Online Supplement for

Robust Procurement Design

Debasis Mishra, Sanket Patil, and Alessandro Pavan*

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This online supplement contains additional results for the article Mishra et al. (2025) (hereafter referred to as *the main text*). The notation is the same as in the original article. All sections, definitions, displayed conditions, and results specific to this document have the suffix "S" to avoid confusion with the corresponding parts in the main text. Section S.1 establishes existence of a robustly optimal mechanism. Section S.2 shows that every robustly optimal mechanism $M = (q, u)$ in which q is left-continuous is undominated. Section S.3 considers an extension in which the designer's short list is more permissive than in the main text and establishes robustness of the key qualitative results. Section S.4 extends Lemma 2 (characterizing the short list of quantity mechanisms) and Proposition 2 (providing a partial characterization of robustly optimal quantity mechanisms) in the main text to a setting in which the set $\mathcal{F}$ of technologies from which the seller's cost is draw is a subset of the entire set CDF(Θ) of cumulative distribution functions supported on $\Theta = [\underline{\theta}, \bar{\theta}]$. Finally, Section S.5 examines comparative statics of robustly optimal mechanisms with respect to the lowest distribution $\underline{F}$ in $\mathcal{F}$ (in the main text such a distribution is a Dirac measure at $\bar{\theta}$). Appendix S.A contains proofs for a few results that are not established earlier in the Supplement.

*Mishra: Indian Statistical Institute, Delhi, dmishra@isid.ac.in. Patil: Indian Institute of Management Bangalore, sanket.patil@iimb.ac.in. Pavan: Northwestern University, alepavan@northwestern.edu.

S.1 Existence of robustly optimal mechanisms

In this section, we show that a robustly optimal mechanism exists. In the main text, Proposition 1 contains necessary and sufficient conditions under which the Baron-Myerson-with-quantity-floor mechanism is robustly optimal. When these conditions are satisfied, existence follows immediately. When they are not, Proposition 2 in the main text identifies properties of robustly optimal mechanisms but does not establish existence. The following proposition shows that a robustly optimal mechanism always exists.

Proposition S.1 *A robustly optimal mechanism exists.*

Proof: Recall that a mechanism $M = (q, u)$ is robustly optimal if and only if u satisfies $u(\theta) = \int_{\theta}^{\bar{\theta}} q(s)ds$ for all $\theta \in \Theta$, with q solving the following program (where we use the fact that $F^{\star}$ is absolutely continuous over $\mathbb{R}$, with density $f^{\star}(\theta) > 0$ if and only if $\theta \in \Theta$):

$$\max_{q:[\underline{\theta},\bar{\theta}]\to[0,\bar{\mathrm{q}}]} \int_{\underline{\theta}}^{\bar{\theta}} \Big[V^{\star}(q(\theta)) - z^{\star}(\theta)q(\theta)\Big] f^{\star}(\theta)d\theta$$

subject to

q weakly decreasing

$$\underline{V}(q(\theta)) - \theta q(\theta) - \int_{\theta}^{\bar{\theta}} q(y)dy \geq G^{*} \qquad \forall\ \theta \in [\underline{\theta}, \bar{\theta}].$$

Hereafter, we refer to the set of schedules q satisfying the restrictions in the above problem as the "feasible set".

Since each q in the feasible set is uniformly bounded, i.e., $0 \leq q(\theta) \leq \bar{\mathrm{q}}$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$, by Helly's selection theorem, the set of weakly decreasing schedules q is sequentially compact under the point-wise convergence topology. Furthermore, since $\underline{V}$ is continuous and q is uniformly bounded, by the dominated convergence theorem, for any $\theta \in \Theta$, the function $H(q;\theta)$ defined by $H(q;\theta) \equiv \underline{V}(\mathrm{q}(\theta)) - \theta q(\theta) - \int_{\theta}^{\bar{\theta}} q(y)dy$ is sequentially continuous in q. Hence, the feasible set is sequentially compact under the point-wise convergence topology. That this set is non-empty is immediate given that the function q satisfying $q(\theta) = \mathrm{q}_{\ell}$ for all $\theta \in \Theta$ meets all the requirements.

Next, observe that the objective function in the above program is sequentially continuous in q. To see this, let ϕ be the function defined, for all $(\theta, \mathrm{q}) \in \Theta \times [0, \bar{\mathrm{q}}]$ by

$$\phi(\theta, \mathrm{q}) = \Big[V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}\Big] f^\star(\theta).$$

Note that, given any function q in the feasible set, the value of the objective function is

$$\int_{\underline{\theta}}^{\bar{\theta}} \phi(\theta, q(\theta)) d\theta.$$

Clearly, for any $\theta \in \Theta$, $\phi(\theta, \mathrm{q})$ is continuous in q. Furthermore, for every $\theta \in \Theta$ and $\mathrm{q} \in [0, \bar{\mathrm{q}}]$,

$$\phi(\theta, \mathrm{q}) \leq \Big[V^\star(D^*(z^\star(\theta))) - z^\star(\theta) D^*(z^\star(\theta))\Big] f^\star(\theta) \equiv \bar{\phi}(\theta),$$

where the inequality follows from the fact that the quantity $D^*(z^\star(\theta)) = q^{\mathrm{BM}}(\theta)$ maximizes the virtual surplus function $V^\star(\cdot) - z^\star(\theta) D^*(\cdot)$ over $[0, \bar{\mathrm{q}}]$. Because $\bar{\phi}$ is continuous, ϕ is uniformly bounded.

Now take a sequence (q_n) of feasible schedules converging to q under the point-wise convergence topology. Then, for every $\theta \in [\underline{\theta}, \overline{\theta}]$, we have that

$$\lim_{n \to \infty} \phi(\theta, q_n(\theta)) = \phi(\theta, q(\theta))$$

by continuity of ϕ in the second argument. Furthermore, for each q_n in the sequence, we have that

$$\phi(\theta, q_n(\theta)) \leq \bar{\phi}(\theta) \qquad \forall\ \theta \in [\underline{\theta}, \overline{\theta}].$$

Then, by the dominated convergence theorem,

$$\lim_{n \to \infty} \int_{\underline{\theta}}^{\overline{\theta}} \phi(\theta, q_n(\theta)) d\theta = \int_{\underline{\theta}}^{\overline{\theta}} \phi(\theta, q(\theta)) d\theta.$$

This establishes the sequential continuity of the objective function under the point-wise convergence topology. Since the range of the objective function is a subset of $\mathbb{R}$, from the extreme value theorem, we conclude that the above optimization program has a solution, i.e., a robustly optimal mechanism exists. ■

S.2 Undomination

In this section, we formally define what it means for a mechanism to be undominated, and then establish that robustly optimal mechanisms are undominated.

Recall that $\mathcal{M}$ is the set of all IC and IR mechanisms.

Definition S.1 *For any pair of mechanisms $M = (q, u)$ and $\widehat{M} = (\hat{q}, \hat{u})$, M* ***dominates*** *$\widehat{M}$ if, for every $(V, F) \in \mathcal{V} \times \mathcal{F}$,*

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V(q(\theta)) - \theta q(\theta) - u(\theta)\Big] F(\mathrm{d}\theta) \geq \int_{\underline{\theta}}^{\overline{\theta}} \Big[V(\hat{q}(\theta)) - \theta \hat{q}(\theta) - \hat{u}(\theta)\Big] F(\mathrm{d}\theta),$$

with the inequality strict for some (V, F).

A mechanism $\widehat{M} \in \mathcal{M}$ is ***undominated*** *if there does not exist a mechanism $M \in \mathcal{M}$ that* ***dominates*** *it.*

The following lemma points to an internal consistency property of the set of robustly optimal mechanisms: each robustly optimal mechanism is either undominated, or it is dominated by another robustly optimal mechanism.

Lemma S.1 *Suppose $M^{\mathrm{OPT}} = (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$ is a robustly optimal mechanism and $M = (q, u) \in \mathcal{M}$ dominates M^{OPT}. Then M is a robustly optimal mechanism.*

Proof: Since M^{OPT} is a robustly optimal mechanism, for all $\theta \in [\underline{\theta}, \overline{\theta}]$,

$$\underline{V}(q^{\mathrm{OPT}}(\theta)) - \theta q^{\mathrm{OPT}}(\theta) - u^{\mathrm{OPT}}(\theta) \geq G^*.$$

Now pick any $\theta \in [\underline{\theta}, \overline{\theta}]$. Since M dominates M^{OPT}, under $V = \underline{V}$ and $F = \delta_\theta$ (where δ_θ is the Dirac distribution that puts unit point mass at θ), we have that

$$\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta) \geq \underline{V}(q^{\mathrm{OPT}}(\theta)) - \theta q^{\mathrm{OPT}}(\theta) - u^{\mathrm{OPT}}(\theta).$$

Combining the two inequalities, we have that $\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta) \geq G^*$. Since this holds for all θ, by Lemma 2 in the main text, we have that $M \in \mathcal{M}^{\mathrm{SL}}$.

Next, pick any $\theta \in [\underline{\theta}, \overline{\theta}]$. Since M dominates M^{OPT}, under $V = V^\star$ and $F = \delta_\theta$, we have that

$$V^\star(q(\theta)) - \theta q(\theta) - u(\theta) \geq V^\star(q^{\text{OPT}}(\theta)) - \theta q^{\text{OPT}}(\theta) - u^{\text{OPT}}(\theta),$$

which implies that

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q(\theta)) - \theta q(\theta) - u(\theta)\Big] F^\star(\mathrm{d}\theta) \geq \int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q^{\text{OPT}}(\theta)) - \theta q^{\text{OPT}}(\theta) - u^{\text{OPT}}(\theta)\Big] F^\star(\mathrm{d}\theta).$$

Since $M \in \mathcal{M}^{\text{SL}}$ and M^{OPT} is robustly optimal, the above inequality implies that M is also robustly optimal. In turn, this implies that the above inequality is an equality, and, therefore, for almost all $\theta \in \Theta$,

$$V^\star(q(\theta)) - \theta q(\theta) - u(\theta) = V^\star(q^{\text{OPT}}(\theta)) - \theta q^{\text{OPT}}(\theta) - u^{\text{OPT}}(\theta).$$

■

We then have the following result:

Proposition S.2 *If the Baron-Myerson-with-quantity-floor mechanism $M^\star = (q^\star, u^\star)$ is robustly optimal, it is undominated.*

Proof: Suppose $M^\star$ is robustly optimal and $M = (q, u)$ dominates it. By Lemma S.1, M is also robustly optimal. By Corollary 1 in the main text, $q(\theta) = q^\star(\theta)$ for all $\theta > \underline{\theta}$. This implies that $u(\theta) = u^\star(\theta)$ for all θ.

Consider the pair $(V^\star, \delta_{\underline{\theta}})$, where $\delta_{\underline{\theta}}$ is the Dirac distribution that puts unit mass at $\underline{\theta}$. Then

$$V^\star(q^\star(\underline{\theta})) - \underline{\theta} q^\star(\underline{\theta}) - u^\star(\underline{\theta}) > V^\star(q(\underline{\theta})) - \underline{\theta} q(\underline{\theta}) - u(\underline{\theta}).$$

The inequality holds because $u(\underline{\theta}) = u^\star(\underline{\theta})$ and because $q^\star(\underline{\theta}) \equiv q^{\text{BM}}(\underline{\theta})$ uniquely maximizes surplus $V^\star(\mathrm{q}) - \underline{\theta}\mathrm{q}$. This inequality, however, contradicts the fact that $M = (q, u)$ dominates $(q^\star, u^\star)$. ■

A consequence of the last Lemma S.1 is that, when the conditions in Proposition 1 in the main text are satisfied, the Baron-Myerson-with-quantity-floor mechanism is not only robustly optimal but also undominated. This result generalizes, albeit under a mild technical condition (which is satisfied by $(q^\star, u^\star)$).

Proposition S.3 *Let $M^{\mathrm{OPT}} = (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$ be a robustly optimal mechanism in which q^{OPT} is left-continuous. Then, M^{OPT} is undominated.*

Proof: See Appendix S.A. ■

S.3 More permissive short list

In this section, we consider a short list that also contains mechanisms that are not worst-case optimal, but for which the guarantee is not too small relative to the maximal one. Formally, let

$$\mathcal{M}_\gamma^{\mathrm{SL}} \equiv \{M \in \mathcal{M} : G(M) \geq \gamma G(M') \ \forall \ M' \in \mathcal{M}\},$$

where $\gamma \in [0,1]$. The analysis in the main text corresponds to the case $\gamma = 1$. Here, we extend the results to $\gamma \in [0,1)$. The proofs of all results in this section are quite straightforward generalizations of the corresponding ones in the main text and are available upon request.

