{1-\varepsilon}}F\exp\left(\gamma_F\log\left(p^{\ell *}\right)\right)\Bigg]
\end{aligned}
$$

$$
p^{\ell *}=\left[(1-\theta)\left(\frac{1-\theta\left(\pi^{\ell}\right)^{\varepsilon-1}}{1-\theta}\right)^{-\frac{\varepsilon}{1-\varepsilon}}+\theta\left(\pi^{\ell}\right)^{\varepsilon}\left(p^{\ell *}\right)^{-1}\right]^{-1}
$$

Given π^{ℓ}, we can get $p^{\ell *}$

$$\left(p^{\ell *}\right)^{-1}=\frac{1-\theta}{1-\theta\left(\pi^{\ell}\right)^{\varepsilon}}\left(\frac{1-\theta\left(\pi^{\ell}\right)^{\varepsilon-1}}{1-\theta}\right)^{-\frac{\varepsilon}{1-\varepsilon}}.$$

Given π^{ℓ} and $p^{\ell *}$ we can get C^{ℓ}

$$\frac{1}{C^{\ell}}=\frac{1-p}{C \exp\left(\gamma_C \log\left(p^{\ell *}\right)\right) \pi \exp\left(\gamma_\pi \log\left(p^{\ell *}\right)\right)}\left(1-\frac{p}{\left(1+r^{\ell}\right) \pi^{\ell}}\right)^{-1}.$$

Note that we already have a restriction on π^{ℓ} here,

$$\pi^{\ell}>\frac{p}{\left(1+r^{\ell}\right)}$$

Given π^{ℓ}, $p^{\ell *}$, and C^{ℓ}, we get F^{ℓ}:

$$F^{\ell}=\left(\frac{C^{\ell}+G^{\ell}}{C^{\ell}}+\theta \frac{1-p}{1+r^{\ell}} F \exp\left(\gamma_F \log\left(p^{\ell *}\right)\right)\right)\left(\left(\pi^{\ell}\right)^{1-\varepsilon}-\theta \frac{p}{1+r^{\ell}}\right)^{-1}$$

Use the final equation as a check to see if the original guess gives us a limit point. We have found that our main results are not sensitive to how this limit point is calculated.

B Analysis of other equilibria than considered in text

B.1 Expectations of returning to negative-inflation steady state

Here, we reproduce the analogue of Figure 5, along with the government purchases multiplier, assuming that agents believe they will return to the negative-inflation steady state after the ZLB ends.

