A Unified Framework for Monetary Theory and Policy Analysis

Ricardo Lagos
New York University

Randall Wright
University of Pennsylvania

January 22, 2001

Abstract

Tractable versions of search-theoretic models of money allow agents to store only very limited inventories of money – e.g., 0 or 1 unit. This makes some policy experiments, especially changes in the money supply, difficult to interpret, at best. Perhaps for this reason, search models have not had much of an impact on economists interested in monetary policy, who typically resort to reduced-form models. In this paper we present a framework that attempts to bridge the gap between the pure theory of money and the relatively applied branch of the literature that emphasizes policy.

Financial support from STICERD and NSF is gratefully acknowledged. First version: May 2000.
1 Introduction

Search-theoretic models provide a foundation for monetary economics based on explicit descriptions of specialization in tastes and technology, the pattern of meetings, the information structure, and so on. The framework allows one to formalize the nature of the frictions that can make monetary exchange an equilibrium or an efficient arrangement, to determine endogenously which objects serve as media of exchange, to show how different types of regimes (commodity versus at money, one currency versus many, etc.) lead to qualitatively different outcomes, and to study many other issues that are not easy to address using reduced-form models. However, tractable versions of these models allow agents to store only very limited inventories of money—e.g., 0 or 1 unit. This makes some things, especially changes in the money supply, difficult to interpret, at best. Generalized versions without these inventory restrictions are extremely complicated, mainly because one needs to keep track of the endogenous distribution of money holdings. For these reasons, search models have not had much of an impact on economists who study monetary policy, who typically resort to reduced-form models like those that involve cash-in-advance constraints, money in the utility function, and so on.

1 A brief review of the literature follows. In the rst generation search models of money, the terms as trade as well as the distribution of money holdings were essentially fixed; examples include Kiyotaki and Wright (1989, 1991, 1993), Aiyagari and Wallace (1991), Matsuyama, Kiyotaki and Matsui (1993), and Williamson and Wright (1994). The second generation of models in this literature introduced bargaining to determine the price level but kept the inventory restrictions that made the distribution of money holdings trivial; examples include Shi (1995), Trejos and Wright (1995), Coles and Wright (1998), and Kocherlakota (1998). The third generation relaxed the inventory restrictions to endogenize
In this paper we present a model that attempts to bridge the gap between
the pure theory of money and the relatively applied branch of the literature
that emphasizes policy. The economy we study has some features of search
models, with their decentralized trading and price setting, and also some
features of more centralized market models. Indeed, this is because in our
model agents sometimes trade as in search theory - with each other, at prices
determined by bargaining - and agents sometimes trade as in Walrasian the-
ory - with the market, at market clearing prices. We specify the preferences
and technology for the goods traded in the Walrasian markets in such a way
that this part of the model is not very interesting, but it makes the other,
more interesting part of the model, tractable. That is, all the action is in
the search markets, where money is used as a medium of exchange, nominal
prices are determined, and gains from trade are realized, while the Walrasian
markets are only used by agents to adjust there money holdings.

In particular, by specifying preferences and technology for the Walrasian
the distribution of money holdings as well as prices; see Green and Zhou (1997), Camera
and Corbae (1999), Taber and Wallace (1999), Zhou (1999), and Molico (1999). (One can
also endogenize the distribution of money holdings but keep prices ...xed, as in Berensten
[1999], e.g.). The ...rst and second generation models have the problem emphasized in
the text: the inventory restrictions make the analysis of changes in the money dubious.
The third generation models are better on this dimension, but they are very đıc cult to
analyze and, at the end of the day, still do not seem all that useful for many policy issues
(with exceptions, such as the interesting policy analysis in Molico [1999]). There is a class
of models that avoids the đıc culty of solving for the distribution of money holdings by
making assumptions that render it degenerate, including Shi (1997, 1998, 1999), Rauch
(2000), Head and Shi (2000), and Berensten and Rocheteau (2000). This branch of the
literature is similar in spirit to what we do here, in the sense that we will also make
assumptions that render the distribution of money holdings degenerate, but the way in
which we do it is very different. In terms of the reduced form approach, it is far too vast
to go into here, but examples of what we have in mind include the work of Cooley and
goods to be linear, agents entering this market with different money holdings all leave with the same money holdings. At least in the simplest version of the model, this makes the distribution of money holdings of agents entering the search sector degenerate: in equilibrium, any agent you meet in this sector has the same amount of money. In other words, the model has a representative agent. Of course, off the equilibrium path, you can hold any amount of money you want. The dynamic programming conditions for individual decisions regarding money balances, which are quite similar to those found in reduced-form models, are combined with the bargaining solution, as typically found in search theory, to generate the equilibrium conditions. In some sense, the method used to solve the model is a hybrid of techniques used to solve different models in the literature. The same can be said of the results.

For example, in the most basic version of the model we find that there always exists a nonmonetary equilibrium, there exists a unique monetary steady state under simple conditions, and there are dynamic equilibria where inflation is a self-fulfilling prophecy, as well as more exotic outcomes such as cycles of any order, very much as in the standard overlapping generations model. Steady state equilibria (although not all dynamic equilibria) are constrained in the sense that agents in the search sector spend all their money as soon as they meet the right person, and would spend more if they had it, off the equilibrium path, reminiscent of the way the cash-in-advance constraint binds in a cash-in-advance model. Moreover, the equilibrium is not efficient, but there is a sense in which we approach the efficient outcome as frictions vanish if the bargaining solution satisfies certain conditions, as in the typical
The model displays classical neutrality: increasing the money supply increases all nominal variables proportionately leaving all real variables unchanged. Neutrality does not hold in the basic search models of money, as an artifact of the restrictions on money holdings that were only intended for tractability.\(^2\) When we introduce policies that involve changing the money supply over time, we find that whether money is super neutral depends on the way in which it is injected; e.g., lump sum transfers have real effects, proportional transfers do not. Under certain conditions, the optimal policy is given by the Friedman rule - deplete at the rate of time preference. However, whether this gets us all the way to the rst best outcome depends on the bargaining solution. There is also a sense in which if we switch to commodity money from fiat money the e cient outcome is the unique (steady state or dynamic) equilibrium.

2 The Model

Time is discrete and continues forever. There is a [0; 1] continuum of agents. There are two types of perfectly divisible, nonstorable commodities - general and special. Each agent produces a subset and consumes a different subset of the special commodities. Specialization is modeled as follows. Given two agents \(i\) and \(j\) drawn at random, there are four possible events. The probability that both consume something the other produces (a double coincidence)
is denoted ±. The probability that \( i \) consumes something \( j \) produces but not vice-versa (a single coincidence) is denoted \( \frac{3}{4} \). Symmetrically, the probability that \( j \) consumes something \( i \) produces but not vice-versa is also \( \frac{3}{4} \). The probability neither wants anything the other can produce is \( 1 - 2\frac{3}{4} \) ± which means \( \frac{3}{4} \cdot (1 ± \frac{1}{2}) = \frac{1}{2} \). This notation allows us to capture the standard specifications in the literature as special cases.\(^3\) In any single coincidence meeting, if \( i \) wants what \( j \) produces, we call the former the buyer and the latter the seller.

Let \( u(q) \) and \( c(q) \) be the utility of consumption and production of special commodities. Assume that \( u(0) = c(0) = 0, u'(q) > 0, c'(q) > 0, u''(q) < 0, c''(q) < 0 \), and that there exists \( \xi > 0 \) such that \( u(\xi) = c(\xi) \). We use \( q^\star \) to denote the efficient quantity, namely the solution to \( u'(q^\star) = c'(q^\star) \). Note that we can always normalize \( c(q) = q \) with no loss in generality, as this merely amounts to measuring output in utils, and we will do so when convenient below. By contrast with special goods, there are two key features of the general commodity: it can be consumed and produced by everyone, and it generates linear utility. In particular, consumption of \( Q \) units of the general commodity yields \( Q \) utils and production of \( Q \) units yields \( -Q \) utils. Clearly, if \( \gg \), 1 there are no gains from trading general commodities for general commodities. For now we assume \( \gg = 1 \). Obviously general

\(^3\)For example, in Kiyotaki and Wright (1989) and the extension in Aiyagari and Wallace (1991), there are \( N \) goods and \( N \) types of agents, and each type \( n \) produces only good \( n \) and consumes only good \( n + 1 \) (mod \( N \)). Then, if \( N > 2 \) we have \( \frac{3}{4} = 1 = N \) and ± = 0, while if \( N = 2 \) we have \( ± = 1 \) and \( \frac{3}{4} = 0 \). As another example, in Kiyotaki and Wright (1993), the event that \( i \) consumes what \( j \) produces is assumed to be independent of the event that \( j \) consumes what \( i \) produces, and each occurs with probability \( x \). Then we have ± = \( x^2 \) and \( \frac{3}{4} = x(1 \pm x) \).
commodities are not very interesting in this model, except for the fact that they mean utility is perfectly transferable between agents. To complete the specification of preferences, let $\tilde{\gamma} = 1/(1 + r)$ be the discount factor.

