Adversarial Coordination and Public Information Design

Online Appendix

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Abstract

This document contains the proof of Examples 2 and 3 in the main text. All numbered items contain the prefix "S." Any numbered reference without the prefix "S" refers to an item in the main text.

Proof of Example 2 in Main Text

Proof of Example 2. The proof is in two steps. Step 1 characterizes the threshold θ^*_σ defining the optimal deterministic monotone rule, whereas Step 2 constructs the non-monotone policy that strictly improves over the optimal deterministic monotone one.

Step 1. The primitives in this example satisfy the conditions in Theorem 2 in the main text. This means that, given any signal s disclosed by any policy Γ, MARP is in threshold strategies, which in turn implies that the default outcome is monotone in θ.

Next recall that, for any default threshold $\theta \in [0,1]$, the corresponding signal threshold $x^*_\sigma(\theta)$ is implicitly defined by $P_\sigma(x^*_\sigma(\theta)|\theta) = \theta$. Using the fact that, for any $\theta \in [-K, 1+K]$ and $x \in [\theta-\sigma, \theta+\sigma]$, $P_\sigma(x|\theta) = (x-\theta+\sigma)/2\sigma$, we have that $x^*_\sigma(\theta) = (1+2\sigma)\theta - \sigma$.

For any $\hat{\theta} \in [0,1]$, let $\Gamma^{\hat{\theta}} \equiv \{\{0,1\}, \pi^{\hat{\theta}}\}$ be the deterministic monotone policy with cutoff $\hat{\theta}$. Next, for any $\theta \in [\hat{\theta}/(1+2\sigma), 1]$, let $V^{\Gamma^{\hat{\theta}}}_\sigma(\theta) \equiv U^{\Gamma^{\hat{\theta}}}_\sigma(x^*_\sigma(\theta), 1|x^*_\sigma(\theta))$ be the expected payoff differential between pledging and not pledging of the marginal investor with signal $x^*_\sigma(\theta)$, when each investor pledges if and only if their signal is above $x^*_\sigma(\theta)$ (and hence default occurs if, and only if, fundamentals are below θ), the quality of the investors' signal is σ, and the policy $\Gamma^{\hat{\theta}}$ announces that $s=1$, thus revealing that $\theta \geq \hat{\theta}$. Note that, for any $0 \leq \theta < \hat{\theta}/(1+2\sigma)$, $x^*_\sigma(\theta) + \sigma < \hat{\theta}$, which implies that the signal $x^*_\sigma(\theta)$ is not consistent with the event that fundamentals are above $\hat{\theta}$. Equivalently, when $\theta \geq \hat{\theta}$, the lowest possible signal that an individual may receive is $\hat{\theta} - \sigma$. When each investor pledges if and only if $x > \hat{\theta} - \sigma$, default occurs if and only if $\theta \leq \hat{\theta}/(1+2\sigma)$. Hence, the lowest default threshold that is consistent with the policy $\Gamma^{\hat{\theta}}$ is $\hat{\theta}/(1+2\sigma)$. The function $V^{\Gamma^{\hat{\theta}}}_\sigma(\theta)$ is thus defined only for $\theta \in [\hat{\theta}/(1+2\sigma), 1]$.

The cutoff θ^*_σ characterizing the optimal deterministic monotone policy is given by

$$\theta^*_\sigma = \inf\{\hat{\theta} \in [0,1] : V^{\Gamma^{\hat{\theta}}}_\sigma(\theta) \geq 0 \text{ for all } \theta \in [\hat{\theta}/(1+2\sigma), 1]\}. \tag{1}$$

Claim S2. *For any $\hat{\theta} \in [0,1]$, $V^{\Gamma^{\hat{\theta}}}_\sigma(\cdot)$ has a unique minimizer. Letting*

$$\theta^{\min}_\sigma(\hat{\theta}) \equiv \arg min_{\theta \in [\hat{\theta}/(1+2\sigma),1]} V^{\Gamma^{\hat{\theta}}}_\sigma(\theta),$$

we have that $\theta^{\min}_\sigma(\hat{\theta})$ satisfies $x^*_\sigma(\theta^{\min}_\sigma(\hat{\theta})) - \sigma = \hat{\theta}$.

Proof of Claim S2. Clearly, for any $\theta \in [\hat{\theta}/(1+2\sigma), \hat{\theta}]$, $V^{\Gamma^{\hat{\theta}}}_\sigma(\theta) = g$. This is because when each investor pledges if and only if $x > x^*_\sigma(\theta)$ default occurs only for fundamentals below θ. Hence the announcement that $\theta > \hat{\theta}$ reveals to the marginal investor with signal $x^*_\sigma(\theta)$ that default will not occur.

Next, observe that for any $\theta \in (\hat{\theta}, (\hat{\theta}+2\sigma)/(1+2\sigma)]$, $x^*_\sigma(\theta) - \sigma < \hat{\theta}$, implying that[1]

$$V_\sigma^{\Gamma^{\hat{\theta}}}(\theta) = g - (g+|b|)\,\mathbb{P}_\sigma\{\tilde{\theta} \leq \theta | \tilde{\theta} \geq \hat{\theta}; x^*_\sigma(\theta)\} = g - (g+|b|)\,\frac{\theta - \hat{\theta}}{(1+2\sigma)\,\theta - \hat{\theta}},$$

which is strictly decreasing in θ. Finally, note that, for any $\theta \in ((\hat{\theta}+2\sigma)/(1+2\sigma), 1]$, $x^*_\sigma(\theta) - \sigma > \hat{\theta}$, implying that

$$V_\sigma^{\Gamma^{\hat{\theta}}}(\theta) = g - (g+|b|)\,\mathbb{P}_\sigma\{\tilde{\theta} \leq \theta | \tilde{\theta} \geq \hat{\theta}; x^*_\sigma(\theta)\} = g + (g+|b|)(\theta - 1),$$

which is strictly increasing in θ. Hence, $V_\sigma^{\Gamma^{\hat{\theta}}}(\cdot)$ has a single minimizer over $[\hat{\theta}/(1+2\sigma), 1]$. The latter is equal to $\theta^{min}_\sigma(\hat{\theta}) = (\hat{\theta}+2\sigma)/(1+2\sigma)$ and is such that $x^*_\sigma(\theta^{min}_\sigma(\hat{\theta})) - \sigma = \hat{\theta}$. $\square$

Next, let $\Gamma^{\theta^*_\sigma} \equiv (\{0,1\}, \pi^{\theta^*_\sigma})$ be the optimal deterministic monotone policy (with cut-off $\hat{\theta} = \theta^*_\sigma$). Using the characterization of θ^*_σ in (1), we thus have that, under $\Gamma^{\theta^*_\sigma}$, at the point $\theta^{min}_\sigma(\theta^*_\sigma)$ at which $V_\sigma^{\Gamma^{\theta^*_\sigma}}$ reaches its minimum, $V_\sigma^{\Gamma^{\theta^*_\sigma}}(\theta^{min}_\sigma(\theta^*_\sigma)) = 0$. Using the fact that

$$V_\sigma^{\Gamma^{\theta^*_\sigma}}\left(\theta^{min}_\sigma(\theta^*_\sigma)\right) = g - (g+|b|)\,\frac{\theta^{min}_\sigma(\theta^*_\sigma) - \theta^*_\sigma}{(1+2\sigma)\,\theta^{min}_\sigma(\theta^*_\sigma) - \theta^*_\sigma},$$

we then have that $\theta^*_\sigma = (1+2\sigma)\frac{|b|}{g+|b|} - 2\sigma$. Next, let $\Gamma_\emptyset$ be the no-disclosure policy and note that, for any $\theta \in [0,1]$,

$$V_\sigma^{\Gamma_\emptyset}(\theta) = g - (g+|b|)\,\mathbb{P}_\sigma\{\tilde{\theta} \leq \theta | x^*_\sigma(\theta)\} = g + (g+|b|)(\theta - 1),$$

which is increasing in θ and has a unique zero at $\theta = |b|/(g+|b|) \equiv \theta^{MS}$.

This means that, in the absence of any disclosure, under the unique rationalizable strategy profile (and hence under MARP), each investor pledges if and only if $x > x^*_\sigma(\theta^{MS})$, and default occurs if and only if fundamentals are below θ^{MS}. The results above then imply that the optimal deterministic policy $\Gamma^{\theta^*_\sigma}$ is defined by a threshold $\theta^*_\sigma = (1+2\sigma)\,\theta^{MS} - 2\sigma = x^*_\sigma(\theta^{MS}) - \sigma$ that coincides with the left end-point of the support of the posterior beliefs of

[1] The notation $\mathbb{P}_\sigma\left\{\tilde{\theta} \leq \theta | \tilde{\theta} \geq \hat{\theta}; x\right\}$ stands for the probability that an investor with signal x assigns to the event that $\tilde{\theta} \leq \theta$ when the quality of his exogenous signal is parametrized by σ and the policy reveals that $\tilde{\theta} \geq \hat{\theta}$.

each agent with signal $x^*_\sigma\left(\theta^{MS}\right)$. In fact, for any truncation point $\hat{\theta} < x^*_\sigma\left(\theta^{MS}\right) - \sigma$, there exists θ close to θ^{MS} such that $V_\sigma^{\Gamma^{\hat{\theta}}}(\theta) < 0$ implying that refraining from pledging for all $x < x^*_\sigma\left(\theta^{MS}\right)$ is rationalizable in the continuation game following the announcement that $\theta \geq \hat{\theta}$, implying that the policy $\Gamma^{\hat{\theta}}$ fails to satisfy PCP. Similarly, for any truncation point $\hat{\theta} > x^*_\sigma(\theta^{MS}) - \sigma$, $V_\sigma^{\Gamma^{\hat{\theta}}}(\theta)$ reaches its minimum at $\theta_\sigma^{min}(\hat{\theta}) > \theta^{MS}$ and is such that $V_\sigma^{\Gamma^{\hat{\theta}}}(\theta_\sigma^{min}(\hat{\theta})) = V_\sigma^{\Gamma_\emptyset}(\theta_\sigma^{min}(\hat{\theta})) > V_\sigma^{\Gamma_\emptyset}(\theta^{MS}) = 0$, where the inequality follows from the monotonicity of $V_\sigma^{\Gamma_\emptyset}(\cdot)$. Hence, $\theta^*_\sigma = x^*_\sigma\left(\theta^{MS}\right) - \sigma$.

