Application of DSGE Model: What Happens if we Switch from Fixed to Flexible Interest Rate Regime?
Two Examples

• Standard monetary policy briefing question:
  – ‘What Happens if We Set the Interest Rate to Fixed Level for $y$ Periods?’

• Policy question relevant in some countries today:
  – ‘What Happens if Shocks Drive the Economy into the Zero Lower Bound and are expected to keep the economy there for a while?’
  – ‘Zero Lower Bound’: lower bound on nominal rate of interest.
Model

• Model in linearized form:

\[ E_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0, \]

– Here,

• \( z_t \) denotes the list of endogenous variables whose values are determined at time \( t \).

• \( s_t \) denotes the list of exogenous variables whose values are determined at time \( t \).

\[ s_t = P s_{t-1} + \varepsilon_t. \]

• Solution: \( A \) and \( B \) in

\[ z_t = A z_{t-1} + B s_t, \]

– where

\[ \alpha_0 A^2 + \alpha_1 A + \alpha_2 I = 0, \]

\[ (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1)B] = 0 \]
Policy Experiment

• The $n^{th}$ equation in the system is a monetary policy rule (Taylor rule). One of the variables in $z_t$ is the policy interest rate, $R_t$, in deviation from its non-stochastic steady state value:

\[ R_t = \tau'z_t. \]

• $\tau$ is composed of 0’s and a single 1

• Policy:
  – it is now time $t=T$ and policy is $R_t = \tilde{d}$ from $t=T+1$ to $t=T+y$.
  – For $t>T+y$, policy follows the Taylor rule again.
Convenient to ‘Stack’ the System to be Conformable with Dynare Notation

• First set of equations is the equilibrium conditions and second set is the exogenous shock process:

\[
E_t \left\{ \begin{bmatrix} A_0 & Z_{t+1} \\ \alpha_0 & \beta_0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A_1 & Z_t \\ \alpha_1 & \beta_1 \\ 0 & I \end{bmatrix} + \begin{bmatrix} A_2 & Z_{t-1} \\ \alpha_2 & 0 \\ 0 & -P \end{bmatrix} + \begin{bmatrix} \epsilon_t \\ 0 \\ -\varepsilon_t \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

− or

\[E_t \{A_0 Z_{t+1} + A_1 Z_t + A_2 Z_{t-1} + \epsilon_t\} = 0.\]

− where \(\epsilon_t\) is independent over time and in time \(t\) information set.

• The \(n^{th}\) row of the above system corresponds to monetary policy rule.
**Fixed R Equilibrium Conditions**

- Delete monetary policy rule (i.e., $n^{th}$ equation) from system and replace it by $R_t = \tilde{d}$:
  
  - Let $\hat{A}_0$ and $\hat{A}_2$ denote $A_0$ and $A_2$ with their $n^{th}$ rows replaced by $0$'s.
  
  - Let $\hat{A}_1 = \begin{bmatrix} \hat{\alpha}_1 & \hat{\beta}_1 \\ 0 & I \end{bmatrix}$

- Where $\hat{\alpha}_1$ is $\alpha_1$ with its $n^{th}$ row replaced by $\tau'$ and $\hat{\beta}_1$ is $\beta_1$ with its $n^{th}$ row replaced by $0$'s.

- Equilibrium conditions:

\[
E_t\{\hat{A}_0Z_{t+1} + \hat{A}_1Z_t + \hat{A}_2Z_{t-1} + \epsilon_t\} = d.
\]

- $d$ is a column vector zero in all but one location and $\tau'd = \tilde{d}$
Problem

• Equilibrium conditions for $t=T+1,...,T+y$:
  \[ E_t \{ \hat{A}_0 Z_{t+1} + \hat{A}_1 Z_t + \hat{A}_2 Z_{t-1} + \epsilon_t \} = d. \]

• Equilibrium conditions for $t>T+y$:
  \[ E_t \{ A_0 Z_{t+1} + A_1 Z_t + A_2 Z_{t-1} + \epsilon_t \} = 0. \]

