1. Consider a two-sector economy with the following preferences:

\[ \sum_{t=0}^{\infty} \beta^t u(c_t, l_t), \]

where

\[ u(c, l) = \begin{cases} \frac{(c(1-l)^\gamma)^{1-\gamma}}{1-\gamma}, & \gamma \neq 1, \gamma > 0 \\ \log c + \eta \log (1 - l), & \gamma = 1 \end{cases} \]

Consumption goods are produced using the following technology:

\[ c_t \leq A k_{ct,t}^{\alpha (1-\alpha)} \]

and investment goods are produced using the following technology:

\[ I_t = b k_{I,t}, \]

where

\[ k_{t+1} = (1 - \delta) k_t + I_t \]
\[ k_t = k_{ct,t} + k_{I,t}, k_{ct,t}, k_{I,t} \geq 0, \]

where \( \delta \in (0, 1) \) and \( b > \delta \). Also, the initial stock of capital, \( k_0 > 0 \), is given. The following condition will later be useful to guarantee boundedness of utility:

\[ \beta (1 - \delta + b)^{(1-\gamma)} < 1. \]

(a) Show that the efficient allocations solve the following problem:

\[ V(k_0) = \max_{k_{t+1}, l_t \in \Gamma(k_t)} \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}, l_t). \]

Display the function, \( F \), and the correspondence, \( \Gamma \).
(b) Show that $V$ has the following form:

$$V(k_0) = k_0^{\alpha(1-\gamma)}w,$$

where $w$ is finite. Establish the finiteness of $w$ for $\gamma > 1$ and $\gamma < 1$.

(c) Show that $w$ is the fixed point of a particular functional equation. Does the functional equation satisfy Blackwell’s conditions to be a contraction? Display formulas which can be used to solve for $w$ as well as the optimal level of employment and the optimal growth rate of capital.

(d) Compute the price of capital, $P_{k,t}$, in this model. Show that along a growth path, $P_{k,t} \rightarrow 0$. Show that the marginal product of capital also converges to zero. Show that the rate of return on capital is constant.

(e) Does the economy satisfy the convergence property?

2. Consider the endogenous growth model with human capital discussed in class. One sector produces a homogeneous output good, which is transformed one-for-one into consumption and investment. The homogeneous output good is itself produced using a Cobb-Douglas production function:

$$c_t + k_{t+1} - (1 - \delta) k_t = k_t^\alpha n_t^{1-\alpha}.$$

Another sector produces human capital according to the following accumulation equation:

$$h_{t+1} = h_t + \lambda(h_t - n_t),$$

where $\lambda > 0$, $c_t \geq 0$, $k_{t+1} \geq (1 - \delta) k_t$, $0 \leq n_t \leq h_t$, and $h_0$, $k_0$ are given. Preferences are:

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma},$$

$\gamma > 0$. To ensure boundedness, we require $\beta(1 + \lambda)^{1-\gamma} < 1$. In class, the problem was reformulated in recursive form. It was shown that there are policy rules of the form, $x_{t+1} = f(x_t)$, $y_t = g(x_t)$, where $x_t = k_t/h_t$ and $y_t = h_{t+1}/h_t$. 

(a) Set $\alpha = 1/3$, $\delta = 0.10$, $\beta = 0.97$, $\lambda = 0.04$, $\gamma = 1.1$. Compute steady state values of $x$, $y$. How do these values change with $\alpha$ and with $\lambda$? Provide intuition.

(b) Develop an expression for the one-period rate of return physical capital that is the analog of the expression for the rate of return on human capital developed in class. Express this rate of return in terms of $x$ and $y$.

(c) Set up the problem of choosing the efficient allocations in Lagrangian form. Must the rate of return on physical and human capital be the same at all dates? Prove your answer.

(d) Write down a set of functional equations that the equilibrium policy functions, $x_{t+1} = f(x_t)$ and $y_t = g(x_t)$, and the value function must satisfy.

   i. Prove that the value function is a fixed point of a particular operator and that that operator satisfies Blackwell’s sufficient conditions to be a contraction.

   ii. Use the perturbation method to develop first-order Taylor series approximations around steady state for $f$ and $g$. Calculate and report the the first order Taylor series approximation.

(e) For initial conditions that are close to steady state, does the physical to human capital ratio converge monotonically to steady state? Explain. What is the rate of return on human and physical capital for $x$ above steady state and for $x$ below steady state? If you conclude that the system always converges to steady state, then report how long does it take to close 90% of the gap to steady state. Explain the formula you use for this calculation.

(f) Describe a competitive decentralization for this economy and show that the equilibrium allocations coincide with the efficient allocations.

3. Consider the OLG model presented in class, in which sustained growth in the aggregate capital stock cannot occur in equilibrium. Suppose that $b$ is large so that there is a strong incentive to save. Display a set of equations that allow you to compute a steady state equilibrium (i.e., one in which all old consume the same amount, all young compute
the same amount, the capital stock is constant and so are the rental rate and wage rate) with positive growth in the consumption of each household.