Pandora's Auctions: Dynamic Matching with Unknown Preferences

Supplementary Material

Daniel Fershtman[*] Alessandro Pavan[†]

January 2017

Abstract

This document contains proofs for the article "Pandora's Auctions: Dynamic Matching with Unknown Preferences," 2017 *American Economic Review* Papers and Proceedings.

1 Proof of Proposition 1

The proof follows from Theorem 1 in Fershtman and Pavan (2016; henceforth FP), adapted to the environment under examination here.

First, observe that the environment is a special case of the one in FP, in which (*i*) the platform's capacity constraint is equal to 1, (*ii*) the platform's costs are identically equal to zero in all periods, and (*iii*) the processes governing the agents' period-$t > 0$ private information are "endogenous," as defined in Section 2 in FP.

Next, observe that the scores $S^t_{ij}(b^t; \beta, m^{<t})$ correspond to the indexes in FP, adapted to the simplified protocol of the Pandora β-auctions, in which agents are not asked to revise their membership status after period one. The reason why the platform does not need to provide the agents with the possibility of revising their membership statuses after period 0 is the nature of the agents' "horizontal types,"

$$\varepsilon^t_{ij} \equiv \Pr(\omega_{ij} = G|\mathcal{I}^t_i) - L\Pr(\omega_{ij} = B|\mathcal{I}^t_i).$$

In particular, note that, in this simple environment, the sign of the bid b^t_{ij} is used by the platform to uncover the sign of ε^t_{ij}. Furthermore, because the history of past interactions $m^{<t}$ is observable, once the sign of ε^t_{ij} is revealed, so is its precise value. Recall, in fact, that

[*]Department of Economics, Northwestern University, 2001 Sheridan Road, Andersen Hall, Evanston, IL 60208-2600. Email: dfershtman@u.northwestern.edu.

[†]Department of Economics, Northwestern University and CEPR, 2001 Sheridan Road, Andersen Hall, Evanston, IL 60208-2600. Email: alepavan@northwestern.edu.

$\varepsilon^t_{ij} \in \{-L, \lambda - (1-\lambda)L, 1\}$, and that, conditional on $\varepsilon^t_{ij} > 0$, $\varepsilon^t_{ij} = 1$ if and only if $m^{<t}$ is such that $m^s_{ij} = 1$ for some $s \leq t-1$. Hence, at any period $t \geq 1$, the vertical types θ_i can be uncovered directly from the bids b^t along with the history of past interactions $m^{<t}$.

The indexes are then calculated as follows. Let $t \geq 1$, and suppose $m^{<t}$ is such that the pair $(i,j) \in \{xz, yz, zx, zy\}$ has never been matched in previous periods. Consider first the case in which the bids satisfy $b^t_{ij}, b^t_{ji} \geq 0$. Such bids reveal that the agents' period-0 signals were both favorable, i.e., $\sigma_{ij} = \sigma_{ji} = G$. Let $v^t_{ij} = \theta_i \varepsilon^t_{ij}$ denote the expected value that agent i derives from interacting with agent j given agent i's information at period $t \geq 1$. The assumption that λ is close to 1 then implies that the period-t index for the match S^t_{ij} is equal to

$$
\begin{aligned}
S^t_{ij}\left(b^t; \beta, m^{<t}\right) &\equiv \max{}_\tau \left\{ \frac{\mathbb{E}\left[\sum_{s=t}^{\tau} \delta^{s-t}\left(\beta_i v^s_{ij} + \beta_j v^s_{ij}\right)\right]}{\mathbb{E}\left[\sum_{s=t}^{\tau} \delta^{s-t}\right]} \right\} \\
&= \frac{\beta_i b^t_{ij} + \beta_j b^t_{ji} + \frac{\delta\lambda^2}{1-\delta}\left(\frac{\beta_i b^t_{ij} + \beta_j b^t_{ji}}{\lambda-(1-\lambda)L}\right)}{1 + \frac{\delta\lambda^2}{1-\delta}} \\
&= \left(\beta_i b^t_{ij} + \beta_j b^t_{ji}\right)\left(\frac{1-\delta+\frac{\delta\lambda^2}{\lambda-(1-\lambda)L}}{1-\delta+\delta\lambda^2}\right),
\end{aligned}
$$

where τ is a stopping time for the process that governs the evolution of $\beta_i v^s_{ij} + \beta_j v^s_{ij}$, and where the expectation is computed using the bids b^t to uncover the agents' "vertical" types θ and the "horizontal" types ε^t_{ij} and ε^t_{ji}, as explained above. Note that the assumption that λ is close to one simplifies the analysis by implying that, no matter the two agents' vertical types θ_i and θ_j, the optimal stopping time in the definition above involves stopping when one of the two agents learns that the quality of the match is bad.