In addition to the above goods, there is another object called money. Money cannot be consumed or produced, but agents can store any non-negative quantity for free (for now). Although it has no intrinsic value, in principle, money could be used to trade for special or general commodities; it is important to emphasize, however, that there is no constraint that says agents must use money to trade, and if they do it will be an equilibrium outcome. In any case, while money can potentially trade for special or general commodities, and special commodities can trade against each other (barter), we make the following assumption to guarantee that there is no trade of general for special commodities: In each period there are two sub-periods, say day and night, and special commodities can only be produced during the day while the general good can only be produced at night. Therefore, the only feasible trades that may occur during the day are barter in special goods and the exchange of money for special goods, and the only feasible trades that may occur at night are the exchange of money for general goods.

During the day, agents participate in a standard, anonymous, bilateral search/matching market. Each period there is a probability $\gamma$ of meeting someone and each meeting is a random draw from the set of agents. In this market the terms of trade are determined by bargaining. At night there is a frictionless or centralized market where one dollar buys $\tilde{A}$ units of the general good (i.e., $p_3 = 1=\tilde{A}$ is the nominal price of the general good). Agents take as
Figure 1: Timing

given \( \hat{A} \), and that they can buy or sell as much as they like at this price. The role of this market is to reallocate liquidity from those with an excess supply to those with an excess demand. Notice that all trades are spot trades (quid pro quo): there is no intertemporal trade in the search market because it is anonymous; there is no intertemporal trade in the general good market since agents are homogeneous in this market. Figure 1 depicts the timing of events.

Let \( V(m) \) be the value function for an agent with \( m \) dollars in the morning before he enters the search market. Let \( F(m) \) be the CDF of money holdings of agents in this market; i.e., \( F(m) \) is the measure of agents holding \( m \cdot \hat{m} \). The total money stock, which is fixed for now, is given by \( M = \int m dF(m) \).

Let \( q(m; m) \) and \( d(m; m) \) be the quantity of goods and dollars that change hands in single coincidence meeting between a buyer with \( m \) dollars and a seller with \( m \) dollars. Let \( B(m; m) \) be the expected net payo from a barter trade when an agent with \( m \) dollars meets an agent with \( m \) and there is a double coincidence of wants. Let \( W(m) \) be the value function of an agent with \( m \) dollars at the end of the first sub-period, between day and night,
Then we have the Bellman equation

\[
V(m) = \sum \int f_u[q(m; \tilde{m}) + W[m + d(m; m)]g dF(\tilde{m}) + \sum \int f_i c[q(m; m)] + W[m + d(m; m)]g dF(\tilde{m}) + \sum B(m; m)dF(\tilde{m}) + (1 - 2\beta^4) W(m)
\]

The first term is the expected gain from a single coincidence meeting where you purchase \(q(m; \tilde{m})\) and then go to the general goods market with \(m + d(m; m)\) dollars. The second term is the expected gain from a single coincidence meeting where you sell \(q(m; m)\) and then go to the general goods market with \(m + d(m; m)\) dollars. The third term is the payoff from a double coincidence meeting, and the final term is the payoff to going to the bank with \(m\) after no trading opportunity.

The first result is that every agent leaves the general goods market and enters the special goods market with the same amount of money. This is so because an agent with \(m\) dollars in the general goods market chooses next period’s money holdings \(m^0\) to solve

\[
W(m) = \max \hat{A}(m_j \cdot m^0) + -V(m^0).
\]

Given \(V(m)\) is strictly concave (which will be seen below), the solution satisfies

\[
\hat{A} = -V^0(m^0) : \quad (2)
\]

Hence, \(m^0\) is independent of \(m\), which means \(F(m)\) is degenerate: all agents in the search market hold \(m = M\). This allows us to replace the random variable \(m\) with \(M\) in (1). Moreover, notice \(W(m + x) = \hat{A}x + W(m)\). Hence,
we have

\[
V(m) = \alpha B(m; M) + \alpha f u[q(m; M)] \hat{A}d(m; M)g \\
+ \alpha f c[q(M; m)] + \hat{A}d(M; m)g + \hat{A}(m; m) + V(m^0).
\]

The next step is to analyze bargaining. There are two types of bargaining situations: barter and monetary. The rst type occurs whenever there is a double coincidence meeting, and in this case we adopt the symmetric Nash bargaining solution. This implies that, regardless of the money holdings of the two agents, each produces \( q^* \) for the other, where \( q^* \) is the solution to \( u(q^*) = c(q^*) \), and no money changes hands, and hence \( B(m; m) | \ W(m) = u(q^*) + c(q^*) = B \); see Lemma 1 in the Appendix.\(^4\) Thus, the Bellman equation reduces to

\[
V(m) = \alpha B + \alpha f u[q(m; M)] \hat{A}d(m; M)g \\
+ \alpha f c[q(M; m)] + \hat{A}d(M; m)g + \hat{A}(m; m) + V(m^0).
\]

Note that the rst term, \( \alpha B \), depends only on exogenous variables.

Now consider bargaining when an agent with \( m \) dollars meets one with \( M \) dollars, and the former consumes the good the latter produces but not

\(^4\)Suppose two agents, call them 1 and 2, have a double coincidence, and assume \( \mu \) has bargaining power \( \mu \) in a generalized Nash problem (not necessarily \( \frac{1}{2} \)). Let \( m_1 \) be the money holdings of \( j \), which is given. Let \( q \) be the consumption \( j \) and \( c \) the dollars that go from agent 1 to agent 2, which are to be determined. Following the method in the Appendix, one can show the solution is \( q = q^* \) and \( c = (1 - 2 \mu_1)[u(q^*) - c(q^*)] = 0 \), as long as the constraint \( c = 2 \mu_1 m_2 ; m_1 \) is not binding. That is, both agents produce \( q^* \) and money is exchanged to give agent 1 the appropriate gains from trade. Moreover, \( B_1 | W_1 = 2 \mu_1 [u(q^*) - c(q^*)] \). In the symmetric case, we have \( \mu = \frac{1}{2} \) and the constraint is obviously not binding, as stated in the text. For \( \mu \) very different form \( \frac{1}{2} \), however, the constraint may bind. One cannot check whether the constraint binds without nding the full equilibrium, in general, which is why we set \( \mu = \frac{1}{2} \).
vice-versa. In this case we use the generalized Nash solution,
\[
\max_{q,d} [u(q) + W(m-d) + T(m)] [1 - \mu [c(q) + W(m+d) + T(m)]^\mu];
\] (4)
subject to \(d \cdot m\) and the incentive conditions
\[
\begin{align*}
  u(q) + W(m-d) &\leq W(m) \\
  c(q) + W(m+d) &\leq W(m);
\end{align*}
\]
In (4), \(\mu\) denotes the bargaining power of the consumer and \(T(m)\) the threat point of an agent with \(m\) units of money. For now we set \(T(m) = W(m)\), which implies the incentive constraints do not bind for any \(\mu \in (0;1)\) and the Nash problem becomes
\[
\max_{q,d} [u(q) + \hat{A}d] [1 - \mu [c(q) + \hat{A}d]^\mu] \text{ subject to } d \cdot m.
\] (5)
Notice that if the constraint is not binding then the solution is independent of money holdings \(m\) and \(\bar{m}\). If the constraint binds, then the solution depends only on \(m\), the money holdings of the buyer. Since the solution never depends on the money holdings of the seller, \(\bar{m}\), in what follows we write \(q = q(m)\) and \(d = d(m)\).

A steady state equilibrium with constant money supply \(M\) can now be defined as a list including the value function \(V(m)\), the terms of trade \([q(m) ; d(m)]\), and the nominal price of the general good \(1 = \hat{A}\), solving the Bellman equation (3), the bargaining problem (4), and the \(\text{rst order condition (2)}\). Implicitly, the equilibrium also specifies the value of going to the general goods market \(W(m) = \hat{A}(m \cdot m^0) + \bar{V}(m^0)\) and money demand
equals money supply, \( m^0 = M \). Also, the distribution \( F(m) \) is part of an equilibrium, but we know it is degenerate at \( m = M \) with probability 1.

3 Steady State Equilibrium

We now proceed to characterize steady state equilibria. In the general goods market, an agent chooses the money holdings he brings into the next period according to (2), \( \dot{A} = -V^0(m^0) \). To compute \( V^0(m^0) \), we differentiate (3) to get

\[
V^0(m) = \frac{\partial^2 u^0[q(m)]}{\partial m} \frac{\partial q}{\partial m} - \frac{\partial u^0}{\partial m} \frac{\partial d}{\partial m} + \dot{A}.
\]

Updating \( V^0(m) \) one period, letting \( \dot{A}^0 \) denote \( \dot{A} \) next period, and substituting into (2), we arrive at the Euler equation

\[
\dot{A} = -\frac{1}{2} \frac{\partial^2 u^0[q(m)]}{\partial m} \frac{\partial q}{\partial m} - \frac{\partial u^0}{\partial m} \frac{\partial d}{\partial m} + \dot{A}^0 : 3/4.
\]

Notice this contains the terms of trade \( q \) and \( d \) as well as their derivatives with respect to the buyer’s money holdings \( m \). To determine these we need to examine the bargaining solution.

