Simple New Keynesian Model without Capital: Implications of Networks

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Objectives

• Provide a rigorous development of the basic New Keynesian model without capital.
  – Previous exposure to the model is helpful, but not absolutely necessary.

• Present a version of the model that incorporates a simple formulation of the ‘network’ nature of production.
  – In standard model, all production is sold directly to final purchasers.
  – In fact (see, e.g., Basu AER1996) about 1/2 of gross production by firms is sold to other firms.

• See Christiano, Trabandt and Walentin (Handbook of Monetary Economics, 2011) for an extended discussion of the approach to networks developed here.
Implications of thinking about networks

- Obtain a quantitatively important theory of the cost of inflation.

- Raise questions about the effectiveness of inflation targetting as a device for stabilizing inflation and the macroeconomy.

- Flatten the slope of the Phillips curve because of strategic complementarities in price setting.
Background Readings on Networks


Households

- Problem:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left( \log C_t - \exp \left( \tau_t \frac{N_t^{1+\varphi}}{1+\varphi} \right) \right), \quad \tau_t = \lambda \tau_{t-1} + \varepsilon_t$$

s.t. \( P_t C_t + B_{t+1} \leq W_t N_t + R_{t-1} B_t + \text{Profits net of taxes}_t \)

- First order conditions:

\[
\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \quad (5)
\]

\[
\exp(\tau_t) C_t N_t^{\varphi} = \frac{W_t}{P_t}.
\]
Goods Production

- A homogeneous final good is produced using the following (Dixit-Stiglitz) production function:

\[ Y_t = \left[ \int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}}. \]

- Each intermediate good, \( Y_{i,t} \), is produced as follows:

\[ Y_{i,t} = \exp(a_t) N_i^{\gamma} I_{i,t}^{1-\gamma}, \quad a_t \sim \text{exogenous shock to technology}, \]

\[ 0 < \gamma \leq 1. \]

- \( I_{i,t} \sim \text{‘materials’ these are purchases of the homogeneous output good (Basu’s simplified way of capturing that firms buy goods from other firms).} \)

- Before discussing the firms that operate these production functions, we briefly investigate the socially efficient (‘First Best’) allocation of resources across \( i \).

  - simplify the discussion with \( \gamma = 1 \) (no materials).
Efficient Sectoral Allocation of Resources Across Sectors

- With Dixit-Stiglitz final good production function, there is a socially optimal allocation of resources to all the intermediate activities, $Y_{i,t}$
  - It is optimal to run them all at the same rate, i.e., $Y_{i,t} = Y_{j,t}$ for all $i, j \in [0, 1]$.
- For given $N_t$, it is optimal to set $N_{i,t} = N_{j,t}$, for all $i, j \in [0, 1]$
- In this case, final output is given by
  $$Y_t = e^{at}N_t.$$  
- Best way to see this is to suppose that labor is *not* allocated equally to all activities.
  - Explore one simple deviation from $N_{i,t} = N_{j,t}$ for all $i, j \in [0, 1]$.  

Suppose Labor *Not* Allocated Equally

• Example:

\[
N_{it} = \begin{cases} 
2\alpha N_t & i \in \left[0, \frac{1}{2}\right] \\
2(1-\alpha)N_t & i \in \left[\frac{1}{2}, 1\right] 
\end{cases}, \quad 0 \leq \alpha \leq 1.
\]

• Note that this is a particular distribution of labor across activities:

\[
\int_0^1 N_{it} di = \frac{1}{2} 2\alpha N_t + \frac{1}{2} 2(1-\alpha)N_t = N_t
\]
Labor Not Allocated Equally, cnt’d

\[ Y_t = \left[ \int_0^1 Y_{t,i}^{\frac{\varepsilon-1}{\varepsilon}} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}} \]

\[ = \left[ \int_0^{\frac{1}{2}} Y_{t,i}^{\frac{\varepsilon-1}{\varepsilon}} \, di + \int_{\frac{1}{2}}^1 Y_{t,i}^{\frac{\varepsilon-1}{\varepsilon}} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}} \]

\[ = e^{a_t} \left[ \int_0^{\frac{1}{2}} N_{t,i}^{\frac{\varepsilon-1}{\varepsilon}} \, di + \int_{\frac{1}{2}}^1 N_{t,i}^{\frac{\varepsilon-1}{\varepsilon}} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}} \]

\[ = e^{a_t} \left[ \int_0^{\frac{1}{2}} (2\alpha N_t)^{\frac{\varepsilon-1}{\varepsilon}} \, di + \int_{\frac{1}{2}}^1 (2(1 - \alpha)N_t)^{\frac{\varepsilon-1}{\varepsilon}} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}} \]

\[ = e^{a_t} N_t \left[ \int_0^{\frac{1}{2}} (2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} \, di + \int_{\frac{1}{2}}^1 (2(1 - \alpha))^\frac{\varepsilon-1}{\varepsilon} \, di \right]^{\frac{\varepsilon}{\varepsilon-1}} \]

\[ = e^{a_t} N_t \left[ \frac{1}{2} (2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} + \frac{1}{2} (2(1 - \alpha))^\frac{\varepsilon-1}{\varepsilon} \right]^{\frac{\varepsilon}{\varepsilon-1}} \]

\[ = e^{a_t} N_t f(\alpha) \]
\[ f(\alpha) = \left[ \frac{1}{2} (2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} + \frac{1}{2} (2(1 - \alpha))^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}} \]