Lemma 1 in the main text remains unchanged. Lemma 2 instead is generalized as follows:

Lemma S.2 (Short-list characterization) *Take any IC and IR mechanism $M \equiv (q, u) \in \mathcal{M}$. Then $M \in \mathcal{M}_\gamma^{\mathrm{SL}}$ if and only if*

$$\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta) \geq \gamma G^* \qquad \forall \ \theta \in \Theta. \tag{S.1}$$

There are two important distinctions from the case $\gamma = 1$ in the main text. First, when $\gamma < 1$ the rent $u(\overline{\theta})$ for the least efficient type is not necessarily zero. Second, $q(\overline{\theta})$ is not restricted to be equal to q_ℓ.

An immediate implication of the last lemma is that the short list for $\gamma' > \gamma$ is a subset of the shortlist for γ.

Below, we focus on mechanisms in $\mathcal{M}_\gamma^{\mathrm{SL}}$ for which $u(\overline{\theta}) = 0$. The reason for focusing on this subset of $\mathcal{M}_\gamma^{\mathrm{SL}}$ is that robustly optimal mechanisms always feature no rent for the highest cost type. The next lemma generalizes Lemma 4 in the main text.

Lemma S.3 *Take any weakly decreasing function $q : \Theta \to \mathbb{R}_+$. The following statements are equivalent:*

1. *for all $\theta \in \Theta$,*
$$\underline{V}(q(\theta)) - \theta q(\theta) - \int_\theta^{\overline{\theta}} q(y)dy \geq \gamma G^*; \tag{S.2}$$

2. *for all $\theta \in \Theta$,*
$$\int_\theta^{\overline{\theta}} q(y)dy \leq \int_\theta^{\overline{\theta}} \underline{D}(y)dy - \int_\theta^{\underline{P}(q(\theta))} \left(\underline{D}(y) - q(\theta)\right) dy + (1-\gamma)G^*; \tag{S.3}$$

3. *Condition (S.3) holds for $\theta \in \{\underline{\theta}, \overline{\theta}\}$ and, for all $\theta \in (\underline{\theta}, \overline{\theta})$,*
$$\int_\theta^{\overline{\theta}} q(y)dy \leq \int_\theta^{\overline{\theta}} \underline{D}(y)dy + (1-\gamma)G^*. \tag{S.4}$$

We now modify the definition of Baron-Myerson-with-quantity-floor mechanism to account for the more permissive short list. Because the function $\underline{V}(\mathrm{q}) - \overline{\theta}\mathrm{q}$ is concave in q, there are two solutions to the equation $\underline{V}(\mathrm{q}) - \overline{\theta}\mathrm{q} = \gamma G^*$. Denote these two solutions by $\underline{\mathrm{q}}_\ell^\gamma$ and $\bar{\mathrm{q}}_\ell^\gamma$, with $\bar{\mathrm{q}}_\ell^\gamma > \underline{\mathrm{q}}_\ell^\gamma$. The generalization of Baron-Myerson-with-quantity-floor mechanism uses the lowest of the two floors, $\underline{\mathrm{q}}_\ell^\gamma$. This is because, as shown in Proposition S.5 below, the relevant constraint imposed by robustness is the one requiring that $q(\theta) \geq \underline{\mathrm{q}}_\ell^\gamma$ for all $\theta \in \Theta$.

Definition S.2 *The **Baron-Myerson-with-quantity-floor** mechanism is the mechanism $M_\gamma^\star \equiv (q_\gamma^\star, u_\gamma^\star)$ where $q_\gamma^\star$ is the quantity schedule defined, for all θ, by*
$$q_\gamma^\star(\theta) \equiv \max\{q^{\mathrm{BM}}(\theta), \underline{\mathrm{q}}_\ell^\gamma\} \tag{S.5}$$
and where $u^\star$ is given by $u_\gamma^\star(\theta) = \int_\theta^{\overline{\theta}} q_\gamma^\star(y)dy$ for all θ.

Proposition S.4 (Optimality of Baron-Myerson-with-quantity-floor) *The mechanism $M^\star_\gamma$ is robustly optimal if and only if*

$$\int_{\underline{\theta}}^{\overline{\theta}} q^\star_\gamma(y)dy \leq \int_{\underline{\theta}}^{\overline{\theta}} \underline{D}(y)dy - \int_{\underline{\theta}}^{\underline{P}(q^\star_\gamma(\underline{\theta}))} \left[\underline{D}(y) - q^\star_\gamma(\underline{\theta})\right]dy + (1-\gamma)G^*, \tag{S.6}$$

and

$$\int_{\theta}^{\overline{\theta}} q^\star_\gamma(y)dy \leq \int_{\theta}^{\overline{\theta}} \underline{D}(y)dy + (1-\gamma)G^* \qquad \forall \theta > \underline{\theta}. \tag{S.7}$$

Let $\theta^\star_\gamma$ be the threshold defined as follows. If $q^{\text{BM}}(\overline{\theta}) \leq \underline{\text{q}}^\gamma_\ell$, by continuity of q^{BM} along with the fact that $q^{\text{BM}}(\underline{\theta}) > \underline{\text{q}}^\gamma_\ell$, let $\theta^\star_\gamma$ be the unique solution to $q^{\text{BM}}(\theta^\star_\gamma) = \underline{\text{q}}^\gamma_\ell$. If, instead, $q^{\text{BM}}(\overline{\theta}) > \underline{\text{q}}^\gamma_\ell$ (i.e., if q^{BM} never crosses $\underline{\text{q}}^\gamma_\ell$), then let $\theta^\star_\gamma \equiv \overline{\theta}$. Next let

$$\theta^m_\gamma \equiv \max\{\theta : \theta \in \arg\min_{y\in\Theta} \underline{W}(y, q^\star_\gamma)\},$$

where, as in the main text, given any function q, $\underline{W}(\theta, q) \equiv \underline{V}(q(\theta)) - \theta q(\theta) - \int_{\theta}^{\overline{\theta}} q(y)dy$.

The following proposition generalizes Proposition 2 in the main text:

Proposition S.5 *Suppose the Baron-Myerson-with-quantity-floor mechanism is not robustly optimal. Then $\theta^m_\gamma < \theta^\star_\gamma$, and every robustly optimal mechanism $M^{\text{OPT}} = (q^{\text{OPT}}, u^{\text{OPT}})$ satisfies the following properties:*

(a) $q^{\text{OPT}}(\theta) = \underline{\text{q}}^\gamma_\ell$ *for all* $\theta \in (\theta^\star_\gamma, \overline{\theta})$*;*

(b) $q^{\text{OPT}}(\theta) \leq q^{\text{BM}}(\theta)$ *for all* $\theta \in (\underline{\theta}, \theta^\star_\gamma)$*, with the inequality strict over a set of types* $I \subseteq [\theta^m_\gamma, \theta^\star_\gamma)$ *of positive Lebesgue measure;*

(c) $q^{\text{OPT}}(\theta) = q^{\text{BM}}(\theta)$ *for all* $\theta \in (\underline{\theta}, \theta^m_\gamma)$*.*

Next, consider price regulation. One can then show that every robustly optimal price regulation has the same price schedule as in the Baron-Myerson-with-price-cap mechanism. The latter is given by $p_\gamma(\theta) = \min(z^\star(\theta), \underline{P}(\underline{q}^\gamma_\ell))$, with $\underline{P}(\underline{q}^\gamma_\ell) > \overline{\theta}$. Finally, one can show that a result analogous to Proposition 4 in the main text holds (which compares price and quantity regulation) holds. However, the conditions for quantity regulation to dominate price regulation are more stringent because of the larger price-cap. Namely,

Proposition S.6 ([quantity vs price regulation) *1. If the Baron-Myerson-with-quantity-floor mechanism $M^\star_\gamma$ is robustly optimal, quantity regulation dominates price regulation (strictly if $D^\star(\underline{P}(\underline{q}^\gamma_\ell)) > q^\star_\gamma(\bar{\theta})$).*

2. If the Baron-Myerson-with-quantity-floor mechanism $M^\star_\gamma$ is not robustly optimal and $D^\star(\underline{P}(\underline{q}^\gamma_\ell)) = q^\star_\gamma(\bar{\theta})$, price regulation strictly dominates quantity regulation.

S.4 More general forms of technological uncertainty

In this section, we consider the case in which the set of feasible technologies $\mathcal{F}$ contains only a subset of $\mathrm{CDF}(\Theta)$, that is, of all cumulative distribution functions with support contained in Θ. Namely, we assume there exists a cdf $\underline{F} \in \mathrm{CDF}(\Theta)$ such that $\mathcal{F}$ is the set of all cdfs $F \in \mathrm{CDF}(\Theta)$ such that $F(\theta) \geq \underline{F}(\theta)$ for all $\theta \in \Theta$. In the main text, we assumed $\underline{F}(\theta)$ is the Dirac distribution that puts unit mass at $\bar{\theta}$. This amounted to $\mathcal{F} = \mathrm{CDF}(\Theta)$. Here, instead, we allow $\underline{F}$ to have support $[\theta_s, \overline{\theta}]$, where $\theta_s \in (\underline{\theta}, \overline{\theta})$. In particular, we assume the following:

Definition S.3 *The cdf $\underline{F}$ is regular with respect to θ_s if it is absolutely continuous over $\mathbb{R}$ with density $f(\theta) > 0$ if only if $\theta \in [\theta_s, \overline{\theta}]$ and with $\underline{z}(\theta) \equiv \theta + \underline{F}(\theta)/\underline{f}(\theta)$ continuous and increasing over $[\theta_s, \overline{\theta}]$.*

Let $\underline{M}^{\mathrm{BM}} = (\underline{q}^{\mathrm{BM}}, \underline{u}^{\mathrm{BM}}) \in \mathcal{M}$ denote an arbitrary IC and IR mechanism that is optimal under the model $(\underline{V}, \underline{F})$. Note that such a mechanism is not unique, but in any such a mechanism $\underline{q}^{\mathrm{BM}}$ is weakly decreasing, $\underline{u}^{\mathrm{BM}}(\bar{\theta}) = 0$, and $\underline{u}^{\mathrm{BM}}(\theta) = \int_\theta^{\bar{\theta}} \underline{q}^{\mathrm{BM}}(y)dy$ for all θ. Then let

$$G^*_s \equiv \int_{\theta_s}^{\bar{\theta}} \underline{W}(\theta, \underline{q}^{\mathrm{BM}})\underline{F}(\mathrm{d}\theta). \tag{S.8}$$

be the buyer's expected welfare under the mechanism $\underline{M}^{\mathrm{BM}}$ when the gross value function is $\underline{V}$, and the technology is $\underline{F}$. When $\theta_s = \bar{\theta}$, as in the main text, $G^*_s = G^*$. Finally, for any IC and IR mechanism $M = (q, u) \in \mathcal{M}$, let

$$\underline{w}_q \equiv \inf_{\theta \leq \theta_s} \underline{W}(\theta, q),$$

where, as in the main text, $\underline{W}(\theta,q) \equiv \underline{V}(q(\theta)) - \theta q(\theta) - \int\limits_{\theta}^{\overline{\theta}} q(y)dy$. The next proposition generalizes Lemma 2 in the main text by providing a complete characterization of the short list.