– Solution for $t>T+y$:
  \[
  \begin{pmatrix}
  z_t \\
  s_t
  \end{pmatrix}
  =
  \begin{bmatrix}
  a & BP \\
  0 & P
  \end{bmatrix}
  \begin{pmatrix}
  z_{t-1} \\
  s_{t-1}
  \end{pmatrix}
  +
  \begin{pmatrix}
  b & 0 \\
  0 & -I
  \end{pmatrix}
  \begin{pmatrix}
  0 \\
  -\epsilon_t
  \end{pmatrix}
  \]
  or,
  \[ Z_t = aZ_{t-1} + b\epsilon_t \]

• How is the solution for $t=T+1,...,T+y$?

\[ Z_t = aZ_{t-1} + b\epsilon_t \]
Solve the Model ‘Backward’

• In period $t=T+y$:

$E_{T+y} \left\{ \hat{A}_0 \underbrace{Z_{T+y+1}}_{=aZ_{T+y}+b\epsilon_{T+y+1}} + \hat{A}_1 Z_{T+y} + \hat{A}_2 Z_{T+y-1} + \epsilon_{T+y} \right\} = d$

- or

$$(\hat{A}_0 a + \hat{A}_1) Z_{T+y} + \hat{A}_2 Z_{T+y-1} + \epsilon_{T+y} = d$$

$\rightarrow Z_{T+y} = a_1 Z_{T+y-1} + b_1 \epsilon_{T+y} + d_1$

$a_1 \equiv -(\hat{A}_0 a + \hat{A}_1)^{-1} \hat{A}_2$

$b_1 \equiv -(\hat{A}_0 a + \hat{A}_1)^{-1}$

$d_1 \equiv (\hat{A}_0 a + \hat{A}_1)^{-1} d$
Backward, cnt’d

• Period $t=T+y-1$:

$$E_{T+y-1}\left\{ \hat{A}_0 Z_{T+y} + \hat{A}_1 Z_{T+y-1} + \hat{A}_2 Z_{T+y-2} + \epsilon_{T+y-1} \right\} = d.$$  

– or

$$(\hat{A}_0 a_1 + \hat{A}_1) Z_{T+y-1} + \hat{A}_0 d_1 + \hat{A}_2 Z_{T+y-2} + \epsilon_{T+y-1} = d.$$  

$$\to Z_{T+y-1} = a_2 Z_{T+y-2} + b_2 \epsilon_{T+y-1} + d_2$$

$$a_2 = -(\hat{A}_0 a_1 + \hat{A}_1)^{-1} \hat{A}_2$$

$$b_2 = -(\hat{A}_0 a_1 + \hat{A}_1)^{-1}$$

$$d_2 = (\hat{A}_0 a_1 + \hat{A}_1)^{-1} (d - \hat{A}_0 d_1)$$

– and so on…..
Backwards, cnt’d

• Solution for $t=T+1,\ldots,T+y$.

$$Z_{T+y-j} = a_{j+1} Z_{T+y-j-1} + b_{j+1} \epsilon_{T+y-j},$$

– for $j=0,1,2,\ldots,y-1$, where

$$a_{j+1} = - (\hat{A}_0 a_j + \hat{A}_1)^{-1} \hat{A}_2,$$

$$b_{j+1} = - (\hat{A}_0 a_j + \hat{A}_1)^{-1}$$

$$d_{j+1} = (\hat{A}_0 a_j + \hat{A}_1)^{-1} (d - \hat{A}_0 d_j),$$

$$a_0 \equiv a, \ b_0 \equiv b, \ d_0 = 0.$$
In Sum

- Future stochastic realization of length, $x$, with interest rate fixed at some specified value for $y < x$ periods. Three steps:

  - Backward step:
    \[ a_0, a_1, \ldots, a_y; b_0, b_1, \ldots, b_y; d_0, d_1, \ldots, d_y \]