We begin with a special case \( \mu = 1 \) (the buyer makes a take-it-or-leave-it offer), and take up the general case later. In this case, the solution to the bargaining problem in (5) is

\[
q(m) = \frac{1}{2} \frac{q}{c^1(\dot{A}m)} \quad \text{if } c(q^* \dot{A}) \cdot \dot{A}m
\]

and \( d(m) = c[q(m)] \dot{A} \). Notice that \( \frac{\partial q}{\partial m} = \dot{A}c^0(q) \) and \( \frac{\partial d}{\partial m} = 1 \) if the constraint binds while \( \frac{\partial q}{\partial m} = \frac{\partial d}{\partial m} = 0 \) if it does not. Updating these
one period and using the equilibrium condition \( m = M \) the Euler equation reduces to

\[
\dot{A} = \begin{cases} 
-\dot{A}^0 & \text{if } c(q^\dagger) \cdot \dot{A}^0 > 1 + \frac{\beta}{\lambda} A^{\mu(q(M))} \\
-\dot{A}^0 & \text{if } c(q^\dagger) \cdot \dot{A}^0 \leq 1 + \frac{\beta}{\lambda} A^{\mu(q(M))} 
\end{cases}
\]

(6)

In steady state \( \dot{A}^0 = 0 \). If \( \dot{A} = 0 \) (a nonmonetary steady state) then \( q = 0 \) in every single coincidence meeting and hence

\[
(1 \cdot \dot{\gamma}) V(m) = \beta [u(q) \cdot c(q)]:
\]

In this case \( V^0(m) = 0 \) for all \( m \) so (2) is satisfied. Hence, there always exists a nonmonetary steady state. We now look for monetary steady states. For any \( \dot{\gamma} < 1 \) there cannot be a monetary steady state equilibrium where the constraint is not binding. In a monetary steady state where the constraint binds, the value of \( q \) solves

\[
\dot{\gamma} A^{\mu(q(M))} = c(q) + 1 \cdot \frac{\beta}{\lambda} = 1, \quad \text{or using } \dot{\gamma} = \frac{1}{(1 + r)},
\]

\[
\frac{u^0(q)}{c^0(q)} = \frac{\beta + r}{\beta}. \tag{7}
\]

Since the left hand side is strictly decreasing, there can be at most one solution to (7). Since \( u^0(q^\dagger) = c^0(q^\dagger) = 1 < (\beta + r) = \frac{\beta}{\lambda} \) there exists a solution if

\[
\frac{u^0(0)}{c^0(0)} > \frac{\beta + r}{\beta}, \tag{8}
\]

which is true if standard Inada conditions hold on \( u \).

Therefore under standard conditions there exists a unique monetary steady state and it is constrained in the sense that agents always spend all their money (and would spend more if they had it on the equilibrium path). Moreover, \( q < q^\dagger \). Hence the equilibrium is not efficient. Also, \( q \) is monotonically
increasing in $\$\$\$\$ and decreasing in $r$. As $r \to 0$, $q \to q^*$; that is, as the frictions vanish $q$ approaches the efficient outcome. Also, as $\$\$\$\$ $0$, $q \to 0$; that is, as the probability of a single coincidence vanishes the value of money goes to 0.

Moreover, the constraint implies $d = M$ and $\hat{A} = c(q) \Rightarrow M$. The nominal price of general goods is $p_g = \frac{1}{\hat{A}} = M \Rightarrow c(q)$ and the nominal price of special goods $p_s = d = q = M \Rightarrow q$. If we normalize $c(q) = q$ then $p_g = p_s = p$. The model displays classical neutrality: increasing $M$ increases all nominal variables proportionately leaving all real variables unchanged. Along the equilibrium path, the value function is given by

$$(1 - \bar{\gamma}) V(M) = \frac{1}{\gamma} [u(q) - c(q)] + \bar{\gamma} B;$$

If a planner could choose $q$, he would choose $q = q^*$ to maximize $V(M)$; the value of $M$ is irrelevant.

Lemma 2 in the Appendix generalizes the above analysis for any $\mu \in (0; 1)$. It shows that in any monetary steady state the constraint $d \cdot m$ must be binding, and that a monetary steady state exists if a generalized version of (8) holds, namely

$$\frac{u^0(0)}{c^0(0)} > \frac{1}{\mu} \frac{1}{\frac{\gamma}{\gamma} + \frac{r}{\gamma} [1 + a(0)] \mu} \frac{3}{4}.$$ (10)

where

$$a(q) = \frac{\mu(1 + \mu) [u(q) - c(q)] [u^0(q) c^0(q) + c^0(q) u^0(q)]}{uc^0(q) + (1 + \mu) c^0(q)}.$$
We also derive the generalized version of (7),

\[
1 + \mu \frac{u(q)}{c(q)} + \frac{1}{1 + \frac{\hat{\theta}}{\theta} (q)} = \frac{\theta}{\theta + \mu};
\]

which can be solved for \( q \). Then \( \hat{A} \) is determined by

\[
\hat{A} M = \frac{\mu u(q) c(q) + (1 - \mu) u(q) c'(q)}{\mu u(q) c(q) + (1 - \mu) u(q) c'(q)};
\]

Note that if \( \mu = 1 \), then (12) becomes \( \hat{A} M = c(q) \) and \( \hat{a}(q) = 0 \), which implies that (11) reduces to (7).

Lemma 2 also implies \( q < q^* \). Moreover, \( q \) is bounded away from \( q^* \) even in the limit as \( r \to 0 \). Interestingly, we only get the efficient outcome when the frictions vanish \( (r = 0) \) and the buyer has all the bargaining power. We will return to this later. This generalized model also exhibits classical neutrality: \( q \) is unaffected by \( M \) while nominal prices \( p_5 = M = q \) and \( p_\delta = \frac{\mu u(q) + (1 - \mu) c(q) + M}{\mu u(q) c(q) + (1 - \mu) u(q) c'(q)} \) are proportional to \( M \).

4 Dynamic Equilibrium

In this section we return to the case with \( \mu = 1 \) and consider dynamic, rather than merely steady state, equilibria. We also introduce a real flow return \( \delta \) per nominal unit of money, since this allows one to make some additional points. This has the interpretation of a storage cost if \( \delta < 0 \) and a real dividend (interest payments on currency) if \( \delta > 0 \). We can also think of \( \delta = 0 \) as corresponding to commodity money, with the case of pure money corresponding to \( \delta = 0 \). We continue to assume the money supply and all other exogenous variables are constant, but since the endogenous \( \hat{A} \),
may change through time we write $V(m_t; \dot{A}_t)$ and $W(m_t; \dot{A}_t)$. From now on we normalize $c(q) = q$, which is really without loss in generality, since it just means interpreting the production cost as measured in utils. Also, unless otherwise indicated we assume $u'(0) > \frac{\partial u}{\partial q}$, which is what we needed for a monetary steady state to exist in the last section, according to (8).

Exactly the same logic used above implies the Bellman equation is

$$V(m_t; \dot{A}_t) = \mathbb{E}B + \mathbb{E}[u(q) \; \dot{A}_t] + A_t(m_t; m_{t+1}) - V(m_{t+1}; \dot{A}_{t+1} + \dot{m}_t);$$

Differentiating with respect to $m_t$ and substituting the result into the analogue of (2), $\dot{A}_t = -V_1 i m_{t+1}; \dot{A}_{t+1}$, we have

$$\dot{A}_t = -\frac{1}{2} \mathbb{E}u'(q_{t+1}) \frac{\partial A_{t+1}}{\partial m_{t+1}} + \frac{3}{4} \mathbb{E} \dot{A}_{t+1} \frac{\partial A_{t+1}}{\partial m_{t+1}} + \dot{A}_{t+1} + \dot{m}_t.$$ 

The bargaining problem and hence $q$ and $d$ and their derivatives are exactly as before. Hence, we have

$$\dot{A}_t = \frac{1}{2} \dot{A}_{t+1} + \frac{3}{4} \mathbb{E} u'(q_{t+1}; M) + 1 \mathbb{E} \dot{A}_{t+1} \frac{\partial A_{t+1}}{\partial m_{t+1}} + \dot{A}_{t+1} + \dot{m}_t$$

which except for the time subscripts, the appearance of $\dot{m}_t$, and the normalization $c(q) = q$, is the same as (6). Denote the right hand of (13) by $G^i \dot{A}_{t+1}$.

A dynamic equilibrium is a bounded\(^5\) sequence $f \dot{A}_t g^i_{t=0}$ satisfying (13) ($q$ and $d_t$ can be recovered from the other equilibrium conditions if needed). A special case is a steady state, which is an equilibrium such that $\dot{A}_t = \dot{A}_s$ for all $t$. Figure 2 shows $\dot{A}_{t+1} = G^i \dot{A}_t$. Notice there is regime switching: inside the shaded region where $\dot{A}_{t+1} < q^i = M$, the constraint $d_{t+1} \cdot M$ binds, while

\(^5\)Lemma 3 in the Appendix shows that we could never have an equilibrium with an unbounded sequence $f \dot{A}_t g$, and so we simply include boundedness in the definition here.
outside this region it does not. Consider the... at money case $\gamma = 0$. It follows
from Lemma 4 in the Appendix that $G(0) = 0$, $G^0(0) > 1$, and that there
exists a unique monetary steady state $\bar{\phi}$, and it satisfies $\bar{\phi} < \frac{M}{\theta}$. Since
$\bar{\phi}$ is unstable, there exists a continuum of dynamic equilibria: starting at any
$\phi_0$ in the interval $(0; \bar{\phi})$, there is a bounded path $\phi_t$ satisfying $\phi_t \neq 0$ (there
may also be other bounded paths that do not converge to the origin; see
below). Exactly as in other models of... at money, including the overlapping
generation model, in particular, expectations of inflation can be self-fulfilling
in this model.