Step 2. Having characterized the optimal deterministic monotone policy $\Gamma^{\theta^*_\sigma}$, we now show that, when σ is small, there exists another policy Γ that also satisfies PCP and guarantees no default for a larger set of fundamentals than $\Gamma^{\theta^*_\sigma}$.

Let $\sigma^\# \equiv \frac{\theta^{MS}}{2(1-\theta^{MS})} > 0$. For any $\sigma \in (0, \sigma^\#)$, $\theta^*_\sigma = (1+2\sigma)\,\theta^{MS} - 2\sigma > 0$. For any $\sigma, \delta, \gamma > 0$ small, let $\theta''_\sigma(\delta,\gamma) \equiv x^*_\sigma\left(\theta^{MS} - \delta\right) - \sigma = (1+2\sigma)(\theta^{MS} - \delta) - 2\sigma$ and $\theta'_\sigma(\delta,\gamma) \equiv \theta''_\sigma(\delta,\gamma) - \gamma$. Note that, for any $\sigma \in (0, \sigma^\#)$, $\delta > 0$ and $\gamma > 0$ can be chosen so that $0 < {\theta'}_\sigma(\delta,\gamma) < \theta''_\sigma(\delta,\gamma) < \theta^*_\sigma$.

Consider the non-monotone deterministic policy $\Gamma_{\delta,\gamma} \equiv \{\{0,1\}, \pi_{\delta,\gamma}\}$ given by

$$\pi_{\delta,\gamma}(1|\theta) \equiv \mathbf{1}\left\{\theta \in \left[{\theta'}_\sigma(\delta,\gamma), \theta''_\sigma(\delta,\gamma)\right] \cup [\theta^*_\sigma, \infty)\right\}.$$

We show that, for any $\sigma \in (0, \sigma^\#)$, there exit $\delta, \gamma > 0$ such that (i) $0 \leq {\theta'}_\sigma(\delta,\gamma) < \theta''_\sigma(\delta,\gamma) < \theta^*_\sigma$, and (ii) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) \geq 0$ for all $\theta > {\theta'}_\sigma(\delta,\gamma)/(1+2\sigma)$, with $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) = 0$ only for $\theta = \theta^{MS}$.[2]

First observe that, for any $\sigma \in (0, \sigma^\#)$, $\delta \in \left(0, \theta^{MS} - \frac{2\sigma}{1+2\sigma}\right)$ and

$$0 < \gamma \leq (1+2\sigma)\left(\theta^{MS} - \delta\right) - 2\sigma \equiv R_0\left(\delta, \theta^{MS}, \sigma\right)$$

guarantee that $0 \leq {\theta'}_\sigma(\delta,\gamma) < \theta''_\sigma(\delta,\gamma) < \theta^*_\sigma$.[3]

Next note that, for any (σ,δ,γ) with $\sigma \in (0,\sigma^\#)$, $\delta \in \left(0, \theta^{MS} - 2\sigma/(1+2\sigma)\right)$ and $0 < \gamma \leq R_0\left(\delta, \theta^{MS}, \sigma\right)$, $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) = V_\sigma^{\Gamma^{\theta^*_\sigma}}(\theta)$ for all $\theta \in \left[\theta^{MS} - \delta, 1\right]$. Indeed, for any $\theta \in \left[\theta^{MS} - \delta, 1\right]$, $x^*_\sigma(\theta) - \sigma > \theta''_\sigma(\delta,\gamma)$ implying that the the posterior beliefs of the marginal investor with signal $x^*_\sigma(\theta)$ under the policy $\Gamma_{\delta,\gamma}$ coincide with those under the policy $\Gamma^{\theta^*_\sigma}$.

[2]Consistently with the notation above, $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta)$ is the expected payoff of the marginal investor with signal $x^*_\sigma(\theta)$ when the policy $\Gamma_{\delta,\gamma}$ announces that $s = 1$ and the quality of the agents' exogenous signals is parametrized by σ. For any $\theta < {\theta'}_\sigma(\delta,\gamma)/(1+2\sigma)$, $x^*_\sigma(\theta) + \sigma < \theta'$, which implies that the signal $x^*_\sigma(\theta)$ is not consistent with the event that fundamentals are above ${\theta'}_\sigma(\delta,\gamma)$. Equivalently, because the lowest signal that is consistent with $\theta \in [{\theta'}_\sigma(\delta,\gamma), \theta''_\sigma(\delta,\gamma)] \cup [\theta^*_\sigma, \infty)$ is ${\theta'}_\sigma(\delta,\gamma) - \sigma$, the lowest default threshold is ${\theta'}_\sigma(\delta,\gamma)/(1+2\sigma)$.

[3]Observe that $\sigma \in (0, \sigma^\#)$ implies that $\theta^{MS} - 2\sigma/(1+2\sigma) > 0$. In turn, $\delta \in \left(0, \theta^{MS} - 2\sigma/(1+2\sigma)\right)$ implies that $0 < \theta''_\sigma(\delta,\gamma) < \theta^*_\sigma$ and that $R_0\left(\delta, \theta^{MS}, \sigma\right) > 0$. Finally, that $0 < \gamma \leq R_0\left(\delta, \theta^{MS}, \sigma\right)$ implies that $0 \leq {\theta'}_\sigma(\delta,\gamma) < \theta''_\sigma(\delta,\gamma)$.

Let $\theta^{\sharp}_{\sigma}(\delta,\gamma)$ be such that $x^{*}_{\sigma}\left(\theta^{\sharp}_{\sigma}(\delta,\gamma)\right)-\sigma=\theta^{'}_{\sigma}(\delta,\gamma)$. Dropping the arguments of $\theta^{\sharp}_{\sigma}(\delta,\gamma)$, $\theta^{'}_{\sigma}(\delta,\gamma)$ and $\theta^{''}_{\sigma}(\delta,\gamma)$ to ease the notation, we have that

$$\theta' = \theta'' - \gamma = x^{*}_{\sigma}\left(\theta^{MS}-\delta\right)-\sigma-\gamma = (1+2\sigma)\left(\theta^{MS}-\delta\right)-2\sigma-\gamma.$$

From the definition of $\hat{\theta}$ we have that $x^{*}_{\sigma}(\hat{\theta})-\sigma = (1+2\sigma)\,\theta^{\sharp}-2\sigma=\theta'$. Combining the above two results we obtain that $\theta^{\sharp}=\theta^{MS}-\delta-\gamma/(1+2\sigma)$. Fixing $\sigma\in(0,\sigma^{\#})$, note that, for $\delta,\gamma>0$ small, $\theta^{\sharp}\geq\theta^{*}_{\sigma}$. Specifically, for any $\sigma\in(0,\sigma^{\#})$ and any $0<\delta<2\sigma\left(1-\theta^{MS}\right)$, $\theta^{\sharp}\geq\theta^{*}_{\sigma}$ if and only if

$$\gamma\leq(1+2\sigma)\left(2\sigma\left(1-\theta^{MS}\right)-\delta\right)\equiv R_1\left(\delta,\theta^{MS},\sigma\right).$$

Next, observe that, for any $\theta\in\left[\theta^{\sharp},\theta^{MS}-\delta\right)$,

$$\begin{aligned} V^{\Gamma_{\delta,\gamma}}_{\sigma}(\theta) &= g-(g+|b|)\,\mathbb{P}_{\sigma}\{\tilde{\theta}\leq\theta|\tilde{\theta}\in[x^{*}_{\sigma}(\theta)-\sigma,\theta'']\cup[\theta^{*}_{\sigma},\infty);x^{*}_{\sigma}(\theta)\} \\ &= g-(g+|b|)\left(\theta''-\theta^{*}_{\sigma}+2\sigma(1-\theta)\right)/\left(\theta''-\theta^{*}_{\sigma}+2\sigma\right), \end{aligned}$$

which is strictly increasing in θ. Similarly, for any $\theta\in\left[\theta^{*}_{\sigma},\theta^{\sharp}\right)$,

$$\begin{aligned} V^{\Gamma_{\delta,\gamma}}_{\sigma}(\theta) &= g-(g+|b|)\,\mathbb{P}_{\sigma}\{\tilde{\theta}\leq\theta|\tilde{\theta}\in[\theta',\theta'']\cup[\theta^{*}_{\sigma},\infty);x^{*}_{\sigma}(\theta)\} \\ &= g-(g+|b|)\,\frac{\theta-\theta^{*}_{\sigma}+\gamma}{x^{*}_{\sigma}(\theta)+\sigma-\theta^{*}_{\sigma}+\gamma}=g-(g+|b|)\,\frac{\theta-\theta^{*}_{\sigma}+\gamma}{(1+2\sigma)\,\theta-\theta^{*}_{\sigma}+\gamma}, \end{aligned}$$

which is strictly deceasing for any $\gamma\leq\theta^{*}_{\sigma}$. Note that $\theta'\geq0$ requires that $\gamma\leq\theta^{*}_{\sigma}$. Next, note that, for $\theta\in[\theta'',\theta^{*}_{\sigma})$,

$$\begin{aligned} V^{\Gamma_{\delta,\gamma}}_{\sigma}(\theta) &= g-(g+|b|)\,\mathbb{P}_{\sigma}\{\tilde{\theta}\leq\theta|\tilde{\theta}\in[\theta',\theta'']\cup[\theta^{*}_{\sigma},\infty);x^{*}_{\sigma}(\theta)\} \\ &= g-(g+|b|)\,\frac{\gamma}{x^{*}_{\sigma}(\theta)+\sigma-\theta^{*}_{\sigma}+\gamma}=g-(g+|b|)\,\frac{\gamma}{(1+2\sigma)\,\theta-\theta^{*}_{\sigma}+\gamma}, \end{aligned}$$

and, therefore, $V^{\Gamma_{\delta,\gamma}}_{\sigma}(\cdot)$ is increasing over the range $[\theta'',\theta^{*}_{\sigma})$. Finally, for $\theta\in[\theta',\theta'')$, we have that

$$\begin{aligned} V^{\Gamma_{\delta,\gamma}}_{\sigma}(\theta) &= g-(g+|b|)\,\mathbb{P}_{\sigma}\{\tilde{\theta}\leq\theta|\tilde{\theta}\in[\theta',\theta'']\cup[\theta^{*}_{\sigma},\infty);x^{*}_{\sigma}(\theta)\} \\ &= g-(g+|b|)\,\frac{\theta-\theta'}{x^{*}_{\sigma}(\theta)+\sigma-\theta^{*}_{\sigma}+\gamma}=g-(g+|b|)\,\frac{\theta-\theta'}{(1+2\sigma)\,\theta-\theta^{*}_{\sigma}+\gamma}. \end{aligned}$$

Hence $V^{\Gamma_{\delta,\gamma}}_{\sigma}(\cdot)$ is decreasing over $[\theta',\theta'')$ if $(1+2\sigma)\,\theta'=x^{*}_{\sigma}(\theta')+\sigma>\theta^{*}_{\sigma}$. Using the fact that $\theta'=\theta''-\gamma$, together with the fact that $\theta''=x^{*}_{\sigma}\left(\theta^{MS}-\delta\right)-\sigma$ and $\theta^{*}_{\sigma}=(1+2\sigma)\,\theta^{MS}-2\sigma$, we have that $(1+2\sigma)\,\theta' > \theta^*_\sigma$ if

$$\gamma \;<\; 2\sigma\left[(1+2\sigma)\,\theta^{MS}-2\sigma\right]/\,(1+2\sigma)-(1+2\sigma)\,\delta \equiv R_2\left(\delta,\theta^{MS},\sigma\right).$$

Lastly, observe that, for any $\theta \in [\theta'/(1+2\sigma),\theta']$, $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta)=g$.

