Application of Log-linearization Methods: Optimal Policy
Log-linear Methods

- Equilibrium conditions:
  \[ v(k_t, k_{t+1}, k_{t+2}) = 0, \]
  \[ t = 0, 1, 2, ... \]

- Solution:
  - compute steady state, \( k^* \) such that \( v(k^*, k^*, k^*) = 0 \).
  - expansion about steady state: \( V_0 \tilde{k}_t + V_1 \tilde{k}_{t+1} + V_2 \tilde{k}_{t+2} = 0 \).
  - solve linearized system.
Log-linear Methods ...

– what is optimal monetary policy?

– drop monetary policy rule

– now we’re short one equation!

– system underdetermined....‘many solutions’

– pick the best one.
Log-linear Methods ...

• Potential problem: time inconsistency of optimal monetary policy:
  – period $t$ announcement about period $t + 1$ policy action, $X$, influenced in part by the impact of $X$ on period $t$ decisions by the public.
  – when $t + 1$ occurs and it is time to actually implement $X$, period $t$ decisions by public are past history.
    * temptation in $t + 1$ to modify $X$ since $X$ no longer influences period $t$ decisions of public.
  – temptation to modify $X$ in $t + 1$ must be avoided, if there is to be any hope to have optimal policy. Bad outcomes could occur otherwise.
    * discipline on the part of policy makers is required, if they are to avoid temptation to deviate.

• Technical implication of potential time inconsistency.
  – $v$ equilibrium conditions seemingly not time invarient: apparently our log-linearization methods do not apply!
  – follow Kydland-Prescott ‘trick’ and put problem in Lagrangian form.
  – problem of avoiding temptation to deviate boils down to the admonition, ‘remember your multipliers!’
Example #1: Optimal Monetary Policy - Toy Example

• Setup
  – Model
    * One equation characterizing private sector behavior:

\[ \pi_t - \beta \pi_{t+1} - \gamma y_t = 0, \ t = 0, 1, 2, .... \]  \hspace{1cm} (1)

* Another equation characterizes policy.

  – Want to do optimal policy, so throw away policy equation.

  – System is now under-determined: one equation in two variables, \( \pi_t \) and \( y_t \).
Example #1: Optimal Monetary Policy - Toy Example ...

- Optimization delivers the other equations.

* Optimize objective:

$$\sum_{t=0}^{\infty} \beta^t u(\pi_t, y_t)$$

subject to (1).

* If objective corresponds to social welfare function, this is called Ramsey optimal problem.

* Objective may be preferences of policy maker.
Example #1: Optimal Monetary Policy - Toy Example

- Lagrangian representation of problem:

\[
\max_{\{\pi_t, y_t; t=0,1,\ldots\}} \sum_{t=0}^{\infty} \beta^t \left\{ u(\pi_t, y_t) + \lambda_t [\pi_t - \beta \pi_{t+1} - \gamma y_t] \right\}
\]

\[
= \max_{\{\pi_t, y_t; t=0,1,\ldots\}} \left\{ u(\pi_0, y_0) + \lambda_0 [\pi_0 - \beta \pi_1 - \gamma y_0] \right\}
+ \beta u(\pi_1, y_1) + \beta \lambda_1 [\pi_1 - \beta \pi_2 - \gamma y_1] + \ldots \}
\]

- First order necessary conditions for optimization:

\[
u_{\pi}(\pi_0, y_0) + \lambda_0 = 0 \quad (*)
\]
\[
u_{\pi}(\pi_1, y_1) + \lambda_1 - \lambda_0 = 0
\]
\[
\ldots
\]
\[
u_y(\pi_0, y_0) - \gamma \lambda_0 = 0
\]
\[
u_y(\pi_1, y_1) - \gamma \lambda_1 = 0
\]
\[
\ldots
\]
\[
\pi_0 - \beta \pi_1 - \gamma y_0 = 0
\]
\[
\pi_1 - \beta \pi_2 - \gamma y_1 = 0
\]
\[
\ldots
\]
Example #1: Optimal Monetary Policy - Toy Example ...

- These equations ‘look’ different than the ones we’ve seen before
  - They are not stationary, (*) is different from the others.
    * reflects that at time 0 there is a constraint ‘missing’
    * no need to respect what people were expecting you to do as of time \(-1\)
    * do need to respect what they expect you to do in the future, because that affects current behavior.
    * that’s the source of the ‘time inconsistency of optimal plans’.

- Can trick the problem into being stationary (see, e.g., Kydland and Prescott (JEDC, 1990s) and Levin, Onatski, Williams, and Williams, Macro Annual, 2005). Then, apply standard log-linearization solution method.
Example #1: Optimal Monetary Policy - Toy Example ...