![Figure 2: Law of Motion for $\phi_t$, for different $\gamma$.](image)
Now suppose \( \theta < 0 \), which means \( G^i_1 \) shifts up uniformly in \((\hat{A}_t; \hat{A}_{t+1})\) space. It is easy to see that there is some \( \theta_0 < 0 \) such that if \( \theta < \theta_0 \) then there exist no monetary equilibria: starting from any \( \hat{A}_0 > 0 \), \( \hat{A}_t \to 1 \). Hence, if money has very bad intrinsic properties (like a storage cost) there is no equilibrium where it is valued. If \( \theta > \theta_0 \) then there are (generically) an even number of monetary steady states; note that \( u^{(0)} < 0 \) implies \( G \) is concave and \( G^i_1 \) is convex, and hence there are exactly 2 monetary steady state equilibria (Lemma 4), but in general there could be more than 2. These steady states are alternatively stable and unstable, and so there exists a continuum of dynamic equilibria that converge to one of the stable steady states, depending on which \( \hat{A}_0 \) we pick. Hence, if the intrinsic properties of money are bad but not too bad there are multiple monetary equilibria where it is valued.

Finally, consider the case \( \theta > 0 \), which means \( G^i_1 \) shifts down uniformly in \((\hat{A}_t; \hat{A}_{t+1})\) space. There still exists a unique steady state \( \hat{A}^s \) (again by Lemma 4), and since it is unstable there are no nonconstant bounded paths satisfying \( \hat{A}_t = G(\hat{A}_{t+1}) \) and therefore no other equilibrium.\(^6\) Moreover, one can easily show that \( \hat{A}^s \to q^* = M \) if and only if \( \theta = r q^* = M \). In other words, the constraint is slack, and hence the unique equilibrium is efficient (\( q = q^* \)) if

\(^6\)Note that \( \hat{A} = 0 \) is not an equilibrium when \( \theta > 0 \). That is, by giving money a strictly positive dividend, we have ruled out the non-monetary equilibrium that always existed when \( \theta = 0 \). Additionally, suppose we relax the assumption \( u^{(0)}(0) > \frac{\partial G(0)}{\partial A} \) so that monetary equilibria do not exist when \( \theta = 0 \). By giving money any strictly positive dividend we guarantee \( G(0) > 0 \) and hence there exists a monetary equilibrium. These results are similar to those that arise in simple search models of money where money is endowed with desirable properties, including the property of being acceptable to government agents (e.g., Li and Wright 1999).
and only if the aggregate dividend \( \delta M \) exceeds \( rq \). It may be surprising that the unique equilibrium is efficient under this condition, although there are related results in the literature on commodity money (e.g. Berentsen, Molico and Wright [2000]). In any case, note that the condition \( \delta M > rq \) can be expensive: e.g., pursuing the commodity money interpretation, we must issue a large number of coins and each coin must have a sufficiently high metallic content, implying we need a sufficiently high quantity of precious metal.

We now return to the case \( \delta = 0 \) (at money) to show how even more complicated outcomes are possible. When \( \delta = 0 \) we know there is a unique monetary steady state \( \hat{A}^S, \hat{A}_{t+1} < q^i-M \) at \( \hat{A}_{t+1} = \hat{A}^S, \) and \( G^i(A^S) = 1 + \frac{-\delta}{M} A^S u(\hat{A}^S M) \). Figure 2 shows a case where \( G^i(A^S) > 0 \). However, it is clear that not only can we have \( G^i(A^S) \) be negative, we can have \( G^i(A^S) < 1 \); indeed the latter condition holds if and only if \( M \hat{A}^S u(\hat{A}^S M) < 1 \). Figure 3 shows \( \hat{A}_t = G(\hat{A}_{t+1}) \) and \( \hat{A}_{t+1} = G^i(\hat{A}_t) \), which obviously intersect on the 45\(^{\circ}\) line, at \( \hat{A}^S \) in a case where \( G^i(A^S) < 1 \).

It should be clear from the figure that this case implies \( G \) and \( G^i \) must also intersect outside the 45\(^{\circ}\) line: since \( G^i < 1 \) is less steep than \( G \) at \( \hat{A}^S \), \( G^i \) is trapped between the two branches of \( G \) to the right of \( \hat{A}^S \). The figure shows \( G^i \) crossing \( G \) at \((\hat{A}_t; \hat{A}_{t+1}) = (\hat{A}_t; \hat{A}_L)\); by symmetry, they also cross at \((\hat{A}_t; \hat{A}_{t+1}) = (\hat{A}_L; \hat{A}_t)\).

This construction generates an equilibrium involving cycles: \( \hat{A}_t = \hat{A}_L \) implies \( \hat{A}_{t+1} = G^i(\hat{A}_L) = \hat{A}_H \) and this implies \( \hat{A}_{t+2} = G(\hat{A}_H) = \hat{A}_L \). Equivalently, \( \hat{A}_L \) and \( \hat{A}_H \) are fixed points of the second iterate \( G^2 \). Note that \( \hat{A}_L < \hat{A}^S < \hat{A}_H \). Also note that in this example \( \hat{A}_H < q^i-M \), so the constraint
Figure 3: Existence of a 2-cycle

actually binds at each date. Figure 4 shows an example where \( \hat{A}_i > q^i = M \), illustrating the fact that the constraint can be slack at some dates. However, the constraint must bind at \( \hat{A}_L \) since \( \hat{A}_L < \hat{A}^s \) and \( \hat{A}^s < q^i = M \). In any case, a two-cycle equilibrium exists if and only if \( G(q^s) < 1 \). See Azariadis (1993) for a simple discussion of the dynamic system theory underlying this sort of result.

To give an explicit example, consider

\[
u(q) = \frac{(b + q)^{1_i}}{1_i} b^{1_i};
\]
Figure 4: A 2-cycle with regime-switching

for constants $b > 0$ and $\gamma > 0$. To reduce notation, set $M = 1$ (with no loss in generality, since money is neutral). The steady state is

$$\tilde{\phi}^s = \frac{\mu}{r + \frac{\gamma}{4}} \beta$$

For $b \leq 0$, it is easy to see that $G(\tilde{\phi}^s) = 1 + \frac{\gamma}{4} + \frac{\gamma}{r}$, and so $G(\tilde{\phi}^s) < 1$ if and only if $\gamma > \frac{1}{2} \frac{1}{1+r}$. More generally, given $b > 0$ there is a critical point such that as we increase $\gamma$ beyond this point the system bifurcates and two-cycle equilibria emerge. As $\gamma$ increases further cycles of other periodicity emerge.

For example, with $\beta = 1$, $\gamma = 0.5$, $b = 0.01$, and $r = 0.1$, cycles of period 3 emerge at $\gamma = 0.39$ (this is verified by looking for fixed points of $G^3$ different

---

7This useful utility function can be thought of as generalizing the more usual constant relative risk aversion specification by introducing $b$ so as to force $u(0) = 0$, which we need in this type of a setup. Note that it actually displays increasing relative risk aversion and decreasing absolute risk aversion.
from $\hat{A}^c$). Once we have cycles of period 3 in this model, we have cycles of all periods (see Azariadis [1993]).

Summarizing, with $\phi = 0$, there is a unique steady state monetary equilibrium, as well as and a unique steady state non-monetary equilibrium. There also exists a continuum of dynamic equilibria starting at any $\hat{A}_0 > 0$ and converging monotonically to 0, where the constraint $\hat{A}_{t+1} M \cdot q^f$ binds at every date. There may also exist other, more complicated dynamic equilibria, such as the cycles seen in Figure 3, and it is possible for the constraint to bind only at some dates, as in Figure 4. With commodity money, in the sense that it has a real return which can either be interpreted as a storage cost or a dividend the situation changes as shown in Figure 2: $\phi < 0$ and big implies there are no monetary equilibria; $\phi < 0$ but not so big implies there are generically an even number of monetary steady state equilibria (exactly 2 with certain assumptions on the utility function), a continuum of dynamic equilibria; and $\phi > 0$ implies there is a unique equilibrium, which is the monetary steady state, and moreover we can guarantee $\hat{A}^c > q^f M$ and hence $q = q^f$ by making $\phi M$ big.

5 Inflation

Suppose the money stock grows at a constant rate $\zeta$; that is, $M_{t+1} = (1 + \zeta) M_t$. The timing of events is as follows. First, there is a monetary injection. Then agents search for the special good, and at the end of the period they trade the general good and adjust their money holdings. We
rst consider the case in which the injections are made in the form of (equal) lump-sum transfers. As in the previous section, we make the normalization \( c(q) = q \), and return to the case \( \circ = 0 \).