We thus have that the function $V_\sigma^{\Gamma_{\delta,\gamma}}$ is such that (1) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta)\geq 0$ for all $\theta \geq \theta'/(1+2\sigma)$, and (2) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta)=0$ only if $\theta=\theta^{MS}$, if and only if the following conditions hold: (a) $V_\sigma^{\Gamma_{\delta,\gamma}}\left(\theta^\sharp\right)>0$, and (b) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta'')>0$. Requiring that $V_\sigma^{\Gamma_{\delta,\gamma}}\left(\theta^\sharp\right)>0$ is equivalent to

$$\begin{aligned} g-(g+|b|)\left(\theta^\sharp-\theta^*_\sigma+\gamma\right)/\left(x^*_\sigma\left(\theta^\sharp\right)+\sigma-\theta^*_\sigma+\gamma\right) &> 0\\ \Leftrightarrow \theta^{MS}\left(\theta^*_\sigma-\gamma\right)-\left((1+2\sigma)\,\theta^{MS}-2\sigma\right)\theta^\sharp &> 0. \end{aligned}$$

Recall that $\theta^*_\sigma=(1+2\sigma)\,\theta^{MS}-2\sigma$. Using the fact that $\theta^\sharp=\theta^{MS}-\delta-\frac{\gamma}{1+2\sigma}$, we conclude that a sufficient condition for $V_\sigma^{\Gamma_{\delta,\gamma}}\left(\theta^\sharp\right)>0$ is that

$$\begin{aligned} \left(\theta^{MS}-\delta-\gamma/\,(1+2\sigma)\right)\theta^*_\sigma &< \theta^{MS}\left(\theta^*_\sigma-\gamma\right)\\ \Leftrightarrow \gamma &< \delta\,(1+2\sigma)\left((1+2\sigma)\,\theta^{MS}-2\sigma\right)/\,(2\sigma)\equiv R_3\left(\delta,\theta^{MS},\sigma\right). \end{aligned}$$

Next, observe that $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta'')>0$ is equivalent to

$$\begin{aligned} \gamma &< \left(1-\theta^{MS}\right)\left((1+2\sigma)\,\theta''-\theta^*_\sigma+\gamma\right)\\ \Leftrightarrow \gamma &< \left(\frac{1-\theta^{MS}}{\theta^{MS}}\right)\left((1+2\sigma)\left[(1+2\sigma)\left(\theta^{MS}-\delta\right)-2\sigma\right]-(1+2\sigma)\,\theta^{MS}+2\sigma\right)\equiv R_4\left(\delta,\theta^{MS},\sigma\right). \end{aligned}$$

We conclude that, for any $\sigma \in (0,\sigma^\#)$, (i) $0 \leq \theta'_\sigma(\delta,\gamma) < \theta''_\sigma(\delta,\gamma) < \theta^*_\sigma$, and (ii) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta)\geq 0$ for all $\theta > \theta'_\sigma(\delta,\gamma)/(1+2\sigma)$, with $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta)=0$ only for $\theta=\theta^{MS}$, if

$$0<\delta<\min\left\{\theta^{MS}-\frac{2\sigma}{1+2\sigma},2\sigma\left(1-\theta^{MS}\right),\frac{2\sigma\left[(1+2\sigma)\,\theta^{MS}-2\sigma\right]}{(1+2\sigma)^2},\frac{2\sigma}{1+2\sigma}\left[\theta^{MS}-\frac{2\sigma}{1+2\sigma}\right]\right\}\equiv\varsigma\left(\theta^{MS},\sigma\right)$$

and $0<\gamma<\min_{0\leq i\leq 4}R_i\left(\delta,\theta^{MS},\sigma\right)$. Note that $\sigma<\sigma^\#$ implies that $\varsigma\left(\theta^{MS},\sigma\right)>0$, whereas $\delta<\varsigma\left(\theta^{MS},\sigma\right)$ implies that $\min_{0\leq i\leq 4}R_i\left(\delta,\theta^{MS},\sigma\right)>0$. Finally note that, for any $\sigma \in (0,\sigma^\#)$, and any $\theta \geq \theta'_\sigma(\delta,\gamma)$, the payoff $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta)$ is continuous in the threshold θ^*_σ. Hence there exists a policy Γ whose rule π is given by $\pi\,(1|\theta)\equiv \mathbf{1}\left\{\theta\in[\theta'_\sigma(\delta,\gamma),\theta''_\sigma(\delta,\gamma)]\cup[\theta^*_\sigma+\varepsilon,\infty)\right\}$ with $\varepsilon>0$ arbitrarily small, such that Γ strictly improves over $\Gamma^{\theta^*_\sigma}$ and is such that $V_\sigma^{\Gamma}(\theta)>0$ for all $\theta>\theta'_\sigma(\delta,\gamma)/(1+2\sigma)$, implying that Γ satisfies PCP. Q.E.D.

Proof of Example 3 in Main Text

Preliminaries. For any $\theta \in (0,1)$, any $\sigma \in \mathbb{R}_+$, note that, in this example $x_\sigma^*(\theta) \equiv \theta + \sigma\Phi^{-1}(\theta)$, where Φ is the cdf of the standard Normal distribution and ϕ its density. Also let $x_\sigma^*(0) \equiv -\infty$ and $x_\sigma^*(1) \equiv +\infty$. For any $(\theta_0, \hat{\theta}, \sigma) \in (0,1) \times \mathbb{R} \times \mathbb{R}_+$, let $\psi(\theta_0, \hat{\theta}, \sigma)$ denote the payoff from pledging of an investor with private signal $x_\sigma^*(\theta_0)$, when default occurs if and only if $\theta \leq \theta_0$, the policy reveals that $\theta \geq \hat{\theta}$, and the precision of private information is σ^{-2}. Then let $\hat{\sigma} \equiv \inf\{\sigma \in \mathbb{R}_+ :\ \psi(\theta_0,0,\sigma) > 0 \text{ all } \theta_0 \in (0,1)\}$ if $\{\sigma \in \mathbb{R}_+ :\ \psi(\theta_0,0,\sigma) > 0 \text{ all } \theta_0 \in (0,1)\} \neq \emptyset$ and else $\hat{\sigma} = +\infty$.[4] Then let $\Psi(\sigma) \equiv \inf_{\theta_0 \in (0,1)} \psi(\theta_0,0,\sigma)$ and note that $\lim_{\sigma \to 0^+} \Psi(\sigma) < 0$, implying that $\hat{\sigma} > 0$. For any $\sigma \in \mathbb{R}_+$ for which $\psi(\theta_0,0,\sigma) > 0$ for all $\theta_0 \in (0,1)$, the policy maker can avoid default for every $\theta > 0$ by using the monotone rule $\pi(\theta) = \mathbf{1}\{\theta > 0\}$. This case is uninteresting. Hereafter, we thus confine attention to the case in which $\sigma < \hat{\sigma}$.

Let $U_\sigma^\Gamma(x, 1|x)$ denote the payoff from pledging of an agent with signal x who expects all other agents to pledge if and only if their signal exceeds x , when the precision of private information is σ^{-2}, and the policy Γ announces that $s = 1$. Also let

$$U_\sigma^\Gamma\left(x_\sigma^*(0), 1|x_\sigma^*(0)\right) \equiv \lim_{x \to -\infty} U_\sigma^\Gamma(x, 1|x)$$

and

$$U_\sigma^\Gamma\left(x_\sigma^*(1), 1|x_\sigma^*(1)\right) \equiv \lim_{x \to +\infty} U_\sigma^\Gamma(x, 1|x).$$

Now let $\mathbb{G}_\sigma$ denote the set of deterministic binary policies $\Gamma = (\{0,1\}, \pi)$ such that, $\pi(\theta) = 0$ for all $\theta \leq 0$, $\pi(\theta) = 1$ for all $\theta > 1$ and $U_\sigma^\Gamma(x, 1|x) \geq 0$ for all $x \in \mathbb{R}$. From the proofs of Theorems 1 and 2, observe that, given any σ, any deterministic binary policy Γ satisfying PCP and such that $\pi(\theta) = 0$ for all $\theta \leq 0$ and $\pi(\theta) = 1$ for all $\theta > 1$ belongs in $\mathbb{G}_\sigma$. However, $\mathbb{G}_\sigma$ contains also policies that do not satisfy PCP.[5]

Proof Structure. The proof is in four steps. Step 1 establishes that, when σ is small, under any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$, any interval $(\theta', \theta''] \subset \left(0, \theta^{MS}\right]$ receiving a pass grade

[4] Recall that, when the announcement that $s = 1$ reveals that $\theta \geq 0$, the unique rationalizable profile features all agents pledging, irrespective of x, if and only if $\psi(\theta_0,0,\sigma) > 0$ for all $\theta_0 \in (0,1)$. This follows directly from Lemma 1 in the main text.

