Simple New Keynesian Model: Foundations

Lawrence J. Christiano
Outline

• Preferences and Technology.
  – First best, efficient allocations (benchmark for later analysis).

• Flexible Price Equilibrium
  – Efficient, as long as a monopoly distortion is eliminated.
    • Complete monetary neutrality.

• Equilibrium in sticky price version of the model (New Keynesian model).
  – Private sector equilibrium conditions.
  – Ramsey Equilibrium.
  – Linearization (first order perturbation)
  – Taylor rule
  – Properties of Taylor rule equilibrium
Preferences and Technology

- Household utility:
  \[ E_0 \sum_{t=0}^{\infty} \beta^t \left( \log C_t - \exp \left( \tau_t \frac{N_t^{1+\varphi}}{1 + \varphi} \right) \right), \quad \tau_t = \lambda \tau_{t-1} + \epsilon_t^\tau \]

- Goods production
  - Final goods:
    \[ Y_t = \int_0^1 Y_{i,t}^{\frac{\varepsilon - 1}{\varepsilon}} \, dj \]
  - Intermediate goods:
    \[ Y_{i,t} = \exp (a_t) N_{i,t}, \quad a_t = \rho a_{t-1} + \epsilon_t^a \]

- Labor:
  \[ N_t = \int_0^1 N_{it} \, di. \]
First Best Allocations

- Must solve two problems:
  - For any given $N_t$, how should employment be allocated across sectors?
  - What should the level of total employment, $N_t$, be?
Efficient Sectoral Allocation of Employment

- Consider a given level of $N_t$.
- Can show: efficiency dictates allocating employment equally across sectors:
  \[ N_{it} = N_t, \ i \in [0, 1]. \]
- Consider, for example, the class of deviations from equal:
  \[ N_{it} = \begin{cases} 
    2\alpha N_t & i \in [0, \frac{1}{2}] \\
    2 (1 - \alpha) N_t & i \in [\frac{1}{2}, 1] 
  \end{cases}, \ 0 \leq \alpha \leq 1. \]
  This allocation is consistent with $N_t$:
  \[
  \int_0^1 N_{it} di = \frac{1}{2} 2\alpha N_t + \frac{1}{2} 2 (1 - \alpha) N_t = N_t
  \]
\[ Y_t = \left[ \int_0^1 Y_{i,t}^\frac{\varepsilon-1}{\varepsilon} \, di \right]^\frac{\varepsilon}{\varepsilon-1} \]

\[ = \left[ \int_0^{\frac{1}{2}} Y_{i,t}^\frac{\varepsilon-1}{\varepsilon} \, di + \int_{\frac{1}{2}}^1 Y_{i,t}^\frac{\varepsilon-1}{\varepsilon} \, di \right]^\frac{\varepsilon}{\varepsilon-1} \]

\[ = e^{at} \left[ \int_0^{\frac{1}{2}} N_{i,t}^\frac{\varepsilon-1}{\varepsilon} \, di + \int_{\frac{1}{2}}^1 N_{i,t}^\frac{\varepsilon-1}{\varepsilon} \, di \right]^\frac{\varepsilon}{\varepsilon-1} \]

\[ = e^{at} \left[ \int_0^{\frac{1}{2}} \left(2\alpha N_t\right)^\frac{\varepsilon-1}{\varepsilon} \, di + \int_{\frac{1}{2}}^1 \left(2 \left(1 - \alpha\right) N_t\right)^\frac{\varepsilon-1}{\varepsilon} \, di \right]^\frac{\varepsilon}{\varepsilon-1} \]

\[ = e^{at} N_t \left[ \int_0^{\frac{1}{2}} \left(2\alpha\right)^\frac{\varepsilon-1}{\varepsilon} \, di + \int_{\frac{1}{2}}^1 \left(2 \left(1 - \alpha\right)\right)^\frac{\varepsilon-1}{\varepsilon} \, di \right]^\frac{\varepsilon}{\varepsilon-1} \]

\[ = e^{at} N_t \left[ \frac{1}{2} \left(2\alpha\right)^\frac{\varepsilon-1}{\varepsilon} + \frac{1}{2} \left(2 \left(1 - \alpha\right)\right)^\frac{\varepsilon-1}{\varepsilon} \right]^\frac{\varepsilon}{\varepsilon-1} = e^{at} N_t f(\alpha) \]
Efficient Resource Allocation Means Equal Labor Across All Sectors

\[ f(\alpha) = \left[ \frac{1}{2} (2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} + \frac{1}{2} (2(1 - \alpha))^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}} \]
Economy with Efficient $N$ Allocation

- Efficiency dictates
  \[ N_{it} = N_t \text{ all } i \]

- So, with efficient production:
  \[ Y_t = e^{a_t} N_t \]

- Resource constraint:
  \[ C_t \leq Y_t \]

- Preferences:
  \[
  E_0 \sum_{t=0}^{\infty} \left( \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right), \quad \tau_t = \lambda \tau_{t-1} + \varepsilon_t^\tau, \varepsilon_t^\tau \sim \text{iid}, \]

Efficient Determination of Labor

• Lagrangian:

\[
\max_{C_t, N_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right\} 
\]

\[
\{ u(C_t, N_t, \tau_t) + \lambda_t [e^{a_t} N_t - C_t] \}
\]

• First order conditions:

\[
u_c(C_t, N_t, \tau_t) = \lambda_t, \quad u_n(C_t, N_t, \tau_t) + \lambda_t e^{a_t} = 0
\]

• or:

\[
u_{n,t} + u_{c,t} e^{a_t} = 0
\]

marginal cost of labor in consumption units
\[
\frac{-du}{dN_t} = \frac{du}{dC_t} = \frac{dC_t}{dN_t}
\]

\[
\frac{-u_{n,t}}{u_{c,t}} = e^{a_t}
\]

marginal product of labor
Efficient Determination of Labor, cont’d

• Solving the fonc’s:

\[
\frac{-u_{n,t}}{u_{c,t}} = e^{a_t}
\]

\[
C_t \exp(\tau_t) N_t^\phi = e^{a_t}
\]

\[
e^{a_t} N_t \exp(\tau_t) N_t^\phi = e^{a_t}
\]

\[
\rightarrow N_t = \exp\left(\frac{-\tau_t}{1 + \phi}\right)
\]

\[
\rightarrow C_t = \exp\left(a_t - \frac{\tau_t}{1 + \phi}\right)
\]

• Note:

  – Labor responds to preference shock, not to tech shock
Response to a Jump in $a$

Budget constraint

Indifference curve

Higher $a$

Leisure, 1-N
Key Features of First Best

- Employment does not respond to $a_t$.
  - This result is only *somewhat* sensitive to the absence of capital.
  - In RBC models response of $N_t$ to $a_t$ positive, but small.
    - One reason for the small response is that income effects mitigate substitution effects.