Efficient Resource Allocation Means Equal Labor Across All Sectors

\[ \varepsilon = 6 \]

\[ \varepsilon = 10 \]
Homogeneous Goods Production

- Competitive firms:
  - maximize profits:

\[
P_tY_t - \int_0^1 P_{i,t} Y_{i,t} dj,
\]

subject to:

\[
Y_t = \left[ \int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{\frac{\varepsilon}{\varepsilon-1}}.
\]

- Foncs:

\[
Y_{i,t} = Y_t \left( \frac{P_t}{P_{i,t}} \right)^{\varepsilon} \rightarrow P_t = \left( \int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}}.
\]

"cross price restrictions"
Intermediate Goods Production

- Demand curve for \( i^{th} \) monopolist:

\[
Y_{i,t} = Y_t \left( \frac{P_t}{P_{i,t}} \right)^\varepsilon.
\]

- Production function:

\[
Y_{i,t} = \exp \left( a_t \right) N_i^\gamma I_{i,t}^{1-\gamma}, \quad a_t \sim \text{exogenous shock to technology}, \quad 0 < \gamma \leq 1.
\]

- \( I_{i,t} \sim \text{‘materials’ these are purchases of the homogeneous output good (Basu’s simplified way of capturing that firms buy goods from other firms).} \)

- Calvo Price-Setting Friction:

\[
P_{i,t} = \begin{cases} 
\tilde{P}_t & \text{with probability } 1 - \theta \\
\frac{P_{i,t-1}}{P_{i,t-1}} & \text{with probability } \theta
\end{cases}.
\]
Cost Minimization Problem

- Price setting by intermediate good firms is discussed later.
  - The intermediate good firm must produce the quantity demanded, $Y_{i,t}$, at the price that it sets.
  - Right now we take $Y_{i,t}$ as given and we investigate the cost minimization problem that determines the firm’s choice of inputs.

Cost minimization problem:

$$\min_{N_{i,t}, I_{i,t}} \bar{W}_t N_{i,t} + \bar{P}_t I_{i,t} + \lambda_{i,t} \left[ Y_{i,t} - A_t N_{i,t}^{\gamma} I_{i,t}^{1-\gamma} \right]$$

with resource costs:

- Subsidy, if $\nu > 0$
- Cost, including finance, of a unit of labor

$$\bar{W}_t = \left( 1 - \nu \right) \times \left( 1 - \psi + \psi R_t \right) W_t$$

- Cost, including finance, of a unit of materials

$$\bar{P}_t = (1 - \nu) \times \left( 1 - \psi + \psi R_t \right) P_t$$
Cost Minimization Problem

- Problem:

\[
\min_{N_{i,t}, I_{i,t}} \bar{W}_{t} N_{i,t} + \bar{P}_{t} I_{i,t} + \lambda_{i,t} \left[ Y_{i,t} - A_{t} N_{i,t}^{\gamma} I_{i,t}^{1-\gamma} \right]
\]

- First order conditions:

\[
\bar{P}_{t} I_{i,t} = (1 - \gamma) \lambda_{i,t} Y_{i,t}, \quad \bar{W}_{t} N_{i,t} = \gamma \lambda_{i,t} Y_{i,t},
\]

so that,

\[
\frac{I_{it}}{N_{it}} = \frac{1 - \gamma}{\gamma} \frac{\bar{W}_{t}}{\bar{P}_{t}} = \frac{1 - \gamma}{\gamma} \exp \left( \tau_{t} \right) C_{t} N_{t}^{\phi}
\]

\[
\rightarrow \frac{I_{it}}{N_{it}} = \frac{I_{t}}{N_{t}}, \text{ for all } i.
\]
Cost Minimization Problem

- Firm first order conditions imply

\[ \lambda_{i,t} = \left( \frac{\bar{P}_t}{1 - \gamma} \right)^{1-\gamma} \left( \frac{\bar{W}_t}{\gamma} \right)^\gamma \frac{1}{A_t}. \]

- Divide marginal cost by \( P_t \):

\[ s_t \equiv \frac{\lambda_{i,t}}{P_t} = (1 - \nu) (1 - \psi + \psi R_t) \left( \frac{1}{1 - \gamma} \right)^{1-\gamma} \]
\[ \times \left( \frac{1}{\gamma} \exp (\tau_t) C_t N_t^\phi \right)^\gamma \frac{1}{A_t} \] (9),

after substituting out for \( \bar{P}_t \) and \( \bar{W}_t \) and using the household’s labor first order condition.

