Attention, Coordination, and Bounded Recall

Online Appendix

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Abstract

This Appendix contains the proofs of Claims 1 and 2 in Section 2 in the main text.

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A Claim 1

Claim 1. *Consider economies in which (a) the sensitivity of the complete-information equilibrium actions to the fundamentals is first-best efficient ($\kappa_1 = \kappa_1^*$), (b) there are no externalities from dispersion ($u_{\sigma\sigma} = 0$), and (c) inefficiencies in the use of information originate in the discrepancy between the equilibrium and the socially optimal degrees of coordination ($\alpha \neq \alpha^*$). There exists $R^* > 0$ such that, when $\alpha > \alpha^*$, starting from $\hat{z}$, forcing the agents to pay more attention to a source n that receives positive attention in equilibrium (i.e., for which $\hat{z}_n > 0$) increases welfare if $t_n/C_n'(\hat{z}) < R^*$, whereas it reduces it if $t_n/C_n'(\hat{z}) > R^*$. The opposite conclusions hold for $\alpha < \alpha^*$. Furthermore, when there exists an increasing, convex, and differentiable function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any $z = (z_1, ..., z_N) \in \mathbb{R}_+^N$, $C(z) = c\left(\sum_{s=1}^N z_s\right)$, the equilibrium total attention is inefficiently low and too few sources receive positive attention if $\alpha > \alpha^*$, whereas the opposite conclusions hold if $\alpha < \alpha^*$.*

Proof of Claim 1. Let $\hat{z}$ denote the equilibrium allocation of attention and z^* the allocation of attention that maximizes welfare when the planner can control the agents' actions. Similarly, let $\#\hat{N}$ denote the number of sources that receive positive attention in equilibrium, and $\#N^*$ the number of sources that receive positive attention when the planner can control both the agents' allocation of attention and the agents' use of information.

Note that, in these economies, for any z, and any n, the discrepancy between the social and the private benefit of increasing the attention allocated to source n is proportional to the difference[1]

$$\frac{(\gamma_n^*(z))^2}{(z_n)^2 t_n} - \frac{(\gamma_n(z))^2}{(z_n)^2 t_n}. \tag{A.1}$$

Using the expressions for γ and γ^*, the difference in (A.1) is equal to

$$m^*(z)\frac{t_n \eta_n^2}{[(1-\alpha^*)z_n t_n + \eta_n]^2} - m(z)\frac{t_n \eta_n^2}{[(1-\alpha)z_n t_n + \eta_n]^2}$$

where

$$m(z) \equiv \frac{(1-\alpha)^2}{\left[\pi_\theta + \sum_{l=1}^N \frac{(1-\alpha)z_l t_l \eta_l}{(1-\alpha)z_l t_l + \eta_l}\right]^2} \text{ and } m^*(z) \equiv \frac{(1-\alpha^*)^2}{\left[\pi_\theta + \sum_{l=1}^N \frac{(1-\alpha^*)z_l t_l \eta_l}{(1-\alpha^*)z_l t_l + \eta_l}\right]^2}.$$

It follows that, starting from the equilibrium allocation of attention $\hat{z}$, the social benefit of increasing the attention allocated to source n exceeds the private benefit if

$$\frac{(1-\alpha)\hat{z}_n t_n + \eta_n}{(1-\alpha^*)\hat{z}_n t_n + \eta_n} > \sqrt{\frac{m(\hat{z})}{m^*(\hat{z})}}, \tag{A.2}$$

whereas if falls short of the private benefit when the inequality in (A.2) is reversed.

Now note that, when $\alpha > \alpha^*$, $m(\hat{z}) < m^*(\hat{z})$, whereas, when $\alpha < \alpha^*$, $m(\hat{z}) > m^*(\hat{z})$. This means that, for any source that receives no attention in equilibrium (i.e., such that $\hat{z}_n = 0$), the

[1] Note that the comparison here applies also to sources that receive no attention, i.e., for which $z_n = 0$.

social benefit exceeds the private benefit when $\alpha > \alpha^*$, whereas the opposite conclusion holds when $\alpha < \alpha^*$.