- First best consumption not a function of intertemporal considerations.
  - Consumption is a function only of current realization of shocks.
    - Time series representations of shocks irrelevant.
    - For example, the impact of $\epsilon_t^a$ and $\epsilon_t^c$ on $E_t a_{t+j}$ and $E_t \tau_{t+j}$, $j > 0$, respectively have no effect on first best consumption.
  - The parameter, $\beta$, does not enter first-best $N_t$ and $C_t$. 
Markets

• First, consider flexible prices.
  – Equilibrium efficient as long as government subsidy neutralizes monopoly power distortion.
  – Monetary policy relevant to real allocations.

• Second, introduce price-setting frictions in goods market.
  – Labor market has flexible wages.
  – We’ll see that equilibrium could be first best, or could be really bad. It all depends on the quality of monetary policy.
Households

- Problem:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left( \log C_t - \exp(\tau_t) \frac{N_t^{1+\phi}}{1+\phi} \right), \quad \tau_t = \lambda \tau_{t-1} + \varepsilon_t$$

s.t. $$P_tC_t + B_{t+1} \leq W_tN_t + R_{t-1}B_t + \text{Profits net of taxes}$$

- First order conditions ($\bar{\pi}_t \equiv P_t/P_{t-1}$):

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}}$$

$$\exp(\tau_t) C_t N_t^\phi = \frac{W_t}{P_t}.$$
Final Goods

• Final good firms:
  
  – maximize profits:

\[ P_t Y_t - \int_0^1 P_{i,t} Y_{i,t} dj, \]

subject to:

\[ Y_t = \left[ \int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{\frac{\varepsilon}{\varepsilon-1}}. \]

– Foncs:

\[ Y_{i,t} = Y_t \left( \frac{P_t}{P_{i,t}} \right)^\varepsilon \rightarrow P_t = \left( \int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}}. \]
Intermediate Good Producers

- Demand curve for monopoly producer of $Y_{i,t}$:

$$Y_{i,t} = Y_t \left( \frac{P_t}{P_{i,t}} \right)^{\varepsilon}.$$  

- Production function:

$$Y_{i,t} = \exp(a_t) N_{i,t}, \quad \Delta a_t = \rho \Delta a_{t-1} + \varepsilon_t^a$$

- Competitive in labor market:
  - Wage rate, $W_t$, taken as given.
  - (Real) marginal cost of production:

$$s_t = \frac{(1 - \nu) W_t / P_t}{\varepsilon^a_t},$$

where $\nu$ is a government subsidy to firms.

- Optimizing monopolist sets price as a markup, $\varepsilon / (\varepsilon - 1)$, over marginal cost:

$$P_{i,t} = \frac{\varepsilon}{\varepsilon - 1} P_t s_t.$$
Features of Flexible Price Equilibrium

- Prices are all equal:
  \[ P_{i,t} = P_t, \text{ for all } t, \]
  so sectoral allocation of employment efficient and:
  \[
  1 = \frac{\varepsilon}{\varepsilon - 1} s_t = \frac{\varepsilon}{\varepsilon - 1} \frac{(1 - \nu) W_t}{P_t} e^{a_t}.
  \]

- Aggregate employment, \( N_t \), might not be efficient:
  \[
  \underbrace{-u_{N,t}}_{u_{c,t}} \quad \text{by household optimization} \quad \Rightarrow \quad \frac{W_t}{P_t} = e^{a_t} \frac{1}{\frac{\varepsilon}{\varepsilon - 1} (1 - \nu)} (*)
  \]

- If \( \nu = 0 \):
  - Actual marginal cost < actual marginal product of labor.
  - Households get ‘wrong’ signal from the market about their actual productivity, \( e^{a_t} \).
Profits, Government Budget Constraint and Aggregate Resources (Walras' Law)

• Government budget constraint and profits:

\[
\text{taxes} = \nu W_t \int_i N_{i,t} \, di \\
\text{profits} = \int_i [P_{i,t} Y_{i,t} - (1 - \nu) W_t N_{i,t}] \, di
\]

• Household budget constraint at equality:

\[
P_t C_t = W_t N_t + \int_i [P_{i,t} Y_{i,t} - (1 - \nu) W_t N_{i,t}] \, di + \nu W_t \int_i N_{i,t} \, di
\]

because \( N_{i,t} = N_t, Y_{i,t} = Y_t \)

\[
\Rightarrow W_t N_t + P_t Y_t - (1 - \nu) W_t N_t + \nu W_t N_t = P_t Y_t \\
\rightarrow C_t = Y_t.
\]

• Using efficiency of sectoral resource allocation: \( C_t = e^{at} N_t \). (***)
Equilibrium

• Sequences, $\{P_t, P_{i,t}, W_t/P_t, R_t, C_t, N_t\}_{t=0}^{\infty}$, such that
  - Households’ problems are solved
  - Firms’ problems are solved
  - Markets clear.

• By equations (*), (**)

\[
C_t = e^{a_t} N_t,
\]

\[
\exp(\tau_t) C_t N_t^\varphi = e^{a_t} \frac{1}{\frac{\varepsilon}{\varepsilon-1} (1 - \nu)}.
\]

• Then,

\[
C_t = e^{a_t} N_t
\]

\[
N_t = \left[ \frac{\exp(-\tau_t)}{\frac{\varepsilon}{\varepsilon-1} (1 - \nu)} \right]^{\frac{1}{1+\varphi}}.
\]
Properties of Equilibrium

- Efficient if, and only if, monopoly power extinguished so that real wage correctly signals marginal product of work:

\[ \frac{\varepsilon}{\varepsilon - 1} (1 - \nu) = 1, \]

so that equilibrium employment and consumption are first best:

\[ N_t = \exp \left( -\frac{\tau_t}{1 + \varphi} \right), \quad C_t = \exp \left( a_t - \frac{\tau_t}{1 + \varphi} \right) \]

\[ \exp (a_t) = \frac{W_t}{P_t} = \exp (\tau_t) C_t N_t^{\varphi}. \]

- Evidently, \( C_t, N_t, W_t/P_t \) are uniquely determined.
  - The objects, \( R_t, \pi_{t+1}, P_t \) are not uniquely determined.
  - Presumably, monetary policy can determine these variables. But, note that monetary policy is ineffective and uninteresting in the flexible price case, since consumption and employment are completely determined without regard to monetary policy.
Real Interest Rate in Equilibrium

- The real interest rate, $R_t / \bar{\pi}_{t+1}$.
  - Absent uncertainty, $R_t / \bar{\pi}_{t+1}$ determined uniquely:
    \[
    \frac{1}{C_t} = \beta \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}}.
    \]
  - With uncertainty, household intertemporal condition simply places a single linear restriction across all the period $t + 1$ values for $R_t / \bar{\pi}_{t+1}$ that are possible given period $t$.

- The real interest rate, $\tilde{r}_t$, on a risk free one-period bond that pays in $t + 1$ is uniquely determined:
  \[
  \frac{1}{C_t} = \tilde{r}_t \beta E_t \frac{1}{C_{t+1}}.
  \]

- By no-arbitrage, the following weighted average of $R_t / \bar{\pi}_{t+1}$ across period $t + 1$ states of nature is uniquely determined:
  \[
  \tilde{r}_t = \frac{E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}}}{E_t \frac{1}{C_{t+1}}}. 
  \]
• The object, $\tilde{r}_t$, plays no role in the computation of the equilibrium allocations.