Figure 9: Learning Equilibrium, Returning to Negative-Inflation SS

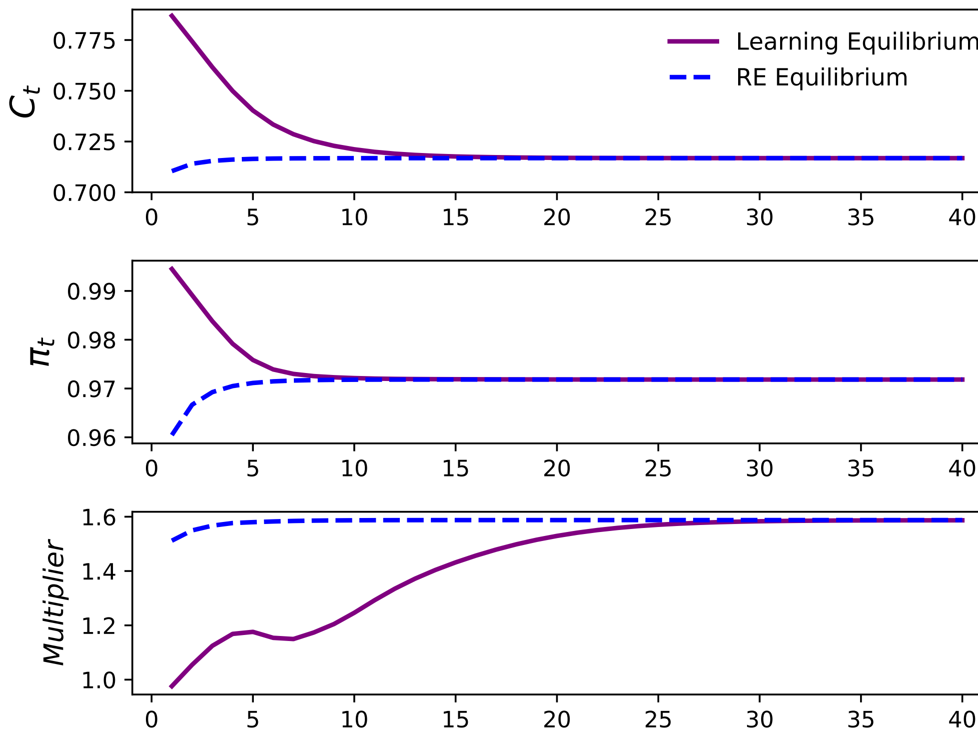
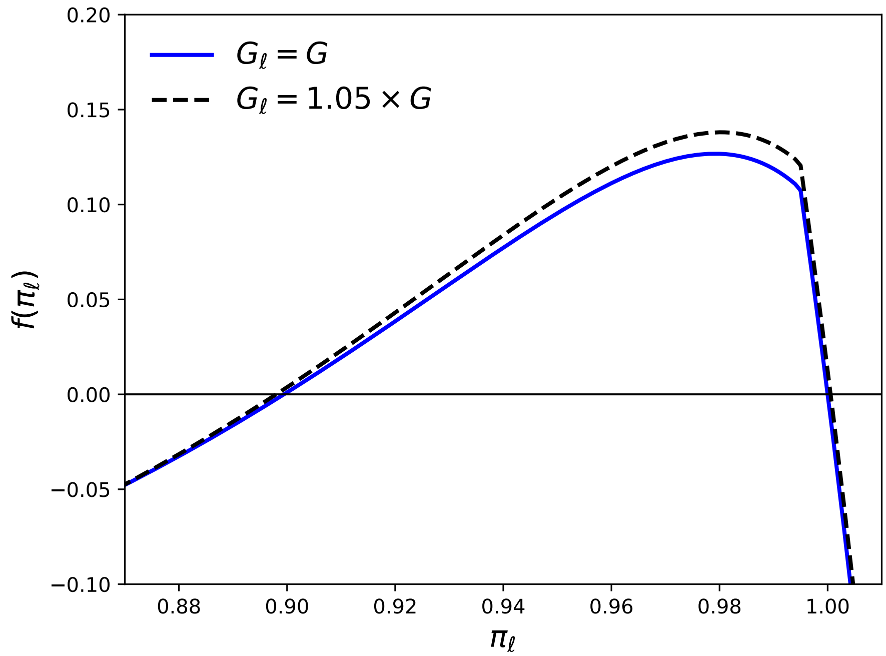


Figure 10: Effects of an Increase in G on $f\left(\pi^{\ell}\right)$, Sunspot Equilibrium



B.2 Sunspot equilibria

Here, we assume that r_t never changes. Instead, we assume that there is a sunspot that take value 1 at time $t = 1$. With probability p, the sunspot remains at the value 1. With complementary probability, the sunspot takes the value 0 and remains at that value forever. Figure 10 is analogous to Figure 4 in the main text. Again, an increase in G shifts $f\left(\pi^{\ell}\right)$ up. Because the sunspot equilibrium is the left crossing, inflation declines, causing real interest rates to rise, and consumption to fall. The multiplier in the sunspot equilibrium is about 0.91.

C Details about non-linear approximations

We approximate equilibrium laws of motion for prices and quantities in the nonlinear NK model using a finite-elements method. In particular, we define a grid on p^*_{t-1} that spans 0.8 to 1 and contains at least the 2001 points at each increment 0.0001. At each grid point, we assign a parameter value that characterizes equilibrium law of motion for the relevant aggregate variable if p^*_{t-1} is equal to the grid point. Between grid points, we linearly interpolate the values at the two closes points.

D Rotember-style adjustment costs

In this section, we consider a version of the model where firms face Rotember-style adjustment costs rather than Calvo-style adjustment costs.

