1. (Boldrin-Montrucchio 1986). Consider the policy rule, \( g : [0, 1] \to [0, 1] : \)
\[ g(x) = 4x(1 - x). \]

Draw this function, along with the 45 degree line, by hand in the unit box. Find an economy, \((F, \Gamma, \beta, X)\), for which the above function is the policy rule, where the economy satisfies all of our assumptions (i.e., assumptions A4.3-A4.9 in Stokey and Lucas). Here, \( x \) is the aggregate stock of capital at the beginning of the period and \( x' \) is its value at the end of the period.

Some hints: Recall, \( \Gamma, \beta \in (0, 1) \) and \( F \) satisfy
\[ g(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta v(y), \]
where
\[ v(x) = F(x, g(x)) + \beta v(g(x)). \]

Recall that the definition of a function or a correspondence must include a specification of the domain and range. Recall too that a function, say \( f(x, y) \), is strictly concave iff:
\[ f_{xx}(x, y) < 0, \quad f_{yy}(x, y) < 0, \quad f_{xx}(x, y)f_{yy}(x, y) - f_{xy}(x, y)^2 > 0, \]
for all \((x, y)\) in the domain of \( f \). Also, it is easy to verify that
\[ g(x) = \arg \max_{y \in \Gamma(x)} \Psi(x, y), \]
when \( \Psi(x, y) \) is defined as follows:
\[ \Psi(x, y) = -\frac{1}{2}y^2 + yg(x) - \frac{1}{2}Lx^2 + ax, \]
and \( L, a \) are known constants. Finally, note that \( v(x) = \Psi(x, g(x)) \), and use this to back out \( F \). You can think of your task as having to identify values of \( a \) and \( L \) that ensure assumptions A4.3-A4.9 are satisfied. How many such values are there?
2. Suppose a planner chooses to maximize, by choice of \( c_0, c_1, c_2, \ldots \), the following expression:

\[
u(c_0) + \delta[\beta u(c_1) + \beta^2 u(c_2) + \ldots], \ u(c_t) = \log(c_t) \tag{1}\]

subject to

\[c_t = k_t^\alpha - k_{t+1}, \ 0 < \alpha < 1, \ c_t, k_{t+1} \geq 0, \ k_0 \text{ given},\]

where \( 0 < \delta < \beta < 1 \). When \( \delta = 1 \), this is the problem studied in exercises 2.2 and 4.9 in SL.

(a) Let \( k_{t+1} = g_t(k_t) \) denote the policy rule that solves this problem, \( t = 0, 1, \ldots \). From the perspective of period 0, the part of the problem from \( t = 1 \) and on looks exactly like the problem with \( \delta = 1 \). As a result, you know that the optimized value of \( u(c_1) + \beta^2 u(c_2) + \ldots \) has the form, \( v(k_1) \), and you know how to compute \( v(k_1) \) because it has a simple log-linear form. Use this to show that the optimal choice of \( k_1 \) has the form:

\[k_1 = g k_0^\alpha,\]

where \( g \) is a scalar. Derive an explicit formula relating \( g \) to the parameters of the model, \( \beta, \alpha, \delta \). How does the saving rate from period \( t = 1 \) and on compare with the date 0 saving rate?

(b) Is there a unique \( k^* \) with the property \( k_t \to k^* \) as \( t \to \infty \) for all \( k_0 \)? Display a formula relating \( k^* \) to the parameters of the model.

(c) Suppose \( \beta = 1/1.03, \alpha = 1/3, \delta = 0.85 \). Suppose \( k_0 = k^* \). Display the values of \( k_0, k_1, k_2, k_3, k_4, k_5 \) that solve the problem as of date zero.