• But, $\tilde{r}_t$ is important from an economic perspective since it plays a key role in the consumption decisions of households in the model.
Real Interest Rate, Continued

- To understand the economics of the model, it is useful to examine response of $\tilde{r}_t$ to shocks:

$$
\tilde{r}_t = \frac{1}{\beta} \frac{1}{E_t \frac{C_t}{C_{t+1}}} \rightarrow \tilde{r}_t = \frac{1}{\beta} E_t \exp \left[ a_t - a_{t+1} + \frac{\tau_{t+1} - \tau_t}{1+\varphi} \right]
$$

$$
= \frac{1}{\beta} \exp \left[ E_t (a_t - a_{t+1}) + E_t \left( \frac{\tau_{t+1} - \tau_t}{1+\varphi} \right) + V \right],
$$

where $c_t \equiv \log C_t$ and assuming $\varepsilon_{t+1}^a$ and $\varepsilon_{t+1}^\tau$ are Normally distributed ($V$ is very small).

- Then,

$$
r_t^* \equiv \log \tilde{r}_t = -\log (\beta) + E_t \left[ a_{t+1} - a_t - \left( \frac{\tau_{t+1} - \tau_t}{1+\varphi} \right) - V \right]
$$
Real Interest Rate, Continued

\[ r_t^* = -\log(\beta) + E_t \left[ a_{t+1} - a_t - \left( \frac{\tau_{t+1} - \tau_t}{1 + \varphi} \right) - V \right] \]

- The response of \( r_t^* \) to \( \varepsilon_t^a \) or \( \varepsilon_t^\tau \) depends how the forecast of future \( a_t \) and \( \tau_t \) responds.
- Get opposite response of \( r_t^* \) to \( \varepsilon_t^a \) in trend stationary case:
  \[ a_t = \rho a_{t-1} + \varepsilon_t^a. \]
- Suppose technology jumps 1 percent (i.e., \( \varepsilon_t^a, a_t \) rise by 0.01).
  - In difference stationary case (with \( \rho > 0 \)), aggregate demand (i.e., \( C_t \)), absent a change in \( r_t^* \), responds by more than 1 percent, for consumption smoothing reasons. A jump in \( r_t^* \) ensures that aggregate demand only rises by the efficient amount, 1 percent.
  - In the trend stationary case, aggregate demand rises by less than 1 percent, absent a change in \( r_t^* \). A fall in \( r_t^* \) helps to boost aggregate demand so that it rises by the efficient amount, 1 percent.
Dynamic Properties of the Model

Response to .01 Technology Shock in Period 1

\[(a_t - a_{t-1}) = 0.75(a_{t-1} - a_{t-2}) + \varepsilon_{a_t}\]

Response to .01 Preference Shock in Period 1

\[\tau_t = 0.5\tau_{t-1} + \varepsilon_{\tau_t}\]

\[\log r_t = -\log \beta + 0.75\Delta a_t\]

\[\log r_t = -\log \beta - (0.5 - 1)\tau_t/(1 + 1)\]
Because of its attractive properties as a guide to aggregate demand for goods (here, just $C_t$), will later refer to $r_t^*$ as the ‘natural rate of interest’.

When we study equilibrium with price setting frictions, welfare-reducing discrepancies between equilibrium and first-best allocations will be traced to deviations between the actual and natural rate of interest.
Outline

• Preferences and Technology. (done)
  – First best, efficient allocations (benchmark for later analysis).

• Flexible Price Equilibrium (done)
  – Efficient, as long as a monopoly distortion is eliminated.
    • Complete monetary neutrality.

• Equilibrium in sticky price version of the model (next).
  – Private sector equilibrium conditions.
  – Ramsey Equilibrium.
  – Linearization (first order perturbation)
  – Taylor rule
  – Properties of Taylor rule equilibrium
The Model with Price Setting Frictions on Intermediate Good Producers

- Household and final good producers unchanged from before:

\[
\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}}
\]

\[
\exp(\tau_t) C_t N^p_t = \frac{W_t}{P_t}.
\]

\[
Y_t = \left[ \int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{\frac{\varepsilon}{\varepsilon-1}}, Y_{i,t} = Y_t \left( \frac{P_t}{P_{i,t}} \right)^{\varepsilon}
\]

\[
\to P_t = \left( \int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}}
\]

- Intermediate good firms’ problem changes.
Intermediate Good Producers

- **Demand curve for** $i^{th}$ **monopolist:**
  \[
  Y_{i,t} = Y_t \left( \frac{P_t}{P_{i,t}} \right)^\varepsilon.
  \]

- **Production function:**
  \[
  Y_{i,t} = \exp (a_t) N_{i,t}, \quad a_t = \rho a_{t-1} + \varepsilon_t^a
  \]

- **Calvo Price-Setting Friction:**
  \[
  P_{i,t} = \begin{cases} 
  \tilde{P}_t & \text{with probability } 1 - \theta \\
  \frac{P_{i,t-1}}{P_{i,t-1}} & \text{with probability } \theta
  \end{cases}
  \]

- **Real marginal cost:**
  \[
  S_t = \frac{d\text{Cost}}{d\text{output}} = \frac{W_t}{P_t} \cdot \frac{(1 - \nu)}{\exp (a_t)}
  \]
  minimize monopoly distortion by setting $\frac{\varepsilon - 1}{\varepsilon}$
Intermediate Good Firm

• Present discounted value of firm profits:

\[ E_t \sum_{j=0}^{\infty} \beta^j \begin{pmatrix} \mathcal{E}^t_0 + \text{marginal value of dividends to household} = u_{c,t+j}/P_{t+j} \\ P_{i,t+j}Y_{i,t+j} - P_{t+j}s_{t+j}Y_{i,t+j} \end{pmatrix} \]

• Each of the \(1 - \theta\) firms that can optimize price choose \(\tilde{P}_t\) to optimize in selecting price, firm only cares about future states in which it can’t reoptimize

\[ E_t \sum_{j=0}^{\infty} \beta^j \begin{pmatrix} \mathcal{E}^t_0 + \text{marginal value of dividends to household} = u_{c,t+j}/P_{t+j} \\ P_{i,t+j}Y_{i,t+j} - P_{t+j}s_{t+j}Y_{i,t+j} \end{pmatrix} \]

\[ v_{t+j}[\tilde{P}_tY_{i,t+j} - P_{t+j}s_{t+j}Y_{i,t+j}]. \]
Intermediate Good Firm Problem

• Substitute out the demand curve:

\[ E_t \sum_{j=0}^{\infty} (\beta \theta)^j v_{t+j} [\tilde{P}_t Y_{i,t+j} - P_{t+j}s_{t+j}Y_{i,t+j}] \]

\[ = E_t \sum_{j=0}^{\infty} (\beta \theta)^j v_{t+j} Y_{t+j}P^\varepsilon_{t+j} [\tilde{P}_t^{1-\varepsilon} - P_{t+j}s_{t+j}\tilde{P}_t^{-\varepsilon}] . \]