- Note from (9) that \( i^{th} \) firm’s marginal cost, \( s_t \), is independent of \( i \) and \( Y_{it} \).
Share of Materials in Intermediate Good Output

- Firm $i$ materials proportional to $Y_{i,t}$:

$$I_{i,t} = \frac{(1 - \gamma) \lambda_{i,t} Y_{i,t}}{\bar{P}_t} = \mu_t Y_{i,t},$$

where

$$\mu_t = \frac{(1 - \gamma) s_t}{(1 - \nu) (1 - \psi + \psi R_t)} \quad (10).$$

- "Share of materials in firm-level gross output", $\mu_t$. 
Decision By Firm that Can Change Its Price

- $i^{th}$ intermediate good firm’s objective:

$$E_t^i \sum_{j=0}^{\infty} \beta^j \nu_{t+j} \begin{bmatrix} \text{revenues} \\ \text{total cost} \end{bmatrix} = \begin{bmatrix} P_{i,t+j}Y_{i,t+j} - P_{t+j}s_{t+j}Y_{i,t+j} \end{bmatrix}$$

$\nu_{t+j}$ - Lagrange multiplier on household budget constraint

- Firm that gets to reoptimize its price is concerned only with future states in which it does not change its price:

$$E_t^i \sum_{j=0}^{\infty} \beta^j \nu_{t+j} \begin{bmatrix} P_{i,t+j}Y_{i,t+j} - P_{t+j}s_{t+j}Y_{i,t+j} \end{bmatrix}$$

$$= E_t \sum_{j=0}^{\infty} (\beta \theta)^j \nu_{t+j} \begin{bmatrix} \tilde{P}_{t}Y_{i,t+j} - P_{t+j}s_{t+j}Y_{i,t+j} \end{bmatrix} + X_t,$$

where $\tilde{P}_t$ denotes a firm’s price-setting choice at time $t$ and $X_t$ not a function of $\tilde{P}_t$. 
Decision By Firm that Can Change Its Price

- Substitute out demand curve:

\[
E_t \sum_{j=0}^{\infty} (\beta \theta)^j v_{t+j} [\tilde{P}_{t} Y_{i,t+j} - P_{t+j} s_{t+j} Y_{i,t+j}]
\]

\[
= E_t \sum_{j=0}^{\infty} (\beta \theta)^j v_{t+j} Y_{t+j} P_{t+j}^\varepsilon \left[\tilde{P}_{t}^{1-\varepsilon} - P_{t+j} s_{t+j} \tilde{P}_{t}^{-\varepsilon}\right].
\]

- Differentiate with respect to \( \tilde{P}_{t} \):

\[
E_t \sum_{j=0}^{\infty} (\beta \theta)^j v_{t+j} Y_{t+j} P_{t+j}^\varepsilon \left[(1 - \varepsilon) (\tilde{P}_{t})^{-\varepsilon} + \varepsilon P_{t+j} s_{t+j} \tilde{P}_{t}^{-\varepsilon - 1}\right] = 0,
\]

or,

\[
E_t \sum_{j=0}^{\infty} (\beta \theta)^j v_{t+j} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[\frac{\tilde{P}_{t}}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j}\right] = 0.
\]

- When \( \theta = 0 \), get standard result - price is fixed markup over marginal cost.
Decision By Firm that Can Change Its Price

- Substitute out the multiplier:

\[
E_t \sum_{j=0}^{\infty} (\beta \theta)^j \frac{u'(C_{t+j})}{P_{t+j}} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[ \frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.
\]

- Using assumed log-form of utility,

\[
E_t \sum_{j=0}^{\infty} (\beta \theta)^j \frac{Y_{t+j}}{C_{t+j}} (X_{t,j})^{-\varepsilon} \left[ \tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0,
\]

\[
\tilde{p}_t \equiv \frac{\tilde{P}_t}{P_t}, \quad \tilde{\pi}_t \equiv \frac{P_t}{P_{t-1}}, \quad X_{t,j} = \left\{ \begin{array}{ll} \frac{1}{\pi_{t+j} \pi_{t+j-1} \cdots \pi_{t+1}}, & j \geq 1 \\ 1, & j = 0. \end{array} \right.
\]

\[
X_{t,j} = X_{t+1,j-1} \frac{1}{\pi_{t+1}}, \quad j > 0
\]
Decision By Firm that Can Change Its Price

- Want $\tilde{p}_t$ in:
  \[
  E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} (X_{t,j})^{-\varepsilon} \left[ \tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0
  \]

- Solving for $\tilde{p}_t$, we conclude that prices are set as follows:
  \[
  \tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+1}} (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t}.
  \]

- Need convenient expressions for $K_t, F_t$. 
Decision By Firm that Can Change Its Price

\[ K_t = E_t \sum_{j=0}^{\infty} (\beta \theta)^j \frac{Y_{t+j}}{C_{t+j}} (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \]

\[ = \frac{\varepsilon}{\varepsilon - 1} \frac{Y_t}{C_t} s_t \]

\[ + \beta \theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \varepsilon \sum_{j=0}^{\infty} (\beta \theta)^j X_{t+1,j}^{-\varepsilon} \frac{Y_{t+j+1}}{C_{t+j+1}} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j} \]

\[ = \frac{\varepsilon}{\varepsilon - 1} \frac{Y_t}{C_t} s_t + \beta \theta E_t \left( \frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} K_{t+1} \]