• Consider:

\[
\begin{align*}
    v(\pi_t, \pi_{t+1}, y_t, \lambda_t, \lambda_{t-1}) &=
    \begin{bmatrix}
        u_{\pi}(\pi_t, y_t) + \lambda_t - \lambda_{t-1} \\
        u_{y}(\pi_t, y_t) - \gamma \lambda_t \\
        \pi_t - \beta \pi_{t+1} - \gamma y_t
    \end{bmatrix},
    \text{ for all } t.
\end{align*}
\]

– time \( t \) ‘endogenous variables’: \( \lambda_t, \pi_t, y_t \)

– time \( t \) ‘state variable’: \( \lambda_{t-1} \).

– ‘solution’:

\[
\lambda_t = \lambda(\lambda_{t-1}), \quad \pi_t = \pi(\lambda_{t-1}), \quad y_t = y(\lambda_{t-1}),
\]

such that

\[
v(\pi(\lambda_{t-1}), \pi(\lambda(\lambda_{t-1})), y(\lambda_{t-1}), \lambda(\lambda_{t-1}), \lambda_{t-1}) = 0, \text{ for all possible } \lambda_{t-1}.
\]
Example #1: Optimal Monetary Policy - Toy Example 

- In general, solving this problem exactly is intractable.
- But, can log-linearize!

- **Step 1**: find $\pi^*, y^*, \lambda^*$ such that following three equations are satisfied:
  \[ v(\pi^*, \pi^*, y^*, \lambda^*, \lambda^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]

- **Step 2**: log-linearly expand $v$ about steady state
  \[ v(\pi_t, \pi_{t+1}, y_t, \lambda_t, \lambda_{t-1}) \approx v_1\pi^*\hat{\pi}_t + v_2\pi^*\hat{\pi}_{t+1} + v_3y^*\hat{y}_t + v_4\Delta\hat{\lambda}_t + v_5\Delta\hat{\lambda}_{t-1}, \]
  where
  \[ \Delta\hat{\lambda}_t \equiv \lambda_t - \lambda^* \] (play it safe, don’t divide by something that could be zero!)

- **Step 3**: Posit
  \[ \Delta\hat{\lambda}_t = A_\lambda\Delta\hat{\lambda}_{t-1}, \quad \hat{\pi}_t = A_\pi\Delta\hat{\lambda}_{t-1}, \quad \hat{y}_t = A_y\Delta\hat{\lambda}_{t-1}, \]
  and find $A_\lambda, A_\pi, A_y$ that solve
  \[ [(v_1\pi^*A_\pi + v_2\pi^*A_\pi A_\lambda + v_3y^*A_y + v_4A_\lambda + v_5) \Delta\hat{\lambda}_{t-1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}] \]
  for all $\Delta\hat{\lambda}_{t-1}$. 

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Example #1: Optimal Monetary Policy - Toy Example ...

- What does the stationary solution have to do with the original non-stationary problem?

  - Do we have a solution to the period 0 problem, (*)?

    \[ u_{\pi}(\pi_0, y_0) + \lambda_0 = 0. \]

  - Yes! Just pretend that this equation really has the following form:

    \[ u_{\pi}(\pi_0, y_0) + \lambda_0 - \lambda_{-1} = 0. \]

    Expression (*) does have this form, if we set \( \lambda_{-1} = 0 \). Then,

    \[ \pi_0 = \pi(0), \ y_0 = y(0), \ \lambda_0 = \lambda(0). \]
Example #1: Optimal Monetary Policy - Toy Example ...

- The situation is exactly what it is in the neoclassical model when we want to know what happens when initial capital is away from steady state.

  – Plug $k_0$ into the stationary rule

  $$k_1 = g(k_0).$$

- Possible computational pitfall: if $\lambda_{-1} = 0$ is far from $\lambda^*$, then linearized solution might be highly inaccurate.
Example #1: Optimal Monetary Policy - Toy Example ...

- Optimal policy in real time.

- Suppose today is date zero.
  
  – Solve for $\lambda(\cdot)$, $y(\cdot)$, $\pi(\cdot)$

  – set $\lambda_{-1} = 0$

  – Compute and present in charts:

    \[
    \begin{align*}
    \lambda_0 &= \lambda(\lambda_{-1}), \quad y_0 = y(\lambda_{-1}), \quad \pi_0 = \pi(\lambda_{-1}) \\
    \lambda_1 &= \lambda(\lambda_0), \quad y_1 = y(\lambda_0), \quad \pi_1 = \pi(\lambda_0) \\
    \ldots
    \\
    \lambda_t &= \lambda(\lambda_{t-1}), \quad y_t = y(\lambda_{t-1}), \quad \pi_t = \pi(\lambda_0) \\
    \ldots
    \end{align*}
    \]
Example #1: Optimal Monetary Policy - Toy Example ...