• Differentiate with respect to \( \tilde{P}_t \):

\[ E_t \sum_{j=0}^{\infty} (\beta \theta)^j v_{t+j} Y_{t+j}P^\varepsilon_{t+j} [(1 - \varepsilon)(\tilde{P}_t)^{-\varepsilon} + \varepsilon P_{t+j}s_{t+j}\tilde{P}_t^{-\varepsilon-1}] = 0, \]

• or

\[ E_t \sum_{j=0}^{\infty} (\beta \theta)^j v_{t+j} Y_{t+j}P^\varepsilon_{t+j} \left[ \frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0. \]
Intermediate Good Firm Problem

- Objective:

\[
E_t \sum_{j=0}^{\infty} \left( \beta \theta \right)^j \frac{u'(C_{t+j})}{P_{t+j}} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[ \frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.
\]

\[
\rightarrow E_t \sum_{j=0}^{\infty} \left( \beta \theta \right)^j P_{t+j}^{\varepsilon} \left[ \frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.
\]

\[
E_t \sum_{j=0}^{\infty} \left( \beta \theta \right)^j (X_{t,j})^{-\varepsilon} \left[ \tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0,
\]

\[
\tilde{p}_t = \frac{\tilde{P}_t}{P_t}, \quad X_{t,j} = \begin{cases} 
\frac{1}{\bar{\pi}_{t+j} \bar{\pi}_{t+j-1} \cdots \bar{\pi}_{t+1}}, & j \geq 1 \\
1, & j = 0.
\end{cases}, \quad X_{t,j} = X_{t+1,j-1} \frac{1}{\bar{\pi}_{t+1}}, \quad j > 0
\]
Intermediate Good Firm Problem

• Want $\tilde{p}_t$ in:

$$E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{-\varepsilon} \left[ \tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon-1} s_{t+j} \right] = 0$$

• Solution:

$$\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t}$$

• But, still need expressions for $K_t$, $F_t$. 
\[ K_t = E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \]

\[ = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta \theta E_t \sum_{j=1}^{\infty} (\beta \theta)^{j-1} \left( \frac{1}{\pi_{t+1}} X_{t+1,j-1} \right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \]

\[ = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta \theta E_t \left( \frac{1}{\pi_{t+1}} \right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta \theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j} \]

\[ = E_t \text{ by LIME} \]

\[ = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta \theta \left( \frac{1}{\pi_{t+1}} \right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta \theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j} \]

\[ \text{exactly } K_{t+1}! \]

\[ = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta \theta E_t \left( \frac{1}{\pi_{t+1}} \right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta \theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j} \]

\[ = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta \theta E_t \left( \frac{1}{\pi_{t+1}} \right)^{-\varepsilon} K_{t+1} \]
• From previous slide:

\[ K_t = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta \theta E_t \left( \frac{1}{\pi_{t+1}} \right)^{-\varepsilon} K_{t+1}. \]

• Substituting out for marginal cost:

\[ \frac{\varepsilon}{\varepsilon - 1} s_t = \frac{\varepsilon}{\varepsilon - 1} (1 - \nu) \left( \frac{W_t}{P_t} \right) \frac{\exp(a_t)}{\exp(\tau_t) N_t^\theta C_t}. \]

\[ = \frac{W_t}{P_t} \text{ by household optimization} \]
In Sum

• solution:

\[ \hat{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t}, \]

• Where:

\[ K_t = (1 - \nu_t) \frac{\varepsilon}{\varepsilon - 1} \frac{\exp(\tau_t) N_t^\theta C_t}{\exp(a_t)} + \beta \theta E_t \left( \frac{1}{\pi_{t+1}} \right)^{-\varepsilon} K_{t+1}. \]

\[ F_t \equiv E_t \sum_{j=0}^{\infty} (\beta \theta)^j (X_{t,j})^{1-\varepsilon} = 1 + \beta \theta E_t \left( \frac{1}{\pi_{t+1}} \right)^{1-\varepsilon} F_{t+1} \]
To Characterize Equilibrium

• Have equations characterizing optimization by firms and households.

• Still need:
  – Expression for all the prices. Prices, $P_{i,t}$, $0 \leq i \leq 1$, will all be different because of the price setting frictions.
  – Relationship between aggregate employment and aggregate output not simple because of price distortions:

$$Y_t \neq e^{a_t}N_t,$$ in general

• This part of the analysis is the reason why it made Calvo famous – it’s not easy.
Going for Prices

- Aggregate price relationship

\[ P_t = \left[ \int_0^1 P_{i,t}^{(1-\varepsilon)} \, di \right]^{\frac{1}{1-\varepsilon}} \]

\[ = \left[ \int \text{firms that reoptimize price} \, P_{i,t}^{(1-\varepsilon)} \, di + \int \text{firms that don’t reoptimize price} \, P_{i,t}^{(1-\varepsilon)} \, di \right]^{\frac{1}{1-\varepsilon}} \]

- All reoptimizers choose same price

\[ \implies \left[ (1 - \theta)\tilde{P}_t^{(1-\varepsilon)} + \int \text{firms that don’t reoptimize price} \, P_{i,t}^{(1-\varepsilon)} \, di \right]^{\frac{1}{1-\varepsilon}} \]

- In principle, to solve the model need all the prices, \( P_t, P_{i,t}, 0 \leq i \leq 1 \)
  - Fortunately, that won’t be necessary.
Key insight

\[ \int_{\text{firms that don't reoptimize price in } t} P_{i,t}^{(1-\varepsilon)} \, di \]

add over prices, weighted by \# of firms posting that price

\[ \frac{\text{number of firms that had price, } P(\omega), \text{ in } t-1 \text{ and were not able to reoptimize in } t}{\int f_{t-1,t}(\omega) \, d\omega} \]

\[ P(\omega)^{(1-\varepsilon)} \]
Applying the Insight

• By Calvo randomization assumption

\[ f_{t-1,t}(\omega) = \theta \times \left( f_{t-1}(\omega) \right) \], for all \( \omega \)

• Substituting:

\[ \int_{\text{firms that don't reoptimize price}} P_{i,t}^{(1-\varepsilon)} di = \int f_{t-1,t}(\omega)P(\omega)^{(1-\varepsilon)} d\omega \]
Applying the Insight

• By Calvo randomization assumption

\[ f_{t-1,t}(\omega) = \theta \times \sqrt{f_{t-1}(\omega)}, \text{ for all } \omega \]

• Substituting:

\[
\int_{\text{firms that don’t reoptimize price}} P_{i,t}^{(1-\varepsilon)} di = \int f_{t-1,t}(\omega)P(\omega)^{(1-\varepsilon)} d\omega \\
= \theta \int f_{t-1}(\omega)P(\omega)^{(1-\varepsilon)} d\omega \\
= \theta P_{t-1}^{(1-\varepsilon)}
\]
Applying the Insight

• By Calvo randomization assumption

\[ f_{t-1,t}(\omega) = \theta \times \frac{f_{t-1}(\omega)}{f_{t-1,t}(\omega)} \text{, for all } \omega \]

• Substituting:

\[ \int_{\text{firms that don’t reoptimize price}} P_{i,t}^{(1-\varepsilon)} di = \int f_{t-1,t}(\omega)P(\omega)^{(1-\varepsilon)} d\omega \]

\[ = \theta \int f_{t-1}(\omega)P(\omega)^{(1-\varepsilon)} d\omega \]

\[ = \theta P_{t-1}^{(1-\varepsilon)} \]
Expression for $\tilde{P}_t$ in terms of aggregate inflation

- Conclude that this relationship holds between prices:

$$P_t = \left[ (1 - \theta)\tilde{P}_t^{(1-\epsilon)} + \theta P_{t-1}^{(1-\epsilon)} \right]^{\frac{1}{1-\epsilon}}.$$

- Only two variables here!

