1. Consider a one period version of the example of an A-D equilibrium under uncertainty discussed in class. There are two types of people, type 1 and type 2. There is an equal number of people belonging to each type and there are many people. There is a source of uncertainty: 

\[ s \in \{0, 1\}, \]

and \( \mu_i \equiv \mu (s = 0) \), \( \mu^h \equiv \mu (s = 1) \). Person 1 receives an endowment, \( e^1(s) = s \) and person 2 receives \( e^2(s) = 1 - s \). Denote type \( i \)'s level of consumption in state \( s \) by \( c^i(s) \), \( s = 0, 1 \). The utility function is

\[ u\left(c^i(s)\right) = \frac{c^i(s)^{1-\gamma} - 1}{1 - \gamma}, \quad \gamma > 0. \]

(a) Describe a constrained optimization problem for this economy, the solution to which represents the efficient allocations. Suppose that the social welfare function weighs everyone equally. Show that the solution to the problem is perfect insurance (equality of the marginal utility of consumption across states of nature) and perfect equality: \( c^i(s) = 1/2, \quad i = 1, 2, \quad s = 0, 1 \).

(b) Now, consider the Arrow-Debreu equilibrium corresponding to this economy. There are markets for consumption goods in the \( s = 0 \) and \( s = 1 \) states. Denote the state contingent price of consumption goods by

\[ P(s), \quad P(0) \equiv P_l, \quad P(1) \equiv P^h. \]

Suppose that people of type 1 are provided with a lump sum transfer of \( \delta \) and people of type 2 are provided with a lump sum transfer of \( -\delta \), where the value of \( \delta \) will be determined by an outside entity, say a government. Define the ‘wealth’ of persons 1 and 2, respectively, as follows:

\[
W^1 = \delta + P_l e^1(0) + P^h e^1(1) \\
W^2 = -\delta + P_l e^2(0) + P^h e^2(1).
\]
(For the problem to be interesting, we require that \( \delta \) not be so large that either \( W^1 \) or \( W^2 \) are negative.) Express the decision problem of each household in Lagrangian form and express \( c^i(s) \) and \( \lambda^i \) as functions of the variables that are exogenous to the household, \( i.e., P_l, P_h, W^i, \mu_l, \mu_h \), for \( s = 0, 1 \). Here, \( \lambda^i \) denotes the multiplier on the type \( i \) person’s budget constraint. Explain why \( \lambda^i \) is the \( i^{th} \) person’s marginal utility of wealth, \( W^i \). Also, derive an expression for a person’s relative consumption across states of nature, \( c^i(1)/c^i(0) \), \( i = 1, 2 \). Will households acquire perfect consumption insurance (\( i.e., \) marginal utility of consumption independent of \( s \)) independent of the value taken on by prices, \( P_l \) and \( P_h \)?

(c) Formally define an Arrow-Debreu equilibrium for this economy. Show that the relative price of consumption across states of nature, \( P_h/P_l \), is proportional to the relative probabilities of the corresponding two states of nature. Show that households receive perfect consumption insurance (in the sense defined above) in an Arrow-Debreu equilibrium, regardless of the value of \( \delta \).

(d) The solution to the efficient allocation problem involves both perfect insurance and perfect equality. Perfect insurance occurs independent of the value of \( \delta \). Display an expression for \( \delta \) as a function only of exogenous variables that ensures perfect equality. Show that when \( \mu_h > \mu_l \), then \( \delta < 0 \) and people of type 2 receive a positive wealth transfer. Provide intuition for why this is so.

2. Consider the neoclassical growth model in the second question of homework 4, in which \( \delta \neq \beta \). Consider two alternative definitions of a sequence of markets equilibrium. Definition 1 is: a sequence of markets equilibrium be a set of sequences,

\[
\begin{align*}
& r_0, w_0, c_0, y_0, k_1, n_0, \\
& r_2, w_2, c_2, y_2, k_2, n_1, \\
& \vdots
\end{align*}
\]

such that (i) for each \( t, c_t, k_{t+1}, n_t \) are the period \( t \) part of the solution to the household problem, (ii) for each \( t, y_t, n_t, k_t \) solve the representative firm problem, and (iii) \( c_t + k_{t+1} - (1 - \delta) k_t \leq k_t^{\alpha} n_t^{1-\alpha} \) for each \( t \).
Definition 2 coincides with definition 1, except that (i) is replaced by (i)': for each $t$, the variables dated $t$ and later solve the household problem.