D.1 Model specifcation

Households and final goods firms are identical to those in the Calvo model. In each period, all monopolists update their price. They maximize

$$E_t \sum_{k=0}^{\infty} d_{t+k} \frac{\lambda_{t+k}}{P_{t+k}} \left((1+\upsilon)P_{j,t+k} - s_{t+k}P_{t+k}\right) \left(\frac{P_{j,t+k}}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k} - \frac{\Phi_{t+1}}{2} \left(\frac{P_{j,t+k}}{P_{j,t+k-1}} - 1\right)^2 P_{t+k} \quad (29)$$

The where all variables are defined as in the main paper, and Φ_t is a potentially endogenous scaling parameter that determines the cost of adjusting prices. The first-order condition is

$$0 = \varepsilon \left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon} Y_t - \varepsilon s_t \left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon-1} Y_t + \Phi_t \left(\frac{P_{j,t}}{P_{j,t-1}} - 1\right) \pi_t - \frac{1}{1+r_t} E_t \frac{\lambda_{t+1}}{\lambda_t} \Phi_{t+1} \left(\frac{P_{j,t+1}}{P_{j,t}} - 1\right) \pi_{t+1} \quad (30)$$

Imposing symmetry, we have that $p^*_t = 0$ for all t and

$$(\pi_t - 1)\,\pi_t = \frac{\varepsilon}{\Phi_t}(s_t - 1)Y_t + \frac{1}{1+r_t} E_t \frac{\lambda_{t+1}}{\lambda_t} \frac{\Phi_{t+1}}{\Phi_t} (\pi_{t+1} - 1)\,\pi_{t+1}. \quad (31)$$

The resource constraint becomes

$$Y_t = C_t + G_t + \frac{\Phi_t}{2} (\pi_t - 1)^2\,. \quad (32)$$

The system of equations determining equilibrium can then be expressed as

$$\frac{1}{C_t} = \frac{1}{1+r_t} R_t E_t \frac{1}{C_{t+1}\pi_{t+1}}$$

$$(\pi_t - 1)\,\pi_t = \frac{\varepsilon}{\Phi_t}\left(\chi\left[C_t + G_t + \frac{\Phi_t}{2}(\pi_t - 1)^2\right] C_t - 1\right) Y_t + \frac{1}{1+r_t} E_t \frac{C_t}{C_{t+1}} \frac{\Phi_{t+1}}{\Phi_t}(\pi_{t+1} - 1)\,\pi_{t+1}$$

along with the monetary policy equation.

D.2 Steady states

The target-inflation steady state of the model is identical to the Calvo model because in both steady states there is no inflation. The negative-inflation steady state can be easily solved for by setting $\pi = \beta$ and computing C from the resource constraint.

D.3 Equilibrium in the nonlinear model

In an EW equilibrium, consumption and inflation jump to their steady state values after the ZLB no longer binds. We initially focus on the target-inflation steady state. We consider the same value for r_t we consider in the Calvo model. We look for a minimum-state, interior equilibrium. Thus, we look for two numbers, C^ℓ and π^ℓ. The monetary policy rule divides the candidate inflation rates into a subinterval in which the ZLB is binding and a second one in which it is not. Define

$$\pi^\ell_{UB} \equiv \frac{1 + \alpha - \frac{1}{\beta}}{\alpha}.$$

Then $R^\ell = 1$ if $\pi^\ell \leq \pi^\ell_{ZLB}$, and $R^\ell = \frac{1}{\beta} + \alpha(\pi^\ell - 1)$ otherwise. Solving the intertemporal Euler equation for $\pi^\ell \leq \pi^\ell_{ZLB}$ gives us consumption as a function of inflation so that

$$C^\ell\left(\pi^\ell\right) = \frac{1 + r^\ell - p\frac{1}{\pi^\ell}}{1-p} C$$

where C is the steady state level of consumption. We have assumed $\pi = 1$. The requirement that $C > 0$ means

$$\pi^\ell > \pi^\ell_{LB} = \frac{p}{1 + r^\ell}.$$

Note that C^ℓ is increasing on the interval from π^ℓ_{LB} to π^ℓ_{ZLB}, which is not surprising, given that the nominal interest rate is fixed.