  - Two forward steps: draw shocks, and simulate

\[
\begin{align*}
Z_{T+1} &= a_y Z_T + b_y \epsilon_{T+1} + d_y \\
Z_{T+2} &= a_{y-1} Z_{T+1} + b_{y-1} \epsilon_{T+2} + d_{y-1} \\
&\quad \vdots \\
Z_{T+y+1} &= a Z_{T+y} + b \epsilon_{T+y+1} \\
&\quad \vdots \\
Z_{T+x} &= a Z_{T+x-1} + b \epsilon_{T+x}
\end{align*}
\]
Simple New Keynesian Model

Net rate of inflation (deviated from natural inflation, which is zero)

\[ \beta E_t \pi_{t+1} + \kappa x_t - \pi_t = 0 \] (Phillips curve)

Nominal net rate of interest

\[ -[r_t - E_t \pi_{t+1} - r^*_t] + E_t x_{t+1} - x_t = 0 \] (IS equation)

\[ \alpha r_{t-1} + (1 - \alpha)\phi \pi_t + (1 - \alpha)\phi_x x_t - r_t = 0 \] (policy rule)

\( a_t \) is log technology shock, which is AR(1) in first difference with ar coefficient \( \rho \)

\[ r^*_t - \rho \Delta a_t - \frac{1}{1 + \varphi} (1 - \lambda)\tau_t = 0 \] (definition of natural rate)

\( \frac{1}{\varphi} \) is Frish labor supply elasticity

Natural rate of interest
Solving the Model

\[
s_t = \begin{pmatrix} \Delta a_t \\ \tau_t \end{pmatrix} = \begin{bmatrix} \rho & 0 \\ 0 & \lambda \end{bmatrix} \begin{pmatrix} \Delta a_{t-1} \\ \tau_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ \varepsilon_t^\tau \end{pmatrix}
\]

\[
s_t = P s_{t-1} + \epsilon_t
\]

\[
\begin{bmatrix}
\beta & 0 & 0 & 0 \\
\frac{1}{\sigma} & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{pmatrix}
\pi_{t+1} \\
x_{t+1} \\
r_{t+1} \\
r_{t+1}^*
\end{pmatrix}
+
\begin{bmatrix}
-1 & \kappa & 0 & 0 \\
0 & -1 & -\frac{1}{\sigma} & \frac{1}{\sigma} \\
(1-\alpha)\phi_{\pi} & (1-\alpha)\phi_{x} & -1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{pmatrix}
\pi_t \\
x_t \\
r_t \\
r_{t}^*
\end{pmatrix}
+
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
s_{t+1} \\
0 \\
0 \\
0
\end{pmatrix}
+
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
-\sigma \psi \rho & -\frac{1}{\sigma+\phi} (1-\lambda)
\end{pmatrix}
\]

\[
E_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0
\]
Model Solution

\[ E_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0 \]

\[ s_t - P s_{t-1} - \epsilon_t = 0. \]

• Solution:

\[ z_t = A z_{t-1} + B s_t \]

• where:

\[ \alpha_0 A^2 + \alpha_1 A + \alpha_2 I = 0, \]

\[ (\beta_0 + \alpha_0 B) P + [\beta_1 + (\alpha_0 A + \alpha_1) B] = 0 \]
Simulation

• Parameter values:

\[ \beta = 0.99, \; \phi_x = 0, \; \phi_\pi = 1.5, \; \alpha = 0, \; \rho = 0, \; \lambda = 0.5, \; \varphi = 1, \; \theta = 0.75 \] (Calvo sticky price parameter)

\[ \text{variance, innovation in preference shock} = 0.01^2, \; \text{variance, innovation in technology growth} = 0.01^2 \]

\[ \kappa = \frac{(1-\theta)(1-\beta \theta)(1+\varphi)}{\theta} = 0.1717. \]

• Experiment:

– From periods -8,-7,...,-1,0, economy is stochastically fluctuating with Taylor rule in place.

– At period 0, monetary authority commits to keeping interest rate fixed in t=1,2,3,4, at the value it took on in t=0. Afterward, return to Taylor rule

– After period 0, economy continues to be hit by shocks.
Under optimal policy, rate would have been low.

Interest rate fixed at a relatively high level.

Under Taylor rule, rate would have been low.