Similarly, if the pair (i,j) has never been matched and the period-t bids are such that $b^t_{ij} \geq 0, b^t_{ji} < 0$, then

$$
\begin{aligned}
S^t_{ij}\left(b^t; \beta, m^{<t}\right) &= \frac{\beta_i b^t_{ij} + \beta_j b^t_{ji} + \frac{\delta\lambda}{1-\delta}\left(\frac{\beta_i b^t_{ij}}{\lambda-(1-\lambda)L} + \beta_j b^t_{ji}\right)}{1 + \frac{\delta\lambda}{1-\delta}} \\
&= \beta_i b^t_{ij}\left(\frac{1-\delta+\frac{\delta\lambda}{\lambda-(1-\lambda)L}}{1-\delta+\delta\lambda}\right) + \beta_j b^t_{ji}.
\end{aligned}
$$

If (i,j) has never been matched and $b^t_{ij}, b^t_{ji} < 0$ then the index is negative; without loss of generality, it can be set equal to -1. Finally, if the pair (i,j) was matched in one of the previous periods, then the agents' match values are expected to remain constant, and therefore

$$
S^t_{ij}\left(b^t; \beta, m^{<t}\right) = \beta_i b^t_{ij} + \beta_j b^t_{ji}.
$$

The matches implemented under the truthful equilibria of the β-auctions are therefore equivalent to those implemented under the "index matching rule" in FP.

Next, observe that the period-t payments, $t > 0$, in Condition (1) in the present paper, as well as the period-0 payments in Condition (2) in the present paper coincide with the respective payments in FP. To see this, it is useful to introduce some additional notation.

Denote by $\chi \equiv (\chi^t)_{t=1}^{\infty}$, the matching rule associated with the Pandora β-auction. As in FP, such a rule describes, for each $t \geq 1$, the match implemented given the agents' membership choices, the current bids, and the history of previous interactions. For any $t \geq 1$, $(b^{<t}, m^{<t})$, any weights $\beta \equiv (\beta_i)_{i=x,y,z}$, let

$$W^t \equiv \mathbb{E}^{\lambda[\chi]|b^t,m^{<t}} \left[\sum_{s=t}^{\infty} \delta^{s-t} \left((\beta_x v_{xz}^s + \beta_z v_{zx}^s)\, \chi_{xz}^s + \left(\beta_y v_{yz}^s + \beta_z v_{zy}^s\right) \chi_{yz}^s \right) \right]$$

denote the *continuation weighted surplus* — the net present value of the sum of the agents' current and future weighted expected match values implemented under the matching rule χ. Here $\lambda[\chi]|b^t, m^{<t}$ denotes the stochastic process over $(b^s, m^{<s})$, $s \geq t$, under the matching rule χ when: (a) the period-t bids are b^t, (b) all agents follow truthful strategies from period $s > t$ onward, and (c) the true "vertical" types θ and period-t "horizontal" types ε_{ij}^t are the ones inferred from the bids (as explained above). Similarly, for any $i \in \{x, y, z\}$, denote by W_{-i}^t the continuation weighted surplus, as defined above, when the scores involving agent i are identically equal to zero at all histories.

Next, let $R_i^t \equiv W^t - W_{-i}^t$ denote the *contribution* of agent $i \in \{x, y, z\}$ to the continuation weighted surplus and

$$r_i^t \equiv R_i^t - \delta \mathbb{E}^{\lambda[\chi]|b^t,m^{<t}} \left[R_i^{t+1} \right]$$

the corresponding *flow marginal contribution.* The formula for the period-t payments, $t \geq 1$, in FP requires that, for each agent i,

$$p_i^t = \sum_{j \in N_{-i}} b_{ij}^t \cdot \chi_{ij}^t - \frac{r_i^t}{\beta_i},$$

where $N_{-i} \equiv \{z\}$ if $i \in \{x, y\}$ and $N_{-i} \equiv \{x, y\}$ if $i = z$. That is, agent i's period-t payment should be equal to the flow value the agent derives from the match implemented in period t (as reflected by the agent's own bids), net of a "discount" proportional to the agent's flow marginal contribution to weighted continuation surplus, with the coefficient of proportionality equal to $1/\beta_i$.

Observe that the flow marginal contribution r_i^t, $i \in \{x, y, z\}$, can be rewritten as

$$\begin{aligned}
r_i^t &= \left\{ \left(\beta_x b_{xz}^t + \beta_z b_{zx}^t\right) \chi_{xz}^t + \left(\beta_y b_{yz}^t + \beta_z b_{zy}^t\right) \chi_{yz}^t + \delta \mathbb{E}^{\lambda[\chi]|b^t, m^{<t}} \left[W^{t+1}\right] \right\} \\
&\quad - W_{-i}^t - \delta \left\{ \mathbb{E}^{\lambda[\chi]|b^t, m^{<t}} \left[W^{t+1}\right] - \mathbb{E}^{\lambda[\chi]|b^t, m^{<t}} \left[W_{-i}^{t+1}\right] \right\} \\
&= \left\{ \left(\beta_x b_{xz}^t + \beta_z b_{zx}^t\right) \chi_{xz}^t + \left(\beta_y b_{yz}^t + \beta_z b_{zy}^t\right) \chi_{yz}^t \right\} - W_{-i}^t + \delta \mathbb{E}^{\lambda[\chi]|b^t, m^{<t}} \left[W_{-i}^{t+1}\right].
\end{aligned}$$