[5] These are those for which there exists x such that $U_\sigma^\Gamma(x, 1|x) = 0$; when Γ announces $s = 1$, in addition to the rationalizable profile under which all agents pledge, there also exists a rationalizable profile under which each agent pledges if and only if his signal exceeds x.

(i.e., such that $\pi(\theta) = 1$ for all $\theta \in (\theta', \theta'']$) has a sufficiently small Lebesgue measure, with the measure vanishing as $\sigma \to 0^+$.

Step 2 then considers an *auxiliary game* G_σ in which the agents play less aggressively than under MARP. Namely, G_σ is the game in which (i) the policy maker's choice set is $\mathbb{G}_\sigma$ and (ii) given *any* policy $\Gamma \in \mathbb{G}_\sigma$, all agents pledge after receiving the signal $s = 1$ and refrain from pledging after receiving the signal $s = 0$.[6] We show that, when σ is small, given any policy $\Gamma \in \mathbb{G}_\sigma$ that gives a fail grade to an interval $(\theta', \theta''] \subseteq (\underline{\theta}, \theta^{MS}]$ of large Lebesgue measure, there exists another policy $\Gamma^{\#} \in \mathbb{G}_\sigma$ that gives a pass grade to a F-positive measure subset of $(\theta', \theta'']$, has a mesh smaller than Γ, and is such that, when agents play as in G_σ, the probability of default under $\Gamma^{\#}$ is strictly smaller than under Γ.

Step 3 then combines the results from Steps 1 and 2 to show that, when σ is small, given any policy $\Gamma \in \mathbb{G}_\sigma$ for which the mesh $M(\Gamma)$ of $(0, \theta^{MS}]$ is larger than ε, there exists another policy $\Gamma' \in \mathbb{G}_\sigma$ with a mesh $M(\Gamma')$ smaller than ε such that, when agents play as in G_σ, the probability of default is strictly smaller under Γ' than under Γ. Starting from $\Gamma' \in \mathbb{G}_\sigma$ one can then construct a "nearby" policy $\Gamma^* \in \mathbb{G}_\sigma$ such that the probability of default under Γ^* is arbitrarily close to that under Γ' (and hence strictly smaller than under the original policy Γ) and such that $U_\sigma^{\Gamma^*}(x, 1|x) > 0$ for all x. The last property implies that Γ^* satisfies PCP also when agents play according to MARP. The policy Γ^* thus strictly improves upon Γ also in the original game, as claimed in the main text.

Finally, step 4 closes the proof by showing how to construct the function $\mathcal{E}$ relating the noise σ in the agents' exogenous private information to the bound $\mathcal{E}(\sigma)$ on the mesh of the policies.

Step 1. We start with the following result:

Lemma S3-A. *For any $\varepsilon \in \mathbb{R}_{++}$, there exists $\sigma(\varepsilon) \in \mathbb{R}_{++}$ such that, for any $\sigma \in (0, \sigma(\varepsilon)]$, the following is true: for any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$ and any cell $(\theta', \theta''] \in D^\Gamma$ with $|\theta'' - \theta'| > \varepsilon$, necessarily $\pi(\theta) = 0$.*

Proof of Lemma S3-A. We first show (Property S3-A below) that, for any $\sigma > 0$, if the policy maker were to replace Γ with the cutoff policy $\Gamma^{\theta'}$, then for any $\theta \leq \theta''$, $U_\sigma^{\Gamma^{\theta'}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) \geq U_\sigma^{\Gamma}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$.[7] Next, we show (Property S3-B below) that, for

[6] The agent's behavior is consistent with MARP only for those $\Gamma \in \mathbb{G}_\sigma$ for which, for all x, $U_\sigma^\Gamma(x, 1|x) > 0$.

[7] For any $\hat{\theta} \in [0,1]$, $\Gamma^{\hat{\theta}} = (\{0,1\}, \pi^{\hat{\theta}})$ is the deterministic monotone policy with cut-off $\hat{\theta}$.

any $\theta > \theta'$, as σ goes to zero, $U_\sigma^{\Gamma^{\theta'}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$ converges uniformly to $\int_0^1 u(\theta, A)dA$. Because $\int_0^1 u(\theta, A)dA < 0$ for $\theta < \theta^{MS}$, the above two properties imply that, for σ small, $U_\sigma^\Gamma(x_\sigma^*(\theta), 1|x_\sigma^*(\theta) < 0$ for some $\theta \in (\theta', \theta'']$, and hence that $\Gamma \notin \mathbb{G}_\sigma$. The result in the lemma then follows by contrapositive.

Property S3-A. *For any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$ and any cell $(\theta', \theta''] \in D^\Gamma$ such that $\pi(\theta) = 1$ for all $\theta \in (\theta', \theta'']$, $U_\sigma^\Gamma(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) \leq U_\sigma^{\Gamma^{\theta'}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$ for all $\theta \leq \theta''$.*

Proof of Property S3-A. The proof follows from Results S3-A-1 and S3-A-2 below.

Result S3-A-1. *Pick any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$. Given the partition $D^\Gamma \equiv \left\{d_i = \left(\underline{\theta}_i, \bar{\theta}_i\right] : i = 1, ..., N\right\}$ of $(0, \theta^{MS}]$ induced by Γ, take any cell $d_i = \left(\underline{\theta}_i, \bar{\theta}_i\right]$ for which $\pi(\theta) = 1$ for all $\theta \in d_i$. Let $\Gamma_L^i = \{\{0,1\}, \pi_L^i\} \in \mathbb{G}_\sigma$ be the policy constructed as follows: (a) $\pi_L^i(\theta) = 0$ for all $\theta \leq \underline{\theta}_i$; and (b) $\pi_L^i(\theta) = \pi(\theta)$ for all $\theta > \underline{\theta}_i$. Then, for all $\theta \in [0,1]$, $U_\sigma^{\Gamma_L^i}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) \geq U_\sigma^\Gamma(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$.*

Proof of Result S3-A-1. Note that, under the new policy, $\pi_L^i(\theta) = \pi(\theta) \times \mathbf{1}\{\theta > \underline{\theta}_i\}$. The posterior beliefs $\Lambda_\sigma^{\Gamma_L^i}(\cdot|x, 1)$ about θ of an agent with exogenous signal x and endogenous signal $s = 1$ under the new policy Γ_L^i thus dominate, in the FOSD sense, the analogous beliefs $\Lambda_\sigma^\Gamma(\cdot|x, 1)$ under the original policy Γ.[8] The result then follows from the fact that, given any default threshold θ, the payoff from pledging when the fundamentals are equal to $\tilde{\theta}$ and default occurs if and only if $\tilde{\theta} \leq \theta$ is nondecreasing in $\tilde{\theta}$. *End of Proof of Result S3-A-1.*

Result S3-A-2. *Pick any policy $\Gamma = \{\{0,1\}, \pi\} \in \mathbb{G}_\sigma$. Given the partition $D^\Gamma \equiv \left\{d_i = \left(\underline{\theta}_i, \bar{\theta}_i\right] : i = 1, ..., N\right\}$ of $(0, \theta^{MS}]$* induced by Γ, *take any cell $d_i = \left(\underline{\theta}_i, \bar{\theta}_i\right]$, $i \geq 2$, for which $\pi(\theta) = 1$ for all $\theta \in d_i$. Let $\Gamma_R^i = \{\{0,1\}, \pi_R^i\} \in \mathbb{G}_\sigma$ be the policy constructed from Γ as follows: (a) $\pi_R^i(\theta) = \pi(\theta)$ for all $\theta \leq \underline{\theta}_i$; and (b) $\pi_R^i(\theta) = 1$ for all $\theta > \underline{\theta}_i$. Then, for all $\theta \leq \bar{\theta}_i$, $U_\sigma^{\Gamma_R^i}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) \geq U_\sigma^\Gamma(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$.*

Proof of Result S3-A-2. Let $\Theta^1 \equiv \{\theta \in \Theta : \pi(\theta) = 1\}$ and $\Theta_i^0 \equiv \{\theta \in (\underline{\theta}_i, 1] : \pi(\theta) = 0\}$.

[8] No matter the shape of the beliefs $\Lambda_\sigma^\Gamma(\cdot|x, 1)$, the announcement that $\theta > \underline{\theta}_i$ is always "good news" in the sense of Milgrom (1981) and hence $\Lambda_\sigma^{\Gamma_L^i}(\cdot|x, 1) \succ_{FOSD} \Lambda_\sigma^\Gamma(\cdot|x, 1)$.

For any $\theta^{\#} \leq \bar{\theta}_i$, and any x,

$$\begin{aligned}
\Lambda_{\sigma}^{\Gamma_R^i}\left(\theta^{\#}|x,1\right) &= Pr\{\theta \leq \theta^{\#}|x, \theta \in (\Theta^1 \cup \Theta_i^0)\} = \frac{Pr\{\theta \leq \theta^{\#} \wedge \theta \in (\Theta^1 \cup \Theta_i^0)|x\}}{Pr\{\theta \in (\Theta^1 \cup \Theta_i^0)|x\}} \\
&= \frac{Pr\{\theta \leq \theta^{\#} \wedge \theta \in \Theta^1|x\}}{Pr\{\theta \in (\Theta^1 \cup \Theta_i^0)|x\}} + \frac{Pr\{\theta \leq \theta^{\#} \wedge \theta \in \Theta_i^0|x\}}{Pr\{\theta \in (\Theta^1 \cup \Theta_i^0)|x\}} = \frac{Pr\{\theta \leq \theta^{\#} \wedge \theta \in \Theta^1|x\}}{Pr\{\theta \in (\Theta^1 \cup \Theta_i^0)|x\}} \\
&\leq Pr\{\theta \leq \theta^{\#}|x, \theta \in \Theta^1\} = \Lambda_{\sigma}^{\Gamma}(\theta^{\#}|x,1).
\end{aligned}$$

Given the above inequality, and the fact that $b < 0 < g$, we then have that, for any $\theta \leq \bar{\theta}_i$,

$$\begin{aligned}
U_{\sigma}^{\Gamma}(x_{\sigma}^*(\theta), 1|x_{\sigma}^*(\theta)) &= b \cdot \Lambda_{\sigma}^{\Gamma}\left(\theta|x_{\sigma}^*(\theta), 1\right) + g \cdot \left(1 - \Lambda_{\sigma}^{\Gamma}\left(\theta|x_{\sigma}^*(\theta), 1\right)\right) \\
&\leq b \cdot \Lambda_{\sigma}^{\Gamma_R^i}\left(\theta|x_{\sigma}^*(\theta), 1\right) + g \cdot \left(1 - \Lambda_{\sigma}^{\Gamma_R^i}\left(\theta|x_{\sigma}^*(\theta), 1\right)\right) = U_{\sigma}^{\Gamma_R^i}(x_{\sigma}^*(\theta), 1|x_{\sigma}^*(\theta)).
\end{aligned}$$

End of Proof of Result S3-A-2.