Next, consider sources that receive strictly positive attention in equilibrium. In the proof of Corollary 1 in the main text, I establish that

$$\hat{z}_n = \frac{\eta_n}{\sqrt{t_n}(1-\alpha)} \left\{ (1-\alpha)\sqrt{\frac{|u_{kk}|\kappa_1^2}{2C_n'(\hat{z})} \frac{1}{M_1(\hat{z})}} - \frac{1}{\sqrt{t_n}} \right\} \tag{A.3}$$

where

$$M_1(z) \equiv \pi_\theta + \sum_{l=1}^{N} \frac{(1-\alpha)\eta_l z_l t_l}{(1-\alpha)z_l t_l + \eta_l} > 0. \tag{A.4}$$

It follows that

$$\hat{z}_n t_n = \frac{\eta_n \sqrt{t_n}}{\sqrt{C_n'(\hat{z})}} Q(\hat{z}) - \frac{\eta_n}{(1-\alpha)} \tag{A.5}$$

where

$$Q(z) \equiv \sqrt{\frac{|u_{kk}|\kappa_1^2}{2} \frac{1}{M_1(z)}}.$$

Using (A.5), I can rewrite the left-hand-side of (A.2) as follows

$$\frac{(1-\alpha)Q(\hat{z})\sqrt{\frac{t_n}{C_n'(\hat{z})}}}{(1-\alpha^*)Q(\hat{z})\sqrt{\frac{t_n}{C_n'(\hat{z})}} + \frac{\alpha^*-\alpha}{1-\alpha}}$$

which is decreasing in $t_n/C_n'(\hat{z})$ for $\alpha > \alpha^*$ and increasing in $\frac{t_n}{C_n'(\hat{z})}$ for $\alpha < \alpha^*$. I conclude that, when $\alpha > \alpha^*$, there exists a critical value $R^* > 0$ such that, starting from the equilibrium allocation of attention $\hat{z}$, the planner would like the agents to locally increase the attention allocated to any source of information that receives positive attention in equilibrium and such that $t_n/C_n'(\hat{z}) < R^*$ and decrease the attention allocated to any source that receives positive attention and for which $t_n/C_n'(\hat{z}) > R^*$. The opposite conclusions hold for $\alpha < \alpha^*$.

Lastly, suppose there exists an increasing, convex, and differentiable function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any $z = (z_1, ..., z_N) \in \mathbb{R}_+^N$, $C(z) = c\left(\sum_{s=1}^N z_s\right)$. Note that, in these economies, the efficient allocation $(z^*, k(\cdot; z^*))$ coincides with the equilibrium allocation of another economy that differs from the original one only in the degree of coordination. It thus suffices to show that the equilibrium total attention $\hat{Z} \equiv \sum_{s=1}^N \hat{z}_s$ as well as the number of sources $\#\hat{N}$ that receive strictly positive attention in equilibrium decrease with α, where $\hat{N}$ denotes the subset of sources that receive strictly positive attention in equilibrium. The attention allocated in equilibrium to each source n is given by

$$\hat{z}_n = \frac{\eta_n}{\sqrt{t_n}(1-\alpha)} \max\left\{ \hat{T} - \frac{1}{\sqrt{t_n}}; 0 \right\} \tag{A.6}$$

where

$$\hat{T} \equiv (1-\alpha)\sqrt{\frac{|u_{kk}|\,\kappa_1^2}{2c'(\hat{Z})} \frac{1}{\hat{M}_1}} \tag{A.7}$$

with

$$\hat{Z} \equiv \sum_{l=1}^{N} \hat{z}_l \tag{A.8}$$

and

$$\hat{M}_1 \equiv M_1(\hat{z}). \tag{A.9}$$

Combining (A.6)-(A.9), I then have that, for any α, $\hat{T}$ is the unique solution to the following equation

$$\begin{aligned} \tfrac{T}{1-\alpha}\sqrt{c'\left(\sum_{l=1}^{N}\tfrac{\eta_l}{\sqrt{t_l}(1-\alpha)}\max\left\{T-\tfrac{1}{\sqrt{t_l}};0\right\}\right)}\left(\pi_\theta+\sum_{l=1}^{N}\frac{\eta_l\sqrt{t_l}\max\left\{T-\frac{1}{\sqrt{t_l}};0\right\}}{\sqrt{t_l}\max\left\{T-\frac{1}{\sqrt{t_l}};0\right\}+1}\right) \\ =\kappa_1\sqrt{\tfrac{|u_{kk}|}{2}}. \end{aligned} \tag{A.10}$$