For a detailed derivation, see, e.g.,
Decision By Firm that Can Change Its Price

- Conclude:

\[
\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{-\varepsilon} Y_{t+j} \frac{\varepsilon}{C_{t+j} \varepsilon - 1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{1-\varepsilon} Y_{t+j} \frac{\varepsilon}{C_{t+j}}} = \frac{K_t}{F_t'}
\]

where

\[
K_t = \frac{\varepsilon}{\varepsilon - 1} \frac{Y_t}{C_t} s_t + \beta \theta E_t \left( \frac{1}{\tilde{\pi}_{t+1}} \right)^{-\varepsilon} K_{t+1} \quad (1)
\]

- Similarly,

\[
F_t = \frac{Y_t}{C_t} + \beta \theta E_t \left( \frac{1}{\tilde{\pi}_{t+1}} \right)^{1-\varepsilon} F_{t+1} \quad (2)
\]
Interpretation of Price Formula

- Note,

\[
\frac{1}{P_{t+j}} = \frac{1}{P_t} X_{t,j}, \quad s_{t+j} = \frac{\lambda_{t+j}}{P_{t+j}} = \frac{\lambda_{t+j}}{P_t} X_{t,j}, \quad \tilde{p}_t = \frac{\tilde{P}_t}{P_t}.
\]

Multiply both sides of the expression for \( \tilde{p}_t \) by \( P_t \) :

\[
\tilde{P}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{1-\varepsilon} \frac{Y_{t+j}}{C_{t+j}}^{\varepsilon/(\varepsilon-1)} \lambda_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{1-\varepsilon} \frac{Y_{t+j}}{C_{t+j}}^{\varepsilon/(\varepsilon-1)}} = \frac{\varepsilon}{\varepsilon - 1} \sum_{j=0}^{\infty} E_t \omega_{t+j} \lambda_{t+j}
\]

where

\[
\omega_{t+j} = \frac{(\beta \theta)^j (X_{t,j})^{1-\varepsilon} \frac{Y_{t+j}}{C_{t+j}}^{\varepsilon/(\varepsilon-1)}}{E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{1-\varepsilon} \frac{Y_{t+j}}{C_{t+j}}^{\varepsilon/(\varepsilon-1)}} \quad \sum_{j=0}^{\infty} E_t \omega_{t+j} = 1.
\]

Evidently, price is set as a markup over a weighted average of future marginal cost, where the weights are shifted into the future depending on how big \( \theta \) is.
Moving On to Aggregates

- Aggregate price level.
- Aggregate measures of production.
  - Value added.
  - Gross output.
Aggregate Price Index

- Rewrite the aggregate price index.
  - let $p \in (0, \infty)$ the set of logically possible prices for intermediate good producers.
  - let $g_t(p) \geq 0$ denote the measure (e.g., ‘number’) of producers that have price, $p$, in $t$
  - let $g_{t-1,t}(p) \geq 0$, denote the measure of producers that had price, $p$, in $t-1$ and could not reoptimize in $t$

- Then,

$$P_t = \left( \int_0^1 P_{i,t}^{(1-\epsilon)} di \right)^{\frac{1}{1-\epsilon}} = \left( \int_0^\infty g_t(p) p^{(1-\epsilon)} dp \right)^{\frac{1}{1-\epsilon}}.$$

- Note:

$$P_t = \left( (1 - \theta) \tilde{P}_t^{1-\epsilon} + \int_0^\infty g_{t-1,t}(p) p^{(1-\epsilon)} dp \right)^{\frac{1}{1-\epsilon}}.$$
Aggrega\text{e Price Index}

- Calvo randomization assumption:

\[
\begin{align*}
\text{measure of firms that had price, } p, \text{ in } t-1 \text{ and could not change} \\
g_{t-1,t} (p) \\
\text{measure of firms that had price } p \text{ in } t-1 \\
= \theta \times g_{t-1} (p)
\end{align*}
\]

- Then,

\[
P_t = \left( (1 - \theta) \tilde{P}_{1-\epsilon}^t + \int_0^\infty g_{t-1,t} (p) p^{(1-\epsilon)} dp \right)^{\frac{1}{1-\epsilon}} = p_{t-1}^{1-\epsilon} \left( 1 - \theta \right) \tilde{P}_{1-\epsilon}^t + \theta \int_0^\infty g_{t-1} (p) p^{(1-\epsilon)} dp
\]
Restriction Between Aggregate and Intermediate Good Prices

• ‘Calvo result’:

\[ P_t = \left( \int_0^1 P_{i,t}^{(1-\epsilon)} \, di \right)^{\frac{1}{1-\epsilon}} = \left[ (1 - \theta) \tilde{P}_t^{(1-\epsilon)} + \theta P_{t-1}^{(1-\epsilon)} \right]^{\frac{1}{1-\epsilon}}. \]