When applied to the environment under examination here, the formula for the period-t payments, $t > 0$, in FT thus reduces to

$$\begin{aligned}
p_i^t &= \sum_{j \in N_{-i}} b_{ij}^t \cdot \chi_{ij}^t - \frac{1}{\beta_i} \left(\left(\beta_x b_{xz}^t + \beta_z b_{zx}^t\right) \chi_{xz}^t + \left(\beta_y b_{yz}^t + \beta_z b_{zy}^t\right) \chi_{yz}^t \right) \\
&\quad - \frac{1}{\beta_i} \left(\delta \mathbb{E}^{\lambda[\chi]|b^t, m^{<t}} \left[W_{-i}^{t+1}\right] - W_{-i}^t \right).
\end{aligned}$$

Since $W_{-z}^t = 0$ all $t > 0$, the formula in FT requires that agent z's period-t payment, $t > 0$, be equal to

$$\begin{aligned}
p_z^t &= b_{zx}^t \cdot \chi_{xz}^t + b_{zy}^t \cdot \chi_{yz}^t - \frac{1}{\beta_z} \left\{ \left(\beta_x b_{xz}^t + \beta_z b_{zx}^t\right) \chi_{xz}^t + \left(\beta_y b_{yz}^t + \beta_z b_{zy}^t\right) \chi_{yz}^t \right\} \\
&= -\frac{1}{\beta_z} \left(\beta_x b_{xz}^t \chi_{xz}^t + \beta_y b_{yz}^t \chi_{yz}^t \right).
\end{aligned}$$

Likewise, when $\chi_{xz}^t = 0$, the formula in FT requires that agent x's period-t payment, $t > 0$, be equal to

$$p_x^t = -\frac{1}{\beta_x} \left\{ \left(\beta_y b_{yz}^t + \beta_z b_{zy}^t\right) \chi_{yz}^t + \delta \mathbb{E}^{\lambda[\chi]|b^t, m^{<t}} \left[W_{-x}^{t+1}\right] - W_{-x}^t \right\} = 0.$$

When, instead, $\chi_{xz}^t = 1$ (in which case $\chi_{yz}^t = 0$), agent x's period-t payment should be equal to

$$p_x^t = -\frac{1}{\beta_x} \left\{ \beta_z b_{zx}^t + \delta \mathbb{E}^{\lambda[\chi]|b^t, m^{<t}} \left[W_{-x}^{t+1}\right] - W_{-x}^t \right\} = \frac{1}{\beta_x} \left\{ (1-\delta) W_{-x}^t - \beta_z b_{zx}^t \right\},$$

where the last equality follows from the observation that $\mathbb{E}^{\lambda[\chi]|b^t, m^{<t}} \left[W_{-x}^{t+1}\right] = W_{-x}^t$. The formula for the period-t payments, $t > 0$, for agent y is analogous to that for agent x, and hence omitted. From the above observations, it is then easy to verify that the period-t payments in (1) in the present paper satisfy the conditions in FP.

Next, consider the period-0 payments. For any profile of membership statuses $\hat{\boldsymbol{\theta}}$, let

$$D_i(\hat{\boldsymbol{\theta}}) \equiv \mathbb{E}^{\lambda[\chi]|\hat{\boldsymbol{\theta}}} \left[\sum_{t=1}^{\infty} \delta^t \sum_{j \in N_{-i}} \varepsilon_{ij}^t \chi_{ij}^t \right]$$

denote agent i's average match quality from participating in the Pandora $\beta-$auction, where $\lambda[\chi]|\hat{\boldsymbol{\theta}}$ denotes the distribution over future bids and matches under truthful strategies. It is then immediate to see that the period-0 payments in (2) in the present paper satisfy the condition required in FP.

Proposition 1 then follows from Theorem 1 in FP.

2 Proof of Proposition 2

First observe that a matching mechanism is welfare (alternatively, profit) maximizing if it is feasible (meaning that it matches at most a pair of agents in each period, with the pair belonging in different sides) and admits a PBE under which welfare (alternatively, profit) is at least as high as under any other PBE of any other feasible mechanism.

The result about the welfare maximizing auction in Proposition 2 in the present paper then follows from Theorem 3 in FP. The result for profit maximization follows from observing that the weights in the profit-maximizing $\beta-$auctions are such that

$$\beta(\theta) = \frac{2\theta - \overline{\theta}}{\theta} = 1 - \frac{1 - F(\theta)}{f(\theta)\theta},$$

where F is the cdf and f is the pdf of the random variable θ (recall that θ is drawn from a Uniform distribution in the present paper). The assumption that $q\lambda > (1 - q\lambda)L$, along with the fact that the probability each agent i interacts with each agent j is weakly higher when $\omega_{ij} = G$ than when $\omega_{ij} = B$, then implies that, when the weights β are the profit-maximizing ones, as defined above, the intertemporal average match quality (as defined above) is non-negative for all vertical types, included the lowest. The result about the optimality of the β-auctions then follows from Theorem 2 in FT.

References

[1] Fershtman, Daniel, and Alessandro Pavan. 2016. “Matching auctions.” Mimeo, Northwestern University.