(a) Explain why an equilibrium exists under definition 1, but not under definition 2.

(b) Explain why the problem for definition 2 disappears with $\delta = \beta$. Why do the difficulties present in the definition 2 equilibrium when $\delta \neq \beta$ not exist in the Arrow-Debreu equilibrium concept?

3. Introducing monopoly power into the neoclassical growth model. In the neoclassical growth model, there is only one good, so monopoly power would entail assuming that there is only one firm. To avoid this silly implication, we introduce a Dixit-Stiglitz technology which allows for the presence of monopoly power, without the implication that there is only one firm. This question is basically a review of simple price theory.

In this discussion, we first discuss the firm sector. Because the date $t$ part of the firm problem is independent of what happens before or after, we will simply consider the problem at one date. It is convenient from the point of notation, to temporarily drop the time subscript, $t$.

In discussing the firm sector, we take the aggregate supply of capital, $k$, and labor, $n$, as given. These cannot be determined until the households have been brought into the picture.

At first, the firm sector may seem algebra-intensive. However, at the very end the whole setup collapses into three simple equations, equations that are very similar to three analogous equations that characterize the firm sector in our decentralization of the neoclassical growth model.

Final goods, $y$, are produced by a representative, competitive firm, using a continuum of intermediate goods, $x(i), i \in (0, 1)$:

$$ y = \left[ \int_0^1 x(i)^{\lambda} di \right]^\frac{1}{\lambda}, \quad 0 < \lambda < 1. \quad (1) $$

Note that in the limiting case where $\lambda \to 1$, the intermediate goods are perfect substitutes in the production of $y$. The price of the $i^{th}$
intermediate good is denoted by \( p(i) \), which the final good producer takes as given. The profits of the final good producer are:

\[
\pi = y - \int_0^1 p(i)x(i)di,
\]

where the price of the final good has been normalized at unity. The problem of the final good producer is to choose \( x(i) \), \( i \in (0,1) \) to maximize profits subject to the budget constraint. The first order necessary condition for profit maximization is:

\[
p(i) = y^{1-\lambda}x(i)^{\lambda-1}, \quad i \in (0,1).
\]

Alternatively, this can be written in the form of a demand curve:

\[
x(i) = yp(i)^{1/(\lambda-1)}.
\]

so that the elasticity of demand of \( x(i) \) with respect to price, \( p(i) \), is \( 1/(1-\lambda) \). Not surprisingly, this elasticity converges to infinity as \( \lambda \to 1 \).

The intermediate good, \( x(i) \), is produced by a single firm which, therefore, is a monopolist. The monopolist sets its price, \( p(i) \), and output, \( x(i) \), treating the first order condition of the final good producer as its demand curve, \( p(i) = P(x(i), y) \), where

\[
P(x(i), y) = y^{1-\lambda}x(i)^{\lambda-1}.
\]

The monopolist knows the value of \( y \) and correctly understands that its choice of \( x(i) \) has no impact on \( y \). The monopolist uses capital and labor to produce \( x(i) \) using the following technology:

\[
x(i) = k(i)^\alpha n(i)^{1-\alpha} = f(k(i), n(i)), \quad 0 < \alpha < 1,
\]

where \( k(i) \) and \( n(i) \) are the capital and labor hired by the monopolist in the capital rental and labor markets, respectively. The monopolist is small in the factor markets, and so it takes the rental rate on capital, \( r \), and the wage rate, \( w \), as given. The monopolist’s problem is to choose \( x(i) \) to maximize profits, subject to (2), the technology, (3), and the
given \( r, w \). It is convenient to derive an expression for the monopolist’s cost function, \( C(x(i), w, r) : \)

\[
C(x(i), w, r) = \min_{k(i), n(i)} rk(i) + wn(i),
\]

subject to (3). In Lagrangian form,

\[
C(x(i), w, r) = \min_{k(i), n(i)} rk(i) + wn(i) + \mu [x(i) - f(k(i), n(i))],
\]

where \( \mu \geq 0 \) is the multiplier. The first order conditions for this problem are:

\[
\begin{align*}
    r &= \mu f_k \\
    w &= \mu f_n \\
    x &= f.
\end{align*}
\]

Here, \( \mu, k \) and \( n \) are to be determined as functions of \( x, r, w \). To obtain an expression for \( \mu \), solve (4) for \( k/n \):

\[
k/n = f_k^{-1} \left( \frac{r}{\mu} \right).
\]