- Divide by $P_t$:

$$1 = \left[ (1 - \theta)\tilde{P}_t^{(1-\epsilon)} + \theta\left(\frac{1}{\pi_t}\right)^{(1-\epsilon)} \right]^{\frac{1}{1-\epsilon}}$$

- Rearrange:

$$\tilde{P}_t = \left[ \frac{1 - \theta\pi_t^{(\epsilon-1)}}{1 - \theta} \right]^{\frac{1}{1-\epsilon}}$$
Relation Between Aggregate Output and Aggregate Inputs

• Technically, there is no ‘aggregate production function’ in this model
  – If you know how many people are working, $N$, and the state of technology, $a$, you don’t have enough information to know what $Y$ is.
  – Price frictions imply that resources will not be efficiently allocated among different inputs.
    • Implies $Y$ low for given $a$ and $N$. How low?
    • Tak Yun (JME) gave a simple answer.
\[ Y_t^* = \int_0^1 Y_{i,t} \, di \quad \text{(labor market clearing)} \]

[= \int_0^1 A_t N_{i,t} \, di \quad \Rightarrow \quad A_t N_t]

\[
 Y_t \int_0^1 \left( \frac{P_{i,t}}{P_t} \right)^{-\varepsilon} \, di
\]

\[
 = Y_t P_t^\varepsilon \int_0^1 (P_{i,t})^{-\varepsilon} \, di
\]

\[
 = Y_t P_t^\varepsilon (P_t^*)^{-\varepsilon}
\]

**Where:**

\[ P_t^* = \left[ \int_0^1 P_{i,t}^{-\varepsilon} \, di \right]^{-\frac{1}{\varepsilon}} = [(1 - \theta)\tilde{P}_t^{-\varepsilon} + \theta(P_{t-1}^*)^{-\varepsilon}]^{-\frac{1}{\varepsilon}} \]
Relationship Between Agg Inputs and Agg Output

• Rewriting previous equation:

\[ Y_t = \left( \frac{P_t^*}{P_t} \right)^\varepsilon Y_t^* \]

\[ = p_t^* e^{a_t N_t}, \]

• ‘efficiency distortion’:

\[ p_t^* : \begin{cases} 
\leq 1 \\
= 1 & P_{i,t} = P_{j,t}, \text{ all } i, j
\end{cases} \]
Example of Efficiency Distortion

\[ P_{j,t} = \begin{cases} 
  P^1 & 0 \leq j \leq \alpha \\
  P^2 & \alpha \leq j \leq 1 
\end{cases} \]

\[ P_t^* = \left( \frac{P_t}{P_t^*} \right)^{\varepsilon} = \left( \frac{\alpha + (1 - \alpha)\left(\frac{p^2}{p^1}\right)^{-\varepsilon}}{\alpha + (1 - \alpha)\left(\frac{p^2}{p^1}\right)^{1-\varepsilon}} \right)^{\frac{1}{1-\varepsilon}} \]

\[ \alpha = 0.5, \varepsilon = 10 \]

\[ \log \frac{P^1}{P^2} \]
Collecting Equilibrium Conditions

• Price setting:

\[ K_t = (1 - \nu) \frac{\epsilon}{\epsilon - 1} \exp(\tau_t) N_t^\phi C_t \frac{A_t}{A_t} + \beta \theta E_t \tilde{\pi}_{t+1} \bar{K}_{t+1} \]  

\[ F_t = 1 + \beta \theta E_t \tilde{\pi}_{t+1} \bar{F}_{t+1} \]  

• Intermediate good firm optimality and restriction across prices:

\[ \frac{K_t}{F_t} = \tilde{\rho}_t \text{ by firm optimality} \]  

\[ \frac{K_t}{F_t} = \tilde{\rho}_t \text{ by restriction across prices} \]

\[ = \tilde{\rho}_t \text{ by firm optimality} \]

\[ = \tilde{\rho}_t \text{ by restriction across prices} \]

\[ \left( \frac{1 - \theta \tilde{\pi}_t^{(\epsilon - 1)}}{1 - \theta} \right) \frac{1}{1 - \epsilon} \]
Equilibrium Conditions

• Law of motion of (Tak Yun) distortion:

\[ p_t^* = \left[ (1 - \theta) \left( \frac{1 - \theta \bar{\pi}_t^{(\varepsilon-1)}}{1 - \theta} \right) \frac{\varepsilon}{\varepsilon-1} + \frac{\theta \bar{\pi}_t^\varepsilon}{p_{t-1}^*} \right]^{-1} \] (4)

• Household Intertemporal Condition:

\[ \frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \] (5)

• Aggregate inputs and output:

\[ C_t = p_t^* e^{a_t} N_t \] (6)

• 6 equations, 8 unknowns:

\[ \nu, \ C_t, \ p_t^*, \ N_t, \bar{\pi}_t, \ K_t, \ F_t, \ R_t \]

• System under determined!
Tak Yun Distortion: a Closer Look

- Law of motion, Tak Yun distortion:

\[
p_t^* = p^* (\bar{\pi}_t, p_{t-1}^*) \equiv \left[ (1 - \theta) \left( \frac{1 - \theta \bar{\pi}_t^{(\varepsilon-1)}}{1 - \theta} \right) \right]^{-1}
\]

- Distortion, \( p_t^* \), increasing function of lagged distortion, \( p_{t-1}^* \).
- Current shocks affect current distortion via \( \bar{\pi}_t \) only.

- Derivatives:

\[
p_1^* (\bar{\pi}_t, p_{t-1}^*) = - (p_t^*)^2 \varepsilon \theta \bar{\pi}_t^{\varepsilon-2} \left[ \frac{\bar{\pi}_t}{p_t^*} - \left( \frac{1 - \theta \bar{\pi}_t^{(\varepsilon-1)}}{1 - \theta} \right) \right]^{\frac{1}{\varepsilon-1}}
\]

\[
p_2^* (\bar{\pi}_t, p_{t-1}^*) = \left( \frac{p_t^*}{p_{t-1}^*} \right)^2 \theta \bar{\pi}_t^{\varepsilon}.
\]
Linear Expansion of Tak Yun Distortion in Undistorted Steady State

• Linearizing about $\bar{\pi}_t = \bar{\pi}, p^*_t = p^*$:

$$dp^*_t = p^*_1 (\bar{\pi}, p^*) d\bar{\pi}_t + p^*_2 (\bar{\pi}, p^*) dp^*_{t-1},$$

where $dx_t \equiv x_t - x$, for $x_t = p^*_t, p^*_{t-1}, \bar{\pi}_t$.