Because the left-hand-side of (A.10) is increasing in both α and T, I then have that $\hat{T}$ is decreasing in α. This means that the critical level of transparency required for each source to receive positive attention in equilibrium increases with α. In turn, this implies that $\#\hat{N}$ decreases with α, as claimed. To see that the total attention $\hat{Z}$ also decreases with α, follow steps similar to those in the proof of Example 1 in the main text to see that, for each source that receives strictly positive attention (i.e., such that $n \in \hat{N}$),

$$\hat{z}_n = \frac{\eta_n}{\sqrt{t_n}(1-\alpha)}\left[\frac{(1-\alpha)\sqrt{\frac{|u_{kk}|\kappa_1^2}{2c'\left(\sum_{s\in\hat{N}}\hat{z}_s\right)}}+\sum_{s\in\hat{N}}\frac{\eta_s}{\sqrt{t_s}}}{\pi_\theta+\sum_{s\in\hat{N}}\eta_s}-\frac{1}{\sqrt{t_n}}\right].$$

Summing over all $n \in \hat{N}$, I then have that

$$\begin{aligned} \sum\nolimits_{n\in\hat{N}}\hat{z}_n = \frac{1}{\sqrt{2c'\left(\sum_{s\in\hat{N}}\hat{z}_s\right)}}\left(\frac{\sqrt{|u_{kk}|\kappa_1^2}\sum_{n\in\hat{N}}\frac{\eta_n}{\sqrt{t_n}}}{\pi_\theta+\sum_{s\in\hat{N}}\eta_s}\right) \\ +\frac{1}{1-\alpha}\left\{\frac{\left[\sum_{s\in\hat{N}}\frac{\eta_s}{\sqrt{t_s}}\right]^2}{\pi_\theta+\sum_{s\in\hat{N}}\eta_s}-\sum\nolimits_{n\in\hat{N}}\frac{\eta_n}{t_n}\right\}. \end{aligned}$$

Holding $\hat{N}$ fixed, I then have that

$$\frac{\partial}{\partial\alpha}\left(\sum\nolimits_{s\in\hat{N}}\hat{z}_s\right)\overset{\text{sgn}}{=}\frac{\left[\sum_{s\in\hat{N}}\frac{\eta_s}{\sqrt{t_s}}\right]^2}{\pi_\theta+\sum_{s\in\hat{N}}\eta_s}-\sum\nolimits_{n\in\hat{N}}\frac{\eta_n}{t_n}. \tag{A.11}$$

Below I show that

$$\left(\sum\nolimits_{s\in\hat{N}}\frac{\eta_s}{\sqrt{t_s}}\right)^2-\left(\sum\nolimits_{s\in\hat{N}}\frac{\eta_s}{t_s}\right)\left(\sum\nolimits_{s\in\hat{N}}\eta_s\right)\le 0,$$

which implies that the sign of the right-hand side of (A.11) is always negative. To see this, it suffices to note that

$$\begin{aligned}\left(\sum_{s\in\hat{N}}\frac{\eta_s}{\sqrt{t_s}}\right)^2-\left(\sum_{s\in\hat{N}}\frac{\eta_s}{t_s}\right)\left(\sum_{s\in\hat{N}}\eta_s\right) &= \sum_{s\in\hat{N}}\frac{\eta_s^2}{t_s}+\sum_{s\in\hat{N}}\sum_{k\in\hat{N},k\neq s}\frac{\eta_s\eta_k}{\sqrt{t_s}\sqrt{t_k}} \\ &\quad -\sum_{s\in\hat{N}}\frac{\eta_s^2}{t_s}-\sum_{s\in\hat{N}}\sum_{k\in\hat{N},k\neq s}\frac{\eta_s\eta_k}{t_s} \\ &= \sum_{s,k\in\hat{N},k\neq s}\left[\eta_s\eta_k\left(\frac{2}{\sqrt{t_st_k}}-\frac{1}{t_s}-\frac{1}{t_k}\right)\right]<0.\end{aligned}$$