Non-negativity of C^ℓ also implies an upper bound on π^ℓ. To see this, solve for C^ℓ using the intertemporal Euler equation for $\pi^\ell > \pi^\ell_{ZLB}$ to get

$$C^\ell(\pi^\ell) \equiv \frac{C}{1-p}\left[\frac{1+r^\ell}{\frac{1}{\beta}+\alpha(\pi^\ell-1)} - p\frac{1}{\pi^\ell}\right] = \frac{C}{1-p}\frac{p(\left(\alpha-\frac{1}{\beta}\right) - \left[p\alpha - 1 - r^\ell\right]\pi^\ell}{\left[\frac{1}{\beta}+\alpha(\pi^\ell-1)\right]\pi^\ell}.$$

Note that the numerator is linear and decreasing in π^ℓ, while the denominator is positive and increasing. It follows that there is exactly one value of π^ℓ for which $C^\ell = 0$, given by

$$\pi^\ell_{UB} = \frac{p(\beta\alpha - 1)}{p\beta\alpha - \beta(1+r^\ell)}.$$

we assume that $\beta(1+r^\ell) > p$ and $p\alpha > 1+r^\ell$ so that $\pi^\ell_{UB} > 1$. By continuity $C^\ell > 0$ on the interval π^ℓ_{LB} to π^ℓ_{UB}. We then have the following proposition.

Proposition 2. *An interior, minimum-state equilibrium exists if and only if for some $\pi^\ell \in \left(\pi^\ell_{LB}, \pi^\ell_{UB}\right)$ the value $C^\ell = C^\ell(\pi^\ell)$ solves*

$$\begin{aligned} f(\pi^\ell) = &- \left(\pi^\ell - 1\right)\pi^\ell + \frac{\varepsilon}{\Phi^\ell}\left(\chi\left[C^\ell + G^\ell + \frac{\Phi^\ell}{2}\left(\pi^\ell-1\right)^2\right]C^\ell - 1\right)\left[C^\ell + G^\ell + \frac{\Phi^\ell}{2}\left(\pi^\ell-1\right)^2\right] \\ &+ \frac{1}{1+r^\ell}p\left(\pi^\ell - 1\right)\pi^\ell \end{aligned}$$

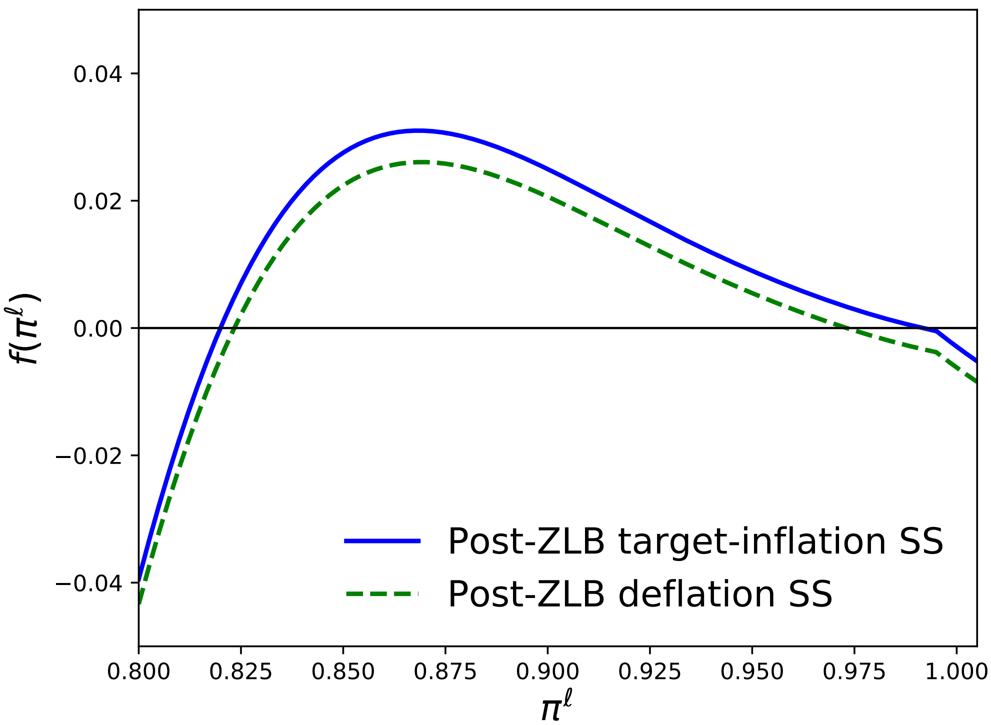
If no such π^ℓ exists, there is no EW equilibrium.