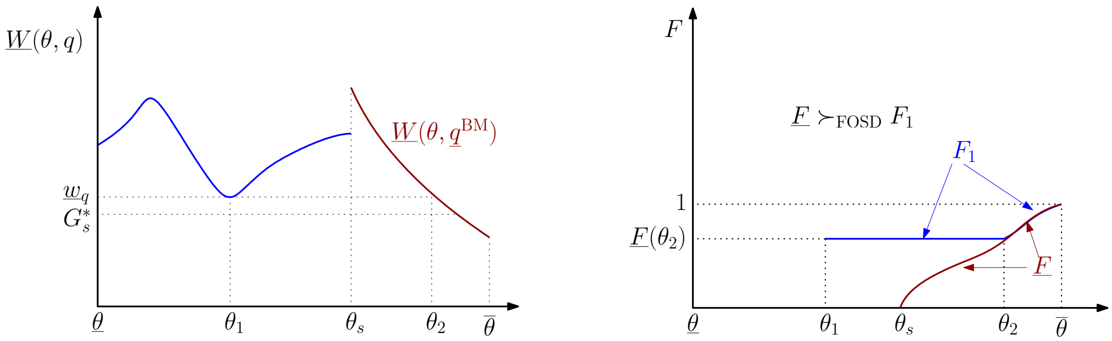
Proposition S.7 *Suppose $\underline{F}$ is regular with respect to θ_s. The following are then true:*

1. *For any $M \in \mathcal{M}^{\mathrm{SL}}$, $G(M) = G_s^*$;*
2. *A mechanism $M \equiv (q,u) \in \mathcal{M}^{\mathrm{SL}}$ if and only if the following conditions jointly hold*
   - *(a) q is weakly decreasing,*
   - *(b) for all θ, $u(\theta) = \int_{\theta}^{\overline{\theta}} q(y)dy$,*
   - *(c) $q(\theta) = \underline{q}^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta_s, \overline{\theta})$,*
   - *(d) $\underline{V}(q(\theta)) - \theta q(\theta) - \int\limits_{\theta}^{\overline{\theta}} q(y)dy \geq G_s^*$ for all $\theta \leq \theta_s$,*
   - *(e) $\underline{w}_q \underline{F}(\theta) + \int\limits_{\theta}^{\overline{\theta}} \underline{W}(y, \underline{q}^{\mathrm{BM}}) \underline{F}(dy) \geq G_s^*$ for all $\theta \in [\theta_s, \overline{\theta}]$.*

Proof: See Appendix S.A. ■

Part (1) follows from the fact that Nature can always pick $(\underline{V}, \underline{F})$, which implies that the guarantee of any IC and IR mechanism is bounded above by the maximal welfare attainable under the lowest gross value function $\underline{V}$ and the worst technology $\underline{F}$. This upper bound on the guarantee can be achieved by offering a mechanism $M \equiv (q,u)$ in which $q(\theta) = \underline{q}^{\mathrm{BM}}(\theta_s)$ for all $\theta \leq \theta_s$, $q(\theta) = \underline{q}^{\mathrm{BM}}(\theta)$ for all $\theta > \theta_s$, and $u(\theta) = \int\limits_{\theta}^{\overline{\theta}} q(y)dy$ for all θ. Against such a mechanism an adversarial Nature cannot do better than selecting $(V,F) = (\underline{V}, \underline{F})$, yielding the buyer a payoff of G_s^*.

Conditions (a)-(d) in Part (2) are generalizations of the robustness constraints in Lemma 2 in the main text. Condition (c) follows from the fact that Nature can always choose the model $(\underline{V}, \underline{F})$ in which case worst-case optimality dictates that the output procured from types $\theta \in (\theta_s, \overline{\theta})$ is the same as under the mechanism $\underline{M}^{\mathrm{BM}} \equiv (\underline{q}^{\mathrm{BM}}, \underline{u}^{\mathrm{BM}})$. Note that Condition (d) applies only to types $\theta \leq \theta_s$ as Nature can now select a Dirac distribution at



A: Function $\underline{W}(\cdot, q)$.

B: Technology F_1 yielding welfare below G_s^*.

Figure S.1: Graphical illustration of new robustness constraint in Condition (e) of Proposition S.7.

θ only when $\theta \leq \theta_s$. Condition (e) stems from the fact that Nature's best response can be a non-Dirac distribution. To understand this constraint, consider Figure S.1. Let $\theta_1 < \theta_s$ be a cost level at which the function $\underline{W}(\cdot, q)$ reaches the minimum over $[\underline{\theta}, \theta_s]$, i.e., $\underline{W}(\theta_1, q) = \underline{w}_q$, and let $\theta_2 > \theta_s$ be a cost level such that $\underline{W}(\theta_2, q) = \underline{W}(\theta_2, \underline{q}^{\mathrm{BM}}) = \underline{w}_q$. Suppose Nature picks a distribution F_1 with an atom at θ_1 equal to $\underline{F}(\theta_2)$ and which agrees with $\underline{F}$ on all $\theta \geq \theta_2$. Because $\underline{W}(\cdot, \underline{q}^{\mathrm{BM}})$ is decreasing, even if $\underline{w}_q \geq G_s^*$, in the absence of the new constraint in Condition (e) it may well be the case that

$$\underline{w}_q F_1(\theta_1) + \int_{\theta_2}^{\overline{\theta}} \underline{W}(\theta, \underline{q}^{\mathrm{BM}}) F_1(\mathrm{d}\theta) = \underline{w}_q \underline{F}(\theta_2) + \int_{\theta_2}^{\overline{\theta}} \underline{W}(\theta, \underline{q}^{\mathrm{BM}}) \underline{F}(\mathrm{d}\theta) < G_s^*,$$

in which case the mechanism's guarantee is below G_s^*. To account for such possibilities, worst-case optimality requires that the constraint in Condition (e) holds.

Next, we establish an analog of Proposition 2 in the main text, which provides a partial characterization of robustly optimal mechanisms when $\mathcal{F} = \mathrm{CDF}(\Theta)$. To do so, we first generalize the definitions of q_ℓ, $q^\star$, $\theta^\star$ and θ^m as follows. Let

$$\mathrm{q}_\ell^s \equiv \underline{D}(\theta_s)$$

denote the efficient output when the inverse demand is $\underline{D}$ and the cost is θ_s. Then let $q_s^\star$ be the quantity schedule defined by

$$q_s^\star(\theta) \equiv \begin{cases} \max\{\underline{q}^{\mathrm{BM}}(\theta), \mathrm{q}_\ell^s\} & \theta < \theta_s \\ \underline{q}^{\mathrm{BM}}(\theta) & \theta \geq \theta_s, \end{cases} \tag{S.9}$$

where q^{BM} continues to denote the optimal quantity schedule of Baron and Myerson (1982) when the model is $(V^\star, F^\star)$, with $F^\star$ regular, whereas $\underline{q}^{\mathrm{BM}}$ is the optimal quantity schedule of Baron and Myerson (1982) when the model is $(\underline{V}, \underline{F})$. The following mechanism is a generalization of Baron-Myerson-with-quantity-floor in the main text.

Definition S.4 *The **Baron-Myerson-with-quantity-bridge** is the mechanism $M_s^\star = (q_s^\star, u_s^\star)$ where $q_s^\star$ is the quantity schedule in (S.9) and $u_s^\star$ is the function given by $u_s^\star(\theta) = \int\limits_{\theta}^{\overline{\theta}} q_s^\star(y)dy$ for all θ.*

Finally, let $\theta_s^\star$ be the threshold cost defined as follows. If $q^{\mathrm{BM}}(\theta_s) \le \mathrm{q}_\ell^s$, by continuity of q^{BM} along with the fact that $q^{\mathrm{BM}}(\underline{\theta}) > \mathrm{q}_\ell^s$ (assured by the regularity of $F^\star$), $\theta_s^\star$ is the unique solution to $q^{\mathrm{BM}}(\theta_s^\star) = \mathrm{q}_\ell^s$. If, instead, $q^{\mathrm{BM}}(\theta_s) > \mathrm{q}_\ell^s$ (i.e., if q^{BM} never crosses q_ℓ^s over the interval $[\underline{\theta}, \theta_s^\star]$), then $\theta_s^\star \equiv \theta_s$. In either case, $\theta_s^\star \le \theta_s$. Similarly, let

$$\theta_s^m \equiv \max\{\theta : \theta \in \arg\min_{y \in [\underline{\theta}, \theta_s]} \underline{W}(y, q_s^\star)\}.$$

Thus, $\underline{w}_{q_s^\star} = \underline{W}(\theta_s^m, q_s^\star)$. Finally, let

$$G_s^{**} \equiv \sup_{\theta \in (\theta_s, \overline{\theta}]} \frac{1}{\underline{F}(\theta)} \int_{\theta_s}^{\theta} \underline{W}(y, \underline{q}^{\mathrm{BM}}) \underline{F}(\mathrm{d}y). \tag{S.10}$$

Proposition S.8 *Suppose $\underline{F}$ is regular with respect to θ_s.*

1. *The Baron-Myerson-with-quantity-bridge mechanism is robustly optimal if and only if $\underline{W}(\theta_s^m, q_s^\star) \ge \max\{G_s^*, G_s^{**}\}$.*

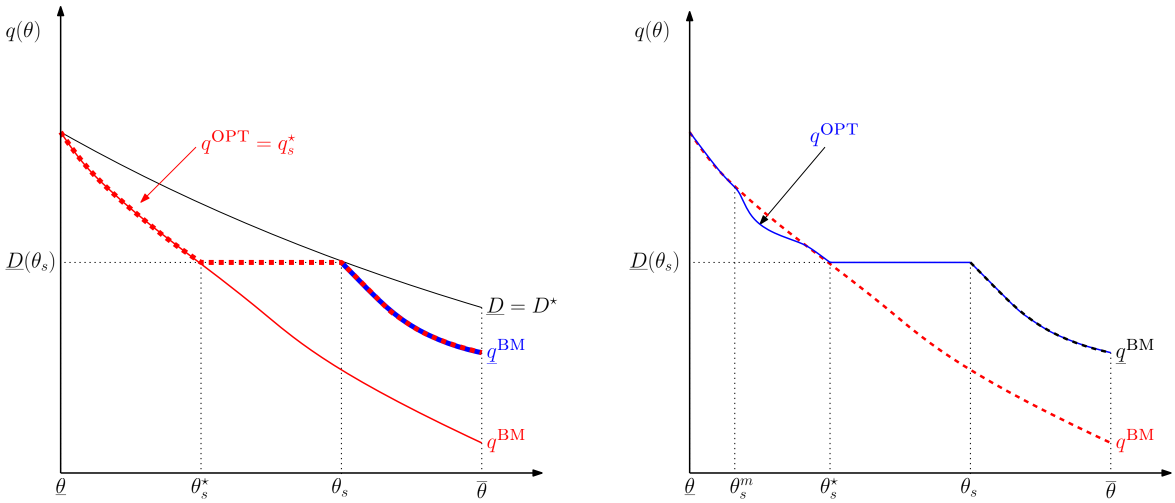
2. *If $\underline{W}(\theta_s^m, q_s^\star) < \max\{G_s^*, G_s^{**}\}$, then $\theta_s^m < \theta_s^\star$ and every robustly optimal mechanism $M^{\mathrm{OPT}} = (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$ satisfies the following properties:*

   (a) $q^{\mathrm{OPT}}(\theta) = \underline{q}^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta_s, \overline{\theta})$,

   (b) $q^{\mathrm{OPT}}(\theta) = \mathrm{q}_\ell^s$ for all $\theta \in (\theta_s^\star, \theta_s)$,

   (c) $q^{\mathrm{OPT}}(\theta) \le q^{\mathrm{BM}}(\theta)$ for almost all $\theta < \theta_s^\star$, with the inequality strict over a Lebesgue positive measure set of types $I \subseteq [\theta_s^m, \theta_s^\star]$, and

   (d) $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta_s^m)$.



A: Baron-Myerson-with-quantity-bridge.

B: Robustly optimal mechanism when $V^\star \neq \underline{V}$.

Figure S.2: Graphical illustration of Proposition S.8.

Proof: See Appendix S.A. ■

Parts (1) and (2) of Proposition S.8 generalize Proposition 1 and Proposition 2, respectively, in the main text.

Remark S.1 *If $V^\star = \underline{V}$ (as when the only uncertainty is over the cost technology), then $\theta_s^m = \theta_s$, and the condition in Part 1 of Proposition S.8 is satisfied. In this case, the Baron-Myerson-with-quantity-bridge mechanism, illustrated in Panel A of Figure S.2, is a robustly optimal mechanism. Panel B illustrates the features of robustly optimal mechanism when, instead, $V^\star \neq \underline{V}$. A comparison of these figures with Figure 1 and Figure 2 in the main text illustrates that the key features of the robustly optimal mechanisms continue extend to general technological uncertainty.*

Remark S.2 *Proposition S.8 further highlights that the key forces identified in Sections 3 and 4 of the main text continue to determine the shape of robustly optimal mechanisms to the left of θ_s. As in the main text, the quantity plateau in robustly optimal mechanisms is determined by $\underline{D}(\theta_s)$, the efficient quantity at θ_s under the lowest possible inverse demand (recall that, in the main text, $\theta_s = \bar{\theta}$, and $\underline{D}(\theta_s) = \mathrm{q}_\ell$).*

S.5 Non-monotonicity of the procured quantity in $\underline{F}$

As Proposition S.8 shows the robustly optimal mechanism depends on $\mathcal{F}$ only through $F^\star$ and $\underline{F}$. In this section, we show that, as $\underline{F}$ changes, the quantity procured under robustly optimal mechanisms may change in a non-monotonic way. To simplify the exposition, we focus on the case $V^\star = \underline{V}$. Proposition S.8 (see Remark S.1), then implies that the robustly optimal mechanism is the Baron-Myerson-with-quantity-bridge. Below, we analyze how the quantity procured under such a mechanism changes as $\underline{F}$ changes.