The value function is now

\[
V(m_t; \hat{A}_t) = \mathbb{E}_t\{u(q_t) \mid \hat{A}_t d_t\} + \mathbb{E}_t v^i m_t^0 + \mathbb{E}_t \hat{A}_t (m_t^0 m_t^0) + \mathbb{E}_t \hat{A}_t \frac{\partial c}{\partial q};
\]

where \( m_{t+1} = m_t^0 + \xi M_t \) is the post-transfer amount of money the agent brings to the search market at \( t + 1 \). The pre-transfer amount of money he chooses to bring into period \( t + 1 \), is denoted \( m_t^0 \) and satisfies

\[
\hat{A}_t = \mathbb{E}_t \{u(q_t) \mid \hat{A}_t d_t\} + \mathbb{E}_t v^i m_t^0 + \mathbb{E}_t \hat{A}_t \frac{\partial c}{\partial q}.
\]

From the previous section we know that in a binding equilibrium this condition is

\[
\hat{A}_t = -\hat{A}_{t+1} \hat{M}_t^1 \mathbb{E}_t \{u(q_t) \mid \hat{A}_t d_t\} + \mathbb{E}_t v^i m_t^0 + \mathbb{E}_t \hat{A}_t \frac{\partial c}{\partial q}.
\]

Moreover, since \( q_t = \hat{A}_t M_t \) in a binding equilibrium with \( \mu = 1 \), we can rewrite (14) as

\[
\frac{q_t}{q_{t+1}} = \frac{1}{1 + \xi} [\mathbb{E}_t \{u(q_t) \mid \hat{A}_t d_t\} + \mathbb{E}_t v^i m_t^0 + \mathbb{E}_t \hat{A}_t \frac{\partial c}{\partial q}].
\]

Along a balanced-path \( q = q_t \) for all \( t \), so

\[
1 + \xi = \frac{1}{\mathbb{E}_t \{u(q_t) \mid \hat{A}_t d_t\} + \mathbb{E}_t v^i m_t^0 + \mathbb{E}_t \hat{A}_t \frac{\partial c}{\partial q}}.
\]

Notice that \( M_{t+1} = M_t = \hat{A}_t \hat{A}_{t+1} = d_{t+1} = d_t = 1 + \xi \) along such a path. In words, the money supply and nominal prices grow at rate \( \xi \). Imposing \( m_t = M_t \), it is easy to see that along the equilibrium path \( V \) is independent of
nominal variables and is again given by (9), so efficiency still requires $q = q^*$. Notice that (16) implies that

$$\frac{\partial q}{\partial \mu} = \frac{1}{\frac{1}{\partial q} / \partial(q^*)} < 0.$$  

And since $q < q^*$ in a monetary equilibrium, inflation reduces welfare. In fact, along a balanced path the optimal rate of inflation is $\hat{\mu} = -1$, $1 = r = (1 + r)$. In other words, output is inefficiently low if

$$(1 + \hat{\mu}) < r$$

namely if the (net) return to holding a unit of money is smaller than the rate of time preference.

For any $\mu \in (0; 1]$, (16) generalizes to

$$1 + \hat{\mu} = \frac{\frac{1}{\partial q} / \partial(q^*)}{1 + \frac{1}{\partial q} / \partial(q^*)} = \frac{\frac{1}{\partial q} / \partial(q^*)}{1 + \frac{1}{\partial q} / \partial(q^*)} + 1.$$  

As we show in the end of the section, for arbitrary $\mu$, efficiency still requires $q = q^*$. So in general, the optimal rate of inflation along the balanced growth path is

$$\hat{\mu}(\mu) = -1 \frac{1 + \frac{1}{\partial q} / \partial(q^*)}{1 + \frac{1}{\partial q} / \partial(q^*)} < 0;$$

where

$$\frac{1}{\partial q} / \partial(q^*) = \frac{\mu(1 - \mu)}{\partial q^*} [u(q^*)] c(q^*)[c^0(q^*)] u^0(q^*);$$

So for any $\mu$ there is a deflation rate that achieves the first best. Moreover, the Friedman rule is the efficient monetary policy when $\mu = 1$, but higher deflation rates are needed to achieve efficiency when $\mu < 1$. Formally,

$$\frac{\partial q}{\partial \mu} = -\frac{1}{\partial q^*} / \partial q^* \frac{\mu(1 - \mu)}{\partial q^*} [u(q^*)] c(q^*)[c^0(q^*)] u^0(q^*).$$
If the monetary injections take the form of proportional instead of lump-sum transfers, then the value function (again assuming $\mu = 1$) becomes

$$V(m_t; A_t) = \oplus [u(q) \cdot A_t] + \oplus B + A_t (m_t \cdot m_0^0) + \frac{1}{m_t+1; A_{t+1}}$$

where $m_{t+1} = (1 + \xi) m_t^0$. The amount of money each agent demands at the end of period $t$ satisfies

$$\hat{A}_t = (1 + \xi) - V_1 (1 + \xi) m_t^0; A_{t+1}.$$  \hspace{1cm} (18)

Hence, we have

$$q_{t+1} = \frac{q_t}{q_0^0}$$

Along a balanced growth path $M_{t+1} = M_t = \hat{A}_t = \hat{A}_{t+1} = 1 + \xi$ and $q_{t+1} = q$, so the growth rate of the money stock has no real effects if monetary injections take the form of proportional transfers: the model exhibits superneutrality.\(^8\)

To close this section, we show how one can also work with real balances, $z_t = \hat{A}_t m_t$. The value function becomes

$$V(z_t) = \oplus [u(q) \cdot q] + \oplus B + z_t \cdot z_0^0 + \frac{1}{z_{t+1}}; V(z_{t+1})$$

where $z_t^0 = \hat{A}_t m_t^0$, $Z_t = \hat{A}_t M_t$ and for the case of lump-sum transfers, $z_{t+1} = (z_0^0 + \xi Z_t) \hat{A}_{t+1} = \hat{A}_t$. The solution to the bargaining problem is

$$q = \begin{cases} q^* & \text{if } q^* \cdot z_t \\ z_t & \text{otherwise}, \end{cases}$$

and the Euler Equation is

$$\hat{A}_t = \frac{1}{2} [\frac{1}{z^*} + \frac{1}{z_{t+1}}] (\hat{A}_{t+1; z_{t+1}}(z_{t+1}) + 1)$$

\hspace{1cm} if $z_{t+1} \cdot q^*$

\hspace{1cm} otherwise.

\(^8\)It is not hard to show that proportional transfers are superneutral for any $\mu \in (0; 1)$.  

25
With this formulation it is easy to show that for arbitrary \( \mu \) efficiency requires that \( q = \varphi^\mu \) along the balanced growth path. First note that as usual, \( m_t = m_t^0 = M_t \), and this implies \( z_t = z_t^0 = Z_t \). Moreover, \( z_t = z_{t+1} \), since real balances are constant along the constant-growth path. Thus the value function of the representative agent is \( (1 - \bar{\gamma}) V(z) = \mathbb{E} \left[ u(q) \right] + \mathbb{E} B \), just as in the case with \( \mu = 1 \) (see (9)).

6 Uncertainty

In this section we extend the model to allow for randomness. We begin by describing an economy where this randomness is match specific: when any two agents meet they draw nonnegative random variables \( \cdot \) and \( \cdot \) drawn from \( H(\cdot;\cdot) \) independently across meetings, implying that the utility of consumption and production are given by \( u(q) \) and \( \cdot q \). Later we will discuss what happens when there are aggregate shocks. Also, below we consider the case where individual money holdings are stochastic. For now we restrict the analysis to steady states and use the simplifying assumption that the buyer has all the bargaining power in a single coincidence meeting, \( \mu = 1 \).

Let \( q(\cdot;\cdot) \) and \( d(\cdot;\cdot) \) denote the solutions to the bargaining problem (the dependence on \( m \) is implicit), let \( B(\cdot;\cdot) \) denote the gain from trade in a double coincidence meeting given \( (\cdot;\cdot) \), and let \( B = \int B(\cdot;\cdot) dH(\cdot;\cdot) \).