A similar set of conditions can be derived using the negative-inflation steady state. Note that the function f in the Rotemberg model is analogous to the function f in the Calvo model. The difference is that in the Calvo model the function f relates to the limit point in the ZLB, while in the Rotemberg model the function f relates to the equilibrium values.

Figure 11 is the analogue of Figure 1, but using our Rotemberg model. Similar to the Calvo model, the relatively-good ZLB equilibrium shifts left when the post-ZLB steady state will be the deflation steady state.

Figure 12 is the analogue of Figure 4, but using our Rotemberg model instead of our Calvo model.

Figure 11: Graph of $f\left(\pi_{\ell}\right)$



Because we know the entire possible support of π^{ℓ}, we can be sure that the only solutions to the model at the ZLB are those displayed in the figures.

D.4 Linearization

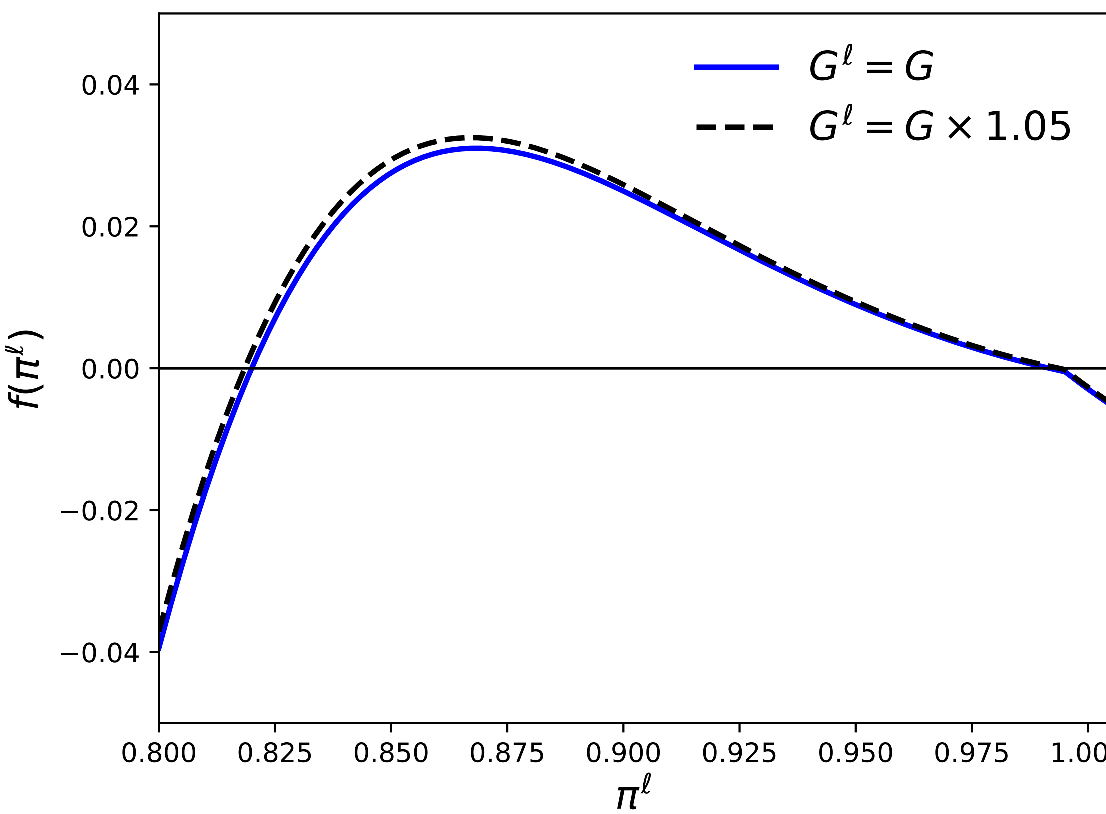
The only equation that needs to be altered relative to the Clavo model is the pricing equation from the monopolists. When we linearize the pricing equation around steady state, we get