Property S3-A follows from Results S3-A-1 and S3-A-2, by taking the cell $d_i = (\theta', \theta'']$. □

Now, fix $\varepsilon \in (0, \theta^{MS})$. For any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, let Γ^{θ^*} be the monotone rule with cut-off θ^*. For any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, any $\sigma \in \mathbb{R}_{++}$, let

$$H_{\sigma}(\theta^*; \varepsilon) \equiv \inf_{\theta \in [\theta^*, \theta^* + \varepsilon]} U_{\sigma}^{\Gamma^{\theta^*}}(x_{\sigma}^*(\theta), 1|x_{\sigma}^*(\theta)).$$

Note that $U_{\sigma}^{\Gamma^{\theta^*}}(x_{\sigma}^*(\theta), 1|x_{\sigma}^*(\theta))$ is continuous in $(\theta^*, \theta, \sigma)$ over $[0,1]^2 \times (0, \hat{\sigma}]$. From Berge's Maximum Theorem, $H_{\sigma}(\theta^*; \varepsilon)$ is thus continuous in (θ^*, σ) over $[0, \theta^{MS} - \varepsilon] \times (0, \hat{\sigma}]$.

For all $\theta^* \in [0, \theta^{MS} - \varepsilon]$, all $\theta \in (\theta^*, \theta^* + \varepsilon]$, $\lim_{\sigma \to 0^+} U_{\sigma}^{\Gamma^{\theta^*}}(x_{\sigma}^*(\theta), 1|x_{\sigma}^*(\theta)) = \int_0^1 u(\theta, A)\mathrm{d}A$. Because $\int_0^1 u(\theta, A)\mathrm{d}A$ is strictly increasing in θ and equal to zero at $\theta = \theta^{MS}$, for any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, $H_{0^+}(\theta^*; \varepsilon) \equiv \lim_{\sigma \to 0^+} H_{\sigma}(\theta^*; \varepsilon) = \lim_{\sigma \to 0^+} \lim_{\theta \to \theta^{*+}} U_{\sigma}^{\Gamma^{\theta^*}}(x_{\sigma}^*(\theta), 1|x_{\sigma}^*(\theta)) = \int_0^1 u(\theta^*, A)\mathrm{d}A$. We show next that $H_{\sigma}(\cdot; \varepsilon)$ converges uniformly to the limit function $H_{0^+}(\cdot; \varepsilon)$ over $[0, \theta^{MS} - \varepsilon]$.

Property S3-B. *Fix $\varepsilon \in (0, \theta^{MS})$. For any $\epsilon < \varepsilon$, there exists $\sigma'(\epsilon) > 0$ such that, for any $\sigma \leq \sigma'(\epsilon)$, and any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, $|H_{\sigma}(\theta^*; \varepsilon) - H_{0^+}(\theta^*; \varepsilon)| < \epsilon$.*

Proof of Property S3-B. The limit function $H_{0^+}(\cdot; \varepsilon)$ is *uniformly* continuous over $[0, \theta^{MS} - \varepsilon]$. As a consequence, there exists $\delta > 0$ such that for any $\theta, \tilde{\theta} \in [0, \theta^{MS} - \varepsilon]$, with $|\tilde{\theta} - \theta| \leq \delta$, necessarily $|H_{0^+}(\tilde{\theta}; \varepsilon) - H_{0^+}(\theta; \varepsilon)| < \epsilon/2$. Next, let $D_{\delta} \equiv \{(\underline{\theta}_i, \bar{\theta}_i] : i = 1, ..., N\}$, $N \in \mathbb{N}$, be any interval partition of $(0, \theta^{MS} - \varepsilon]$ with the property that every cell $(\underline{\theta}_i, \bar{\theta}_i] \in D_{\delta}$ is such that $|\bar{\theta}_i - \underline{\theta}_i| \leq \delta$. For any $i = 1, ..., N$, any $\sigma > 0$, let $\hat{\theta}^i_\sigma \equiv \sup\{\arg\max_{\theta\in[\underline{\theta}_i,\bar{\theta}_i]} H_\sigma(\theta;\varepsilon)\}$. That $H_\sigma(\theta;\varepsilon)$ is continuous in (σ,θ) implies that the hypothesis of Berge's Maximum Theorem hold and, hence, the correspondence $\arg\max_{\theta\in[\underline{\theta}_i,\bar{\theta}_i]} H_\sigma(\theta;\varepsilon)$ is compact-valued and upper hemi-continuous in σ. As a result, for any $\sigma > 0$, $\hat{\theta}^i_\sigma = \max\{\arg\max_{\theta\in[\underline{\theta}_i,\bar{\theta}_i]} H_\sigma(\theta;\varepsilon)\}$. Moreover, $\lim_{\sigma\to 0^+} H_\sigma(\hat{\theta}^i_\sigma;\varepsilon) = H_{0^+}(\hat{\theta}^i_{0^+};\varepsilon)$, where $\hat{\theta}^i_{0^+} \equiv \lim_{\sigma\to 0^+}\hat{\theta}^i_\sigma$.

For any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, let $\left(\underline{\theta}_j, \bar{\theta}_j\right] \in D_\delta$ be the partition cell containing θ^*. Then,

$$H_\sigma(\theta^*;\varepsilon) - H_{0^+}(\theta^*;\varepsilon) \leq H_\sigma(\hat{\theta}^j_\sigma;\varepsilon) - H_{0^+}(\theta^*;\varepsilon)$$
$$= H_\sigma(\hat{\theta}^j_\sigma;\varepsilon) - H_{0^+}(\hat{\theta}^j_{0^+};\varepsilon) + H_{0^+}(\hat{\theta}^j_{0^+};\varepsilon) - H_{0^+}(\theta^*;\varepsilon) < H_\sigma(\hat{\theta}^j_\sigma;\varepsilon) - H_{0^+}(\hat{\theta}^j_{0^+};\varepsilon) + \epsilon/2 < \epsilon$$

for all $\sigma < \bar{\sigma}_j(\epsilon)$, for some $\bar{\sigma}_j(\epsilon) > 0$. The first inequality is by definition of $\hat{\theta}^i_\sigma$. The second inequality follows from the fact that $|\hat{\theta}^j_{0^+} - \theta^*| < \delta$. The last inequality follows from the fact that $\lim_{\sigma\to 0^+} H_\sigma(\hat{\theta}^j_\sigma) = H_{0^+}(\hat{\theta}^j_{0^+})$. Similar arguments imply that $H_\sigma(\theta^*;\varepsilon) - H_{0^+}(\theta^*;\varepsilon) > -\epsilon$ for all $\sigma < \underline{\sigma}_j(\epsilon)$, for some $\underline{\sigma}_j(\epsilon) > 0$.

Now let $\sigma'(\epsilon) \equiv \min\{\min_{i\in N}\{\bar{\sigma}_i(\epsilon)\}, \min_{i\in N}\{\underline{\sigma}_i(\epsilon)\}\}$. For any $\sigma \leq \sigma'(\epsilon)$, and any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, we thus have that $|H_\sigma(\theta^*;\varepsilon) - H_{0^+}(\theta^*;\varepsilon)| < \epsilon$, thus proving that $H_\sigma(\cdot;\varepsilon)$ converges uniformly to $H_{0^+}(\cdot;\varepsilon)$ as $\sigma \to 0^+$. This completes the proof of Property S3-B. □

Next, given $\varepsilon \in (0, \theta^{MS})$, pick an arbitrary $\eta \in (\int_0^1 u(\theta^{MS} - \varepsilon, A)\mathrm{d}A, 0)$. Because $H_{0^+}(\theta^*;\varepsilon) \leq \eta$ for all $\theta^* \in [0, \theta^{MS} - \varepsilon]$, and because $H_\sigma(\cdot;\varepsilon)$ converges uniformly to $H_{0^+}(\cdot;\varepsilon)$, there exists $\sigma(\varepsilon) > 0$ such that, for any $\sigma < (\varepsilon)$, and any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, $H_\sigma(\theta^*;\varepsilon) \leq \eta < 0$. Therefore, for any $\sigma < \sigma(\varepsilon)$, and any monotone policy Γ^{θ^*} with cut-off $\theta^* \in [0, \theta^{MS} - \varepsilon]$, there exists $\theta \in [\theta^*, \theta^* + \varepsilon]$ such that $U^{\Gamma^{\theta^*}}_\sigma(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)) \leq \eta$.

Together, Properties S3-A and S3-B then imply that, for any $\sigma < \sigma(\varepsilon)$, and any policy Γ such that $\pi(\theta) = 1$ for all $\theta \in (\theta', \theta'']$ for some $(\theta', \theta''] \in D^\Gamma$ with $|\theta'' - \theta'| > \varepsilon$, necessarily $U^\Gamma_\sigma(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)) < 0$ for some $\theta \in (\theta', \theta'']$. Hence $\Gamma \notin \mathbb{G}_\sigma$. The claim in Lemma S3-A then follows by contrapositive. This completes the proof of Lemma S3-A. ■

Step 2. Next, we show that, for any policy $\Gamma = (\{0, 1\}, \pi) \in \mathbb{G}_\sigma$ that gives a fail grade to an interval $(\theta', \theta''] \subseteq (0, \theta^{MS}]$ of large Lebesgue measure, there exists another policy $\Gamma^\# \in \mathbb{G}_\sigma$ with a mesh $M(\Gamma^\#) < M(\Gamma)$ such that, when agents play as in G_σ, the probability of default under $\Gamma^\#$ is strictly smaller than under Γ. The result follows from Lemmas S3-B, S3-C and S3-D below.