The value function for an individual entering the search market with \( m \)

---

9Since \( (\cdot;\cdot) \) applies to both agents in the meeting, if there is a double coincidence the symmetric Nash solution is for both agents to produce \( \varphi^\mu(\cdot;\cdot) \), where \( \varphi^\mu(\cdot;\cdot) \) solves \( u(\varphi^\mu q) = \cdot \), and no money changes hands. Notice \( \varphi^\mu \) depends only on the ratio \( \cdot = \cdot \).
dollars is now
\[ V(m) = \text{\( \mathcal{Z} \)} \int \text{\( f(q^{"}; \cdot) \)} \text{\( g(d^{"}; \cdot) \)} \text{\( m \cdot \phi \)} \text{\( dH^{"}; \cdot \)} + \text{\( \mathcal{B} \)} + \text{\( W(m) \)} ; \]

while \( W(m) \) is as before. Differentiating, we get
\[ V^{0}(m) = \text{\( \mathcal{Z} \)}^{1/2} \int \text{\( u^{0}[q^{"}; \cdot] \)} \text{\( \frac{\partial q}{\partial H} \)} \text{\( dH^{"}; \cdot \)} + \text{\( \mathcal{A} \)} ; \]

The bargaining solution is
\[ q^{"}; \cdot = \begin{cases} \frac{1}{2} q^{a^{"}; \cdot} & \text{if } q^{a^{"}; \cdot} \cdot \text{\( \mathcal{A} \)} > 0 \\ \text{\( \mathcal{A} \)} & \text{otherwise} \end{cases} \]

and \( d^{"}; \cdot = \cdot q^{"}; \cdot = \text{\( \mathcal{A} \)} \). Substituting \( \frac{\partial q}{\partial H} \) and \( \frac{\partial d}{\partial H} \) into \( V^{0}(m) \), then substituting the result into \( \text{\( \mathcal{A} \)} = \text{\( \mathcal{Z} \)} V^{0}(m) \) and using \( m = M \), we arrive at
\[ \text{\( \mathcal{Z} \)} \cdot \text{\( \mathcal{A} \)} = \text{\( \mathcal{Z} \)} V^{0}(m) = \int_{C} \text{\( u^{0}(\mathcal{A}m=\cdot) \)} \text{\( \frac{1}{\text{\( \mathcal{Z} \)}^{1/4}} \)} \text{\( \mathcal{B} \)} \text{\( \text{\( \int \)} \text{\( dH^{"}; \cdot \)} = 1 ; \tag{19} \)

where \( C \) is the set of realizations of \( ("; \cdot) \) that render the constraint \( \text{\( \mathcal{A} \)} \cdot m \) binding:

\[ C = f("; \cdot) : q^{a^{"}; \cdot} > \text{\( \mathcal{A} \)} m = \text{\( g \)} ; \]

\[ \frac{\partial}{\partial \text{\( \mathcal{A} \)}} = \frac{1}{\text{\( \mathcal{Z} \)}^{1/4}} \text{\( \frac{\partial}{\partial \text{\( \mathcal{A} \)}} \)} \left( \frac{\partial}{\partial \text{\( \mathcal{A} \)}} \text{\( u^{0}(\mathcal{A}m=\cdot) \)} \right) ; \]

where \( R(q) = \int \text{\( u^{0}(q) = u^{0}(q) \)} \) is the coefficient of relative risk aversion, and so \( \text{\( \mathcal{A} \)} \) is an increasing or decreasing function of \( \cdot \) as \( R < 1 \) or \( R > 1 \). To understand this, note that \( \cdot \) is the relative price of special goods in terms of real balances (or general goods). Whether an increase in the relative price raises or lowers total expenditure depends on \( R > 1 \); hence, whether the constraint is more likely to bind when \( \cdot \) increases also depends on \( R > 1 \). When \( R > 1 \), an increase in \( \cdot \) raises total expenditure in the special good and hence \( \text{\( \mathcal{A} \)} \) falls with \( \cdot \). As a special case, if \( u(q) = \log(q) \) then \( \text{\( \mathcal{A} \)} \) is independent of \( \cdot \).
A steady state equilibrium is characterized by a $\hat{A}$ that solves (19). We show in Lemma 5 in the Appendix that there exists a unique solution to (19) provided $u$ satisfies the Inada conditions. In equilibrium the constraint must bind with positive probability since otherwise $C$ is empty and (19) could not hold. It is possible for the constraint to bind with probability 1 (as in the deterministic case), or to bind with probability less than 1. That is, for a given realization of $\cdot$, buyers with relatively high realizations of " will hit the constraint and spend all their cash, but those with low realizations will not.\footnote{For example, consider the case where $\cdot = 1$ with probability 1 and " is uniform on $[0,1]$. Then the constraint binds if $" > \hat{A} = 1=\hat{A}$ (\hat{A}M). In equilibrium " must satisfy $1 + \frac{\theta}{\mu}(1;\hat{A})^2 = 1$. It is easy to see that there is a unique " that solves the equation, and moreover, that it lies in $(0;1)$, meaning that the constraint $d \cdot m$ will bind in some meetings and not others.} Given $\hat{A}$, $q(\cdot;\cdot)$ and $d(\cdot;\cdot)$ are obtained from the solution to the bargaining problem. The nominal price of the special good in a $(\cdot;\cdot)$-meeting is $p_s = d(\cdot;\cdot) = q(\cdot;\cdot) = \cdot = \hat{A}$ (it is independent of " for this case with $\mu = 1$).

So far we have been interpreting the random variables $(\cdot;\cdot)$ as a shock that is idiosyncratic to each match. However, it is not hard to recast the whole analysis letting $\cdot$ and $\cdot$ be aggregate shocks to utility and production cost respectively. To study this case, we adapt the notation as follows:

$$W (m; s) = \max_{m^0} \hat{A}(s)(m \cdot m^0) + \hat{V}(m^0; s); \text{ and}$$

$$V (m; s) = \hat{W} + \left( \hat{\psi} \hat{u}[q(s)] \hat{A}d(s)g + W (m; s) \right) dH(s)$$

where $s = (\cdot;\cdot)$ denotes the aggregate state and the dependence of $q$ and $d$...
on $m$ is still implicit. The first order condition (after imposing $m^0 = M$) is
\[
\hat{A}(s) = - \hat{A}(s^0) (1 + I(s^0) \otimes \#(s^0 = 0) u_0(\hat{A}(s^0) M = 0) [1 \cdot g] dH(s^0)); \quad (20)
\]
where $I(s)$ is an indicator function which equals 1 if $s \in C$ and 0 otherwise. For the special case of iid shocks, $\hat{A}$ is independent of $s$ and (20) reduces to (19). The current state affects the value of money today only to the extent it enters the forecast of next period’s state.

Let us now return to the case where $s' = 0 = 1$ with probability 1 and consider random transfers of money across agents. In particular, suppose that each period, before entering the search market, money is randomly redistributed across agents (by policy, say). Thus, an agent who chose $m$ dollars at the end of the previous period enters the search market this period with $m + \frac{1}{2}$ dollars, where $\frac{1}{2}$ has CDF $H(\frac{1}{2})$. Assume $E\frac{1}{2} = 0$, so that the total money stock is constant. The support of $\frac{1}{2}$ is $-\frac{1}{2}$ and $\frac{1}{2}$, so that $m > 0$ with probability 1 in equilibrium. For this case the value function is
\[
V(m) = \otimes_{\mathbb{B}} + \otimes_{\mathbb{Q}} \ f_u(q(m + \frac{1}{2}) \otimes \hat{A}(m + \frac{1}{2}) dH(\frac{1}{2} + W(m))
\]
and as usual, the first order condition for money holdings is $\hat{A} = - \check{V^0}(m)$. These imply
\[
\check{V^0}(m) = \otimes_{\mathbb{B}} + \otimes_{\mathbb{Q}} \ f_u(q(m + \frac{1}{2}) \otimes \hat{A}(m + \frac{1}{2}) [1 \cdot g] dH(\frac{1}{2} = 1)
\]
(21)
where $\check{\#(\hat{A}) = \#(\hat{A}) \otimes \#(M + \frac{1}{2}) \otimes [1 \cdot g] dH(\frac{1}{2} = 1)$ where $\#(\hat{A}) = q^- \hat{A} \otimes \#(M)$ is the minimum transfer that makes the constraint slack. Lemma 6 in the Appendix shows that Inada conditions on $u$ imply there exists a unique $\hat{A}$ satisfying this equation.

\footnote{The last term is $E_{\mathbb{Q}} W(m + \frac{1}{2} = W(m)$ because $W(\#)\otimes \#(1)$ is linear and $\frac{1}{2}$ has mean 0.}
We now want to analyze the effect of a change in risk. Consider a family of distributions $H(\frac{1}{2}; \varepsilon)$ where $\varepsilon_2 > \varepsilon_1$ implies $H(\frac{1}{2}; \varepsilon_2)$ is a mean preserving spread of $H(\frac{1}{2}; \varepsilon_1)$: that is, $\int \varepsilon \cdot H_2(\frac{1}{2}; \varepsilon) \, d\frac{1}{2}$ for any $\varepsilon_2$ with equality at $\varepsilon_2 = \frac{1}{2}$. Notice that $H_2(\frac{1}{2}; \varepsilon) = H_2(\frac{1}{2}; \frac{1}{2}) = 0$. Then $\frac{\partial \varphi}{\partial \varepsilon}$ is equal in sign to

$$
\frac{\partial \varphi}{\partial \varepsilon} = \int \varepsilon u^0(M + \frac{1}{2} \hat{\varphi}) \, d\frac{1}{2} H_2(\frac{1}{2}; \varepsilon)
$$

where the last expression results from integrating by parts. The first term vanishes because $u^0 = 1$ at $\frac{1}{2}$ and $H_2(\frac{1}{2}; \varepsilon) = 0$. Integrating by parts again yields

$$
\frac{\partial \varphi}{\partial \varepsilon} = \int \varepsilon u^0(M + \frac{1}{2} \hat{\varphi}) \, d\frac{1}{2} H_2(\frac{1}{2}; \varepsilon)
$$

The first term is unambiguously positive but the sign of the second depends on $u^{\infty}$; if $u^{\infty} > 0$ then $\frac{\partial \varphi}{\partial \varepsilon} > 0$ and a mean-preserving spread of $H$ unambiguously increases the value of money.