• Divide by \( P_t \):

\[ 1 = \left[ (1 - \theta) \tilde{p}_t^{(1-\epsilon)} + \theta \left( \frac{1}{\bar{\pi}_t} \right)^{(1-\epsilon)} \right]^{\frac{1}{1-\epsilon}}. \]

• Rearrange:

\[ \tilde{p}_t = \left[ \frac{1 - \theta}{1 - \theta \bar{\pi}_t^{(\epsilon-1)}} \right]^{\frac{1}{\epsilon-1}}. \]
Aggregate inputs and outputs

- **Gross output** of firm $i$:

$$Y_{i,t} = \exp(a_t) N_{i,t}^{\gamma} I_{i,t}^{1-\gamma}.$$

  - Net output or *value-added* would subtract out the materials that were bought from other firms.

- Economy-wide gross output: sum of value of $Y_{i,t}$ across all firms:

$$\int_0^1 P_{i,t} Y_{i,t} di = \int_0^1 P_t \left( \frac{Y_t}{Y_{i,t}} \right)^{\frac{1}{\epsilon}} Y_{i,t} di = Y_t^{\frac{\epsilon-1}{\epsilon}} Y_t^{\frac{1}{\epsilon}} = P_t Y_t.$$  

- **Gross output production function**: relation between $Y_t$ and non-produced inputs, $N_t$. 

Aggregate inputs and outputs, cnt’d

- Gross output, $Y_t$, is not a good measure of economic output, because it double counts.
  - Some of the output that firm $i$ ‘produced’ is materials produced by another firm, which is counted in that firm’s output.
  - If wheat is used to make bread, wrong to measure production by adding all wheat and all bread. That double counts the wheat.

- Want aggregate value-added: sum of firm-level gross output, minus purchases of materials from other firms.

- Value-added production function: expression relating aggregate value-added in period $t$ to inputs not produced in period $t$.
  - capital and labor.
Gross Output vs Agg Materials and Labor

- Approach developed by Tack Yun (JME, 1996).
- Define $Y^*_t$:

\[
Y^*_t \equiv \int_0^1 Y_{i,t} \, di
\]

\[
\text{demand curve} \quad Y_t \int_0^1 \left( \frac{P_{i,t}}{P_t} \right)^{-\epsilon} \, di = Y_t P_t^\epsilon \int_0^1 (P_{i,t})^{-\epsilon} \, di
\]

\[
= Y_t P_t^\epsilon (P_t^*)^{-\epsilon}
\]

where, using ‘Calvo result’:

\[
P_t^* \equiv \left[ \int_0^1 P_{i,t}^{-\epsilon} \, di \right]^{-\frac{1}{\epsilon}} = \left[ (1 - \theta) \tilde{P}_t^{-\epsilon} + \theta (P_{t-1}^*)^{-\epsilon} \right]^{-\frac{1}{\epsilon}}
\]

- Then

\[
Y_t = p^*_t Y^*_t, \quad p^*_t = \left( \frac{P_t^*}{P_t} \right)^{\epsilon}.
\]
Gross Output vs Agg Materials and Labor

- Relationship between aggregate inputs and outputs:

\[ Y_t = p^*_t Y^*_t = p^*_t \int_0^1 Y_{i,t} di \]
\[ = p^*_t A_t \int_0^1 N_{i,t}^{\gamma} I_{i,t}^{1-\gamma} di = p^*_t A_t \int_0^1 \left( \frac{N_{i,t}}{I_{i,t}} \right)^\gamma I_{i,t} di, \]
\[ = p^*_t A_t \left( \frac{N_t}{I_t} \right)^\gamma I_t, \]

or,

\[ Y_t = p^*_t A_t N_t^{\gamma} I_t^{1-\gamma} \] (6).

- Note that \( p^*_t \) is a function of the ratio of two averages (with different weights) of \( P_{i,t}, i \in (0,1) \)
  - So, when \( P_{i,t} = P_{j,t} \) for all \( i, j \in (0,1) \), then \( p^*_t = 1 \).
  - But, what is \( p^*_t \) when \( P_{i,t} \neq P_{j,t} \) for some \( i, j \in (0,1) \)?
Tack Yun Distortion

- Consider the object,

\[ p_t^* = \left( \frac{P_t^*}{P_t} \right)^\varepsilon, \]

where

\[ P_t^* = \left( \int_0^1 P_{i,t}^{-\varepsilon} di \right)^{-\frac{1}{\varepsilon}}, \]

\[ P_t = \left( \int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}} \]

- In following slide, use Jensen’s inequality to show:

\[ p_t^* \leq 1. \]
Tack Yun Distortion

- Let \( f(x) = x^4 \), a convex function. Then,

\[
\text{convexity: } \alpha x_1^4 + (1 - \alpha) x_2^4 > (\alpha x_1 + (1 - \alpha) x_2)^4
\]

for \( x_1 \neq x_2, \ 0 < \alpha < 1. \)