Lemma S3-B. *For any* $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$ *such that* $\inf_{\theta\in[0,1]}\ U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right), 1|x_\sigma^*\left(\theta\right)\right) > 0$*, there exists another policy* $\hat{\Gamma} = (\{0,1\}, \hat{\pi}) \in \mathbb{G}_\sigma$*, with* $M(\hat{\Gamma}) \leq M(\Gamma)$*, such that, in the auxiliary game* G_σ*, the probability of default under* $\hat{\Gamma}$ *is strictly smaller than under* Γ*.*

Proof of Lemma S3-B. That $\inf_{\theta\in[0,1]}\ U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right), 1|x_\sigma^*\left(\theta\right)\right) > 0$ implies that, starting from $\Gamma = (\{0,1\}, \pi)$, one can construct another policy $\hat{\Gamma} = (\{0,1\}, \hat{\pi})$ sufficiently close to Γ (in the L_1 norm) and such that $\hat{\pi}(\theta) \geq \pi(\theta)$ for all θ, with the inequality strict over some positive F-measure set $(\tilde{\theta}', \tilde{\theta}'') \subseteq (0,1]$, and such that (a) $\hat{\pi}(\theta) = 0$ for all $\theta \leq 0$, (b) $\hat{\pi}(\theta) = 1$ for all $\theta > 1$, (c) $U_\sigma^{\hat{\Gamma}}(x, 1|x) \geq 0$ all x, and (d) $M(\hat{\Gamma}) \leq M(\Gamma)$. By definition of $\mathbb{G}_\sigma$, $\hat{\Gamma} \in \mathbb{G}_\sigma$. That, in the auxiliary game G_σ, the probability of default under $\hat{\Gamma}$ is strictly smaller than under Γ, then follows from the fact that all agents pledge when they receive the signal $s = 1$. This completes the proof of Lemma S3-B. ■

For any $\sigma > 0$, and any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$, $U_\sigma^\Gamma\left(x_\sigma^*\left(\cdot\right), 1|x_\sigma^*\left(\cdot\right)\right)$ is continuous over $[0,1]$. Hence $\inf_{\theta\in[0,1]}\ U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right), 1|x_\sigma^*\left(\theta\right)\right) = \min_{\theta\in[0,1]}\ U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right), 1|x_\sigma^*\left(\theta\right)\right)$.

Lemma S3-C. *Let* $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$ *be such that* $min_{\theta\in[0,1]}\ U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right), 1|x_\sigma^*\left(\theta\right)\right) = 0$*. For any* $\theta_\sigma^\sharp \in \arg min_{\theta\in[0,1]}\ U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right), 1|x_\sigma^*\left(\theta\right)\right)$*, there exists* $\gamma_\sigma^\Gamma > 0$ *such that* $\pi\left(\theta\right) = 1$ *for* $F-$*almost all* $\theta \in \left(\theta_\sigma^\sharp - \gamma_\sigma^\Gamma, \theta_\sigma^\sharp\right)$.

Proof of Lemma S3-C. The proof is by contraposition. Suppose there exists $\delta > 0$ such that $\pi\left(\theta\right) = 0$ for F-almost all $\theta \in \left(\theta_\sigma^\sharp - \delta, \theta_\sigma^\sharp\right)$. Observe that the sign of

$$U_\sigma^\Gamma\left(x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right), 1|x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right)\right)$$

is the same as the sign of

$$b\int_{-\infty}^{\theta_\sigma^\sharp - \delta} \phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right) - \theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta) + g\int_{\theta_\sigma^\sharp - \delta}^{+\infty} \phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right) - \theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta).$$

Next observe that

$$
\begin{aligned}
0 &= U_\sigma^\Gamma\left(x_\sigma^*\left(\theta_\sigma^\sharp\right), 1|x_\sigma^*\left(\theta_\sigma^\sharp\right)\right)\int_{-\infty}^{+\infty}\phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp\right)-\theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta)\\
&= \int_{-\infty}^{\infty}\left(b\mathbf{1}\left\{\theta\leq\theta_\sigma^\sharp\right\}+g\mathbf{1}\left\{\theta>\theta_\sigma^\sharp\right\}\right)\phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp\right)-\theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta)\\
&> \int_{-\infty}^{\infty}\left(b\mathbf{1}\left\{\theta\leq\theta_\sigma^\sharp\right\}+g\mathbf{1}\left\{\theta>\theta_\sigma^\sharp\right\}\right)\phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp-\delta\right)-\theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta)\\
&= \int_{-\infty}^{\infty}\left(b\mathbf{1}\left\{\theta\leq\theta_\sigma^\sharp-\delta\right\}+g\mathbf{1}\left\{\theta>\theta_\sigma^\sharp-\delta\right\}\right)\phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp-\delta\right)-\theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta)\\
&= U_\sigma^\Gamma\left(x_\sigma^*\left(\theta_\sigma^\sharp-\delta\right), 1|x_\sigma^*\left(\theta_\sigma^\sharp-\delta\right)\right)\int_{-\infty}^{+\infty}\phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp-\delta\right)-\theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta)
\end{aligned}
$$

The first equality follows from the assumptions of the lemma. The second equality follows from the definition of the function $U_\sigma^\Gamma\left(x_\sigma^*\left(\theta_\sigma^\sharp\right), 1|x_\sigma^*\left(\theta_\sigma^\sharp\right)\right)$. The inequality follows from the monotonicity of $x_\sigma^*(\cdot)$, the fact that $\phi\left((x-\theta)/\sigma\right)$ is log-supermodular in (x,θ), and Property SCB in the proof of Theorem 2 in the main text. The third equality follows from the fact that $\pi\left(\theta\right)=0$ for F-almost all $\theta\in\left(\theta_\sigma^\sharp-\delta,\theta_\sigma^\sharp\right)$. The last equality follows from the definition of the function $U_\sigma^\Gamma(x_\sigma^*(\theta_\sigma^\sharp-\delta),1|x_\sigma^*(\theta_\sigma^\sharp-\delta))$. Hence, $U_\sigma^\Gamma(x_\sigma^*(\theta_\sigma^\sharp-\delta),1|x_\sigma^*(\theta_\sigma^\sharp-\delta))<0$, thus contradicting the assumption that $\Gamma\in\mathbb{G}_\sigma$. This completes the proof of Lemma S3-C. ■

Lemma S3-D. *For any $\varepsilon>0$, there exists $\sigma^\#(\varepsilon)\in(0,\hat\sigma)$ such that, for any $\sigma\in(0,\sigma^\#(\varepsilon)]$, and any policy $\Gamma=(\{0,1\},\pi)\in\mathbb{G}_\sigma$ for which there exists $(\theta',\theta'']\in D^\Gamma$ such that (a) $|\theta''-\theta'|>\varepsilon$ and (b) $\pi(\theta)=0$ for all $\theta\in(\theta',\theta'']$, there exists another policy $\Gamma^\#=\left(\{0,1\},\pi^\#\right)\in\mathbb{G}_\sigma$, with $M(\Gamma^\#)\leq M(\Gamma)$, such that, in the auxiliary game G_σ, the probability of default under $\Gamma^\#$ is strictly smaller than under Γ.*

Proof of Lemma S3-D. For any $\theta\in(0,1)$, $\lim\limits_{\sigma\to0^+}x_\sigma^*\left(\theta\right)\equiv x_{0^+}^*\left(\theta\right)=\theta$. Furthermore, for any $\varepsilon\in(0,\min\{\theta^{MS},1-\theta^{MS}\})$, the function $x_{0^+}^*:\left[\frac{\varepsilon}{4},1-\frac{\varepsilon}{4}\right]\to\mathbb{R}$ is uniformly continuous. Hence, for any $\delta<\varepsilon/4$, there exists $\tilde\sigma\left(\delta\right)>0$ such that, for any $\sigma\in(0,\tilde\sigma(\delta)]$, and any $\theta\in\left[\frac{\varepsilon}{4},1-\frac{\varepsilon}{4}\right]$, we have that $|x_\sigma^*\left(\theta\right)-\theta|\leq\delta$.[9] In turn, this implies that, for any $\varepsilon>0$ small, there exists $\sigma^\#(\varepsilon)\in(0,\hat\sigma]$ such that, for any $\sigma\in(0,\sigma^\#(\varepsilon)]$, and any $(\theta',\theta'']\in D^\Gamma$ such that $|\theta''-\theta'|>\varepsilon$, we have that, for any $\theta\in[\theta'',1-\frac{\varepsilon}{4}]$, $|\theta-x_\sigma^*\left(\theta\right)|<|(\theta'+\theta'')/2-x_\sigma^*\left(\theta\right)|$. Likewise, for any $\theta\in[\varepsilon/4,\theta']$, and any $\hat\theta\geq\theta''$, we have that $|\theta-x_\sigma^*\left(\theta\right)|<|x_\sigma^*\left(\theta\right)-\hat\theta|$ when

[9] The proof for the existence of a sequence $\left\{x_{\sigma_n}^*(\cdot)\right\}_n$ with domain $\left[\frac{\varepsilon}{4},1-\frac{\varepsilon}{4}\right]$ converging uniformly to its limit function $x_{0^+}^*(\cdot)$ follows from the same arguments that establish the uniform convergence of $\{H_{\sigma_n}(\cdot)\}_n$ to $H_{0^+}(\cdot)$ in Step 1.

$\sigma \in (0, \sigma^{\#}(\varepsilon)]$.

Next, pick any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$ for which there exists $d \equiv (\theta', \theta''] \in D^\Gamma$ such that (a) $|\theta'' - \theta'| > \varepsilon$ and (b) $\pi(\theta) = 0$ for all $\theta \in (\theta', \theta'']$. If $\min_{\theta\in[0,1]} U_\sigma^\Gamma\left(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)\right) > 0$, the result follows directly from Lemma S3-B. Thus assume that $\min_{\theta\in[0,1]} U_\sigma^\Gamma\left(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)\right) = 0$.

Suppose that $\min_{\theta\in[\theta'',1]} U_\sigma^\Gamma\left(x_\sigma^*(\theta), 0|x_\sigma^*(\theta)\right) > 0$. By Lemma S3-C, $U_\sigma^\Gamma\left(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)\right) > 0$ for all $\theta \in (\theta', \theta'']$. Hence, $\min_{\theta\in[\theta',1]} U_\sigma^\Gamma\left(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)\right) > 0$.