• In an undistorted steady state (i.e., $\bar{\pi}_t = p^*_t = p^*_{t-1} = 1$):

$$p^*_1 (1, 1) = 0, \quad p^*_1 (1, 1) = \theta.$$

so that

$$dp^*_t = 0 \times d\bar{\pi}_t + \theta dp^*_{t-1}$$

$$\rightarrow \quad p^*_t = 1 - \theta + \theta p^*_{t-1}$$

• Often, people that linearize NK model ignore $p^*_t$.

  – Reflects that they linearize the model around a price-undistorted steady state.
Current Period Tack Yun Distortion as a Function of Current Inflation
Graph conditioned on two alternative values for $p^*_{t-1}$ and $\theta = 0.75$, $\varepsilon = 6.00$

lagged Tack Yun distortion = 1 (i.e., no distortion)
lagged Tack Yun distortion = 0.9
Ignoring Tak Yun Distortion, a Mistake?

Tack Yun Distortion and Quarterly US CPI Inflation (Gross)
Underdetermined System

• Not surprising: we added a variable, the nominal rate of interest.

• Also, we’re counting subsidy as among the unknowns.

• Have two extra policy variables.

• One way to pin them down: compute optimal policy.
Ramsey-Optimal Policy

• 6 equations in 8 unknowns.....
  – Many configurations of the 8 unknowns that satisfy the 6 equations.
  – Look for the best configurations (Ramsey optimal)
    • Value of tax subsidy and of $R$ represent optimal policy

• Finding the Ramsey optimal setting of the 6 variables involves solving a simple Lagrangian optimization problem.
Ramsey Problem

\[
\max_{v,p^*_t,C_t,N_t,R_t,\bar{\pi}_t,F_t,K_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left( \log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1 + \varphi} \right) \right. \\
+ \lambda_{1t} \left[ \frac{1}{C_t} - E_t \frac{\beta}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \right] \\
+ \lambda_{2t} \left[ \frac{1}{p^*_t} - \left( 1 - \theta \right) \left( \frac{1 - \theta(\bar{\pi}_t)^{\epsilon-1}}{1 - \theta} \right)^{\frac{\epsilon}{\epsilon-1}} + \frac{\theta \bar{\pi}_t^\epsilon}{p^*_{t-1}} \right] \\
+ \lambda_{3t} \left[ 1 + E_t \bar{\pi}_{t+1}^{\epsilon-1} \beta \theta F_{t+1} - F_t \right] \\
+ \lambda_{4t} \left[ (1 - \nu) \frac{\epsilon}{\epsilon - 1} C_t \exp(\tau_t) N_t^{\varphi} e^{a_t} + E_t \beta \theta \bar{\pi}_{t+1}^\epsilon K_{t+1} - K_t \right] \\
+ \lambda_{5t} \left[ F_t \left( \frac{1 - \theta \bar{\pi}_t^{\epsilon-1}}{1 - \theta} \right)^{\frac{1}{1-\epsilon}} - K_t \right] \\
+ \lambda_{6t} \left[ C_t - p_t^* e^{a_t} N_t \right] \left\} \right.
\]
Solving the Ramsey Problem
(surprisingly easy in this case)

• First, substitute out consumption everywhere

\[
\max_{v,p^*,N_t,R_t,\bar{\pi}_t,F_t,K_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left( \log N_t + \log p^*_t - \exp(\tau_t) \frac{N_t^{1+\phi}}{1+\phi} \right) \right. \\
+ \lambda_{1t} \left[ \frac{1}{p^*_t N_t} - E_t \frac{e^{a_t} \beta}{p^*_{t+1} e^{a_{t+1}} N_{t+1} \bar{\pi}_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \right] \\
+ \lambda_{2t} \left[ \frac{1}{p^*_t} - \left( (1 - \theta) \left( \frac{1 - \theta(\bar{\pi}_t)^{\varepsilon-1}}{1 - \theta} \right) \frac{\varepsilon}{\varepsilon-1} + \frac{\theta \bar{\pi}_t^\varepsilon}{p^*_{t-1}} \right) \right] \\
+ \lambda_{3t} \left[ 1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta \theta F_{t+1} - F_t \right] \\
+ \lambda_{4t} \left[ (1 - \nu) \frac{\varepsilon}{\varepsilon-1} \exp(\tau_t) N_t^{1+\phi} p^*_t + E_t \beta \theta \bar{\pi}_{t+1}^\varepsilon K_{t+1} - K_t \right] \\
+ \lambda_{5t} \left[ F_t \left( \frac{1 - \theta \bar{\pi}_t^{\varepsilon-1}}{1 - \theta} \right)^{\frac{1}{1-\varepsilon}} - K_t \right] \left\} \right.
\]

defines $R$

defines $F$

defines tax

defines $K$
Solving the Ramsey Problem, cnt’d

- Simplified problem:

\[ \max_{\pi_t, p_t^*, N_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \left( \log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1 + \varphi} \right) \right. \]

\[ \left. + \lambda_{2t} \left[ \frac{1}{p_t^*} - \left( (1 - \theta) \left( \frac{1 - \theta (\bar{\pi}_t)^{\varepsilon-1}}{1 - \theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^\varepsilon}{p_t^{\varepsilon}} \right) \right] \right\} \]

- First order conditions with respect to \( p_t^*, \bar{\pi}_t, N_t \)

\[ p_t^* + \beta \lambda_{2,t+1} \theta \bar{\pi}_{t+1}^\varepsilon = \lambda_{2t}, \quad \bar{\pi}_t = \left[ \frac{(p_{t-1}^*)^{\varepsilon-1}}{1 - \theta + \theta (p_{t-1}^*)^{\varepsilon-1}} \right]^{\frac{1}{\varepsilon-1}}, \quad N_t = \exp \left( -\frac{\tau_t}{\varphi + 1} \right) \]

- Substituting the solution for inflation into law of motion for price distortion:

\[ p_t^* = \left[ (1 - \theta) + \theta (p_{t-1}^*)^{(\varepsilon-1)} \right]^{\frac{1}{\varepsilon-1}}. \]
Solution to Ramsey Problem

Eventually, price distortions eliminated, regardless of shocks

\[ p_t^* = \left[ (1 - \theta) + \theta(p_{t-1}^*)^{(\varepsilon - 1)} \right]^{\frac{1}{(\varepsilon - 1)}} \]

When price distortions gone, so is inflation.

\[ \bar{\pi}_t = \frac{p_{t-1}^*}{p_t^*} \]

Efficient ('first best') allocations in real economy

\[ N_t = \exp \left( -\frac{\tau_t}{1 + \varphi} \right) \]

\[ 1 - \nu = \frac{\varepsilon - 1}{\varepsilon} \]

\[ C_t = p_t^* e^{a_t} N_t. \]

Consumption corresponds to efficient allocations in real economy, eventually when price distortions gone.
Eventually, Optimal (Ramsey) Equilibrium and Efficient Allocations in Real Economy Coincide

\[ \theta = 0.75, \ \varepsilon = 10 \]

\[ p_t^* = \left[ (1 - \theta) + \theta (p_{t-1}^*)^{(\varepsilon - 1)} \right]^{\frac{1}{(\varepsilon - 1)}} \]
The Ramsey allocations are eventually the best allocations in the economy without price frictions (i.e., ‘first best allocations’)

Refer to the Ramsey allocations as the ‘natural allocations’. ....