The effect of risk on $\hat{\varphi}$ (and the way it depends on $u^{\infty}$) is due to the so-called precautionary demand for money: an increase in risk makes agents want to hold more cash. But since the money stock is fixed, in equilibrium each must hold the same amount and hence the real value of money rises to clear the market. It can also be shown that an increase in risk unambiguously
reduces welfare. To see this, write the Bellman equation as

\[ V(M) = \beta B + W(M) + \int_{\Omega} \left( I\left( \frac{1}{2} f u[A(M + \frac{1}{2})] \right) + \frac{1}{2} g + [1 \cdot I(\frac{1}{2})][u(q^i) i \cdot q^i] \right) dH(\frac{1}{2} \xi); \]

where \( I(\frac{1}{2}) \) is an indicator function which equals 1 if \( \frac{1}{2} < \frac{1}{2} = q^i = A_i M \) and 0 otherwise. Since the integrand is strictly concave in \( \frac{1}{2} \), a mean-preserving spread of \( H \) unambiguously reduces \( V(M) \). This is to be contrasted with some other search models of money, where the distribution of money holdings \( F \) is non-degenerate in equilibrium. In those models, random transfers of money across agents can be welfare improving because they make the distribution of real balances less unequal (and in this way they provide partial insurance); see Molico (1999) or Berentsen (1999). This is the reason lump sum nominal money injections can be beneficial in Molico (1999). Here, the distribution of real balances is degenerate, which is as good as it gets; so random transfers make agents worse off.\(^{13}\)

7 Stochastic Inflation

In this section we extend the analysis of Section 5 by letting the growth rate of the money supply at time \( t \) be a random variable \( \xi_t \) drawn from \( H(\xi_j \xi_{t-1}) \). For the case in which the random injections take the form of lump-sum transfers right before search, we have

\[ W(m; \xi_t; \xi_{t+1}) = \hat{A}(m; m^0) + \frac{1}{2} E \left[ V[m_{t+1}; \xi_{t+1} \xi_{t+1}] \right]; \]

\(^{13}\)The simplest example is a one-time, unanticipated, random redistribution of money across agents, which reduces welfare since \( \int_{\Omega} \int_{\Omega} \int_{\Omega} [u(q^i) i \cdot q^i] dH(\xi) \cdot dH(\xi) = V(M). \)
where \( m_{t+1} = m_0^t + \tilde{\varepsilon}_{t+1}M_t \) and \( \text{EV}^{i} m_{t+1}; \tilde{\varepsilon}_{t+1} \) is given by

\[
\sum_{t \in \mathcal{D}B} \{ u(q_{t+1}) \} \quad \text{if} \quad \tilde{\varepsilon}_{t+1} \land m_t \geq q^i \quad \text{otherwise.}
\]

As usual, \( q_{t+1} = q^i \) if \( \tilde{\varepsilon}_{t+1}m_{t+1} \geq q^i \) and \( q_{t+1} = \tilde{\varepsilon}_{t+1}m_{t+1} \) otherwise. The first order condition for money holdings \( m_0^t \) is

\[
\tilde{\varepsilon}_t = -\text{EV}^{i} m_{t+1}; \tilde{\varepsilon}_{t+1}; \quad \text{where}
\]

\[
\text{EV}_t^i (m_t; \tilde{\varepsilon}_t) = \frac{\tilde{\varepsilon}_t}{A_t} \left[ u^i(q_{t+1}) + 1 \right] \quad \text{if} \quad \tilde{\varepsilon}_t m_t \geq q^i
\]

\[
\text{EV}_t^i (m_t; \tilde{\varepsilon}_t) = \frac{\tilde{\varepsilon}_t}{A_t} \left[ u^i(q_{t+1}) + 1 \right] \quad \text{otherwise.}
\]

Let \( C \) be the set of realizations of the shock that render the constraint binding: \( C = \{ \tilde{\varepsilon}_{t+1} : \tilde{\varepsilon}_{t+1} (m_0^t + \tilde{\varepsilon}_{t+1}M_t) < q^i \} \) and let \( C^c \) denote its complement. With this notation the first order condition becomes

\[
1 = -\sum_{C^c} Z \frac{\tilde{\varepsilon}_{t+1}}{A_t} dH (\tilde{\varepsilon}_{t+1}J_t) + \sum_{C} \frac{\tilde{\varepsilon}_{t+1}}{A_t} \left[ u^i(q_{t+1}) + 1 \right] \; \text{dH} (\tilde{\varepsilon}_{t+1}J_t);
\]

This condition can be conveniently restated in terms of real money balances:

\[
1 = -\sum_{C^c} \frac{z_{t+1}}{(1+z_{t+1})z_t} dH (\tilde{\varepsilon}_{t+1}J_t) + \sum_{C} \frac{z_{t+1}}{(1+z_{t+1})z_t} \left[ u^i(q_{t+1}) + 1 \right] \; \text{dH} (\tilde{\varepsilon}_{t+1}J_t), \quad (22)
\]

where \( C = \{ z_{t+1} : z_{t+1} < q^i \} \). If we consider the case of iid shocks, then \( z_t = z(\tilde{\varepsilon}_t) \) is independent of \( \tilde{\varepsilon}_t \) and \( z_t = z \) for all \( t \). Hence either the constraint binds at all times or it never does. The Euler equation (22) cannot hold if the constraint is slack at all times, so it must bind in every period. Thus in the iid case (22) simplifies to

\[
u^0(z) = \frac{1 + r \; (1 + (1 + (1 + (1 + (1 + (1 + \, \text{etc.}) \right) \right) \right) \right) \right) \right) \right)
\]

32
where $^3 = R(1 + \zeta)^{1} dH(\zeta)$ is the expected (gross) return to holding a unit of money for a period. Notice that $z < q^i \implies$ 

\[ ^3 i \quad 1 < r \]

namely if the expected (net) return to holding a dollar is smaller than the rate of time preference. This is the natural generalization of (17).

When the growth rate of the money supply is persistent, the inflation forecast depends on the current state and hence so will real balances, which will then typically not be constant through time. To see this consider the following example. Suppose $\zeta \in \{ \zeta_1, \zeta_2 \}$, with $\zeta_1 > \zeta_2$, $\text{prob}(\zeta = \zeta_1|\zeta_1) = p$, $\text{prob}(\zeta = \zeta_1|\zeta_2) = s_2$ and $\text{prob}(\zeta = \zeta_2|\zeta_1) = s_1$. The process is iid if $p_1 = s_2$, and persistent if $p_1 > s_2$. Specializing (22) to this case, and conjecturing an equilibrium $(z_1^i, z_2^i)$ where $z_1^i < q^i$, we get: $z_1 = z_1(z_2)$ and $z_2 = z_2(z_1)$, where

\[ z_1(z_2) = \frac{p_1}{s_2} \int (1 + \zeta)^{1} \frac{dH(\zeta)}{s_2(1 + \zeta)} z_2 \int \frac{dH(\zeta)}{s_2(1 + \zeta)} u_0(z_2) z_2 \]

\[ z_2(z_1) = \frac{p_2}{s_1} \int (1 + \zeta)^{1} \frac{dH(\zeta)}{s_1(1 + \zeta)} z_1 \int \frac{dH(\zeta)}{s_1(1 + \zeta)} u_0(z_1) z_1 \]

It is easy to verify that $z_1 = z_2$ for the iid case. Also, $z_i(0) = 0$, $\lim_{x \to 1} z_i(x) = 1$, and

\[ z_1^0(z_2) = \frac{p_1}{s_2} \int (1 + \zeta)^{1} \frac{dH(\zeta)}{s_2(1 + \zeta)} [1 + R(z_2)] u_0(z_2) \]

\[ z_2^0(z_1) = \frac{p_2}{s_1} \int (1 + \zeta)^{1} \frac{dH(\zeta)}{s_1(1 + \zeta)} [1 + R(z_1)] u_0(z_1) \]

\[ \text{Notice that } p_1 s_2 = p_2 s_1 \text{ is always the case since } p_1 + s_1 = 1. \]
where $R(z) = i \ u^0(z) z = u^0(z)$. So $z_0^0(z) > 0$ if $R(z) > 1$. Moreover, the unique pair $(z_1; z_2)$ satisfying $z_i(z_i) = z_i$ is characterized by

$$1 + \Theta [u^0(z_i)] = \frac{(1 + \xi_1) (p_1 i s_2)}{(p_1 p_2 i s_1 s_2)}$$

$$1 + \Theta [u^0(z_i)] = \frac{(1 + \xi_2) (p_2 i s_1)}{(p_1 p_2 i s_1 s_2)}$$

From these it is immediate that $\xi_1 > \xi_2$ implies $z_2 < z_1$; which provided $z_0^0(\phi) > 0$, in turn implies that $z_1^w < z_2^w$. When the shocks to the money supply are persistent, equilibrium real balances are smaller in periods of high inflation in anticipation of high inflation. This is illustrated in Figure 5.