Fix $F^\star$ and consider a sequence $(\underline{F}_n)$ of cdfs corresponding to the lowest elements of the set $\mathcal{F}$ of feasible technologies with the following properties:

(a) for every n there exists $\underline{\theta}_n \in \Theta$ and $\delta_n \geq 0$ such that,

  (1) $\underline{F}_n$ is absolutely continuous over $(-\infty, \overline{\theta})$, with density $\underline{f}_n(\theta) > 0$ for all $\theta \in [\underline{\theta}_n, \overline{\theta})$,

  (2) $\underline{F}_n(\theta) = 0$ for all $\theta < \underline{\theta}_n$, $\underline{F}_n(\theta) = 1$ for all $\theta \geq \overline{\theta}$,

  (3) $\lim_{\theta \uparrow \overline{\theta}} \underline{F}_n(\theta) = 1 - \delta_n$,

(b) for every n, $\underline{\theta}_{n+1} \geq \underline{\theta}_n$, and for every $\theta \in (\underline{\theta}, \overline{\theta})$, there exists n such that $\theta < \underline{\theta}_n < \overline{\theta}$,

(c) for every n, $\delta_{n+1} \geq \delta_n$,

(d) there exists $\overline{n}, \overline{\overline{n}} \in \mathbb{N} \cup \{+\infty\}$ with $\overline{\overline{n}} > \overline{n}$ such that $\underline{\theta}_n = \underline{\theta}$ if, and only if, $n \leq \overline{n}$, and $\delta_n > 0$ if, and only if, $n \geq \overline{\overline{n}}$.

(e) for every n, the function $\underline{z}_n : [\underline{\theta}_n, \overline{\theta}] \to \mathbb{R}$ defined by

$$\underline{z}_n(\theta) \equiv \begin{cases} \theta + \underline{F}_n(\theta)/\underline{f}_n(\theta) & \text{if } \theta \in [\underline{\theta}_n, \overline{\theta}) \\ \overline{\theta} + 1/\underline{f}_n(\overline{\theta}) & \text{if } \theta = \overline{\theta} \text{ and } \delta_n = 0 \\ \overline{\theta} + (1-\delta_n)/\delta_n & \text{if } \theta = \overline{\theta} \text{ and } \delta_n > 0 \end{cases}$$

is increasing over $[\underline{\theta}_n, \overline{\theta}]$ and continuous over $[\underline{\theta}_n, \overline{\theta})$.

(f) for all $\theta \in [\underline{\theta}_{n+1}, \overline{\theta}]$,

$$\frac{\underline{F}_{n+1}(\theta)}{\underline{f}_{n+1}(\theta)} \leq \frac{\underline{F}_n(\theta)}{\underline{f}_n(\theta)}, \tag{S.11}$$

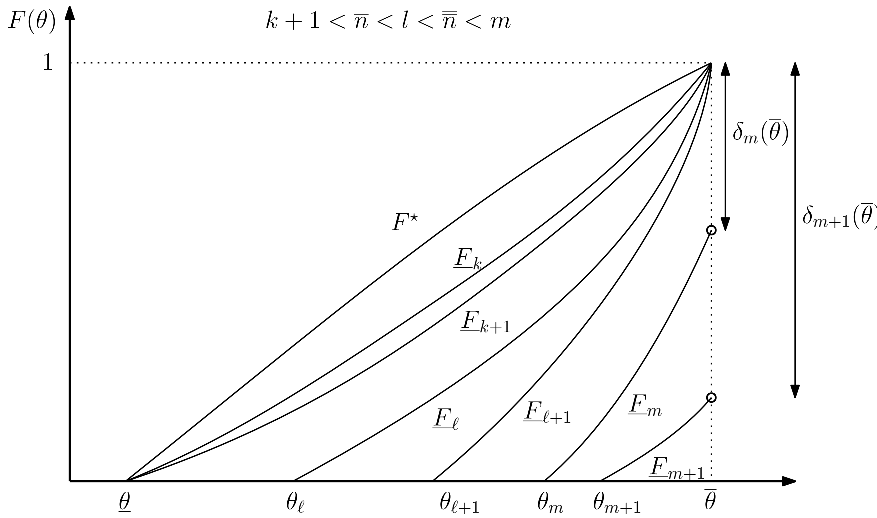


Figure S.3: Pictorial depiction of the sequence $(\underline{F}_n)$.

and for every n and every $\theta \in [\underline{\theta}_n, \overline{\theta}]$,

$$\underline{z}_n(\theta) < z^\star(\theta), \tag{S.12}$$

which is the case when $\frac{\underline{F}_n(\theta)}{\underline{f}_n(\theta)} < \frac{F^\star(\theta)}{f^\star(\theta)}$.

Figure S.3 provides an illustration of the sequence $(\underline{F}_n)$. Note that property (c) above means that the technologies are ranked in the reverse-hazard-rate order. The sequence can thus be interpreted as capturing an increase in the severity of the buyer's (downside) uncertainty over the technology that determines the seller's cost.

Let q_n^{OPT} be a robustly optimal quantity schedule when the lowest technology in $\mathcal{F}$ is $\underline{F}_n$. The following proposition establishes that the quantity procured under a robustly optimal mechanism is not monotone in the lowest distribution $\underline{F}_n$. This property holds despite the fact that, as is well known, the Baron-Myerson quantity schedule $\underline{q}_n^{\mathrm{BM}}$ defined, for all $\theta \in [\underline{\theta}_n, \overline{\theta})$, by

$$\underline{q}_n^{\mathrm{BM}}(\theta) \equiv \arg\max_{\mathrm{q}\in[0,\bar{\mathrm{q}}]} \{V^\star(\mathrm{q}) - \underline{z}_n(\theta)\mathrm{q}\}$$

is increasing in the inverse-hazard rare order: for any $n, n' \in \mathbb{N}$, with $n' > n$ and any $\theta \geq \underline{\theta}_{n'}$, $\underline{q}_{n'}^{\mathrm{BM}}(\theta) \geq \underline{q}_n^{\mathrm{BM}}(\theta)$. That is, when the buyer's model over the technology of the seller's cost coincides with the distribution $\underline{F}_n$, an increase in the distribution (in the inverse-hazard-rate order) leads to an increase in the output procured.

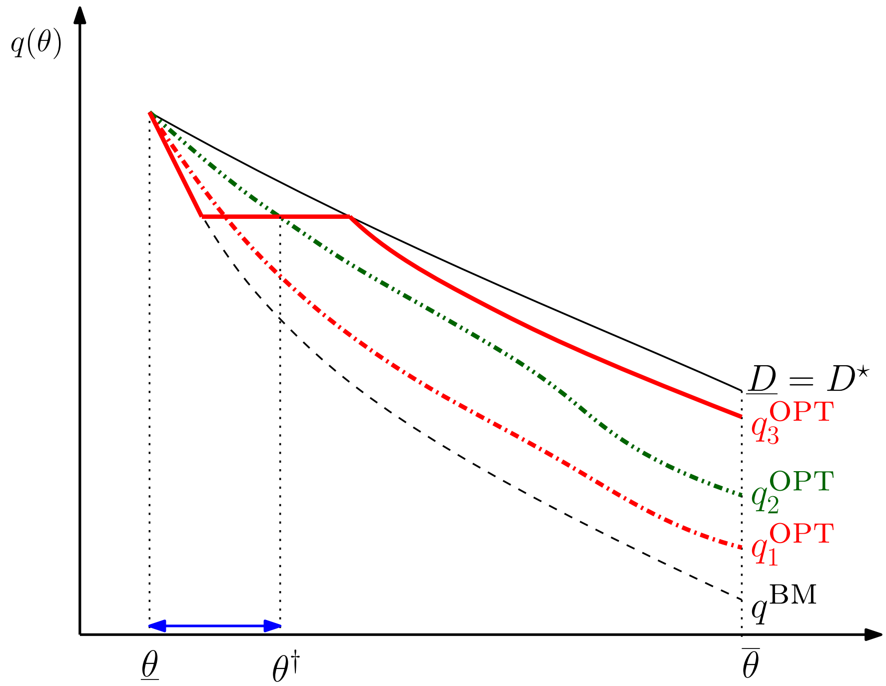


Figure S.4: Illustration of Proposition S.9.

Proposition S.9 (Non-mon. of output in cost uncertainty) *Suppose that $V^\star = \underline{V}$. Let $(\underline{F}_n)$ be any sequence of cdfs satisfying properties (a)-(f) above and let (M_n^{OPT}) be any sequence of mechanisms such that, for each n, $M_n^{\text{OPT}} \equiv (q_n^{\text{OPT}}, u_n^{\text{OPT}})$ is a robustly optimal mechanism when the lowest distribution in $\mathcal{F}$ is $\underline{F}_n$. Then, for every $\theta \in (\underline{\theta}, \overline{\theta})$,*

1. *there exists $n(\theta) \in \mathbb{N}$ such that $q_n^{\text{OPT}}(\theta)$ is weakly increasing in n (respectively, weakly decreasing in n) over $n \leq n(\theta) - 1$ (respectively, over $n > n(\theta)$).*
2. *there exists $j, k \in \mathbb{N}$ with $j < k$ such that $q_j^{\text{OPT}}(\theta) > q_k^{\text{OPT}}(\theta)$.*

Proof: See Appendix S.A. ■

Figure S.4 illustrates the result in Proposition S.9. For any $\theta \in [\underline{\theta}, \theta^\dagger]$, as the lowest technology changes from $\underline{F}_1$ to $\underline{F}_2$, the quantity procured increases. In fact, the robustly optimal quantity schedule changes from the dash-dotted line to the dash-double-dotted line. Note that both $\underline{F}_1$ to $\underline{F}_2$ have support Θ; a reduction in the inverse of reverse hazard rate then implies a reduction in the value of reducing the rents paid to the most efficient types and hence an increase in the output procured under the optimal mechanism. When the lowest technology changes from $\underline{F}_2$ to $\underline{F}_3$, the robustly optimal quantity schedule changes from the dash-double-dotted line to the solid line and the quantity procured from types in the range $[\underline{\theta}, \theta^\dagger]$ goes down. This is because the support of $\underline{F}_3$ no longer contains low-cost types. The buyer can then afford to procure less output from these types without jeopardizing robustness. Thus, the quantity procured from types in the range $[\underline{\theta}, \theta^\dagger]$ is not monotone in n, equivalently, in the worst possible technology.

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S.A Proofs of Propositions S.3, S.7, S.8, and S.9

Proof of Proposition S.3: Let $M^{\text{OPT}} = (q^{\text{OPT}}, u^{\text{OPT}})$ be a robustly optimal mechanism in which q^{OPT} is left-continuous. Towards a contradiction, assume that M^{OPT} is dominated by another mechanism $\widehat{M} = (\hat{q}, \hat{u})$. Lemma S.1 implies that $\widehat{M}$ is also robustly optimal, and therefore, by Proposition 2 in the main text, $\hat{q}$ differs from q only on the interval $(\theta^m, \theta^\star)$, with the thresholds θ^m and $\theta^\star$ as defined in the main text.

Let

$$\Theta_{\hat{q},q^{\text{OPT}}} \equiv \{\theta \in (\theta^m, \theta^\star) : \hat{q}(\theta) < q^{\text{OPT}}(\theta)\}.$$

Lemma S.4, which is stated and proved below after this proof, establishes that $\Theta_{\hat{q},q^{\text{OPT}}}$ has positive Lebesgue measure. Proposition 2 in the main text then implies that

$$\hat{q}(\theta) < q^{\text{OPT}}(\theta) \leq q^{\text{BM}}(\theta) < D^\star(\theta), \qquad \forall\, \theta \in \Theta_{\hat{q},q^{\text{OPT}}}. \tag{S.13}$$

Next, let

$$\theta_h \equiv \sup\{\theta \in \Theta_{\hat{q},q^{\text{OPT}}}\}.$$

The proof below establishes a contradiction by showing that there exists θ at which welfare, under the mechanism M^{OPT}, is strictly larger than under the mechanism $\widehat{M}$, when $(V, F) = (V^\star, \delta_\theta)$.