Below we show that, starting from Γ, we can then construct a policy $\Gamma^\eta \in \mathbb{G}_\sigma$, with $M(\Gamma^\eta) \le M(\Gamma)$ such that, when agents play as in G_σ, the probability of default under Γ^η is strictly smaller than under Γ. Γ^η is obtained from Γ by giving a pass grade to a positive-measure interval of types in the middle of $(\theta', \theta'']$. Formally, take $\eta \in (0, (\theta'' - \theta')/2)$ and let $\Gamma^\eta = (\{0,1\}, \pi^\eta)$ be the policy whose rule π^η is given by (a) $\pi^\eta(\theta) = \pi(\theta)$ for all $\theta \notin [(\theta'+\theta'')/2, (\theta'+\theta'')/2+\eta]$, and (b) $\pi^\eta(\theta) = 1$ for all $\theta \in [(\theta'+\theta'')/2, (\theta'+\theta'')/2+\eta]$. Below we show that $U^{\Gamma^\eta}\left(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)\right) \ge 0$ for all $\theta \in [0,1]$. To see this, let $\Theta^1 \equiv \{\theta \in \Theta : \pi(\theta) = 1\}$ be the collection of fundamentals receiving a pass grade under the original policy Γ. For any $\theta \in [0, \theta']$, and any x,

$$
\begin{aligned}
&\Lambda_\sigma^{\Gamma^\eta}(\theta|x,1) = Pr\{\tilde\theta \le \theta | x, \tilde\theta \in (\Theta^1 \cup [(\theta'+\theta'')/2, (\theta'+\theta'')/2+\eta])\} \\
&= \tfrac{Pr\{\tilde\theta \le \theta \wedge \tilde\theta \in \Theta^1 | x\}}{Pr\{\tilde\theta \in (\Theta^1 \cup [(\theta'+\theta'')/2, (\theta'+\theta'')/2+\eta]) | x\}} \le Pr\{\tilde\theta \le \theta | x, \tilde\theta \in \Theta^1\} = \Lambda_\sigma^\Gamma(\theta|x,1).
\end{aligned}
$$

The first equality follows from the fact that, under Γ^η, the signal $s = 1$ carries the same information as the announcement that $\tilde\theta \in (\Theta^1 \cup [(\theta'+\theta'')/2, (\theta'+\theta'')/2+\eta])$. The inequality follows from the fact that $Pr\left\{\tilde\theta \in (\Theta^1 \cup [(\theta'+\theta'')/2, (\theta'+\theta'')/2+\eta])\, | x\right\} > Pr\left\{\tilde\theta \in \Theta^1 | x\right\}$. The last equality follows from fact that, under the original policy Γ, the signal $s = 1$ carries the same information as the announcement that $\tilde\theta \in \Theta^1$.

Given the above inequality, and the fact that, $b < 0 < g$, we then have that, for any $\theta \in [0, \theta']$,

$$
\begin{aligned}
&U_\sigma^\Gamma(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) = b \cdot \Lambda_\sigma^\Gamma\left(\theta|x_\sigma^*(\theta), 1\right) + g \cdot \left[1 - \Lambda_\sigma^\Gamma\left(\theta|x_\sigma^*(\theta), 1\right)\right] \\
&\le b \cdot \Lambda_\sigma^{\Gamma^\eta}\left(\theta|x_\sigma^*(\theta), 1\right) + g \cdot \left[1 - \Lambda_\sigma^{\Gamma^\eta}\left(\theta|x_\sigma^*(\theta), 1\right)\right] = U_\sigma^{\Gamma^\eta}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)).
\end{aligned}
$$

Hence $U^{\Gamma^\eta}\left(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)\right) \ge 0$, all $\theta \le \theta'$. That $\min_{\theta\in[\theta',1]} U_\sigma^\Gamma\left(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)\right) > 0$, along with the continuity of $U_\sigma^{\Gamma^\eta}\left(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)\right)$ in η implies that $\min_{\theta\in[0,1]} U_\sigma^{\Gamma^\eta}\left(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)\right) \ge 0$ for η small. Hence $\Gamma^\eta \in \mathbb{G}_\sigma$.

Next, consider the more interesting case in which $\min_{\theta\in[\theta'',1]} U^{\Gamma}_{\sigma}\left(x^*_{\sigma}\left(\theta\right),0|x^*_{\sigma}\left(\theta\right)\right) = 0$. Let $\theta^{\#}_{\sigma} \equiv \inf\left\{\theta \geq \theta'' : U^{\Gamma}_{\sigma}(x^*_{\sigma}(\theta),1|x^*_{\sigma}(\theta)) = 0\right\}$. An implication of Lemma S3-C is that that $\theta^{\#}_{\sigma} > \theta''$. Also let $(\theta''',\theta'''']\subset[0,1]$ be the first interval to the immediate right of $(\theta',\theta'']$ such that $\pi(\theta) = 1$ for all $\theta\in(\theta''',\theta'''']$ and let $\hat{\theta} = \min\left\{\theta'''',\theta^{\#}_{\sigma}\right\}$.[10]

Now, pick $\xi > 0$ small and let $\delta(\xi)$ be implicitly defined by

$$F((\theta'+\theta'')/2+\xi) - F((\theta'+\theta'')/2) = F((\theta'''+\hat{\theta})/2+\delta(\xi)) - F((\theta'''+\hat{\theta})/2). \tag{S7}$$

Consider the policy $\Gamma^{\xi} = (\{0,1\},\pi^{\xi})$ defined by (a) $\pi^{\xi}(\theta) = \pi(\theta)$ for all $\theta \notin [(\theta'+\theta'')/2,(\theta'+\theta'')/2+\xi]\cup[(\theta'''+\hat{\theta})/2,(\theta'''+\hat{\theta})/2+\delta(\xi)]$, (b) $\pi^{\xi}(\theta) = 1$ for all $\theta\in[(\theta'+\theta'')/2,(\theta'+\theta'')/2+\xi]$, and (c) $\pi^{\xi}(\theta) = 0$ for all $\theta\in[(\theta'''+\hat{\theta})/2,(\theta'''+\hat{\theta})/2+\delta(\xi)]$. Below we establish that, when $\xi > 0$ is small, such a policy is such that $\min_{\theta\in[0,1]} U^{\Gamma^{\xi}}\left(x^*_{\sigma}\left(\theta\right),1|x^*_{\sigma}\left(\theta\right)\right) > 0$ and hence $\Gamma^{\xi}\in\mathbb{G}_{\sigma}$. To see this, for any arbitrary policy $\tilde{\Gamma} = \{\{0,1\},\tilde{\pi}\}$, any $\theta\in[0,1]$, let

$$V^{\tilde{\Gamma}}_{\sigma}\left(\theta\right) \equiv U^{\tilde{\Gamma}}_{\sigma}\left(x^*_{\sigma}\left(\theta\right),1|x^*_{\sigma}\left(\theta\right)\right)p^{\tilde{\Gamma}}_{\sigma}\left(x^*_{\sigma}\left(\theta\right),1\right),$$

where, for any x, $p^{\tilde{\Gamma}}_{\sigma}(x,1) \equiv \int_{\Theta}\tilde{\pi}(\theta)p_{\sigma}(x|\theta)\mathrm{d}F(\theta)$, with $p_{\sigma}(x|\theta) \equiv \frac{1}{\sigma}\phi((x-\theta)/\sigma)$.

By definition of $\theta^{\#}_{\sigma}$, we must have that, for all θ, $0 = V^{\Gamma}_{\sigma}\left(\theta^{\#}_{\sigma}\right) \leq V^{\Gamma}_{\sigma}\left(\theta\right)$. Next, for any $\xi > 0$, define $\varphi_R\left(\xi\right) \equiv \min_{\theta\in[\theta'',1]} V^{\Gamma^{\xi}}_{\sigma}\left(\theta\right)$. Let $\overline{u}(\tilde{\theta},\theta) \equiv g\mathbf{1}\{\tilde{\theta} > \theta\} + b\mathbf{1}\{\tilde{\theta}\leq\theta\}$ and note that, for any θ,

$$V^{\Gamma^{\xi}}_{\sigma}\left(\theta\right) = V^{\Gamma}_{\sigma}\left(\theta\right) + \int_{(\theta'+\theta'')/2}^{(\theta'+\theta'')/2+\xi}\overline{u}(\tilde{\theta},\theta)p_{\sigma}(x^*_{\sigma}\left(\theta\right)|\tilde{\theta})\mathrm{d}F(\tilde{\theta}) - \int_{(\theta'''+\hat{\theta})/2}^{(\theta'''+\hat{\theta})/2+\delta(\xi)}\overline{u}(\tilde{\theta},\theta)p_{\sigma}(x^*_{\sigma}\left(\theta\right)|\tilde{\theta})\mathrm{d}F(\tilde{\theta}).$$

Using the envelope theorem, we have that, for any $\theta^{\xi}_{\sigma}\in\arg\min_{\theta\in[\theta'',1]} V^{\Gamma^{\xi}}_{\sigma}\left(\theta\right)$,

$$\begin{aligned}
\varphi'_R\left(\xi\right) &= f\left((\theta'+\theta'')/2+\xi\right)\overline{u}\left((\theta'+\theta'')/2+\xi,\theta^{\xi}_{\sigma}\right)p_{\sigma}\left(x^*_{\sigma}(\theta^{\xi}_{\sigma})|(\theta'+\theta'')/2+\xi\right) \\
&\quad -f\left((\theta'''+\hat{\theta})/2+\delta(\xi)\right)\overline{u}((\theta'''+\hat{\theta})/2+\delta(\xi),\theta^{\xi}_{\sigma})p_{\sigma}(x^*_{\sigma}(\theta^{\xi}_{\sigma})|(\theta'''+\hat{\theta})/2+\delta(\xi))\delta'(\xi) \\
&= f\left((\theta'+\theta'')/2+\xi\right)[\overline{u}((\theta'+\theta'')/2+\xi,\theta^{\xi}_{\sigma})p_{\sigma}(x^*_{\sigma}(\theta^{\xi}_{\sigma})|(\theta'+\theta'')/2+\xi) \\
&\quad -\overline{u}((\theta'''+\hat{\theta})/2+\delta(\xi),\theta^{\xi}_{\sigma})p_{\sigma}(x^*_{\sigma}(\theta^{\xi}_{\sigma})|(\theta'''+\hat{\theta})/2+\delta(\xi))],
\end{aligned}$$

where the second equality uses the implicit function theorem applied to (S7) to obtain that

[10] The existence of such an interval follows from the fact that $\pi(\theta) = 1$ in a left neighborhood of $\theta^{\#}_{\sigma}$ by virtue of Lemma S3-C. Also observe that, when $\theta'' < \theta^{MS}$, such an interval is adjacent to $(\theta',\theta'']$ and hence $\theta''' = \theta''$.