(d) Now suppose that when date 1 occurs, the planner decides to reoptimize with respect to \( k_2, k_3, \ldots \). The initial condition for this problem is \( k_1 \), the decision implemented by the planner last period. From the perspective of \( t = 1 \), the planner’s preferences over \( c_t, t \geq 1 \) are as follows:

\[
u(c_1) + \delta[\beta u(c_2) + \beta^2 u(c_3) + \ldots]\]
and the resource constraint is as before. (Note how different the problem for \( t \geq 1 \) looks from the point of view of period 1 than it does from the point of view of period 0.) What values will the planner choose for \( k_1, k_2, k_3, k_4, k_5 \)? If the planner chooses to re-optimize in this way every period, to what value will \( k_t \) actually tend?

(e) Why are the values for \( k_2, k_3, k_4, k_5 \) chosen by the planner in date 1 different from the values planned for these variables as of date 0? Because of this difference, the problem is said to be time inconsistent. If \( \delta \) had been set to one, we would not have had this problem. Why not?

Basically, the attitude of the planner is ‘I’m very impatient today (the discount rate from period 0 to period 1 is \( \beta \delta \)), but I’ll be less impatient tomorrow (the discount rate from period 1 to period 2 is \( \beta \)), so I’ll consume a lot today and save a lot tomorrow.’ Such an attitude is not time consistent because when tomorrow rolls around the planner says the same thing. In the end, the planner just ends up with a low capital stock. This type of model has been used to explain the behavior of smokers, who resolve that ‘tomorrow I’ll quit smoking, but tonight I’ll just have one or two more’. It also has been used to explain the low US saving rate. The notion is that many people say, ‘today I’ll spend, and tomorrow I’ll save’, day after day. (See the papers of David Laibson, of Harvard.) Does the solution that we have used in (d) make any sense? Would a rational person really make decisions in the time-inconsistent way described there?

3. Consider the neoclassical growth model studied in class, with \( \beta = 1/1.03, \alpha = 1/3, \delta = 0.10, \gamma = 1 \), where preferences are given by:

\[
\sum_{t=0}^{\infty} \beta^t u(c_t), \quad u(c) = \frac{c^{1-\gamma}}{1-\gamma},
\]

and the aggregate resource constraint is given by:

\[
c_t + k_{t+1} - (1 - \delta)k_t \leq k_t^\alpha.
\]

(a) What is the steady state value of \( k \)?
(b) Let $k' = g(k)$ denote the policy rule in the recursive formulation of the model. Compute $g'$, the derivative of $g$ at the steady state value of $k$.

(c) According to the first order Taylor expansion of $g$ about steady state,

$$\frac{k_t - k}{k_0 - k} = (g')^t,$$

where $k$ denotes the steady state value of $k$, $k_0$ is the initial value of the capital stock, and $k_t$ is the value of the capital stock in period $t \geq 0$. From the above expression, one can compute how much time, $t$, it takes to close, say, 95 percent of an initial gap, $k_0 - k$, between the initial capital stock and its steady state. That is, one can compute the value of $t$ required for the value of $k_t - k$ to be 5 percent of the value of $k_0 - k$. Compute how much time (i.e., the value of $t$) is required to close 95 percent of a gap between an initial value of the capital stock, $k_0$, and the steady state value.

(d) Recompute the time needed to close 95 percent of the gap when the value of $\delta$ is changed to 0.99. Then, set $\delta$ back to 0.10 and instead change the value of $\gamma$ to 4. In each case, provide the economic intuition behind the change in the time needed to close 95 percent of the gap.

(e) Consider the Solow model, which uses the resource constraint described above, but which assumes that people save and invest a fixed fraction, $s$, of gross output, $k^\alpha$:

$$k_{t+1} - (1 - \delta)k_t = sk_t^\alpha$$

What value of $s$ is required in order for the steady states of the neoclassical and Solow models to coincide? How much time does it take for 95 percent of the gap between $k_0$ and steady state capital to be closed in the Solow model? Provide economic intuition behind the different amounts of time required, in the Solow and neoclassical growth models, to close 95 percent of the gap.