– Natural consumption, natural rate of interest, etc.
Linearizing around Efficient Steady State

• In steady state (assuming $\bar{\pi} = 1, 1 - \nu = \frac{\varepsilon - 1}{\varepsilon}$)

$$p^* = 1, K = F = \frac{1}{1 - \beta\theta}, s = \frac{\varepsilon - 1}{\varepsilon}, \Delta a = \tau = 0, N = 1$$

• Linearizing the Tack Yun distortion, (4), about steady state:

$$\hat{p}_t^* = \theta \hat{p}^*_{t-1} \rightarrow \hat{p}_t^* = 0, t \text{ large}$$

• Denote the output gap in ratio form by $X_t$:

$$X_t \equiv \frac{C_t}{A_t \exp \left( - \frac{\tau_t}{1 + \varphi} \right)} = p^*_tN_t \exp \left( \frac{\tau_t}{1 + \varphi} \right),$$

so that (using $x_t \equiv \hat{X}_t$):

$$x_t = \hat{N}_t + \frac{d\tau_t}{1 + \varphi}$$


**NK IS Curve**

- The intertemporal Euler equation, (5), after substituting for \( C_t \) in terms of \( X_t \): 

\[
\frac{1}{X_t A_t \exp \left( -\frac{\tau_t}{1+\phi} \right)} = \beta E_t \frac{1}{X_{t+1} A_{t+1} \exp \left( -\frac{\tau_{t+1}}{1+\phi} \right)} R_t \frac{\bar{\pi}_{t+1}}{\bar{\pi}_{t+1}},
\]

where

\[
R_{t+1}^* \equiv \frac{1}{\beta} \exp \left( a_{t+1} - a_t - \frac{\tau_{t+1} - \tau_t}{1 + \phi} \right)
\]

then, use

\[
\widehat{z_t}u_t = \widehat{z_t} + \widehat{u}_t, \quad \left( \frac{u_t}{z_t} \right) = \widehat{u}_t - \widehat{z}_t
\]

to obtain:

\[
\hat{X}_t = E_t \left[ \hat{X}_{t+1} - (\hat{R}_t - \hat{\pi}_{t+1} - \hat{R}_{t+1}^*) \right]
\]
NK IS Curve

- Let:

\[ R_t \equiv \exp(r_t) \]

\[ \Rightarrow \hat{R}_t = \frac{d\exp(r_t)}{\exp(r)} = \frac{\exp(r)dr_t}{\exp(r)} = dr_t \equiv r_t - r. \]

- Also,

\[ R_{t+1}^* = \exp(\log R_{t+1}^*) \]

\[ \Rightarrow \hat{R}_{t+1} = \frac{d\exp(\log R_{t+1}^*)}{\exp(\log R^*)} = d\log R_{t+1}^* = r \text{ in efficient steady state, with } \bar{\pi} = 1 \]

\[ = \log R_{t+1}^* - \sqrt[6]{\log R^*} \]

- So, (letting \( r_t^* \equiv E_t \log R_{t+1}^* \))

\[ E_t (\hat{R}_t - \hat{R}_{t+1}^*) = dr_t - E_t d\log R_{t+1}^* = r_t - r_t^*. \]
NK IS Curve

- Substituting
  \[ E_t (\hat{R}_t - \hat{R}_{t+1}^*) = r_t - r_t^* \]
  into
  \[ \hat{X}_t = E_t [\hat{X}_{t+1} - (\hat{R}_t - \hat{\pi}_{t+1} - \hat{R}_{t+1}^*)] \]
  \[ x_t \equiv \hat{X}_t, \]
  and using
  \[ \hat{\pi}_{t+1} = \pi_{t+1}, \text{ when } \bar{\pi} = 1, \]
  we obtain NK IS curve:
  \[ x_t = E_t x_{t+1} - E_t [r_t - \pi_{t+1} - r_t^*] \]

- Also,
  \[ r_t^* = -\log(\beta) + E_t \left[ a_{t+1} - a_t - \frac{\tau_{t+1} - \tau_t}{1 + \phi} \right]. \]
Linearized Marginal Cost

- Marginal cost (using $da_t = a_t$, $d\tau_t = \tau_t$ because $a = \tau = 0$):

$$s_t = (1 - \nu) \frac{\bar{w}_t}{A_t}, \quad \bar{w}_t \equiv \frac{W_t}{P_t} = \exp(\tau_t) N_t^\varphi C_t$$

$$\rightarrow \ \hat{w}_t = \tau_t + a_t + (1 + \varphi) \hat{N}_t$$

- Then,

$$\hat{s}_t = \hat{w}_t - a_t = (\varphi + 1) \left[ \frac{\tau_t}{\varphi + 1} + \hat{N}_t \right] = (\varphi + 1) x_t$$
Linearized Phillips Curve

• Log-linearize equilibrium conditions, (1)-(3), around steady state:

\[
\begin{align*}
\hat{K}_t &= (1 - \beta \theta) \hat{s}_t + \beta \theta (\varepsilon \hat{\pi}_{t+1} + \hat{K}_{t+1}) \quad (1) \\
\hat{F}_t &= \beta \theta (\varepsilon - 1) \hat{\pi}_{t+1} + \beta \theta \hat{F}_{t+1} \quad (2) \\
\hat{K}_t &= \hat{F}_t + \frac{\theta}{1 - \theta} \hat{\pi}_t \quad (3)
\end{align*}
\]

• Substitute (3) into (1)

\[
\hat{F}_t + \frac{\theta}{1 - \theta} \hat{\pi}_t = (1 - \beta \theta) \hat{s}_t + \beta \theta \left( \varepsilon \hat{\pi}_{t+1} + \hat{F}_{t+1} + \frac{\theta}{1 - \theta} \hat{\pi}_{t+1} \right)
\]

• Simplify the latter using (2), to obtain the NK Phillips curve:

\[
\pi_t = \frac{(1 - \theta)(1 - \beta \theta)}{\theta} \hat{s}_t + \beta \pi_{t+1}
\]
The Equilibrium Conditions

\[ x_t = x_{t+1} - [r_t - \pi_{t+1} - r^*_t] \]
\[ \pi_t = \frac{(1-\theta)(1-\beta\theta)}{\theta} \hat{s}_t + \beta \pi_{t+1} \]
\[ \hat{s}_t = (\varphi + 1) x_t \]
\[ r^*_t = -\log(\beta) + E_t \left[a_{t+1} - a_t - \frac{\tau_{t+1} - \tau_t}{1+\phi}\right] \]

Taylor rule (in deviation from steady state):

\[ r_t = \alpha r_{t-1} + (1 - \alpha) \left[\phi_{\pi} \pi_t + \phi_x x_t\right] \]