- Applying this idea to prices:

\[
\text{convexity: } \int_0^1 \left( P_{i,t}^{(1-\varepsilon)} \right)^{\frac{\varepsilon}{\varepsilon-1}} \, di \geq \left( \int_0^1 P_{i,t}^{(1-\varepsilon)} \, di \right)^{\frac{\varepsilon}{\varepsilon-1}}
\]

\[
\iff \left( \int_0^1 P_{i,t}^{-\varepsilon} \, di \right)^{\frac{-1}{\varepsilon}} \leq \left( \int_0^1 P_{i,t}^{(1-\varepsilon)} \, di \right)^{\frac{1}{1-\varepsilon}}
\]

\[
\left\{ \frac{p^*}{p_t} \right\} \leq \left\{ \frac{P_t}{P_{t}} \right\}
\]
Law of Motion of Tack Yun Distortion

- We have
  \[ P_t^* = \left[ (1 - \theta) \tilde{P}_t^{-\varepsilon} + \theta (P_{t-1}^*)^{-\varepsilon} \right]^{-1/\varepsilon} \]

- Then,
  \[ p_t^* \equiv \left( \frac{P_t^*}{P_t} \right)^\varepsilon = \left[ (1 - \theta) \tilde{p}_t^{-\varepsilon} + \theta \frac{\tilde{\pi}_t^\varepsilon}{p_{t-1}^*} \right]^{-1} \]
  \[ = \left[ (1 - \theta) \left( \frac{1 - \theta \tilde{\pi}_t^{(\varepsilon-1)}}{1 - \theta} \right)^{\varepsilon / (\varepsilon - 1)} + \theta \frac{\tilde{\pi}_t^\varepsilon}{p_{t-1}^*} \right]^{-1} \]
  \[ \text{using the restriction between } \tilde{p}_t \text{ and aggregate inflation developed earlier.} \]
Gross Output Production Function

- Recall

\[ I_{i,t} = \mu_t Y_{i,t}, \]

so,

\[ I_t \equiv \int_0^1 I_{i,t} di = \mu_t \int_0^1 Y_{i,t} d = \mu_t Y^*_t = \frac{\mu_t}{p_t^*} Y_t. \]

- Then, the gross output production function is:

\[
Y_t = p_t^* A_t N_t^{\gamma} I_t^{1-\gamma} \\
= p_t^* A_t N_t^{\gamma} \left( \frac{\mu_t}{p_t^*} Y_t \right)^{1-\gamma} \\
\rightarrow Y_t = \left( p_t^* A_t \left( \frac{\mu_t}{p_t^*} \right)^{1-\gamma} \right)^{\frac{1}{\gamma}} N_t
\]
Value Added (GDP) Production Function

- We have

\[ GDP_t = Y_t - I_t = \left(1 - \frac{\mu_t}{p_t^*}\right) Y_t \]

\[ = \left(1 - \frac{\mu_t}{p_t^*}\right) \left(p_t^* A_t \left(\frac{\mu_t}{p_t^*}\right)^{1-\gamma}\right)^{\frac{1}{\gamma}} N_t \]

= Total Factor Productivity (TFP)

\[ = \left(p_t^* A_t \left(1 - \frac{\mu_t}{p_t^*}\right)^{\gamma} \left(\frac{\mu_t}{p_t^*}\right)^{1-\gamma}\right)^{\frac{1}{\gamma}} N_t \]

- Note how an increase in technology at the firm level, by \( A_t \), gives rise to a bigger increase in TFP by \( A_t^{1/\gamma} \).
  - In the literature on networks, \( 1/\gamma \) is referred to as a ‘multiplier effect’ (see Jones, 2011).

- The Tack Yun distortion, \( p_t^* \), is associated with the same multiplier phenomenon.
Decomposition for TFP

• To maximize GDP for given aggregate $N_t$ and $A_t$:

$$\max_{0<p^*_t \leq 1, \ 0 \leq \lambda_t \leq 1} \left( p^*_t A_t (1 - \lambda_t)^\gamma (\lambda_t)^{1-\gamma} \right)^{\frac{1}{\gamma}}$$

$$\rightarrow \lambda_t = 1 - \gamma, \ p^*_t = 1.$$  

• So,

$$\text{TFP}_t = \left( p_t \left( \frac{1 - \frac{\mu_t}{p^*_t}}{\gamma} \right)^\gamma \left( \frac{\mu_t}{p^*_t} \right)^{1-\gamma} \right)^{\frac{1}{\gamma}}$$

Component due to market distortions $\equiv \chi_t$

$$\times \left( A_t (\gamma)^\gamma (1 - \gamma)^{1-\gamma} \right)^{\frac{1}{\gamma}}$$