$\delta'(\xi) = f\left((\theta' + \theta'')/2 + \xi\right)/f((\theta''' + \hat{\theta})/2 + \delta(\xi))$. As a consequence,

$$\lim_{\xi\to 0^+} \varphi_R'(\xi) = f\left((\theta' + \theta'')/2\right)[\overline{u}\left((\theta' + \theta'')/2, \theta_\sigma^\#\right) p_\sigma\left(x_\sigma^*(\theta_\sigma^\#)|(\theta' + \theta'')/2\right) \\ - \overline{u}((\theta''' + \hat{\theta})/2, \theta_\sigma^\#) p_\sigma(x_\sigma^*(\theta_\sigma^\#)|(\theta''' + \hat{\theta})/2)]. \tag{S8}$$

That $\sigma < \sigma^\#(\varepsilon)$ implies that $|x_\sigma^*\left(\theta_\sigma^\#\right) - (\theta''' + \hat{\theta})/2| < |x_\sigma^*\left(\theta_\sigma^\#\right) - (\theta' + \theta'')/2|$. That $p_\sigma(x|\theta)$ is single-peaked in turn implies that $p_\sigma(x_\sigma^*(\theta_\sigma^\#)|(\theta' + \theta'')/2) < p_\sigma(x_\sigma^*(\theta_\sigma^\#)|(\theta''' + \hat{\theta})/2)$ and hence that

$$\overline{u}\left((\theta' + \theta'')/2, \theta_\sigma^\#\right) p_\sigma\left(x_\sigma^*\left(\theta_\sigma^\#\right)|(\theta' + \theta'')/2\right) - \overline{u}\left((\theta''' + \hat{\theta})/2, \theta_\sigma^\#\right) p_\sigma\left(x_\sigma^*\left(\theta_\sigma^\#\right)|(\theta''' + \hat{\theta})/2\right) \\ = b \times \left(p_\sigma\left(x_\sigma^*\left(\theta_\sigma^\#\right)|(\theta' + \theta'')/2\right) - p_\sigma\left(x_\sigma^*\left(\theta_\sigma^\#\right)|(\theta''' + \hat{\theta})/2\right)\right) > 0.$$

Thus, $\lim_{\xi\to 0^+} \varphi_R'(\xi) > 0$. By continuity of $U_\sigma^{\Gamma^\xi}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$ in ξ, we then have that, for $\xi > 0$ small, $\min_{\theta\in[\theta'',1]} U_\sigma^{\Gamma^\xi}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) > 0$.

Next, we prove that, under the policy Γ^ξ, $\min_{\theta\in[0,\theta'']} U_\sigma^{\Gamma^\xi}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) > 0$. For any $\xi > 0$, define $\varphi_L(\xi) \equiv \min_{\theta\in[0,\theta']} V_\sigma^{\Gamma^\xi}(\theta)$. Arguments similar to those used above to compute $\lim_{\xi\to 0^+} \varphi_R'(\xi)$ imply that, for any $\theta_\sigma^{\#\#} \in \arg\min_{\theta\in[0,\theta']} V_\sigma^\Gamma(\theta)$, when $\sigma \leq \sigma^\#(\varepsilon)$,

$$\begin{aligned} \lim_{\xi\to 0^+} \varphi_L'(\xi) &= f\left((\theta' + \theta'')/2\right)[\overline{u}((\theta' + \theta'')/2, \theta_\sigma^{\#\#}) p_\sigma(x_\sigma^*(\theta_\sigma^{\#\#})|(\theta' + \theta'')/2) \\ &\quad - \overline{u}((\theta''' + \hat{\theta})/2, \theta_\sigma^{\#\#}) p_\sigma(x_\sigma^*(\theta_\sigma^{\#\#})|(\theta''' + \hat{\theta})/2)] \\ &= f\left((\theta' + \theta'')/2\right) g\left[p_\sigma(x_\sigma^*(\theta_\sigma^{\#\#})|(\theta' + \theta'')/2) - p_\sigma(x_\sigma^*(\theta_\sigma^{\#\#})|(\theta''' + \hat{\theta})/2))\right] > 0. \end{aligned}$$

The first equality follows from steps analogous to those used to establish (S8). The second equality follows from the fact that, by assumption $\theta_\sigma^{\#\#} \leq \theta'$. The inequality is a consequence of the fact that, for $\sigma \leq \sigma^\#(\varepsilon)$, $|x_\sigma^*\left(\theta_\sigma^{\#\#}\right) - (\theta' + \theta'')/2| < |x_\sigma^*\left(\theta_\sigma^{\#\#}\right) - (\theta''' + \hat{\theta})/2)|$, which, together with the fact that the noise distribution is single-peaked, implies that

$$p_\sigma(x_\sigma^*(\theta_\sigma^{\#\#})|(\theta' + \theta'')/2) > p_\sigma(x_\sigma^*(\theta_\sigma^{\#\#})|(\theta''' + \hat{\theta})/2).$$

Hence, for $\xi > 0$ small, $U_\sigma^{\Gamma^\xi}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) > 0$ for all $\theta \in [0, \theta']$. Furthermore, by Lemma S3-C, $U_\sigma^\Gamma(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) > 0$ for all $\theta \in (\theta', \theta'']$. Hence, provided that ξ is small, the continuity of $U_\sigma^{\Gamma^\xi}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$ in ξ implies that $U_\sigma^{\Gamma^\xi}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) > 0$ also for $\theta \in (\theta', \theta'']$. Combining all the properties above, we thus conclude that, for $\xi > 0$ small, $\min_{\theta\in[0,1]} U_\sigma^{\Gamma^\xi}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) > 0$. Hence $\Gamma^\xi \in \mathbb{G}_\sigma$.

By construction, $M(\Gamma^\xi) < M(\Gamma)$. Furthermore, when agents play according to G_σ, the probability of default under Γ^{ξ} is the same as under Γ. Lemma S3-B then implies that, starting from Γ^{ξ}, one can construct a policy $\Gamma^{\#} \in \mathbb{G}_{\sigma}$, close to Γ^{ξ} in the L_1 norm, such that (1) $M(\Gamma^{\#}) \leq M(\Gamma^{\xi})$ and (2), when agents play according to G_{σ}, the probability of default under $\Gamma^{\#}$ is strictly smaller than under Γ. This completes the proof of Lemma S3-D. ■

Step 3. Steps 1 and 2 imply that there exists a function $\bar{\sigma} : \left(0, \min\{\theta^{MS}, 1 - \theta^{MS}\}\right) \to \mathbb{R}_{++}$, with $\bar{\sigma}(\varepsilon) \leq \min\{\sigma(\varepsilon), \sigma^{\#}(\varepsilon)\}$ for all $\varepsilon \in \left(0, \min\{\theta^{MS}, 1 - \theta^{MS}\}\right)$ and with $\bar{\sigma}(\varepsilon) \to 0^{+}$ as $\varepsilon \to 0^{+}$, such that the following is true: For any $\varepsilon \in \left(0, \min\{\theta^{MS}, 1 - \theta^{MS}\}\right)$, any $\sigma \in (0, \bar{\sigma}(\varepsilon)]$, and any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_{\sigma}$ with $M(\Gamma) > \varepsilon$, there exists another policy $\Gamma' = (\{0,1\}, \pi') \in \mathbb{G}_{\sigma}$ with $M(\Gamma') \leq \varepsilon$ such that, when the agents play as in the auxiliary game G_{σ}, the probability of default under Γ' is strictly smaller than under Γ.[11]

Furthermore, the arguments establishing Lemma S3-D reveal that the policy Γ' can be constructed so that $U_{\sigma}^{\Gamma'}(x, 1|x) > 0$ for all x. The policy Γ' thus satisfies PCP also when agents play according to MARP. The claim in the Example then follows by taking $\Gamma^{*} = \Gamma'$ with Γ' satisfying the above properties.

Step 4. We now complete the proof by showing how to construct the function $\mathcal{E}$ in the example. Let (ε_n) be a non-increasing sequence satisfying $\lim_{n\to\infty} \varepsilon_n = 0$. For each $n \in \mathbb{N}$, then let $\sigma_n = \overline{\sigma}(\varepsilon_n)$, with the function $\overline{\sigma}(\cdot)$ as defined in Step 3. The results in Steps 1-3 above imply that, given $(\varepsilon_n, \sigma_n)$, there exist strictly decreasing subsequences $(\tilde{\varepsilon}_n)$ and $(\tilde{\sigma}_n)$ satisfying $\lim_{n\to\infty} \tilde{\varepsilon}_n = \lim_{n\to\infty} \tilde{\sigma}_n = 0$ such that, for any $n \in \mathbb{N}$, the conclusions in Step 3 hold for $\varepsilon = \tilde{\varepsilon}_n$ and $\bar{\sigma}(\varepsilon_n) = \tilde{\sigma}_n$. Then let $\bar{\sigma} = \tilde{\sigma}_0 > 0$ and $\mathcal{E} : (0, \bar{\sigma}] \to \mathbb{R}_{+}$ be the function defined by $\mathcal{E}(\sigma) = \varepsilon_n$ for all $\sigma \in (\sigma_{n+1}, \sigma_n]$. The result in the example then follows from Steps 1-3, by letting $\mathcal{E}(\cdot)$ be the function so constructed. Q.E.D.

[11] Observe that the thresholds $\sigma(\varepsilon)$ and $\sigma^{\#}(\varepsilon)$ identified in Steps 1 and 2 above are invariant to the initial policy Γ. The same arguments used to arrive at a policy $\Gamma^{\#}$ with mesh $M(\Gamma^{\#}) < M(\Gamma)$ can then be iterated till one arrives at a policy Γ' with mesh $M(\Gamma') \leq \varepsilon$.