‘Actual’ Taylor rule followed in each period.
‘Unconditional’ follow Taylor rule
‘Conditional’ fix interest rate in t=1,2,3,4.
‘Nonstochastic’ set shocks in t>1 to zero (gives mean prediction as of t=0)
‘Stochastic’ shocks drawn from Normal, mean zero, variance indicated, in all periods.

Because we consider a high interest gap and inflation are low.
The Zero Lower Bound

• Monetary policy:

\[ Z_t = R + \rho (R_{t-1} - R) + (1 - \rho) [\alpha_\pi (\pi_t - \pi) + \alpha_y (y_t - y)] \]

\[ R_t = \begin{cases} 
Z_t & Z_t \geq 0 \\
0 & Z_t < 0 
\end{cases} \]

• Here, \( Z_t \) (the ‘shadow interest rate’) is the value to which they would ideally like to set the interest rate.

• When \( Z_t < 0 \), then the zero bound ‘binds’.
  – If \( Z_t \) is very negative then the zero bound binds a lot.
  – In this case, \( R_t \) remains zero even with fluctuations in inflation and output.
ZLB, cnt’d

• Ideal way to model ZLB
  – Sometimes binding, sometimes not.
  – Projection method ideal for this case, but difficult.

• Alternative approaches to ZLB.
  – Eggertsson and Woodford: assume we’re in the zero bound, use very simple model and slightly unrealistic assumption about how you get out.
    • Can solve what happens while you’re in, trivially.
  – With empirically realistic models: assume we’re in the zero bound and that you will leave forever at a specific date in the future.
    • That’s the case we can handle easily with the preceding approach.
Representation of Equilibrium Conditions

- Now, the vector, $z_t$, contains $Z_t$ and $R_t$ as variables.
  - One equation is the Taylor rule, determining the shadow interest rate, $Z_t$.

- There is a second equation, which is one thing when zlb is binding and another when it is not.
  - When zlb not binding, $R_t=Z_t$.
  - When zlb binds, latter equation replaced by $R_t=0$.

- Simulation
  - Difficult to incorporate stochastic shocks, because it makes exit from zero bound stochastic and this is hard to deal with outside the E-W example.
  - Will want deterministic simulation in response to sequence of deterministic $s_t$’s that push economy into binding zlb.
  - For this, must revert to initial notation for characterizing equilibrium conditions.
Model

- Equilibrium conditions after zlb ceases to bind and shocks are back to deterministic steady state values:

\[ \alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} = 0 \]

- Here,
  - \( z_t \) denotes the list of endogenous variables whose values are determined at time \( t \).

- Solution: \( A \) in

\[ z_t = Az_{t-1} \]

- where

\[ \alpha_0 A^2 + \alpha_1 A + \alpha_2 I = 0, \]
Model

• Exogenous shocks take on values, $s_1, s_2, ..., s_T$ and $s_t = 0$ for $t > T$.

• For $t=1, ..., T$, equilibrium conditions are

$$\alpha_{0,t} z_{t+1} + \alpha_{1,t} z_t + \alpha_{2,t} z_{t-1} + \beta_{0,t} s_{t+1} + \beta_{1,t} s_t = d_t$$

$$\alpha_{i,t} = \begin{cases} 
\alpha_i & Z_t \geq 0 \\
\hat{\alpha}_i & Z_t < 0 
\end{cases}, \quad i = 0, 1, 2, \quad d_t = \begin{cases} 
0 & Z_t \geq 0 \\
d & Z_t < 0 
\end{cases}$$

• Use same algorithm as before to solve ‘backwards’ for policy rule governing evolution of $z_t$’s for $t=1, 2, 3, ...$.
  – Do this based on a conjecture about when zlb is binding.

• With the sequence of policy rules in hand, simulate $z_t$’s, $t=1, 2, ...$
  – Evaluate conjecture about when zlb is binding. If conjecture verified, stop. Otherwise, change conjecture and redo backward solution and forward simulation.

• For an example, see Christiano-Eichenbaum-Rebelo JPE, 2011.