Along with the property that the set $\hat{N}$ of sources that receive strictly positive attention in equilibrium decreases with α (in the set inclusion order), the fact that, for given $\hat{N}$, $\sum_{s\in\hat{N}}\hat{z}_s$ is decreasing in α implies that $\hat{Z}$ decreases with α. Q.E.D.

B Claim 2

Claim 2. *Suppose the planner can not dictate to the agents how to respond to the signals they receive from the various sources of information. (a) Consider economies that are either efficient in their use of information ($\kappa=\kappa^*$ and $\alpha=\alpha^*$) or in which the inefficiency in the use of information originates in the sensitivity of the complete-information equilibrium actions to the fundamentals ($u_{\sigma\sigma}=0$, $\alpha=\alpha^*$, but $\kappa_1\neq\kappa_1^*$). Starting from the equilibrium allocation of attention $\hat{z}$, the social benefit of expanding the agents' attention to any source n exceeds the private benefit when $(\kappa_1^*-\kappa_1)/\kappa_1>0$, whereas the opposite is true when $(\kappa_1^*-\kappa_1)/\kappa_1<0$. (b) Consider economies in which the sensitivity of the complete-information equilibrium actions to the fundamentals is first-best efficient and there are no externalities from dispersion ($\kappa_1=\kappa_1^*$ and $u_{\sigma\sigma}=0$). There exists a threshold $M>0$ such that, starting from the equilibrium allocation of attention $\hat{z}$, forcing the agents to increase their attention to any source for which $\hat{z}_n>0$ increases welfare if*

$$\alpha-\alpha^*\overset{\text{sgn}}{=}\left\{\frac{C_n'(\hat{z})}{t_n}-M\right\}$$

and decreases it otherwise.

Proof of Claim 2. The proof is in two steps. Step 1 characterizes the gross marginal benefit of inducing the agents to increase the attention to any given source of information, accounting for the effects of such an increase on the subsequent usage of information. Step 2 uses the result in step 1 to establish the claim.

Step 1. First observe that, for any given z, welfare under the equilibrium strategy $k(\cdot;z)$ is given by[2]

$$w(z)\equiv\mathbb{E}[u(k,K,\sigma_k,\theta)\mid z,k(\cdot;z)]-C(z)=\mathbb{E}[W(\kappa,0,\theta)]-\mathcal{L}(z)-C(z),\tag{B.1}$$

[2] The representation of equilibrium welfare in (B.1) follows from the same steps as in Angeletos and Pavan (2007); the proof is thus omitted.

where $W(K,0,\theta) \equiv u(K,K,0,\theta)$ is the payoff that each agent obtains when all agents take the same action ($W(\kappa,0,\theta)$ is thus welfare under the complete-information equilibrium allocation $\kappa = \kappa_0 + \kappa_1\theta$), whereas

$$\mathcal{L}(z) \equiv \frac{|u_{kk}+u_{\sigma\sigma}|}{2} \cdot Var[k-K \mid z, k(\cdot;z)] + \frac{|u_{kk}+2u_{kK}+u_{KK}|}{2} \cdot Var[K-\kappa \mid z, k(\cdot;z)] \\ - Cov\left[K-\kappa, W_K(\kappa,0,\theta) \mid z, k(\cdot;z)\right]$$

are the welfare losses due to incomplete information. The first two terms in $\mathcal{L}$ measure the welfare losses due to, respectively, the dispersion of individual actions around the aggregate action and the volatility of the aggregate action around its complete-information counterpart. The last term captures losses (or gains) due to the correlation between the 'aggregate error' due to incomplete information, $K-\kappa$, and W_K, the social return to aggregate activity. Following steps similar to those in Angeletos and Pavan (2007) one can show that