First, consider the case in which $q^{\text{OPT}}(\theta_h) > \hat{q}(\theta_h)$. Because $V^\star(\mathrm{q}) - \theta_h \mathrm{q}$ is quasi-concave in q, with a maximum at $D^\star(\theta_h)$, and because $q^{\text{OPT}}(\theta_h) < D^\star(\theta_h)$, $V^\star(\hat{q}(\theta_h)) - \theta_h(\hat{q}(\theta_h)) < V^\star(q^{\text{OPT}}(\theta_h)) - \theta_h q^{\text{OPT}}(\theta_h)$. Furthermore, because, for all $y > \theta_h$, $\hat{q}(y) \geq q^{\text{OPT}}(y)$, $\int_\theta^{\bar{\theta}} \hat{q}(y)dy \geq \int_\theta^{\bar{\theta}} q^{\text{OPT}}(y)dy$. This means that $W(\widehat{M}; V^\star, \delta_{\theta_h}) < W(M^{\text{OPT}}; V^\star, \delta_{\theta_h})$, a contradiction to the fact that $\widehat{M}$ dominates M^{OPT}.

Next, consider the case in which $q^{\text{OPT}}(\theta_h) = \hat{q}(\theta_h)$. The left continuity of q^{OPT}, along with the monotonicity of $\hat{q}$ and $D^\star$, the definition of θ_h, and Condition (S.13) above, imply that there exists a non-empty left-neighborhood $\mathcal{N} \equiv [\theta_\ell, \theta_h)$ of θ_h, with $\mathcal{N} \subseteq \Theta_{\hat{q},q^{\text{OPT}}}$, such that, for any $\theta \in \mathcal{N}$, $q^{\text{OPT}}(\theta) < D^\star(\theta_h)$.

Then let

$$\Delta \equiv \sup_{y \in \mathcal{N}} \left[q^{\text{OPT}}(y) - \hat{q}(y)\right]$$

and observe that, by definition of $\mathcal{N}$, $\Delta > 0$. Thus, there exists $\theta \in \mathcal{N}$ such that $q^{\text{OPT}}(\theta) - \hat{q}(\theta) = \Delta - \epsilon$, where

$$0 \leq \epsilon < \Delta \frac{P^\star(q^{\text{OPT}}(\theta_\ell)) - \theta_h}{P^\star(q^{\text{OPT}}(\theta_\ell)) - \theta^m} < \Delta.$$

Note that, by definition of $\mathcal{N}$, $q^{\text{OPT}}(\theta_\ell) < D^\star(\theta_h)$, which implies that $P^\star(q^{\text{OPT}}(\theta_\ell)) > \theta_h > \theta^m$.

Next, observe that

$$\begin{aligned}
&W(M^{\text{OPT}}; V^\star, \delta_\theta) - W(\widehat{M}; V^\star, \delta_\theta) \\
&= \int\limits_{\hat{q}(\theta)}^{q^{\text{OPT}}(\theta)} \big(P^\star(z) - \theta\big)dz - \int\limits_{\theta}^{\bar{\theta}} \big(q^{\text{OPT}}(y) - \hat{q}(y)\big)dy \\
&= \int\limits_{\hat{q}(\theta)}^{q^{\text{OPT}}(\theta)} \big(P^\star(z) - \theta\big)dz - \int\limits_{\theta}^{\theta_h} \big(q^{\text{OPT}}(y) - \hat{q}(y)\big)dy - \int\limits_{\theta_h}^{\bar{\theta}} \big(q^{\text{OPT}}(y) - \hat{q}(y)\big)dy
\end{aligned}$$

$$
\begin{aligned}
&\geq \int\limits_{\hat{q}(\theta)}^{q^{\mathrm{OPT}}(\theta)} \big(P^\star(z)-\theta\big)dz - \int\limits_{\theta}^{\theta_h} \big(q^{\mathrm{OPT}}(y)-\hat{q}(y)\big)dy \\
&\geq \big(P^\star(q^{\mathrm{OPT}}(\theta_\ell))-\theta\big)\big(q^{\mathrm{OPT}}(\theta)-\hat{q}(\theta)\big)-\Delta(\theta_h-\theta) \\
&= \big(P^\star(q^{\mathrm{OPT}}(\theta_\ell))-\theta\big)(\Delta-\epsilon)-\Delta(\theta_h-\theta) \\
&= \Delta\big(P^\star(q^{\mathrm{OPT}}(\theta_\ell))-\theta_h\big)-\epsilon\big(P^\star(q^{\mathrm{OPT}}(\theta_\ell))-\theta\big) \\
&\geq \Delta\big(P^\star(q^{\mathrm{OPT}}(\theta_\ell))-\theta_h\big)-\epsilon\big(P^\star(q^{\mathrm{OPT}}(\theta_\ell))-\theta^m\big) \\
&> 0.
\end{aligned}
$$

The first equality is immediate. The first inequality follows from the fact that, by definition of θ_h, $q^{\mathrm{OPT}}(y) \leq \hat{q}(y)$ for all $y > \theta_h$. The second inequality follows from the monotonicity of q^{OPT}, the fact that $\theta > \theta_\ell$, and the definition of Δ. The second equality follows from the definition of ϵ. The third equality is immediate. The third inequality follows from the fact that $\theta > \theta^m$. The last inequality follows from the fact that $\epsilon < \Delta \frac{P^\star(q^{\mathrm{OPT}}(\theta_\ell))-\theta_h}{P^\star(q^{\mathrm{OPT}}(\theta_\ell))-\theta^m}$. ■

Lemma S.4 *The set $\Theta_{\hat{q},q^{\mathrm{OPT}}}$ in the proof of Proposition S.3 above has positive Lebesgue measure.*

Proof: Assume, toward a contradiction, that $\Theta_{\hat{q},q^{\mathrm{OPT}}}$ has zero Lebesgue measure. Then, either (1) $\hat{q}(\theta) \geq q^{\mathrm{OPT}}(\theta)$ for all $\theta \in (\theta^m, \theta^\star)$, or (2) $q^{\mathrm{OPT}}(\theta) > \hat{q}(\theta)$ only on countably many $\theta \in (\theta^m, \theta^\star)$. Below, we establish a contradiction to the fact that $M^{\mathrm{OPT}} = (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$ is dominated by $\widehat{M} = (\hat{q}, \hat{u})$ in each of these two cases.

Case 1: Suppose that $\hat{q}(\theta) \geq q^{\mathrm{OPT}}(\theta)$ for all $\theta \in (\theta^m, \theta^\star)$. If this last inequality holds as equality for all $\theta \in (\theta^m, \theta^\star)$, then $\hat{q}$ coincides with q^{OPT} at all $\theta \in \Theta$, a contradiction to the fact that $\widehat{M}$ dominates M^{OPT}. Thus suppose that $\hat{q}(\theta) > q^{\mathrm{OPT}}(\theta)$ for some $\theta \in (\theta^m, \theta^\star)$. Because q^{OPT} is left-continuous and $\hat{q}$ is weakly decreasing, there must exist a positive Lebesgue measure subset of $(\theta^m, \theta^\star)$ over which $\hat{q}(\theta) > q^{\mathrm{OPT}}(\theta)$. This implies that $\hat{u}(\theta^m) > u^{\mathrm{OPT}}(\theta^m)$. Moreover, Lemma 10 in the main text, along with the fact that both M^{OPT} and $\widehat{M}$ are robustly optimal, implies that $q^{\mathrm{OPT}}(\theta^m) = \hat{q}(\theta^m) = \underline{D}(\theta^m)$. Therefore,

$$
V^\star(\hat{q}(\theta^m)) - \theta^m \hat{q}(\theta^m) - \int\limits_{\theta^m}^{\bar{\theta}} \hat{q}(y)dy < V^\star(q^{\mathrm{OPT}}(\theta^m)) - \theta^m q^{\mathrm{OPT}}(\theta^m) - \int\limits_{\theta^m}^{\bar{\theta}} q^{\mathrm{OPT}}(y)dy.
$$

That is, $W(\widehat{M}; V^\star, \delta_{\theta^m}) < W(M^{\mathrm{OPT}}; V^\star, \delta_{\theta^m})$, a contradiction to the fact that $\widehat{M}$ dominates M^{OPT}.

Case 2: Now suppose that $\hat{q}(\theta) < q^{\mathrm{OPT}}(\theta)$ for some $\theta \in (\theta^m, \theta^\star)$ but the subset of $(\theta^m, \theta^\star)$ for which the inequality holds has zero Lebesgue measure (i.e., the inequality holds only over a countable set). Because q^{OPT} is left-continuous, and both q^{OPT} and $\hat{q}$ are weakly decreasing, these points must be the ones where q^{OPT} is discontinuous. Let θ' be one such point of discontinuity of q^{OPT} in $(\theta^m, \theta^\star)$. Because $\hat{q}(\theta) \geq q^{\mathrm{OPT}}(\theta)$ for almost all $\theta \in \Theta$, we have that $\hat{u}(\theta') \geq u^{\mathrm{OPT}}(\theta')$. Furthermore, by Proposition 2 in the main text, $\hat{q}(\theta') < q^{\mathrm{OPT}}(\theta') < D^\star(\theta')$. Combining these facts with the quasi-concavity of the function $V^\star(\mathrm{q}) - \theta'\mathrm{q}$ in q (reaching a maximum at $D^\star(\theta')$), we have that

$$V^\star(\hat{q}(\theta')) - \theta'\hat{q}(\theta') - \int_{\theta'}^{\overline{\theta}} \hat{q}(y)dy < V^\star(q^{\mathrm{OPT}}(\theta')) - \theta' q^{\mathrm{OPT}}(\theta') - \int_{\theta'}^{\overline{\theta}} q^{\mathrm{OPT}}(y)dy.$$

That is, $W(\widehat{M}; V^\star, \delta_{\theta'}) < W(M^{\mathrm{OPT}}; V^\star, \delta_{\theta'})$, a contradiction to the fact that $\widehat{M}$ dominates M^{OPT}.

Combining the two cases, we conclude that the set $\Theta_{\hat{q}, q^{\mathrm{OPT}}}$ has positive Lebesgue measure. ■

Proof of Proposition S.7. **Part 1**. For any IC and IR mechanism $M = (q, u) \in \mathcal{M}$,

$$G(M) = \inf_{(V,F)\in\mathcal{V}\times\mathcal{F}} W(M; V, F) \leq W(M; \underline{V}, \underline{F}) \leq_{(a)} W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F}) = G_s^*. \tag{S.14}$$

Inequality (a) holds because $\underline{M}^{\mathrm{BM}}$ maximizes $W(\cdot; \underline{V}, \underline{F})$ over $\mathcal{M}$. We now show that there exists an IC and IR mechanism $\underline{M} \in \mathcal{M}$ such that $G(\underline{M}) = G_s^*$. Let $\underline{M} \equiv (\underline{q}, \underline{u})$ be the mechanism in which

$$\underline{q}(\theta) = \begin{cases} \underline{q}^{\mathrm{BM}}(\theta_s) & \text{if } \theta < \theta_s \\ \underline{q}^{\mathrm{BM}}(\theta) & \text{if } \theta \geq \theta_s, \end{cases}$$

and $\underline{u}(\theta) = \int_\theta^{\overline{\theta}} \underline{q}(y)dy$ for all θ. Clearly, $\underline{M} \in \mathcal{M}$. Furthermore, since $\underline{F}$ has support $[\theta_s, \overline{\theta}]$, it follows that $W(\underline{M}; \underline{V}, \underline{F}) = W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F})$. Now, recall that $\underline{M}^{\mathrm{BM}} = (\underline{q}^{\mathrm{BM}}, \underline{u}^{\mathrm{BM}})$ is the optimal mechanism for the model $(\underline{V}, \underline{F})$. When $\underline{F}$ is regular with respect to θ_s, $\underline{q}^{\mathrm{BM}}$ is such that, for all $\theta \geq \theta_s$,

$$\underline{q}^{\mathrm{BM}}(\theta) = \underline{D}(\underline{z}(\theta)),$$

where, for all $\theta \geq \theta_s$, $\underline{z}(\theta) \equiv \theta + \underline{F}(\theta)/\underline{f}(\theta)$. Thus, $\underline{q}(\theta) \leq \underline{D}(\theta)$ for all θ, with the inequality strict for $\theta \neq \theta_s$.[1] Part A of Lemma 9 in the main text then implies that $\underline{W}(\cdot, \underline{q})$ is weakly decreasing over Θ. Furthermore, because, for all $F \in \mathcal{F}$, $\underline{F} \succ_{FOSD} F$,

$$W(\underline{M}; \underline{V}, F) \geq W(\underline{M}; \underline{V}, \underline{F}).$$

Because, for any $V \in \mathcal{V}$ and any $F \in \mathcal{F}$, $W(\underline{M}; V, F) \geq W(\underline{M}; \underline{V}, F)$, we thus have that $W(\underline{M}; V, F) \geq W(\underline{M}; \underline{V}, \underline{F})$. We conclude that $G(\underline{M}) = W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F}) = G_s^*$.