- The optimal policy program may break down if policy makers succumb to the temptation to restart the Ramsey problem at a later date.

  – there is a temptation in period 1 when $\pi_1$ is determined, to ignore a constraint that went into determining the announcement made about $\pi_1$ in period 0:

    $$\pi_0 - \beta \pi_1 - \gamma y_0 \quad (*)$$

  – If (*) is ignored at date 1, then $\pi_1$ computed in date 1 solves a different problem than $\pi_1$ computed at date 0 and there will be time inconsistency.
Example #1: Optimal Monetary Policy - Toy Example ...

- Honoring past announcements is equivalent to ‘always respect the past multipliers’.
  - ‘Remembering $\lambda_0$’ in period 1 ensures that constraint

\[
\pi_0 - \beta \pi_1 - \gamma y_0 \quad (*)
\]

is incorporated in period 1. In this case, $\pi_1$ solves the same problem in period 1 that it did in period 0.

- Practical implication of the admonition, ‘always respect your multipliers’:
  - Charts released after later meetings will be consistent with the continuation of charts released after later meetings.
Example #1: Optimal Monetary Policy - Toy Example ...

– Example:

date 0 meeting : $y_0 = y(0), y_1 = y(\lambda(\lambda_{-1})), y_2 = y(\lambda(\lambda(\lambda_{-1})))$, ...

date 1 meeting :  

YES - $y_1 = y(\lambda(\lambda_{-1})), y_2 = y(\lambda(\lambda(\lambda_{-1})))$, ...

NO - $y_1 = y(0), y_2 = y(\lambda_1(0))$, ...

– If Central Bank selects the bad (‘NO’) option people will see the temporal inconsistency of policy, and CB will lose credibility.

– Any differences in charts from one meeting to the next must be fully explicable in terms of new information.
Example #2: Optimal Monetary Policy - More General Discussion

- The equilibrium conditions of a model

\[ E_t f (z_{t-1}, z_t, z_{t+1}, s_t, s_{t+1}) = 0, \text{ for all } z_{t-1} \text{ (endogenous), } s_t \text{ (exogenous)} \]

\[ s_t = P s_{t-1} + \varepsilon_t. \]

- Preferences:

\[ E_t \sum_{t=0}^{\infty} \beta^t U (z_t, s_t). \]

- Could include discounted utility in \( f \):

\[ v (z_{t-1}, z_t, s_t) = U (z_t, s_t) + \beta E_t v (z_t, z_{t+1}, s_{t+1}) \]
Example #2: Optimal Monetary Policy - More General Discussion ...

• Optimum problem:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left\{ U(z_t, s_t) + \lambda'_{t} \underbrace{E_t f(z_{t-1}, z_{t}, z_{t+1}, s_t, s_{t+1})}_{(N-1) \times 1} \right\}.$$

• \( N \) first order conditions:

$$\begin{align*}
\underbrace{U_1(z_t, s_t)}_{1 \times N} + \underbrace{\lambda'_{t}}_{1 \times (N-1)} \underbrace{E_t f_2(z_{t-1}, z_{t}, z_{t+1}, s_t, s_{t+1})}_{(N-1) \times N} \\
+ \beta^{-1} \underbrace{\lambda'_{t-1}}_{1 \times (N-1)} \underbrace{f_3(z_{t-2}, z_{t-1}, z_{t}, s_{t-1}, s_{t})}_{(N-1) \times N} \\
+ \beta \underbrace{\lambda'_{t+1}}_{1 \times (N-1)} \underbrace{E_t f_1(z_t, z_{t+1}, z_{t+2}, s_{t+1}, s_{t+2})}_{(N-1) \times N} = \underbrace{0}_{1 \times N}
\end{align*}$$

– Endogenous variables: \( z_t \ (N) \), \( \lambda_t \ (N - 1) \)

– Equations: Ramsey optimality conditions \( (N) \), equilibrium condition \( (N - 1) \)
Example #2: Optimal Monetary Policy - More General Discussion ...

• First order conditions of optimum problem have exactly the same form as the type of problem we solved using linearization methods.

  – must differentiate $f$ (includes private first order conditions that have already involved differentiation!)