$$Cov\left[K-\kappa, W_K(\kappa,0,\theta) \mid z, k(\cdot;z)\right] = |u_{kk}+2u_{kK}+u_{KK}|\kappa_1^2 \left(\frac{\kappa_1^*-\kappa_1}{\kappa_1}\right)\frac{\sum_n \gamma_n(z)-1}{\pi_\theta},$$

$$Var[K-\kappa \mid z, k(\cdot;z)] = \kappa_1^2 \frac{\left(\sum_{s=1}^N \gamma_s(z)-1\right)^2}{\pi_\theta} + \kappa_1^2 \sum_{s=1}^N \frac{(\gamma_s(z))^2}{\eta_s},$$

and

$$Var[k-K \mid z, k(\cdot;z)] = \kappa_1^2 \sum_{s=1}^N \frac{(\gamma_s(z))^2}{z_s t_s}.$$

Welfare under the equilibrium strategy $k(\cdot;z)$ can thus be expressed as

$$w(z) = \mathbb{E}[W(\kappa,0,\theta)] - \frac{|u_{kk}+u_{\sigma\sigma}|\kappa_1^2}{2}\sum_{s=1}^N \frac{(\gamma_s(z))^2}{z_s t_s} \\ - \frac{|u_{kk}+2u_{kK}+u_{KK}|\kappa_1^2}{2}\left[\frac{\left(\sum_{s=1}^N \gamma_s(z)-1\right)^2}{\pi_\theta} + \sum_{s=1}^N \frac{(\gamma_s(z))^2}{\eta_s}\right] \\ + |u_{kk}+2u_{kK}+u_{KK}|\kappa_1^2 \cdot \left(\frac{\kappa_1^*-\kappa_1}{\kappa_1}\right) \cdot \frac{\sum_{s=1}^N \gamma_s(z)-1}{\pi_\theta} - C(z).$$

The (gross) marginal effect on welfare of an increase in the attention allocated to the n-th source is thus equal to

$$\frac{|u_{kk}+u_{\sigma\sigma}|}{2}\frac{(\kappa_1\gamma_n(z))^2}{(z_n)^2 t_n} - |u_{kk}+u_{\sigma\sigma}|\kappa_1^2\left(\sum_{s=1}^N \frac{\gamma_s(z)}{z_s t_s}\frac{\partial\gamma_s(z)}{\partial z_n}\right) \\ - |u_{kk}+2u_{kK}+u_{KK}|\kappa_1^2\left[\frac{\left(\sum_{s=1}^N \gamma_s(z)-1\right)\left(\sum_{s=1}^N \frac{\partial\gamma_s(z)}{\partial z_n}\right)}{\pi_\theta} + \sum_{s=1}^N \frac{\gamma_s(z)}{\eta_s}\frac{\partial\gamma_s(z)}{\partial z_n}\right] \\ + \frac{|u_{kk}+2u_{kK}+u_{KK}|\kappa_1^2}{\pi_\theta}\left(\frac{\kappa_1^*-\kappa_1}{\kappa_1}\right)\left(\sum_{s=1}^N \frac{\partial\gamma_s(z)}{\partial z_n}\right). \tag{B.2}$$

Substituting $|u_{kk}+2u_{kK}+u_{KK}|=(1-\alpha^*)|u_{kk}+u_{\sigma\sigma}|$, I can rewrite the sum of the second and third addendum in (B.2) as

$$
\begin{aligned}
&-|u_{kk}+u_{\sigma\sigma}|\kappa_1^2\left[\frac{(1-\alpha^*)\left(\sum_{s=1}^N\gamma_s(z)-1\right)\left(\sum_{s=1}^N\frac{\partial\gamma_s(z)}{\partial z_n}\right)}{\pi_\theta}+\sum_{s=1}^N\left(\frac{1-\alpha^*}{\eta_s}+\frac{1}{z_st_s}\right)\gamma_s\frac{\partial\gamma_s(z)}{\partial z_n}\right]\\
&=-|u_{kk}+u_{\sigma\sigma}|\kappa_1^2\left[\sum_{s=1}^N\left(\frac{\left(\frac{1-\alpha^*}{\eta_s}+\frac{1}{z_st_s}\right)\frac{(1-\alpha)\pi_s(z)}{1-\alpha\rho_s(z)}-(1-\alpha^*)}{\pi_\theta+\sum_{n=1}^N\frac{(1-\alpha)\pi_n(z)}{1-\alpha\rho_n(z)}}\right)\frac{\partial\gamma_s(z)}{\partial z_n}\right].
\end{aligned}
$$