Part 2: Necessity. If $M = (q, u) \in \mathcal{M}^{\mathrm{SL}}$, then M is IC and IR, and, therefore, q is weakly decreasing and $u(\theta) = u(\overline{\theta}) + \int_\theta^{\overline{\theta}} q(y)dy$ for all θ. Furthermore, by the result in Part 1, it must be that $G(M) = G_s^*$. Hence, $u(\overline{\theta}) = 0$.

Recall that, for any $\theta \geq \theta_s$, $\underline{q}^{\mathrm{BM}}(\theta) = \arg\max_{\mathrm{q} \in [0, \bar{\mathrm{q}}]}\{\underline{V}(\mathrm{q}) - \underline{z}(\theta)\mathrm{q}\}$. If $q(\theta) \neq \underline{q}^{\mathrm{BM}}(\theta)$ for a positive Lebesgue measure subset of $[\theta_s, \bar{\theta}]$, then inequality (a) in (S.14) is strict and hence $W(M; \underline{V}, \underline{F}) < W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F}) = G_s^\star$. This means that $G(M) < G_s^*$ and hence $M \notin \mathcal{M}^{\mathrm{SL}}$, a contradiction. Because $\underline{q}^{\mathrm{BM}}$ is decreasing and continuous over $[\theta_s, \bar{\theta}]$, we conclude that $q(\theta) = \underline{q}^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta_s, \bar{\theta})$.

Next, observe that, for any $\theta < \theta_s$, $\mathcal{F}$ contains a distribution F corresponding to a Dirac measure at $\theta < \theta_s$ (indeed, $\underline{F} \succ_{\mathrm{FOSD}} F$). Welfare under the lowest gross value function $\underline{V}$ and such an F is $\underline{V}(q(\theta)) - \theta q(\theta) - \int\limits_\theta^{\bar{\theta}} q(y)dy = \underline{W}(\theta, q)$. Hence, it must be that $\underline{W}(\theta, q) \geq G_s^*$.

Finally, observe that the inequality in the constraint in Condition (e) is an equality for $\theta = \theta_s$. Suppose there exists $\tilde{\theta} \in (\theta_s, \overline{\theta}]$ such that

$$\underline{w}_q \underline{F}(\tilde{\theta}) + \int\limits_{\tilde{\theta}}^{\bar{\theta}} \underline{W}(y, \underline{q}^{\mathrm{MB}}) \underline{F}(\mathrm{d}y) < G_s^*. \tag{S.15}$$

By definition of $\underline{w}_q$, there exists $\theta' \leq \theta_s$ such that $\underline{W}(\theta', q)$ is arbitrarily close to $\underline{w}_q$. Let $\widetilde{F}$ be the cdf given by

$$\widetilde{F}(\theta) = \begin{cases} 0 & \text{if } \theta < \theta' \\ \underline{F}(\tilde{\theta}) & \text{if } \theta \in [\theta', \tilde{\theta}) \\ \underline{F}(\theta) & \text{if } \theta \geq \tilde{\theta}. \end{cases}$$

[1] This property holds even if $\underline{F}$ is not regular. In fact, any undominated mechanism $M = (q, u)$ is such that $q(\theta) \leq \underline{D}(\theta)$ for all θ (Mishra and Patil, 2025).

Clearly, $\widetilde{F}(\theta) \geq \underline{F}(\theta)$ for all θ, and hence, $\widetilde{F} \in \mathcal{F}$. Welfare under the mechanism $M = (q, u)$ when Nature selects the model $(\underline{V}, \widetilde{F})$ is equal to

$$W(M; \underline{V}, \widetilde{F}) = \underline{W}(\theta', q)\underline{F}(\tilde{\theta}) + \int_{\tilde{\theta}}^{\bar{\theta}} \underline{W}(y, \underline{q}^{\text{MB}})\underline{F}(\text{d}y) < G_s^*,$$

where the inequality follows from Condition (S.15) and the fact that $\underline{W}(\theta', q)$ is arbitrarily close to $\underline{w}_q$. This, however, is a contradiction to $M \in \mathcal{M}^{\text{SL}}$.

We conclude that properties (a)-(e) are jointly necessary for any $M \in \mathcal{M}^{\text{SL}}$.

Part 2: Sufficiency. Take any mechanism M satisfying properties (a)-(e). By virtue of (a) and (b), M is IC and IR. By virtue of the result in Part 1 of the proposition, it thus suffices to show that $W(M; V, F) \geq G_s^*$ for any model $(V, F) \in \mathcal{V} \times \mathcal{F}$. First, suppose F is a Dirac distribution on some $\theta \leq \theta_s$. Then, Condition (c) implies that

$$W(M; V, F) \geq W(M; \underline{V}, F) = \underline{V}(q(\theta)) - \theta q(\theta) - \int_{\theta}^{\bar{\theta}} q(y)dy \geq G_s^*.$$

Next consider any model $(V, F) \in \mathcal{V} \times \mathcal{F}$ in which F puts a positive probability on $\theta < \theta_s$. Then,

$$\begin{aligned} W(M; V, F) &\geq W(M; \underline{V}, F) \\ &\geq \underline{w}_q F(\theta_s) + \int_{\theta_s}^{\bar{\theta}} \underline{W}(\theta, q)F(\text{d}\theta) \qquad \text{(by definition of } \underline{w}_q\text{)} \\ &= \underline{w}_q F(\theta_s) + \int_{\theta_s}^{\bar{\theta}} \underline{W}(\theta, \underline{q}^{\text{BM}})F(\text{d}\theta) \qquad \text{(because } q(\theta) = \underline{q}^{\text{MB}}(\theta) \text{ for all } \theta \in (\theta_s, \bar{\theta})\text{).} \end{aligned} \tag{S.16}$$

Partition $[\theta_s, \overline{\theta}]$ into $\Theta_1 \equiv \{\theta \in [\theta_s, \overline{\theta}] : \underline{W}(\theta, \underline{q}^{\text{BM}}) \leq \underline{w}_q\}$ and $\Theta_2 \equiv [\theta_s, \overline{\theta}] \setminus \Theta_1$. Note that $\underline{W}(\cdot, \underline{q}^{\text{BM}})$ is decreasing over $[\theta_s, \overline{\theta}]$ and hence Θ_1 is a (possibly empty) interval. If Θ_1 is empty, let $\hat{\theta} \equiv \theta_s$. Else, let $\hat{\theta}$ be the left endpoint of Θ_1. Using (S.16), we have that

$$W(M; V, F) \geq \underline{w}_q F(\theta_s) + \int_{\theta_s}^{\hat{\theta}} \underline{W}(\theta, \underline{q}^{\text{BM}})F(\text{d}\theta) + \int_{\hat{\theta}}^{\bar{\theta}} \underline{W}(\theta, \underline{q}^{\text{BM}})F(\text{d}\theta)$$

$$\geq \underline{w}_q F(\theta_s) + \underline{w}_q \left(F(\hat{\theta}) - F(\theta_s) \right) + \int_{\hat{\theta}}^{\bar{\theta}} \underline{W}(\theta, \underline{q}^{\mathrm{MB}}) F(\mathrm{d}\theta)$$

$$\text{(because } \underline{W}(\cdot, \underline{q}^{\mathrm{MB}}) \geq \underline{w}_q \text{ on } [\theta_s, \hat{\theta}])$$

$$= \underline{w}_q F(\hat{\theta}) + \int_{\hat{\theta}}^{\bar{\theta}} \underline{W}(\theta, \underline{q}^{\mathrm{MB}}) F(\mathrm{d}\theta). \tag{S.17}$$

Now, let $h : \Theta \to \mathbb{R}$ be the weakly decreasing function defined by

$$h(\theta) = \begin{cases} \underline{w}_q & \text{if } \theta \leq \hat{\theta} \\ \underline{W}(\theta, \underline{q}^{\mathrm{MB}}) & \text{otherwise.} \end{cases}$$

From (S.17), we then have that

$$W(M; V, F) \geq \int_{\underline{\theta}}^{\hat{\theta}} h(\theta) F(\mathrm{d}\theta) + \int_{\hat{\theta}}^{\bar{\theta}} h(\theta) F(\mathrm{d}\theta) = \int_{\underline{\theta}}^{\bar{\theta}} h(\theta) F(\mathrm{d}\theta)$$

$$\geq \int_{\underline{\theta}}^{\bar{\theta}} h(\theta) \underline{F}(\mathrm{d}\theta) \qquad (\text{since } \underline{F} \succ_{\mathrm{FOSD}} F)$$

$$= \underline{w}_q \underline{F}(\hat{\theta}) + \int_{\hat{\theta}}^{\bar{\theta}} \underline{W}(\theta, \underline{q}^{\mathrm{MB}}) \underline{F}(\mathrm{d}\theta) \qquad (\text{by definition of } h)$$

$$\geq G_s^* \qquad (\text{by Condition (e)})$$

Hence, $G(M) \geq G_s^*$. Part 1 establishes that, for any $M \in \mathcal{M}^{SL}$, $G(M) = G_s^*$. We conclude that any mechanism satisfying Conditions (a)-(e) is in the short list. ■

Proof of Proposition S.8. The proof is in several parts, each corresponding to a part of the proposition. Before proceeding with the proof, observe that for any mechanism $M = (q, u) \in \mathcal{M}^{\mathrm{SL}}$, Condition (d) in Part 2 of Proposition S.7 implies that $\underline{w}_q \geq G_s^*$. Condition (e) in the same proposition in turn implies that, for all $\theta \geq \theta_s$,

$$\underline{w}_q \underline{F}(\theta) + \int_{\theta}^{\bar{\theta}} \underline{W}(y, \underline{q}^{\mathrm{BM}}) \underline{F}(\mathrm{d}y) \geq G_s^* \equiv \int_{\theta_s}^{\bar{\theta}} \underline{W}(y, \underline{q}^{\mathrm{BM}}) \underline{F}(\mathrm{d}y),$$

which is equivalent to

$$\underline{w}_q \geq \sup_{\theta \in (\theta_s, \bar{\theta}]} \frac{1}{\underline{F}(\theta)} \int_{\theta_s}^{\theta} \underline{W}(y, \underline{q}^{\mathrm{BM}}) \underline{F}(\mathrm{d}y) \equiv G_s^{**}.$$

Thus, Condition (d) and (e) in Proposition S.7 can be equivalently written as $\underline{w}_q \geq \max\{G_s^*, G_s^{**}\}$, or alternatively, $\underline{W}(\theta, q) \geq \max\{G_s^*, G_s^{**}\}$ for all $\theta \leq \theta_s$.

Part 1. If $M_s^\star$ is robustly optimal, then $M_s^\star \in \mathcal{M}^{\mathrm{SL}}$. Observe that $\underline{w}_{q_s^\star} = \underline{W}(\theta_s^m, q_s^\star)$. Thus, as argued above, Conditions (d) and (e) in Part 2 of Proposition S.7 imply that $\underline{W}(\theta_s^m, q_s^\star) \geq \max\{G_s^*, G_s^{**}\}$.