  – good news:
    Dynare code for solving the system
Optimal Monetary Policy - CGG

\[
\max_{\nu_t, p_t^*, N_t, R_t, \pi_t, F_t, K_t} \sum_{t=0}^{\infty} E_0 \beta^t \left\{ \left( \log N_t + \log p_t^* - \exp (\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right) \right.

+ \lambda_{1t} \left[ \frac{1}{p_t^* N_t} - E_t \frac{A_t \beta}{p_{t+1}^* A_{t+1} N_{t+1} \pi_{t+1}} R_t \right]

+ \lambda_{2t} \left[ \frac{1}{p_t^*} - \left( (1 - \theta) \left( \frac{1 - \theta (\bar{\pi}_t)^{\varepsilon-1}}{1 - \theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right) \right]

+ \lambda_{3t} \left[ 1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta \theta F_{t+1} - F_t \right]

+ \lambda_{4t} \left[ (1 - \nu_t) \frac{\varepsilon}{\varepsilon - 1} \exp (\tau_t) N_t^{1+\varphi} p_t^* (1 - \psi + \psi R_t) + E_t \bar{\pi}_{t+1}^{\varepsilon} \beta \theta K_{t+1} - K_t \right]

+ \lambda_{5t} \left[ F_t \left( \frac{1 - \theta \bar{\pi}_t^{\varepsilon-1}}{1 - \theta} \right)^{\frac{1-\varepsilon}{1-\varepsilon}} - K_t \right] \left\} \right.

\cdot 'two degree of freedom' 7 variables, 5 equilibrium conditions
• Law of motion of technology:
  \[ A_t = \rho A_{t-1} + u_t. \]

• We only consider the case,
  \[ (1 - \nu) \frac{\varepsilon}{\varepsilon - 1} = 1. \]

• First consider the case, $\psi = 0$
  
  – Conjecture: restrictions 1, 3, 4, 5 nonbinding (i.e., $\lambda_{1t} = \lambda_{3t} = \lambda_{4t} = \lambda_{5t} = 0$)

    * Step 1: Optimize w.r.t. $p_t^*, \bar{\pi}_t, N_t$ ignoring restrictions 1, 3, 4, 5.

    * Step 2: Solve for $\nu_t, R_t, F_t, K_t$, to satisfy restrictions 1, 3, 4, 5.

  – If this can be done, then the conjecture is verified.
• Simplified problem under conjecture:

\[
\max_{\bar{\pi}_t, p_t^*, N_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left( \log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right) + \lambda_{2t} \left[ \frac{1}{p_t^*} - \left( 1 - \theta \right) \left( \frac{1 - \theta (\bar{\pi}_t)^{e-1}}{1 - \theta} \right) \frac{e}{e-1} + \frac{\theta \bar{\pi}_t^{e}}{p_{t-1}^*} \right] \right\}
\]
• Bottom line. Optimality under state-contingent $\nu_t$ implies:

$$p_t^* = \left[ (1 - \theta) + \theta \left( p_{t-1}^* \right)^{(\varepsilon - 1)} \right]^{\frac{1}{(\varepsilon - 1)}}$$

$$\bar{\pi}_t = \frac{p_{t-1}^*}{p_t^*}$$

$$N_t = \exp \left( -\frac{\tau_t}{1 + \varphi} \right)$$

$$1 - \nu = \frac{\varepsilon - 1}{\varepsilon}$$

$$C_t = \frac{p_t^* \varepsilon A_t N_t}{p_t^* N_t}.$$

• Ramsey-optimal policy is time consistent (no forward-looking constraints on core problem).

• If $\psi > 0$ and $\nu_t$ not state-contingent must work out Ramsey solution numerically.
• Example - no working capital channel ($\psi = 0$):

\[ \theta = 0.75, \ \varepsilon = 2, \ \beta = 0.99, \ \rho = 0.5, \ \varphi = 1. \]

• In this case:

\[
\begin{align*}
N_t &= 1 + 0.45(\lambda_{1,t-1} - \lambda_1) + 0.06(\lambda_{3,t-1} - \lambda_3) + 0.63(\lambda_{4,t-1} - \lambda_4) \\
r_t &= 0.01 - 0.50(\lambda_{1,t-1} - \lambda_1) + 0.10(\lambda_{3,t-1} - \lambda_3) - 0.02(\lambda_{4,t-1} - \lambda_4) - 0.25a_{t-1} \\
    & \quad - 0.51u_t \\
\pi_t &= 1 + 0.07(\lambda_{1,t-1} - \lambda_1) + 0.09(\lambda_{3,t-1} - \lambda_3) + 0.31(\lambda_{4,t-1} - \lambda_4) + 0.25(p^*_t - 1) \\
\lambda_{1,t} &= 0, \\
\lambda_{2,t} &= 3.88 + 0.82(\lambda_{1,t-1} - \lambda_1) + 1.46(\lambda_{3,t-1} - \lambda_3) + 3.65(\lambda_{4,t-1} - \lambda_4) \\
    & \quad + 4.13(p^*_t - 1) \\
\lambda_{3,t} &= 0.05(\lambda_{1,t-1} - \lambda_1) + 0.69(\lambda_{3,t-1} - \lambda_3) + 0.12(\lambda_{4,t-1} - \lambda_4) \\
\lambda_{4,t} &= -0.05(\lambda_{1,t-1} - \lambda_1) + 0.06(\lambda_{3,t-1} - \lambda_3) + 0.63(\lambda_{4,t-1} - \lambda_4) \\
\lambda_{5,t} &= 0.05(\lambda_{1,t-1} - \lambda_1) - 0.06(\lambda_{3,t-1} - \lambda_3) + 0.12(\lambda_{4,t-1} - \lambda_4) \\
\lambda_1 &= \lambda_3 = \lambda_4 = \lambda_5 = 0, \ \lambda_2 = 3.88
\end{align*}
\]

• ‘Resetting multipliers’ makes no difference: no time inconsistency problem.
• Example with $\psi = 0.7$:

$$ N_t = 1 + 0.50\lambda_{1,t-1} + 0.03\lambda_{3,t-1} + 0.40\lambda_{4,t-1} + 0.02a_{t-1} + 0.03u_t $$

$$ r_t = 0.01 - 0.51\lambda_{1,t-1} + 0.12\lambda_{3,t-1} + 0.30\lambda_{4,t-1} - 0.24a_{t-1} - 0.49u_t $$

$$ \pi_t = 1 + 0.05\lambda_{1,t-1} + 0.10\lambda_{3,t-1} + 0.31\lambda_{4,t-1} - 0.01a_{t-1} + 0.25(p^*_t - 1) - 0.02u_t $$

$$ p^*_t = 1 + 0.75(p^*_{t-1} - 1) $$

$$ \lambda_{1,t} = -0.01\lambda_{1,t-1} + 0.04\lambda_{3,t-1} + 0.44\lambda_{4,t-1} + 0.02A_{t-1} + 0.03u_t $$

$$ \lambda_{2,t} = 3.88 + 0.95\lambda_{1,t-1} + 1.42\lambda_{3,t-1} + 3.63\lambda_{4,t-1} + 0.09A_{t-1} + 0.18u_t + 4.13(p^*_{t-1} - 1) $$

$$ \lambda_{3,t} = 0.01\lambda_{1,t-1} + 0.70\lambda_{3,t-1} + 0.13\lambda_{4,t-1} - 0.02a_{t-1} - 0.05u_t $$

$$ \lambda_{4,t} = -0.01\lambda_{1,t-1} + 0.05\lambda_{3,t-1} + 0.62\lambda_{4,t-1} + 0.02a_{t-1} + 0.05u_t $$

$$ \lambda_{5,t} = 0.015\lambda_{1,t-1} - 0.05\lambda_{3,t-1} + 0.13\lambda_{4,t-1} - 0.02a_{t-1} - 0.05u_t $$

$$ \lambda_1 = \lambda_3 = \lambda_4 = \lambda_5 = 0, \lambda_2 = 3.88 $$

• Properties: all multipliers respond to $u_t$; optimal plan not time consistent; employment and inflation respond to $u_t$; $r_t$ drops a little less than before (it’s a tax now); $N_t$ falls somewhat because of the interest rate ‘tax’.
• **Experiment:**

  – Economy is in steady state of optimal plan up to period $t$.

  – A positive shock to technology occurs.

  – Monetary authority computes optimal policy and displays it in a set of charts.

  – Redo charts one period later.
Discussion of the results

- In the absence of a working capital channel (i.e., $\psi = 0$) it is optimal to cut the interest rate, to encourage households not smooth consumption away from what is optimal.

- In the presence of a working capital channel, (i.e., $\psi > 0$), the cut in the interest rate reduces the marginal cost of labor and expands output and employment. By reducing marginal cost, inflation drops.

- The rise in employment and fall in inflation are both costly, and so:
  
  * it is optimal when $\psi > 0$ to cut the interest rate by less.

  * it is optimal to manage expectations so that the incentive to cut prices in the present is reduced.
    - announce inflation close to zero in the next period
    - announce relatively small interest rate drop in the next period.