Using

$$\pi_s(z)\equiv\frac{\eta_sz_st_s}{z_st_s+\eta_s}\text{ and }\rho_s(z)=\frac{z_st_s}{z_st_s+\eta_s},$$

I then have that

$$\left(\frac{1-\alpha^*}{\eta_s}+\frac{1}{z_st_s}\right)\frac{(1-\alpha)\pi_s(z)}{1-\alpha\rho_s(z)}-(1-\alpha^*)=-\frac{\eta_s(\alpha-\alpha^*)}{[(1-\alpha)z_st_s+\eta_s]}.$$

The sum of the second and third addendum in (B.2) can thus be rewritten as

$$|u_{kk}+u_{\sigma\sigma}|\kappa_1^2(\alpha-\alpha^*)\left\{\sum_{s=1}^N\left[\frac{\eta_s}{[(1-\alpha)z_st_s+\eta_s]\left[\pi_\theta+\sum_{n=1}^N\frac{(1-\alpha)\pi_n(z)}{1-\alpha\rho_n(z)}\right]}\right]\frac{\partial\gamma_s(z)}{\partial z_n}\right\}.$$

Next, note that

$$\frac{\eta_s}{[(1-\alpha)z_st_s+\eta_s]\left[\pi_\theta+\sum_{n=1}^N\frac{(1-\alpha)\pi_n(z)}{1-\alpha\rho_n(z)}\right]}=\frac{\gamma_s(z)}{(1-\alpha)z_st_s}.$$

I conclude that, given any z, the net social marginal benefit of forcing the agent to pay more attention to any source n is equal to

$$
\begin{aligned}
\frac{\partial w(z)}{\partial z_n}=&\frac{|u_{kk}+u_{\sigma\sigma}|}{2}\frac{(\kappa_1\gamma_n(z))^2}{(z_n)^2t_n}+|u_{kk}+u_{\sigma\sigma}|\kappa_1^2(\alpha-\alpha^*)\left[\sum_{s=1}^N\left(\frac{\gamma_s(z)}{(1-\alpha)z_st_s}\right)\frac{\partial\gamma_s(z)}{\partial z_n}\right]\\
&+\frac{|u_{kk}+2u_{kK}+u_{KK}|\kappa_1^2}{\pi_\theta}\left(\frac{\kappa_1^*-\kappa_1}{\kappa_1}\right)\left(\sum_{s=1}^N\frac{\partial\gamma_s(z)}{\partial z_n}\right)-C_n'(z).
\end{aligned}
\tag{B.3}
$$

The first term in (B.3) is the direct marginal effect of a reduction in the cross-sectional dispersion of individual actions that obtains as a result of an increase in the attention z_n, holding fixed the equilibrium use of information $k(\cdot;\hat{z})$. The second term combines the marginal effects of changing the equilibrium rule $k(\cdot;\hat{z})$ on (a) the volatility of the aggregate action K around its complete-information counterpart κ and (b) the dispersion of individual actions around the mean action. The third term, which is relevant only in economies that are inefficient under complete information, captures the effect of changing the rule $k(\cdot;\hat{z})$ on the way the "error" due to incomplete information $K-\kappa$ covaries with the inefficiency of the complete-information allocation. Clearly, by usual envelope arguments, these last two terms are equal to zero in economies in which the equilibrium use of information is efficient (that is, in which $k(\cdot;z)=k^*(\cdot;z)$, which happens if, and only if,

$\alpha = \alpha^*$ and $\kappa = \kappa^*$) or, alternatively, when the planner can dictate to the agents how to use their information.