Exogenous, technology component $\equiv \tilde{A}_t$
Evaluating the Distortions

- The equations characterizing the TFP distortion, $\chi_t$:

$$
\chi_t = \left( p_t^* \left( \frac{1 - \frac{\mu_t}{p_t^*}}{\gamma} \right)^\gamma \left( \frac{\frac{\mu_t}{p_t^*}}{1 - \gamma} \right)^{1 - \gamma} \right)^{\frac{1}{\gamma}}
$$

$$
p_t^* = \left[ (1 - \theta) \left( \frac{1 - \theta \bar{\pi}_t^{(\epsilon-1)}}{1 - \theta} \right) + \frac{\theta \bar{\pi}_t^\epsilon}{p_{t-1}^*} \right]^{-1}
$$

- Potentially, NK model provides an ‘endogenous theory of TFP’.
- Standard practice in NK literature is to set $\chi_t = 1$ for all $t$.
  - Set $\gamma = 1$ and linearize around $\bar{\pi}_t = p_t^* = 1$.
  - With $\gamma = 1$, $\chi_t = p_t^*$, and first order expansion of $p_t^*$ around $\bar{\pi}_t = p_t^* = 1$ is:

$$
p_t^* = p^* + 0 \times \bar{\pi}_t + \theta (p_{t-1}^* - p^*), \text{ with } p^* = 1,
$$

so $p_t^* \to 1$ and is invariant to shocks.
Empirical Assessment of the Distortions

- First, do ‘back of the envelope’ calculations in a steady state when inflation is constant and $p^*$ is constant.

\[
p^* = \left[ (1 - \theta) \left( \frac{1 - \theta \pi^{(\varepsilon-1)}}{1 - \theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \pi^{\varepsilon}}{p^*} \right]^{-1}
\]

\[
\rightarrow p^* = \frac{1 - \theta \pi^{\varepsilon}}{1 - \theta} \left( \frac{1 - \theta}{1 - \theta \pi^{(\varepsilon-1)}} \right)^{\frac{\varepsilon}{\varepsilon-1}}
\]

- Approximate TFP distortion, $\chi$:

\[
\chi_t = \left( p_t^* \left( \frac{1 - \frac{\mu_t}{p_t^*}}{\gamma} \right) \left( \frac{\mu_t}{p_t^*} \right)^{1-\gamma} \right)^{\frac{1}{\gamma}} \sim (p^*)^{1/\gamma}
\]
Three Inflation Rates:

- Average inflation in the 1970s, 8 percent APR.
- Several people have suggested that the US raise its inflation target to 4 percent to raise the nominal rate of interest and thereby reduce the likelihood of the zero lower bound on the interest rate becoming binding again.
  - http://www.voxeu.org/article/case-4-inflation
- Two percent inflation is the average in the recent (pre-2008) low inflation environment.
quarterly gross CPI inflation mean, 1972Q1-1983Q4 = 1.02
Cost of Three Alternative Permanent Levels of Inflation

\[ p^* = \frac{1 - \theta \pi^\varepsilon}{1 - \theta} \left( \frac{1 - \theta}{1 - \theta \pi^{(\varepsilon-1)}} \right)^{\varepsilon}, \quad \chi = (p^*)^{1/\gamma} \]

<table>
<thead>
<tr>
<th>Table 1: Percent of GDP Lost(^1) Due to Inflation, 100(1 – (\chi_t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Without networks ((\gamma = 1))</td>
</tr>
<tr>
<td>----------------------------------</td>
</tr>
<tr>
<td><strong>a: Steady state inflation: 8 percent per year</strong></td>
</tr>
<tr>
<td>2.41(^2) (3.92) [10.85]</td>
</tr>
<tr>
<td><strong>b: Steady state inflation: 4 percent per year</strong></td>
</tr>
<tr>
<td>0.46 (0.64) [1.13]</td>
</tr>
<tr>
<td><strong>c: Steady state inflation: 2 percent per year</strong></td>
</tr>
<tr>
<td>0.10 (0.13) [0.21]</td>
</tr>
</tbody>
</table>

Note: number not in parentheses assumes a markup of 20 percent; number in parentheses: 15 percent; number in square brackets: 10 percent
Next: Assess Costs of Inflation Using Non-Steady State Formulas
**Figure 1a:** Percent loss of GDP due to Inflation, assumed markup is 1.2

**Figure 1b:** Percent loss of GDP due to Inflation, assumed markup is 1.15
Inflation Distortions Displayed are Big

- With $\varepsilon = 6$,
  - mean($\chi_t$) = 0.98, a 2% loss of GDP.
  - frequency, $\chi_t < 0.955$, is 10% (i.e., 10% of the time, the output loss is greater than 4.5 percent).

- With more competition (i.e., $\varepsilon$ higher), the losses are greater.
  - with higher elasticity of demand, given movements in inflation imply much greater substitution away from high priced items, thus greater misallocation (caveat: this intuition is incomplete since with greater $\varepsilon$ the consequences of a given amount of misallocation are smaller).