Substitute this into (5) and obtain:

\[
w = \mu f_n \left[ f_k^{-1} \left( \frac{r}{\mu} \right) \right].
\]

Solving this for \( \mu \) and taking into account our functional form assumption, we find:

\[
\mu = \mu(r, w) = \left( \frac{1}{1 - \alpha} \right)^{1-\alpha} \left( \frac{1}{\alpha} \right)^\alpha (r)^\alpha (w)^{1-\alpha}.
\]

To interpret \( \mu \), note that it is the derivative of \( C \) with respect to \( x(i) \) in the Lagrangian expression of the firm minimization problem. Thus, \( \mu(r, w) \) is the marginal cost of producing \( x(i) \). Another intuitive way to see this interpretation of \( \mu \) is to consider (4), which can be rewritten, \( \mu = r/f_k \). Note that \( r \) is the change in cost with respect to a unit change in capital, while \( f_k \) is a change in output with respect to a unit
change in capital, so that the ratio is the change in cost with respect to a change in output. That is,

\[ \mu = \frac{r}{f_k} = \frac{d\text{Cost}}{dk} = \frac{d\text{Cost}}{dx} \times \frac{d\text{Cost}}{dx} = \frac{d\text{Cost}}{dx}. \]

This establishes that the marginal cost of producing \( x(i) \) is independent of the value of \( x(i) \). (The linear homogeneity of the production function is what guarantees this.) From this observation, together with the fact that \( C(0, r, w) = 0 \), we conclude that the cost function of the \( i^{th} \) intermediate good firm has the following representation:

\[ C(x(i), r, w) = \mu(r, w) \times x(i). \]

Taking into account that the intermediate good firm must respect its demand curve, the profit maximization problem is:

\[ \pi(i) = \max_{x(i)} P(x(i), y)x(i) - \mu(r, w)x(i). \]

The first order condition equates marginal revenue, \( P'(x(i), y)x(i) + P(x(i), y) \), to marginal cost, \( \mu(r, w) \). Taking into account our functional form assumptions,

\[ \lambda \left( \frac{y}{x(i)} \right)^{1-\lambda} = \mu(r, w). \]

Note that the value of \( x(i) \) that solves this is independent of \( i \). Similarly, the choice of \( p(i) \) is independent of \( i \). These observations are not surprising in view of the symmetry of the intermediate good firm problems. Denote the optimal values of \( x(i) \) and \( p(i) \) by \( x \) and \( p \), respectively. From the fact that (1) must be satisfied in equilibrium, we have that in equilibrium, \( x = y \), so that, using the previous expression and the demand curve:

\[ \lambda = \mu(r, w) \]

\[ p = 1. \]

That is, in equilibrium the marginal cost is equated to \( \lambda \), the parameter in the final good production function. In addition, the price of the
intermediate input is unity. It is interesting to see the value of profits in equilibrium:

\[ \pi = P(x, y)x - \mu(r, w)x = x[1 - \lambda]. \]

Note that when \( \lambda = 1 \), then profits are zero. This reflects that, in this case, the intermediate inputs are perfect substitutes so that the ‘monopolist’ does not have any exploitable monopoly power. In this case, marginal revenue is just the price of the intermediate good and \( P' = 0 \). For lower values of \( \lambda \), each intermediate good firm becomes less substitutable with the others, and the monopolist has more monopoly power. As a result, the elasticity of demand falls and profits per unit of sales, \( \pi/x \), rises. A crucial thing to note is that, according to (4)-(5), the monopolist pays factors of production less than their marginal products:

\[ r = \lambda f_k < f_k \quad \text{(6)} \]
\[ w = \lambda f_n < f_n. \]

The equations that summarize the firm sector are these, together with (1) which reduces to

\[ y = k^\alpha n^{1-\alpha}, \]

given that \( y = x(i) \) for all \( i \). These equations are conditional on the values of \( k \) and \( n \), which we have taken as given in our discussion of the firm sector. Notice that these three equations look exactly like the equations corresponding to the firm sector in our competitive decentralization of the neoclassical model, which the exception of the appearance of \( \lambda \) in the firm first order conditions.