Step 2. First consider part (a) in the claim. Clearly, in economies that are efficient in their usage of information, the impossibility to dictate to the agents how to map the signals they receive from the sources of information into their actions is inconsequential. The welfare effects of changing the agents' attention are thus the same as in economies in which the planner controls the agents' usage of information. Next consider economies in which the inefficiency in the allocation of attention originates in the discrepancy between the complete-information actions and the first-best actions ($\alpha = \alpha^*$, $U_{\sigma\sigma} = 0$, but $\kappa \neq \kappa^*$). Using (B.3), I have that, in these economies, the social net marginal effect of inducing the agents to increase their attention to any source n, starting from the equilibrium level $\hat{z}$, is equal to[3]

$$\frac{\partial w(\hat{z})}{\partial z_n} = \frac{|u_{kk}|}{2} \frac{(\kappa_1 \gamma_n(\hat{z}))^2}{(\hat{z}_n)^2 t_n} + \frac{|u_{kk} + 2u_{kK} + u_{KK}|\kappa_1^2}{\pi_\theta} \left(\frac{\kappa_1^* - \kappa_1}{\kappa_1} \right) \left\{ \sum_{s=1}^N \frac{\partial \gamma_s(\hat{z})}{\partial z_n} \right\} \\ - C_n'(\hat{z}). \tag{B.4}$$

The result in part (a) then follows from the fact that, for any z,

$$\sum_{s=1}^N \gamma_s(z) = \frac{1}{\frac{\pi_\theta}{\sum_{s=1}^N \frac{(1-\alpha)\eta_s z_s t_s}{(1-\alpha) z_s t_s + \eta_s}} + 1}$$

is increasing in z_n.

Next, consider part (b) in the claim. Using (B.3), I have that, in economies in which $\kappa_1 = \kappa_1^*$ and $u_{\sigma\sigma} = 0$, the net benefit of inducing the agents to increase their attention to any source n is equal to

$$\frac{\partial w(\hat{z})}{\partial z_n} = \frac{|u_{kk}|}{2} \frac{(\kappa_1 \gamma_n(\hat{z}))^2}{(\hat{z}_n)^2 t_n} + |u_{kk}|\kappa_1^2 (\alpha - \alpha^*) \left[\sum_{s=1}^N \left(\frac{\gamma_s(\hat{z})}{(1-\alpha)\hat{z}_s t_s} \right) \frac{\partial \gamma_s(\hat{z})}{\partial z_n} \right] - C_n'(\hat{z}).$$

Using the fact that, for any source n, irrespective of whether $\hat{z}_n > 0$ or $\hat{z}_n = 0$, the private net marginal benefit of increasing the attention to source n is equal to

$$\frac{|u_{kk}|}{2} \frac{(\kappa_1 \gamma_n(\hat{z}))^2}{(\hat{z}_n)^2 t_n} - C_n'(\hat{z}),$$

I then have that the social benefit exceeds the private benefit if and only if

$$\alpha - \alpha^* \overset{\text{sgn}}{=} \sum_{s=1}^N \left(\frac{\gamma_s(\hat{z})}{(1-\alpha)\hat{z}_s t_s} \right) \frac{\partial \gamma_s(\hat{z})}{\partial z_n}.$$

[3]If $\hat{z}_n = 0$, then interpret the derivative as the right-hand derivative.

Next, observe that

$$\sum_{s=1}^{N}\left(\frac{\gamma_s(\hat{z})}{(1-\alpha)\hat{z}_s t_s}\right)\frac{\partial\gamma_s(\hat{z})}{\partial z_n}$$

$$= -\sum_{s=1}^{N}\left[\frac{\gamma_s(\hat{z})}{(1-\alpha)\hat{z}_s t_s}\frac{\frac{(1-\alpha)\eta_s\hat{z}_s t_s}{(1-\alpha)\hat{z}_s t_s+\eta_s}}{\left(\pi_\theta+\sum_{l=1}^{N}\frac{(1-\alpha)\eta_l\hat{z}_l t_l}{(1-\alpha)\hat{z}_l t_l+\eta_l}\right)^2}\frac{\partial}{\partial\hat{z}_n}\left(\frac{(1-\alpha)\eta_n\hat{z}_n t_n}{(1-\alpha)\hat{z}_n t_n+\eta_n}\right)\right]$$