Shocks:

\[ \Delta a_t = \rho \Delta a_{t-1} + \varepsilon_t^a, \quad \tau_t = \lambda \tau_{t-1} + \varepsilon_t^\tau. \]
Solving the Model

\[ s_t = \begin{pmatrix} \Delta a_t \\ \tau_t \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \Delta a_{t-1} \\ \tau_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ \varepsilon_t^\tau \end{pmatrix} \]

\[ s_t = P s_{t-1} + \varepsilon_t \]

\[
\begin{bmatrix}
\beta & 0 & 0 & 0 \\
\frac{1}{\sigma} & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{pmatrix} \pi_{t+1} \\ x_{t+1} \\ r_{t+1} \\ r^*_{t+1} \end{pmatrix} + \begin{bmatrix}
-1 & \kappa & 0 & 0 \\
0 & -1 & -\frac{1}{\sigma} & \frac{1}{\sigma} \\
(1 - \alpha)\phi_{\pi} & (1 - \alpha)\phi_{\pi} & -1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{pmatrix} \pi_t \\ x_t \\ r_t \\ r^*_t \end{pmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{pmatrix} \pi_{t-1} \\ x_{t-1} \\ r_{t-1} \\ r^*_{t-1} \end{pmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\sigma\psi & -\frac{1}{\sigma + \phi} (1 - \lambda)
\end{bmatrix}
\begin{pmatrix} s_{t+1} \\ s_t \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\[ E_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0 \]
Solving the Model

\[ E_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0 \]

\[ s_t - Ps_{t-1} - \epsilon_t = 0. \]

• Solution:

\[ z_t = Az_{t-1} + Bs_t \]

• As before:

\[ \alpha_0 A^2 + \alpha_1 A + \alpha_2 I = 0, \]

\[ F = (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1)B] = 0 \]
\( \phi_x = 0, \phi_\pi = 1.5, \beta = 0.99, \varphi = 1, \rho = 0.2, \theta = 0.75, \alpha = 0, \delta = 0.2, \lambda = 0.5. \)
Dynamic Response to a Preference Shock

- Inflation
- Output gap
- Nominal rate
  - Natural real rate
  - Actual nominal rate
- Preference shock
- Output
  - Natural output
  - Actual output
- Employment
  - Natural employment
  - Actual employment
Why Is Output Inefficiently High or Low Sometimes?

• Brief answer: the Taylor rule sets the wrong interest rate (should be the natural rate).

• Households equate costs with the *private* benefit of working, $W_t$:

$$-\frac{u_{N,t}}{u_{c,t}} = \exp(\tau_t) C_t N_t^\phi = \frac{W_t}{P_t}.$$  

  - So, the only possible reason efficiency may not occur is if $W_t/P_t$ does not correspond to the actual marginal product of labor.

• The relationship between $W_t/P_t$ and labor productivity may be understood by studying the markup of price over marginal cost.
Price over Marginal Cost (Markup)

• $i^{th}$ firm sets $P_{i,t}$ as a markup, $\mu_{i,t}$, over marginal cost

\[ P_{i,t} = \left(\mu_{i,t}\right) \times \frac{(1 - \nu) W_t}{e^{a_t}}, \]

where we have been setting $1 - \nu = \frac{\varepsilon - 1}{\varepsilon}$.

• In the flexible price version of the model the markup is trivial, $\mu_{i,t} = \frac{\varepsilon - 1}{\varepsilon}$ so

\[ P_{i,t} = P_t = \frac{W_t}{e^{a_t}}, \]

and $W_t/P_t$ corresponds to the marginal product of labor.
In the sticky price version of the model, the markup is more complicated.

- firms currently setting prices, do so to get the markup to be \( \frac{\varepsilon}{(\varepsilon - 1)} \) on average in the current and future periods (see earlier discussion)
- for firms not able to set prices in the current period, the markup is

\[
\mu_{i,t} = \frac{P_{i,t-1}}{(1-\nu)W_t}e^{a_t}.
\]

- for these firms the markup moves inversely with a shock to marginal cost.

- Need to look at some aggregate of all markups.
Price over Marginal Cost (Markup)

- Weighted average markup, $\mu_t$:

$$\mu_t \equiv \left( \int_0^1 \mu_{i,t}^{(1-\varepsilon)} \, di \right)^{\frac{1}{1-\varepsilon}} = \frac{P_t}{(1-\nu)W_t} = \frac{1}{\frac{1-\nu}{e^{a_t}}e^{\tau_t}C_t N_t^\varphi}$$

$$= \frac{1}{(1-\nu)e^{\tau_t}p_*^t N_t^{1+\varphi}} = \frac{1}{(1-\nu)p_*^t} \left( \frac{e^{-\frac{\tau_t}{1+\varphi}}}{N_t} \right)^{1+\varphi}.$$

- Consider the output gap:

$$X_t = \frac{C_t}{e^{a_t-\tau_t/(1+\varphi)}} = \frac{p_*^t e^{a_t} N_t}{e^{a_t-\tau_t/(1+\varphi)}} = p_*^t \frac{N_t}{e^{-\tau_t/(1+\varphi)}}$$

- Use this to substitute into the markup

$$\mu_t = \frac{(p_*^t)^{\varphi}}{(1-\nu)} X_t^{-(1+\varphi)}.$$

$\mu_t$ moves inversely with the output gap.
Price over Marginal Cost (Markup)

- The preceding implies:

\[ \frac{-u_{N,t}}{u_{c,t}} \text{ source of efficiency} = \frac{W_t}{P_t} = \frac{e^{at}}{(1 - v) \mu_t} = e^{at} (p_t^*)^{-\varphi} X_t^{1+\varphi}. \]

- Suppose \( p_t^* = 1. \)
- When the output gap is high, \( X > 0, \) then the markup is low and the real wage exceeds \( e^a. \)
  - There is too much employment in this case because the private benefit to the workers of working exceeds the actual benefits (the difference comes out of lump sum profits).

- Why are firms willing to produce with a low markup? Because on average it is high \( (\mu = \varepsilon / (\varepsilon - 1)) \) and so cutting it (not too much!) allows them to still get some profits out of the workers.

- Interpretation: model implies markups are countercyclical.
  - Nekarda and Ramey (‘The Cyclical Behavior of the Price-Cost Markup’, UCSD 2013) argue that the evidence does not support this.
Wrap Up

• Suppose monetary policy puts the interest rate below the natural rate, driving up the output gap.
• The low interest rate gives people an incentive to spend more.
• Marginal cost rises and since prices sticky, markups

$$\mu_t = \frac{P_t}{MC_t},$$

go down ($MC_t = (1 - \nu) W_t / e^{at}$).

• Low markup means high wage when there is a monetary shock, encouraging more work.
• Profits are reduced, but presumably remain positive:

$$\text{profits} = P_{i,t} Y_{i,t} - (1 - \nu) W_t N_{i,t}$$

$$= P_{i,t} Y_{i,t} - e^{at} MC_t \times N_{i,t}$$

$$= [P_{i,t} - MC_t] Y_{i,t}$$

$$= [\mu_{i,t} - 1] MC_t \times Y_{i,t}$$