![Figure 5: Equilibrium real balances with stochastic persistent inflation](image)

To conclude the example, recall that the equilibrium was constructed conjecturing that $z_1^w < q^w$. Since $z_1^w < z_1$; for the conjecture to be correct
it is sufficient to ensure that $z_1 \cdot q^t$, which holds if $(1 + \xi_2)(p_1 s_2 - s_1 s_2) > 0$.

## Appendix

### Lemma 1

In a double coincidence meeting each agent produces $q^t$ for the other and no money changes hands.

**Proof.** In double coincidence meetings, the Nash problem is

$$\max_{q_1, q_2, \xi} [u(q_1) - c(q_2) - \xi] [u(q_2) - c(q_1) + \xi]$$

subject to $m_2 \cdot \xi \cdot m_1$. Where $q_1$ and $q_2$ denote the quantities consumed by agents 1 and 2 respectively and $\xi$ is the amount of dollars 1 pays 2. The problem has a unique solution which is characterized by the first order conditions:

$$u^i(q_1) [u(q_1) - c(q_2) - \xi] = c^i(q_1) [u(q_1) - c(q_2) + \xi]$$

$$c^i(q_2) [u(q_1) - c(q_2) + \xi] = u^i(q_2) [u(q_1) - c(q_2) + \xi]$$

$$u(q_1) - u(q_2) + c(q_1) - c(q_2) + 2\xi = \frac{(2 \xi + \lambda_1 \lambda_2)}{f[u(q_1), c(q_2); A_1, u(q_1), c(q_2); A_2, \xi]}$$

where $\lambda_i$ is the multiplier on agent $i$’s cash constraint. It is easy to verify that the proposed solution, namely $q_1 = q_2 = q^t$ with $\xi = \lambda_1 = \lambda_2 = 0$ satisfies these conditions. ■

### Lemma 2

For any $\mu 2 (0; 1)$ a monetary steady state exists if (10) holds. In a monetary steady state the constraint $d \cdot m$ binds, and $q < q^t$.
Proof. The solution to (5) is characterized by

\[
\begin{align*}
\mu [i \ c(q) + \dot{A}d] u^0(q) &= (1 + \mu) [u(q) i \ \dot{A}d] c^0(q) \\
\mu [i \ c(q) + \dot{A}d] \dot{A} &= (1 + \mu) [u(q) i \ \dot{A}d] \dot{A} \\
\end{align*}
\]

(23)

(24)

where \( \lambda \) is the Lagrange multiplier on \( d \cdot m \). If the constraint does not bind, then \( \lambda = 0 \), \( q = q^\ast \) and \( d = (1 - \dot{A}) [\mu c(q^\ast) + (1 + \mu) u(q^\ast)] \). If the constraint binds then \( \lambda > 0 \), \( d = m \), and (23) and (24) combine to yield

\[
\frac{u^0(q)}{c^0(q)} = 1 + \frac{\mu (u(q) i \ \dot{A}d) \lambda^{-1 + 1}}{\mu c(q) + \dot{A}d}.
\]

(25)

This shows that \( q < q^\ast \). If the constraint does not bind, \( \dot{a} = \dot{a} = 0 \); if it does, \( \dot{a} = \dot{a} = 1 \) and

\[
\begin{align*}
\dot{a} &= \frac{\dot{A} [\mu u^0(q) + (1 + \mu) c^0(q)]}{c^0(q) u^0(q) \mu [i \ c(q) + Am] u^0(q) + (1 + \mu) [u(q) i \ Am] c^0(q)}.
\end{align*}
\]

Using (23) to substitute for \( \dot{A}m \) we arrive at

\[
\begin{align*}
\dot{a} &= \frac{\dot{A} [\mu u^0(q) + (1 + \mu) c^0(q)]}{c^0(q) u^0(q) [1 + \dot{a} (q)]};
\end{align*}
\]

with \( \dot{a} (q) \) as defined in the body of the paper. Updating one period and using the equilibrium condition \( m = M \) we have

\[
\dot{a} = \begin{cases} 
-\dot{A} n & \quad \text{if } \mu c(q^\ast) + (1 + \mu) u(q^\ast) \cdot \dot{A}M \\
-\dot{A}^0 \otimes \mu^0(q) \otimes \frac{1}{\dot{a}} & \quad \text{otherwise}.
\end{cases}
\]

Steady state implies \( \dot{A} = \dot{A}^0 \). Provided \( - < 1 \), if a monetary equilibrium exists the constraint must be binding, and \( q \) is characterized by

\[
\begin{align*}
\dot{a} \otimes \mu^0(q) \otimes \frac{1}{\dot{a}} + (1 + \mu^0) \dot{A} = \dot{A};
\end{align*}
\]

36
Inserting $\xi = @m$ and simplifying we arrive at (11). A monetary steady state is a solution $q > 0$ to (11). At $q = q^f$ the left hand side of (11) is less than the right hand side. Hence, a solution exists if the left hand side is above right hand side at $q = 0$, which holds if (10) holds. This completes the proof. 

Lemma 3 Any equilibrium $f\bar{A}g$ must be bounded.

Proof. Suppose we have an equilibrium and $f\bar{A}g$ is unbounded. Then $\bar{A} \cdot q^f = M$ for all $t$ greater than some $T$. Hence for all $t > T$, the constraint $d_t \cdot m_t$ is slack and all single coincidence meetings result in $q^f$ units of output being exchanged for $d_t = q^f \bar{A}_t$ dollars. At the end of each period, those who were buyers will buy back $d_t$ dollars for $\bar{A}_t d_t = q^f$ units of the general good, and those who were sellers reduce their money holdings by $d_t$ dollars in exchange for $\bar{A}_t d_t = q^f$ units of the general good. Hence in any given period buyers and sellers enjoy utility $u(q^f) \cdot q^f$ and 0 respectively. Consequently, at $t > T$ utility is

$$V(M) = \frac{\partial^{9/4+} [u(q^f) \cdot c(q^f)] + \circ M}{1 \downarrow}$$

But if $\bar{A}_t$ diverges then eventually $\bar{A}_t M > V(M)$, which means all agents would want to deviate from the candidate equilibrium by trading all their money for general goods. 

Lemma 4 Let $g_i \bar{A}_{t+1} q = -\bar{A}_{t+1} \circ u(q(\bar{A}_{t+1} M)) + 1 i \circ \bar{A}$. Then: $g(0) = 0$; $g^0(0) > 1$; $g^f(\bar{A}^\circ) < 1$ at any steady state $\bar{A}^\circ > 0$ as long as $\circ > 0$; and $g^{\text{m}}(\bar{A}) < \circ u^{\text{m}}(q) < i 2 u^{\text{m}}(q) = q$
Proof. By concavity, \( 0 \cdot u^0(x) \cdot u(x) \) for all \( x \). Since \( u(0) = 0 \) we have \( \lim_0 u^0(x) x = 0 \). Thus \( g(0) = 0 \). It is easy to show that \( g^0(\hat{A}) = g(\hat{A}) = \hat{A} + \theta_{\hat{A}} \hat{M} u^0(\hat{A}) \) and hence \( g^0(\hat{A}^s) = 1 \). \( A^s + \theta_{\hat{A}} \hat{M} u^0(\hat{A}^s) < 1 \) at any steady state \( \hat{A}^s > 0 \), when \( \theta > 0 \). This implies \( g(\hat{A}) + \theta^0 \) cuts the \( 45^\circ \) line from above at \( \hat{A}^s \), as shown in Figure 2, and therefore there can be at most one positive steady state, when \( \theta > 0 \). This also that this implies \( g^0(0) > 1 \).

Lemma 5 There exists a unique solution to (19) if \( u \) satisfies Inada conditions.

Proof. Rewrite (19) as

\[
T(\hat{A}) = - - \theta_{\hat{A}} \hat{M} u^0(\hat{A}) = \int_0^{\hat{A}} \int_0^\infty h dH. (\cdot) = 1.
\]

Differentiation implies

\[
T^0(\hat{A}) = \int_0^{\hat{A}} \int_0^\infty u^0(\hat{A}) = \int_0^{\hat{A}} \int_0^\infty \frac{M}{2} dH. (\cdot) < 0.
\]

Also, \( \lim_{\hat{A} \to 0} T(\hat{A}) = 1 \) and \( \lim_{\hat{A} \to 1} T(\hat{A}) = \theta \) by the Inada conditions.

Lemma 6 There exists a unique solution to (21) if \( u \) satisfies Inada conditions.

Proof. Rewrite (21) as

\[
T(\hat{A}) = - - \theta_{\hat{A}} \hat{M} f u^0(M + \frac{1}{2} \hat{A}) dH. (\cdot) = 1.
\]

\[
\frac{1}{2}
\]
Differentiation implies

\[
T^0(\dot{\lambda}) = - \Theta \frac{\partial}{\partial \lambda} u^0(M + \frac{1}{2} \dot{\lambda})(M + \frac{1}{2}) d\lambda \left( \frac{1}{2} < 0 \right);
\]

Finally, \( \lim_{\lambda \to 0} T(\dot{\lambda}) = 1 \) and \( \lim_{\lambda \to 1} T(\dot{\lambda}) = - \) by the Inada conditions.
References