$$+\frac{\gamma_n(\hat{z})}{(1-\alpha)\hat{z}_n t_n}\frac{\frac{\partial}{\partial\hat{z}_n}\left(\frac{(1-\alpha)\eta_n\hat{z}_n t_n}{(1-\alpha)\hat{z}_n t_n+\eta_n}\right)}{\pi_\theta+\sum_{l=1}^{N}\frac{(1-\alpha)\eta_l\hat{z}_l t_l}{(1-\alpha)\hat{z}_l t_l+\eta_l}}$$

$$=\frac{\frac{\partial}{\partial\hat{z}_n}\left(\frac{(1-\alpha)\eta_n\hat{z}_n t_n}{(1-\alpha)\hat{z}_n t_n+\eta_n}\right)}{\pi_\theta+\sum_{l=1}^{N}\frac{(1-\alpha)\eta_l\hat{z}_l t_l}{(1-\alpha)\hat{z}_l t_l+\eta_l}}\left[\frac{\gamma_n(\hat{z})}{(1-\alpha)\hat{z}_n t_n}-\sum_{s=1}^{N}\left(\frac{(\gamma_s(\hat{z}))^2}{(1-\alpha)\hat{z}_s t_s}\right)\right].$$

Clearly,

$$\frac{\frac{\partial}{\partial\hat{z}_n}}{\pi_\theta+\sum_{l=1}^{N}\frac{(1-\alpha)\eta_l\hat{z}_l t_l}{(1-\alpha)\hat{z}_l t_l+\eta_l}}>0.$$

Hence,

$$\sum_{s=1}^{N}\left(\frac{\gamma_s(\hat{z})}{(1-\alpha)\hat{z}_s t_s}\right)\frac{\partial\gamma_s(\hat{z})}{\partial z_n}\overset{\text{sgn}}{=}\frac{\gamma_n(\hat{z})}{(1-\alpha)\hat{z}_n t_n}-\sum_{s=1}^{N}\left(\frac{(\gamma_s(\hat{z}))^2}{(1-\alpha)\hat{z}_s t_s}\right).$$

This means that the social benefit exceeds the private benefit if and only if

$$\alpha-\alpha^*\overset{\text{sgn}}{=}\frac{\gamma_n(\hat{z})}{(1-\alpha)\hat{z}_n t_n}-M_0(\hat{z}),$$

where, for any z, $M_0(z)\equiv\sum_{s=1}^{N}\left(\frac{(\gamma_s(z))^2}{(1-\alpha)z_s t_s}\right)>0$ does not depend on the specific source of information under consideration. Now observe that

$$\frac{\gamma_n(\hat{z})}{(1-\alpha)\hat{z}_n t_n}=\frac{\eta_n}{(1-\alpha)\hat{z}_n t_n+\eta_n}\frac{1}{M_1(\hat{z})}$$

where $M_1(\cdot)$ is the function defined in (A.4). Lastly, use the fact that, for any n such that $\hat{z}_n>0$,

$$C_n'(\hat{z})=\frac{|u_{kk}|}{2}\frac{(\kappa_1\gamma_n(\hat{z}))^2}{(\hat{z}_n)^2 t_n}$$

to note that, for any source that receives strictly positive attention in equilibrium,

$$(1-\alpha)\hat{z}_n t_n+\eta_n=M_2(\hat{z})\sqrt{\frac{t_n\eta_n^2}{C_n'(\hat{z})}}$$

where, for any z, $M_2(z)\equiv\sqrt{\frac{|u_{kk}|\kappa_1^2(M_1(z))^2(1-\alpha)^2}{2}}$. I conclude that there exists a constant

$$M(\hat{z})\equiv[M_0(\hat{z})M_1(\hat{z})M_2(\hat{z})]^2>0$$

such that

$$\frac{\gamma_n(\hat{z})}{(1-\alpha)\hat{z}_n t_n}-M_0(\hat{z})\overset{\text{sgn}}{=}\frac{C_n'(\hat{z})}{t_n}-M(\hat{z}).$$

Part (b) of the claim then follows from the above results. Q.E.D.