Next, suppose that $M_s^\star$ is such that $\underline{W}(\theta_s^m, q_s^\star) \geq \max\{G_s^*, G_s^{**}\}$. We now show that $M_s^\star$ is robustly optimal. By definition, $M_s^\star$ satisfies Conditions (a)-(c) in Part 2 of Proposition S.7. As argued above, that $\underline{W}(\theta_s^m, q_s^\star) \geq \max\{G_s^*, G_s^{**}\}$ implies that Conditions (d) and (e) in Part 2 of Proposition S.7 are satisfied. This means that $M_s^\star \in \mathcal{M}^{\mathrm{SL}}$. To see that $M_s^\star$ maximizes the buyer's payoff under the conjectured model $(V^\star, F^\star)$ over $\mathcal{M}^{\mathrm{SL}}$, first observe that $q_s^\star$ is weakly decreasing because $F^\star$ is regular. Second note that every mechanism $M = (q, u)$ in the short list has a weakly decreasing quantity schedule q that agrees with $\underline{q}^{BM}$ over $(\theta_s, \bar{\theta})$ (by virtue of Condition (c) in Proposition S.7). This means that, in any such a mechanism, $q(\theta) \geq \mathrm{q}_\ell^s$ for all $\theta \in [\underline{\theta}, \theta_s)$. Because, for every $\theta \in [\underline{\theta}, \theta_s)$,

$$q_s^\star(\theta) = \arg \max_{\mathrm{q} \in [\mathrm{q}_\ell^s, \bar{\mathrm{q}}]} \{V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}\}$$

we conclude that, for any $M = (q, u) \in \mathcal{M}^{\mathrm{SL}}$,

$$\int_{\underline{\theta}}^{\bar{\theta}} \Big[V^\star(q(\theta)) - z^\star(\theta) q(\theta)\Big] F^\star(d\theta) \leq \int_{\underline{\theta}}^{\bar{\theta}} \Big[V^\star(q_s^\star(\theta)) - z^\star(\theta) q_s^\star(\theta)\Big] F^\star(d\theta)$$

implying that indeed $M_s^\star$ maximizes the buyer's payoff (under the conjectured model $(V^\star, F^\star)$) over $\mathcal{M}^{\mathrm{SL}}$.

Part 2. We start with the following lemma:

Lemma S.5 *Suppose $\underline{w}_{q_s^\star} < \max\{G_s^*, G_s^{**}\}$. Then the following are true:*

1. $\theta_s^m < \theta_s^\star$,

2. if $\theta_s^m > \underline{\theta}$, then $q_s^\star(\theta_s^m) = \underline{D}(\theta_s^m)$.

Proof: *Part 1 of Lemma S.5.* We consider two cases. First, suppose $q^{\mathrm{BM}}(\theta_s) > \mathrm{q}_\ell^s = \underline{D}(\theta_s)$. Then, $\theta_s^\star = \theta_s$. Since $\underline{D}$ and q^{BM} are decreasing and continuous, there exists a non-empty left neighborhood of θ_s where $q^{\mathrm{BM}}(\theta) > \underline{D}(\theta)$. Part B of Lemma 9 in the main text then implies that $\underline{W}(\cdot, q)$ is increasing on this interval implying $\theta_s^m < \theta_s = \theta_s^\star$.

Next, suppose that $q^{\mathrm{BM}}(\theta_s) \leq \mathrm{q}_\ell^s = \underline{D}(\theta_s)$. Then, $\theta_s^\star \leq \theta_s$, and for every $\theta \in [\theta_s^\star, \theta_s]$, $q_s^\star(\theta) = \mathrm{q}_\ell^s$ and $\underline{W}(\theta, q_s^\star) = \underline{W}(\theta_s, q_s^\star) \geq \max\{G_s^*, G_s^{**}\} > \underline{w}_{q_s^\star}$. The first inequality follows from the definitions of G_s^* and G_s^{**} along with Part A of Lemma 9 in the main text (which implies that $\underline{W}(\cdot, q)$ is weakly decreasing over $[\theta_s, \bar{\theta}]$). The second inequality follows from the assumption of Lemma S.5. This implies that $\underline{W}(\theta_s^\star, q_s^\star) > \underline{w}_{q_s^\star}$. Hence, $\theta_s^m < \theta_s^\star$.

Part 2 of Lemma S.5. Since $\theta_s^m < \theta_s^\star$, we have that $q_s^\star(\theta_s^m) = q^{\mathrm{BM}}(\theta_s^m)$. This property, together with Lemma 10 in the main text, then implies the result. ■

Equipped with this last lemma, we now establish Parts 2(a) and 2(b) of the proposition.

Parts 2(a) and 2(b). Part 2(a) follows from Part (c) of Proposition S.7 because any robustly optimal mechanism belongs to $\mathcal{M}^{\mathrm{SL}}$. Thus consider Part 2(b). We consider two cases. First, suppose that $q^{\mathrm{BM}}(\theta_s) \geq \mathrm{q}_\ell^s$. By the definition of $\theta_s^\star$, we then have that $\theta_s^\star = \theta_s$, and hence the interval $(\theta_s^\star, \theta_s)$ is empty and the result applies vacuously. Next, suppose that $q^{\mathrm{BM}}(\theta_s) < \mathrm{q}_\ell^s$. Then, $\theta_s^\star < \theta_s$. Now, assume, toward a contradiction, that there exists $\theta' \in (\theta_s^\star, \theta_s)$ such that $q^{\mathrm{OPT}}(\theta') > \mathrm{q}_\ell^s$. Monotonicity of q^{OPT} then implies that $q^{\mathrm{OPT}}(\theta) > \mathrm{q}_\ell^s$ for all $\theta \in [\theta_s^\star, \theta']$. This means that there exists a non-zero Lebesgue measure of types such that $q^{\mathrm{OPT}}(\theta) > \mathrm{q}_\ell^s$. Then, consider the mechanism $\widetilde{M} = (\tilde{q}, \tilde{u})$ where the quantity schedule is given by

$$\tilde{q}(\theta) = \begin{cases} q^{\mathrm{OPT}}(\theta) & \text{if } \theta < \theta_s^\star \\ \mathrm{q}_\ell^s & \text{if } \theta \in [\theta_s^\star, \theta_s] \\ \underline{q}^{\mathrm{BM}}(\theta) = q^{\mathrm{OPT}}(\theta) & \text{if } \theta \geq \theta_s \end{cases}$$

and where the rents $\tilde{u}$ are given by $\tilde{u}(\theta) = \int_\theta^{\bar{\theta}} \tilde{q}(y)dy$ for all θ. Because $\tilde{q}$ is weakly decreasing, this ensures that $\widetilde{M}$ is IC and IR. The buyer's payoff from $\widetilde{M}$ (under the conjectured model) is equal to

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(\tilde{q}(\theta)) - z^\star(\theta)\tilde{q}(\theta)\Big] F^\star(\mathrm{d}\theta)$$

which is strictly higher than under M^{OPT}. This follows from the fact that, for any $\theta \in [\theta_s^\star, \theta_s]$, q_ℓ^s maximizes $V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}$ over $\mathrm{q} \geq \mathrm{q}_\ell^s$, along with the fact that $F^\star$ is absolutely continuous. Thus, to produce a contradiction to the robust optimality of M^{OPT}, it suffices to show that $\widetilde{M} \in \mathcal{M}^{\mathrm{SL}}$.

By definition, $\widetilde{M}$ satisfies Conditions (a)-(c) in Part 2 of Proposition S.7. As for Conditions (d) and (e), recall that these conditions are equivalent to $\underline{w}_{\tilde{q}} \geq \max\{G_s^*, G_s^{**}\}$, as established above. That $\underline{w}_{\tilde{q}} \geq G_s^*$ follows from arguments analogous to those in the proof of Lemma 6 in the main text, along with the fact that $\tilde{q}(\theta) \leq q^{\mathrm{OPT}}(\theta)$ for all θ. To establish that $\underline{w}_{\tilde{q}} \geq G_s^{**}$, notice that, for all $\theta \in [\theta_s^\star, \theta_s]$, $\underline{W}(\theta, \tilde{q}) = \underline{W}(\theta_s, \underline{q}^{\mathrm{BM}}) \geq G_s^{**}$. Thus, it suffices to focus on $\theta < \theta_s^\star$. Observe that

$$\inf_{\theta<\theta_s^\star} \underline{W}(\theta, \tilde{q}) \geq \inf_{\theta<\theta_s^\star} \underline{W}(\theta, q^{\mathrm{OPT}}) \geq G_s^{**},$$

where the first inequality follows from the fact that $\tilde{q}(\theta) = q^{\mathrm{OPT}}(\theta)$ for $\theta < \theta_s^\star$, along with the fact that $\tilde{q}(\theta) \leq q^{\mathrm{OPT}}(\theta)$ for all θ, which implies that $\tilde{u}(\theta) \leq u^{\mathrm{OPT}}(\theta)$ for all θ. The second inequality holds because $M^{\mathrm{OPT}} \in \mathcal{M}^{\mathrm{SL}}$.

Part 2(c). From Part 2(b), $q^{\mathrm{OPT}}(\theta) = \mathrm{q}_\ell^s$ for all $\theta \in (\theta_s^\star, \theta_s)$. Now suppose there is a positive-Lebesgue-measure set $I \subseteq [\underline{\theta}, \theta_s^\star)$ such that $q^{\mathrm{OPT}}(\theta) > q_s^\star(\theta) = q^{\mathrm{BM}}(\theta)$. Consider the mechanism $\widetilde{M} = (\tilde{q}, \tilde{u})$ where the quantity schedule is given by

$$\tilde{q}(\theta) = \min\{q_s^\star(\theta), q^{\mathrm{OPT}}(\theta)\} \qquad \forall\, \theta \in \Theta,$$

and where $\tilde{u}(\theta) = \int_{\theta}^{\overline{\theta}} \tilde{q}(y)dy$ for all θ. Clearly, because $\tilde{q}$ is weakly decreasing and $\tilde{u}$ satisfies the above properties, the mechanism $\widetilde{M}$ is IC and IR. Parts 2(a) and 2(b) in the proposition then imply that $\tilde{q}(\theta) = q^{\mathrm{OPT}}(\theta) = q_s^\star(\theta)$ for all $\theta \geq \theta_s^\star$.

The buyer's payoff under $\widetilde{M}$ is strictly higher than under M^{OPT} ; the arguments are similar to those in the proof of Lemma 7 in the main text. Clearly, $\widetilde{M}$ satisfies Conditions (a)-(c) of Part 2 of Proposition S.7. The next two claims establish that $\widetilde{M}$ also satisfies Conditions (d) and (e), that is, $\underline{w}_{\tilde{q}} \geq \max\{G_s^*, G_s^{**}\}$ or, equivalently, $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$

for all $\theta \leq \theta_s$. First, observe that, for every $\theta \in [\theta_s^\star, \theta_s]$, $\tilde{q}(\theta) = q_s^\star(\theta) = \mathrm{q}_\ell^s$ and $\underline{W}(\theta, \tilde{q}) = \underline{W}(\theta_s, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$. Below we establish that $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$ also for $\theta < \theta_s^\star$. The result is a consequence of the following two claims.

Claim S.1 *Suppose $\theta < \theta_s^\star$ is such that either $\tilde{q}(\theta) = q^{\mathrm{OPT}}(\theta)$ or $\underline{D}(\theta) \leq \tilde{q}(\theta) = q_s^\star(\theta) < q^{\mathrm{OPT}}(\theta)$. Then $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$.*

Proof: Pick $\theta < \theta_s^\star$. It suffices to show that $\underline{W}(\theta, \tilde{q}) \geq \underline{W}(\theta, q^{\mathrm{OPT}})$; because $\underline{W}(\theta, q^{\mathrm{OPT}}) \geq \max\{G_s^*, G_s^{**}\}$, the claim then follows.

Note that $q_s^\star(\theta) = q^{\mathrm{BM}}(\theta)$. For any θ such that $\tilde{q}(\theta) = q^{\mathrm{OPT}}(\theta)$, since $\tilde{q}(y) \leq q^{\mathrm{OPT}}(y)$ for all $y \geq \theta$, we have that $\underline{W}(\theta, \tilde{q}) \geq \underline{W}(\theta, q^{\mathrm{OPT}})$. Thus, consider a θ for which $\underline{D}(\theta) \leq \tilde{q}(\theta) = q_s^\star(\theta) = q^{\mathrm{BM}}(\theta) < q^{\mathrm{OPT}}(\theta)$. The quasi-concavity of the function $\underline{V}(\mathrm{q}) - \theta \mathrm{q}$ in q implies that

$$\underline{V}(q_s^\star(\theta)) - \theta q_s^\star(\theta) > \underline{V}(q^{\mathrm{OPT}}(\theta)) - \theta q^{\mathrm{OPT}}(\theta).$$

Together with the fact that $\tilde{q}(y) \leq q^{\mathrm{OPT}}(y)$ for all $y \geq \theta$, this means that $\underline{W}(\theta, \tilde{q}) \geq \underline{W}(\theta, q^{\mathrm{OPT}})$. ■

Claim S.2 *Suppose $\theta < \theta_s^\star$ is such that $q_s^\star(\theta) < \min\{\underline{D}(\theta), q^{\mathrm{OPT}}(\theta)\}$. Then, $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$.*

Proof: The proof considers two cases to establish the existence of $\theta' > \theta$ such that $W(\cdot, \tilde{q})$ is weakly decreasing on $[\theta, \theta']$ with $W(\theta', \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$.