To determine \( k \), and \( n \), we have to bring in the households. Their problem is to solve, at each date, \( t \):

\[ \max \sum_{j=0}^{\infty} \beta^j u(c_{t+j}), \]

subject to

\[ c_{t+j} + k_{t+1+j} - (1 - \delta)k_{t+j} \leq r_{t+j}k_{t+j} + w_{t+j}n_{t+j} + \pi_{t+j} + \int_0^1 \pi_{t+j}(i)di, \]
and $n \leq 1$. Note that households are treated as receiving profits from all firms, final and intermediate. The household optimally chooses $n = 1$ and its first order condition for capital is:

$$u'(c_{t+j}) = \beta u'(c_{t+1+j}) [r_{t+1+j} + 1 - \delta], \ j = 0, 1, 2, \ldots .$$

Substituting out for the equilibrium value of the rental rate of capital,

$$u'(c_{t+j}) = \beta u'(c_{t+1+j}) [\lambda f_{k,t+j+1} + 1 - \delta], \ j = 0, 1, 2, \ldots .$$

Notice that this first order condition resembles the first order condition of the efficient allocations in the neoclassical model, with a crucial exception. Instead of $f_k$ appearing here, it is something less. A consequence of this is that when households make their decisions about saving, they treat the payoff from capital as $r + 1 - \delta$, which is less than the actual payoff, $f_k + 1 - \delta$. Because, in effect, the equilibrium offers households insufficient incentive to save, it will produce a socially inefficiently low level of capital.

A sequence of markets equilibrium for this economy is a sequence of prices, $\{p_t(i), r_t, w_t\}_{t=0}^\infty$, and quantities, $\{n_t(i), k_t(i), n_t, k_{t+1}, c_t\}_{t=0}^\infty$, such that the household and firm problems are satisfied given the prices and the quantities. In addition, we require market clearing, $\int n_t(i) di = n_t$, $\int k_t(i) di = k_t$, for each $t$.

(a) Use the linear homogeneity of the production function, and expressions for profits to show that, in equilibrium,

$$r_t k_t + w_t n_t + \pi_t + \int_0^1 \pi_t(i) di = f(k_t, n_t),$$

so that the resource constraint is satisfied.

(b) Show that, in a steady state,

$$k = \left[\frac{\lambda \alpha}{\frac{1}{\beta} - (1 - \delta)}\right]^{\frac{1}{1-n}}.$$
(c) Show that the efficient allocations of the economy with monopolists solve:
\[
\max_{k_{t+1}, n_t} \sum_{t=0}^{\infty} \beta^t u(c_t),
\]
subject to
\[
c_t + k_{t+1} - (1 - \delta) k_t \leq f(k_t, n_t).
\]
(Hint: note that I dropped the whole thing about (1). In effect, you need to show that this not a binding restriction on the problem of determining the efficient allocations.) Thus, the efficient allocations in the monopoly economy coincide with those of the neoclassical economy. However, because of the presence of monopoly power, the equilibrium allocations are not efficient. This is consistent with the result for the steady state capital stock in (b) above, which establishes that when \( \lambda < 1 \), the equilibrium steady state capital stock in the economy is less than the efficient level of capital, which we derived in class.

(d) Modify the household budget constraint in the following way:
\[
c_{t+j} + k_{t+1+j} - (1 - \delta) k_{t+j} \leq (1 + \tau_{t+j}) r_{t+j} k_{t+j} + w_{t+j} n_{t+j} + \pi_{t+j} + \int_0^1 \pi_{t+j}(i) di - T_{t+j}
\]
Here, \( \tau_{t+j} \) is a tax subsidy on the household’s capital income. In addition, \( T_{t+j} \) is a lump sum transfer from the household to the government. By ‘lump sum’ I mean that the magnitude of the transfer is independent of any choice by the household. We require that the government balance its books in each period:
\[
\tau_t r_t k_t = T_t,
\]
where \( k_t \) is the economy-wide average stock of capital, and not the quantity of capital chosen by any particular household (otherwise, we could not maintain our assumption that from the perspective of the individual household, \( T_t \) is lump sum). Show that there is a value of \( \tau_t \), \( t = 0, 1, 2, \ldots \) such that the allocations in a sequence of market equilibrium are efficient. Note that this value is positive. This reflects that, to steer households towards the efficient level of investment, they need an additional incentive beyond what the
market gives them. The market systematically gives them too little incentive because of the presence of monopoly power. In thinking about whether a tax policy can be found which selects the efficient allocations, be sure to also think about the allocations in date 0.