$$\hat{\pi}_t = \frac{\varepsilon}{\Phi}\left((2-\eta_G)\hat{C}_t + \eta_G \hat{G}_t\right) + \frac{1}{1+r}E_t\hat{\pi}_{t+1}$$

which is identical to the Calvo model with $\hat{p}_t^* = 0$ if

$$\frac{\varepsilon}{\Phi} = \frac{(1-\beta\theta)(1-\theta)}{\theta}.$$

Figure 12: RE Multiplier in ZLB



We work with

$$\Phi_t = \frac{\theta}{(1-\beta\theta)(1-\theta)}\varepsilon\left(C_t + G_t\right) \equiv \phi\left(C_t + G_t\right)$$

where the value of $\theta = 0.85$ comes from our calibration of the Calvo model.

D.5 E-stability and learning

D.5.1 A simple framework of expectation formation

The first-order condition of an individual firm can be written as

$$\begin{aligned}0 = &\left(\varepsilon\left(\frac{\pi_{j,t}}{\pi_t}\right)^{-\varepsilon} - \varepsilon\chi\left[C_t + G_t + \frac{\Phi_t}{2}\left(\pi_t - 1\right)^2\right]C_t\left(\frac{\pi_{j,t}}{\pi_t}\right)^{-\varepsilon-1}\right)\left[C^\ell + G^\ell + \frac{\Phi^\ell}{2}\left(\pi^\ell - 1\right)^2\right] \\ &+ \left(C_t + G_t\right)\phi\left(\pi_{j,t} - 1\right)\pi_t - \frac{1}{1+r_t}E_t\frac{C_t}{C_{t+1}}\phi\left(C_{t+1} + G_{t+1}\right)\left(\pi_{j,t+1} - 1\right)\pi_{t+1}\end{aligned}$$

where $\pi_{j,t} \equiv P_{j,t}/P_{j,t-1}$. Because r_t is the only state variable, when $r_t = r^\ell$, it is natural to assume that firms believe that they will set their inflation rate tomorrow equal to the value they set today. This assumption is true in a rational expectations equilibrium, but not necessarily in a learning equilibrium. Further, it is natural to assume that firms believe that consumption will be the same value in each period that $r_t = r^\ell$. Under these assumptions, and further assuming that the economy returns to the target-inflation steady state when $r_t = r^\ell$, the first-order condition of the firm becomes

$$\begin{aligned}0 = &\left(\varepsilon\left(\frac{\pi_t}{\pi_t^{\ell e}}\right)^{-\varepsilon} - \varepsilon\chi\left[C_t^{\ell e} + G_t^\ell + \phi\frac{C_t^{\ell e} + G_t^\ell}{2}\left(\pi_t^{\ell e} - 1\right)^2\right]C_t^{\ell e}\left(\frac{\pi_t}{\pi_t^{\ell e}}\right)^{-\varepsilon-1}\right)\left[C_t^{\ell e} + G_t^\ell + \phi\frac{C_t^{\ell e} + G_t^\ell}{2}\left(\pi_t^{\ell e} - 1\right)^2\right] \\ &+ \left[1 - \frac{1}{1+r^\ell}p\right]\phi\left(C_t^{\ell e} + G_t^{\ell e}\right)\left(\pi_t - 1\right)\pi_t^{\ell e}\end{aligned}$$

where $C_t^{\ell e}$ and $\pi_t^{\ell e}$ are the firm's time-t belief about consumption and inflation when $r_t = r^\ell$. We have subsituted $\pi_t = \pi_{j,t}$ because each firm makes the same decision, which determins π_t.