Case 1. Suppose $q_s^\star(\theta_s) = \mathrm{q}_\ell^s = \underline{D}(\theta_s)$. Because $q_s^\star$ and $\underline{D}$ are both continuous, there exists $\theta < \theta' \leq \theta_s$ such that $q_s^\star(y) \leq \underline{D}(y)$ for all $y \in [\theta, \theta']$, with $q_s^\star(\theta') = \underline{D}(\theta')$. Thus,

$$\tilde{q}(\theta') = \min\{\underline{D}(\theta'), q^{\mathrm{OPT}}(\theta')\}.$$

Further, for all $y \in [\theta, \theta']$,

$$\tilde{q}(y) = \min\{q^{\mathrm{OPT}}(y), q_s^\star(y)\} \leq \underline{D}(y).$$

Part A of Lemma 9 in the main text implies that $\underline{W}(\cdot, \tilde{q})$ is weakly decreasing over $[\theta, \theta']$ whereas Claim S.1 implies that $\underline{W}(\theta', \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$. Hence, $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$.

Case 2. Now suppose $q_s^\star(\theta_s) = q^{\mathrm{BM}}(\theta_s) > \mathrm{q}_\ell^s = \underline{D}(\theta_s)$. Then, because $q_s^\star(\theta) < \underline{D}(\theta)$, and $\underline{D}$ and $q_s^\star$ are continuous (the latter due to the regularity of $F^\star$), there exists $\theta < \hat{\theta} < \overline{\theta}$ such that $q_s^\star(\hat{\theta}) = \underline{D}(\hat{\theta})$ and $q_s^\star(y) > \underline{D}(y)$ for all $y > \hat{\theta}$. Again, just like we argued in Case 1, there exists $\theta < \theta' \leq \hat{\theta}$ such that $q_s^\star(y) \leq \underline{D}(y)$ for all $y \in [\theta, \theta']$ with $q_s^\star(\theta') = \underline{D}(\theta')$. The same arguments as in Case 1 then imply that $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$. ■

The above two claims establish that $\tilde{q}$ is such that $\underline{w}_{\tilde{q}} \geq \max\{G_s^*, G_s^{**}\}$. From Proposition S.7, we thus conclude that $\widetilde{M} = (\tilde{q}, \tilde{u}) \in \mathcal{M}^{\mathrm{SL}}$.

We complete the proof by showing that there must exist a set of types $I \subseteq [\underline{\theta}, \theta_s^\star]$ of positive Lebesgue measure such that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in I$. To do that assume, towards a contradiction, that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ almost everywhere on $[\underline{\theta}, \theta_s^\star]$. Because $q_s^\star$ is continuous and $q^{\mathrm{BM}}(\theta)$ is the unique maximizer of $V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}$, this means that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta < \theta_s^\star$. This however implies that $M_s^\star$ is robustly optimal, a contradiction.

Part 2(d). Assume, toward a contradiction, that there exists a $\theta \in (\underline{\theta}, \theta_s^m)$ such that $q^{\mathrm{OPT}}(\theta) \neq q^{\mathrm{BM}}(\theta)$. Because q^{BM} is continuous and decreasing, this means that there exists a positive Lebesgue measure set of types $I \subseteq [\underline{\theta}, \theta_s^m)$ such that $q^{\mathrm{OPT}}(\theta) \neq q^{\mathrm{BM}}(\theta)$ for all $\theta \in I$. By Part (c) of the Proposition, we have that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in I$ (as q^{BM} is continuous and both q^{BM} and q^{OPT} are weakly decreasing). Then, let $\widetilde{M} = (\tilde{q}, \tilde{u})$ be the mechanism where the quantity schedule is given by

$$\tilde{q}(\theta) = \begin{cases} q^{\mathrm{BM}}(\theta) & \text{if } \theta \in [\underline{\theta}, \theta_s^m] \\ q^{\mathrm{OPT}}(\theta) & \text{otherwise} \end{cases}$$

and where $\tilde{u}(\theta) = \int\limits_{\theta}^{\overline{\theta}} \tilde{q}(y)dy$ for all θ. Clearly, $\widetilde{M}$ is IC and IR. Below, we show that $\widetilde{M}$ yields a higher payoff to the buyer than M^{OPT} and $\widetilde{M} \in \mathcal{M}^{\mathrm{SL}}$, contradicting the optimality of M^{OPT}.

Because, for any θ, $q^{\mathrm{BM}}(\theta)$ is the unique maximizer of $V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}$,

$$\int\limits_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(\tilde{q}(\theta)) - z^\star(\theta)\tilde{q}(\theta)\Big] F^\star(\mathrm{d}\theta) > \int\limits_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q^{\mathrm{OPT}}(\theta)) - z^\star(\theta)q^{\mathrm{OPT}}(\theta)\Big] F^\star(\mathrm{d}\theta).$$

We now show that $\tilde{q}$ is such that $\underline{w}_{\tilde{q}} \geq \max\{G_s^*, G_s^{**}\}$. To do so, it suffices to show that $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$ for $\theta \leq \theta_s^m$ (in fact, for $\theta > \theta_s^m$, $\underline{W}(\theta, \tilde{q}) = \underline{W}(\theta, q^{\mathrm{OPT}}) \geq$

$\max\{G_s^*, G_s^{**}\}$, where the inequality follows from the fact that $M^{\text{OPT}} \in \mathcal{M}^{\text{SL}}$). Thus consider $\theta \in [\underline{\theta}, \theta_s^m]$. For any such θ, $\tilde{q}(\theta) = q^{\text{BM}}(\theta) = q_s^\star(\theta)$. The latter equality follows from Lemma S.5, which establishes that $\theta_s^m < \theta_s^\star$. Moreover,

$$
\begin{aligned}
\underline{W}(\theta, \tilde{q}) - \underline{W}(\theta, q_s^\star) = \quad & \int_{\theta_s^m}^{\overline{\theta}} q_s^\star(y)dy - \int_{\theta_s^m}^{\overline{\theta}} q^{\text{OPT}}(y)dy \\
\geq_{(a)} & \left[\underline{V}(q^{\text{OPT}}(\theta_s^m)) - \theta_s^m q^{\text{OPT}}(\theta_s^m)\right] - \left[\underline{V}(q^{\text{BM}}(\theta_s^m)) - \theta_s^m q^{\text{BM}}(\theta_s^m)\right] \\
& + \int_{\theta_s^m}^{\overline{\theta}} q_s^\star(y)dy - \int_{\theta_s^m}^{\overline{\theta}} q^{\text{OPT}}(y)dy \\
=_{(b)} & \ \underline{W}(\theta_s^m, q^{\text{OPT}}) - \underline{W}(\theta_s^m, q_s^\star) \\
\geq_{(c)} & \max\{G_s^*, G_s^{**}\} - \underline{W}(\theta_s^m, q_s^\star) \\
\geq_{(d)} & \max\{G_s^*, G_s^{**}\} - \underline{W}(\theta, q_s^\star),
\end{aligned}
$$

Inequality (a) follows from the fact that $\underline{D}(\theta_s^m)$ maximizes $\underline{V}(\mathrm{q}) - \theta_s^m \mathrm{q}$ over all q and $q^{\text{BM}}(\theta_s^m) = \underline{D}(\theta_s^m)$ (Lemma S.5). Equality (b) follows from the fact that $q^{\text{BM}}(\theta) = q_s^\star(\theta)$. Inequality (c) follows from the fact that $M^{\text{OPT}} \in \mathcal{M}^{\text{SL}}$ which implies that $q^{\text{OPT}}(y)$ satisfies the robustness constraints $\underline{W}(\theta_s^m, q^{\text{OPT}}) \geq \max\{G_s^*, G_s^{**}\}$. Inequality (d) follows from the definition of θ_s^m. Hence, $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$. We conclude that $\widetilde{M} \in \mathcal{M}^{\text{SL}}$ and yields a higher payoff to the buyer than M^{OPT} contradicting the optimality of M^{OPT}. ■

This completes the proof of Proposition S.8. ■

Proof of Proposition S.9. Each part below establishes the corresponding part in the proposition.

Part 1. For any $\theta \in (\underline{\theta}, \overline{\theta})$, let $n(\theta)$ be the largest $n > \overline{n}$ such that $\underline{\theta}_n \leq \theta < \underline{\theta}_{n+1}$. Existence of such an $n(\theta)$ is guaranteed by Condition (b) in the definition of the sequence $(\underline{F}_n)$, which implies that $\underline{\theta}_n \leq \underline{\theta}_{n+1} < \overline{\theta}$ and $\lim_{n\to\infty} \underline{\theta}_n = \overline{\theta}$. For any $n \leq n(\theta) - 1$, we have $\theta \in [\underline{\theta}_n, \overline{\theta}]$ and $\theta \in [\underline{\theta}_{n+1}, \overline{\theta}]$. Thus, by Proposition S.8, $q_n^{\text{OPT}}(\theta) = \underline{q}_n^{\text{BM}}(\theta)$ and $q_{n+1}^{\text{OPT}}(\theta) = \underline{q}_{n+1}^{\text{BM}}(\theta)$. Condition (S.11) in turn implies that $\underline{q}_n^{\text{BM}}(\theta) \leq \underline{q}_{n+1}^{\text{BM}}(\theta)$, that is, $q_n^{\text{OPT}}(\theta)$ is weakly increasing in n for $n \leq n(\theta) - 1$.

For any $n > n(\theta)$, $\theta < \underline{\theta}_n$, and therefore, $q_n^{\text{OPT}}(\theta) = \max\{q^{\text{BM}}(\theta), \underline{D}(\underline{\theta}_n)\}$. The quantity $\underline{D}(\underline{\theta}_n)$ is weakly decreasing in n because $\underline{\theta}_n \leq \underline{\theta}_{n+1} < \overline{\theta}$ for every n. Consequently, $q_n^{\text{OPT}}(\theta)$

is also weakly decreasing in n.

Part 2. To establish the second part of the proposition, it suffices to exhibit a pair $j, k \in \mathbb{N}$, with $j < k$, such that $q_j^{\mathrm{OPT}}(\theta) > q_k^{\mathrm{OPT}}(\theta)$. To do so, consider the following two cases.

Case 1. Suppose $q^{\mathrm{BM}}(\theta) \geq \underline{D}(\underline{\theta}_{n(\theta)+1})$. Then let $j = n(\theta)$ and $k = n(\theta)+1$, and observe that

$$q_j^{\mathrm{OPT}}(\theta) = \underline{q}_j^{\mathrm{BM}}(\theta) = \underline{D}\left(\theta + \frac{\underline{F}_j(\theta)}{\underline{f}_j(\theta)}\right) > \underline{D}\left(\theta + \frac{F^\star(\theta)}{f^\star(\theta)}\right) = q^{\mathrm{BM}}(\theta) = q_k^{\mathrm{OPT}}(\theta),$$

where the inequality follows from (S.12).

Case 2. Suppose $q^{\mathrm{BM}}(\theta) < \underline{D}(\underline{\theta}_{n(\theta)+1})$. Then let $j = n(\theta)+1$ and let k be such that $\underline{\theta}_k > \underline{\theta}_j$. Existence of such an k is ensured by Condition (b) in the definition of the sequence $(\underline{F}_n)$. Then

$$q_k^{\mathrm{OPT}}(\theta) = \max\{q^{\mathrm{BM}}(\theta), \underline{D}(\underline{\theta}_k)\} < \underline{D}(\underline{\theta}_j) = q_j^{\mathrm{OPT}}(\theta).$$

To see this, observe that $\underline{\theta}_k > \underline{\theta}_j$ implies that $\underline{D}(\underline{\theta}_k) < \underline{D}(\underline{\theta}_j)$. Hence, if $q_k^{\mathrm{OPT}}(\theta) = \underline{D}(\underline{\theta}_k)$, then $q_k^{\mathrm{OPT}}(\theta) = \underline{D}(\underline{\theta}_k) < \underline{D}(\underline{\theta}_j) = q_j^{\mathrm{OPT}}(\theta)$. If, instead, $q_k^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$, the result follows from the fact that, by assumption, $q^{\mathrm{BM}}(\theta) < \underline{D}(\underline{\theta}_j)$. ■