- Distortions with $\gamma = 1/2$ are roughly twice the size of distortions in standard case, $\gamma = 1$.
  - To see this, note

\[
1 - \chi_t \simeq 1 - (p^*)^{\frac{1}{\gamma}} \quad \Rightarrow \quad \frac{1}{\gamma} (1 - p^*).
\]
Comparison of Steady State and Dynamic Costs of Inflation in 1970s

- Results

<table>
<thead>
<tr>
<th>Table 1: Fraction of GDP Lost, 100(1 − χ), During High Inflation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>No networks, γ = 1</td>
</tr>
<tr>
<td>Steady state lost output</td>
</tr>
<tr>
<td>Mean, 1972Q1-1982Q4</td>
</tr>
</tbody>
</table>

Note * number not in parentheses - markup of 20 percent (i.e., ε = 6)

   number in parentheses - markup of 15 percent. (i.e., ε = 7.7)

- Evidently, distortions increase rapidly in inflation,

\[ E[\text{distortion (inflation)}] > \text{distortion (Einflation)} \]
• Collect the equilibrium conditions.
  – For careful comparison of NK model with RBC model, see http://faculty.wcas.northwestern.edu/~lchrist/course/China_Chengdu_2016/NewKeynesian_model_handout.pdf
  – In RBC model, markets obtain socially efficient allocations independent of monetary policy.
  – In NK model, markets don’t necessarily work well and good monetary policy essential.

• Solve the model.
• Break up the equilibrium conditions into three sets:

1. Conditions (1)-(4) for prices: $K_t, F_t, \bar{\pi}_t, p^*_t, s_t$
2. Conditions (6)-(10) for: $C_t, Y_t, N_t, I_t, \mu_t$
3. Conditions (5) and (11) for $R_t$ and $\chi_t$.

• Consider

  – conditions for the sticky price case.
  – conditions for RBC case: equilibrium allocations are *first best*,
    they are what a benevolent planner would choose.
Equilibrium Conditions Associated with Price Setting

\[ K_t = \frac{\varepsilon}{\varepsilon - 1} \frac{\gamma_t}{c_t} s_t + \beta \theta E_t \bar{\pi}_{t+1}^\varepsilon K_{t+1} \]  
\[ F_t = \frac{\gamma_t}{c_t} + \beta \theta E_t \bar{\pi}_{t+1}^{\varepsilon-1} F_{t+1} \]  
\[ K_t \frac{F_t}{F_t} = \left[ \frac{1 - \theta \bar{\pi}_t^{(\varepsilon-1)}}{1 - \theta} \right]^{\frac{1}{1-\varepsilon}} \]  
\[ p_t^* = \left[ (1 - \theta) \left( \frac{1 - \theta \bar{\pi}_t^{(\varepsilon-1)}}{1 - \theta} \right) + \frac{\theta \bar{\pi}_t^\varepsilon}{p_{t-1}^*} \right]^{-1} \]
Equilibrium Conditions Associated With Gross Output

- Equations:

\[ Y_t = p_t^* A_t N_t^\gamma I_t^{1-\gamma} \quad (6), \quad C_t + I_t = Y_t \quad (7), \quad I_t = \mu_t \frac{Y_t}{p_t^*} \quad (8) \]

\[ s_t = (1 - \nu) (1 - \psi + \psi R_t) \left( \frac{1}{1 - \gamma} \right)^{1-\gamma} \]

\[ \times \left( \frac{1}{\gamma} \exp (\tau_t) C_t N_t^\phi \right) \gamma \frac{1}{A_t} \quad (9) \]

\[ \mu_t = \frac{(1 - \gamma) s_t}{(1 - \nu) (1 - \psi + \psi R_t)} \quad (10), \]
Other Equilibrium Conditions

- Allocative distortion:

\[ \chi_t = \left( p_t^* \left( \frac{1 - \mu_t}{p_t^*} \right)^\gamma \left( \frac{\mu_t}{p_t^*} \right)^{1-\gamma} \right)^{\frac{1}{\gamma}} \]  \hspace{1cm} (11)

- Intertemporal equation

\[ \frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \]  \hspace{1cm} (5)
• Monetary Policy Rule

\[ R_t/R = (R_{t-1}/R)^\rho \exp [(1 - \rho) \phi_\pi (\bar{\pi}_t - \bar{\pi}) + u_t] \]

• Smoothing parameter: \( \rho \).
  - Bigger is \( \rho \) the more persistent are policy-induced changes in the interest rate.

• Monetary policy shock: \( u_t \).
Conclusion About Networks

- Networks alter the New Keynesian model’s implications for inflation.
  - Doubles the cost of inflation.
  - Phillips curve is flatter because of strategic complementarities (when there are price frictions, this makes materials prices inertial which makes marginal costs inertial, which reduces firms’ interest in changing prices).

- For the result on the Taylor principle, see my 2011 handbook chapter and Christiano (2015).
  - When the smoothing parameter in Taylor rule is set to zero and $\psi = 1$, then the model has indeterminacy, even when the coefficient on inflation is 1.5.
  - So, the likelihood of the Taylor principle breaking down goes up when $\gamma$ is reduced, consistent with intuition.
  - When the smoothing parameter is at its empirically plausible value of 0.8, then the solution of the model does not display indeterminacy.