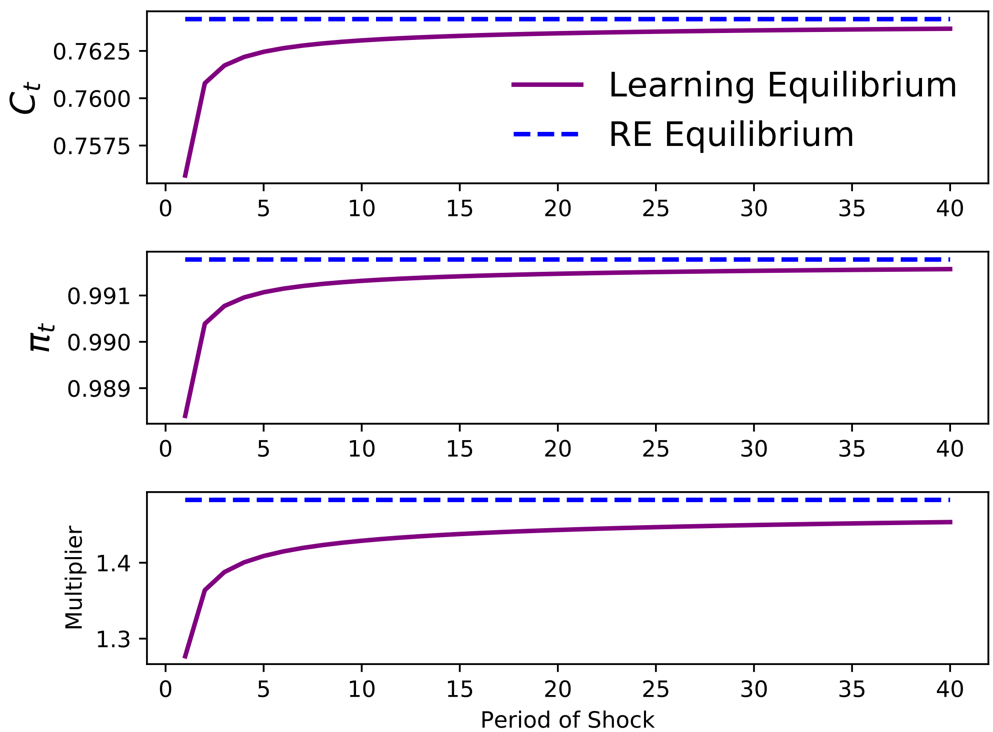
Households see current inflation. It is natural for households to assume that inflation in the next period will the same as in the current period if $r_{t+1} = r^\ell$. Thus, the intertemporal Euler equation is

$$\frac{1}{C_t} = \frac{1}{1+r^\ell}\max\left(1, \frac{1}{\beta} + \alpha\left(\pi_t - 1\right)\right)\left[\frac{p}{\pi_t} + \frac{(1-p)}{C^h}\right].$$

Here, C is the value of consumption in the target-inflation steady state.

These equations define a mapping from beliefs about ocnsumption and inflation, to outcomes for

Figure 13: Learning Equilibrium, Starting at Steady State RE Equilibrium



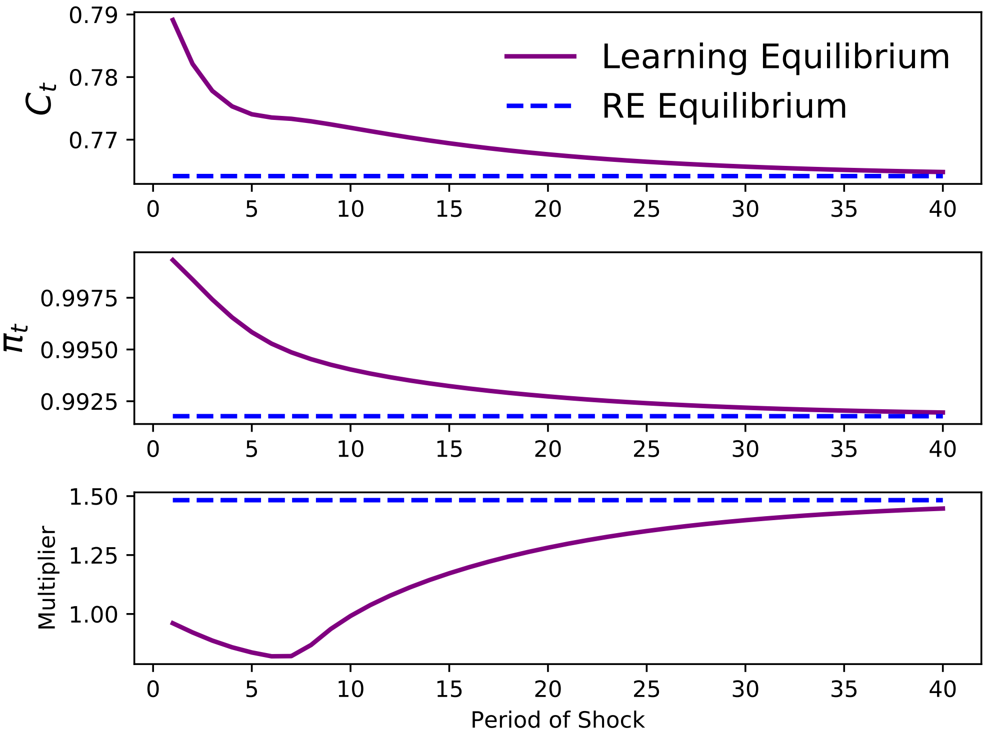
consumption and inflation at the ZLB.

$$\begin{bmatrix} C_t \\ \pi_t \end{bmatrix} = T\left(\begin{bmatrix} C_t^{\ell e} \\ \pi_t^{\ell e} \end{bmatrix}\right).$$

We if we define $\Psi^{PLM} \equiv \begin{bmatrix} C_t^{\ell e} & \pi_t^{\ell e} \end{bmatrix}$, we again have a mapping from the perceived law of motion to the actual law of motion, $\Psi^{ALM} \equiv \begin{bmatrix} C_t & \pi_t \end{bmatrix}$. When we compute $\frac{\partial T(\Psi)}{\partial \Psi}$, we find that only the relatively-good ZLB equilibrium is E-stable. The relatively-bad ZLB equilibrium is not E-stable.

We next compute a learning equilibrium. We assume that $\pi_t^{\ell e} = \pi_{t-1}$. That is, firms expect inflation to be what it was in the previous period. Consistent with this rule, we assume that the eocnomy had been in steady state and firms initially believe that $\pi_t = 1$. The outcomes for consumption, inflation, and the government purchases multiplier are shown in Figure 13. Note that the outcomes eventually converges to the values in the relatively-good ZLB rational expectations equilibrium.

Figure 14: Learning Equilibrium, Starting at Steady State RE Equilibrium



D.5.2 Learning as in Evans and Honkapohja (2003)

We can alternatively specify expectations as in Evans and Honkopohja (2003) so that

$$\frac{1}{C_t} = \frac{1}{1+r^\ell} \max\left(1, \frac{1}{\beta} + \alpha\left(\pi_t - 1\right)\right)\left[\frac{p}{C_t^{\ell e}\pi_t^{\ell e}} + \frac{1-p}{C^h}\right]$$

$$\left(\pi_t - 1\right)\pi_t = \frac{\varepsilon}{\Phi_t}\left(\chi\left[C_t + G_t + \frac{\Phi_t}{2}\left(\pi_t - 1\right)^2\right]C_t - 1\right)Y_t + \frac{1}{1+r^\ell}\left[\frac{C_t}{C_t^{\ell e}}\frac{\Phi_t^{\ell e}}{\Phi_t}\left(\pi_t^{\ell e} - 1\right)\pi_t^{\ell e}\right].$$

We again assume that expectations are updated so that agents think tomorrow's values will be the same as yesterday's values. In this case, we also find that only the relatively-good ZLB equilibrium is E-stable. This form of learning is slower than the other form of learning we consider in the Rotemberg model. Figure 14 shows the equilibrium outcomes.