Robust Procurement: Bayesian Design under Worst-Case Approval Constraints*

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Abstract

We study optimal procurement when a Bayesian designer must obtain approval from a non-Bayesian authority that shares the designer's objective but is uncertain about the value of the good and the supplier's cost. The designer uses a conjectured model to compute expected payoffs but is constrained to select among mechanisms delivering the largest payoff guarantee to the authority. This robustness requirement reshapes the tradeoff between efficiency and rent extraction: it reduces procurement from sellers with intermediate costs but may increase it from those with a high cost. When the good is sold in a market, we show that quantity regulation dominates price regulation if markups under the conjectured model are large, whereas price regulation dominates when demand uncertainty is substantial.

KEYWORDS: Bayesian mechanism design, worst-case scenarios, robustness, uncertainty, procurement, regulation.

JEL CLASSIFICATION: D82, L51

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1 Introduction

The standard approach to contract design assumes that the designer has a conjectured model of the environment, derived from data or subjective beliefs, and selects the mechanism maximizing expected payoff subject to incentive and participation constraints (the so-called Bayesian approach). In many applications, however, the designer must obtain approval from an authority that is skeptical of this model. Such authorities often require contracts to perform satisfactorily across a range of scenarios. For example, procurement contracts in the oil and gas sector must remain viable under conservative price assumptions.

In this paper, we study contract design when a Bayesian designer must secure approval from a max-min authority. The designer first identifies mechanisms that maximize the authority's worst-case payoff over a set of admissible models and then selects among them using her conjectured model. We refer to the mechanisms solving this problem as *robustly optimal*. When the designer and the authority share the same objective, this formulation also captures a designer who seeks robustness against her own model mis-specification.

We conduct our analysis in a procurement setting, where buyers (governments but also private institutions) typically face suppliers with private information about costs, and where contracts typically require approval from authorities—regulators, supervisors, or senior management—who often do not share the designer's confidence in the model used to estimate the value of the good and the technology determining the supplier's cost.[1]

We show that robustness reshapes the tradeoff between efficiency and rent extraction. We begin with environments where uncertainty only concerns the seller's cost distribution. When the conjectured model satisfies standard regularity conditions, the robustly optimal mechanism is as in Baron and Myerson (1982) but with a quantity floor: all types must supply at least the efficient output of the highest-cost type. Relative to the Bayesian benchmark, robustness increases the procurement from high-cost sellers and entails efficiency at both extremes of the cost distribution. These changes protect against the possibility that high-cost firms occur more frequently than conjectured. Because downward distortions serve to limit rents for low-cost types, their value diminishes under cost uncertainty. Procurement from low-cost types, instead, is the same as under the Bayesian optimum, as welfare when contracting with these types exceeds the worst-case level even when the conjectured cost

[1]While the need for approval is particularly salient in procurement, the robustness approach developed in this paper whereby the designer uses a conjectured model to select among worst-case optimal mechanisms applies also to other design problems.

distribution is wrong.

When uncertainty also affects demand, this mechanism remains optimal under suitable conditions, which we characterize. Otherwise, robustness leads to reduced procurement from intermediate-cost types, reflecting the need to guard against demand overestimation. At these cost levels, the screening motive and the need for approval work in the same direction, leading to further downward distortions relative to the Bayesian benchmark.

We then study environments with downstream markets, where demand can be learned by asking sellers to post prices. We characterize robustly optimal price regulations, under which sellers can choose from a menu of markups computed under the designer's conjectured model, and where the final transfer to the seller responds to the realized demand, making the mechanism ex-post individually rational and incentive compatible. The markups are the same as under the Bayesian optimum but with a cap binding for high-cost types. The cap protects the buyer against the possibility that high-cost sellers are more prevalent than conjectured by the designer.

Finally, we compare price and quantity regulation. Under standard Bayesian analysis, price regulation is preferred because it allows quantities to adjust to realized demand. Under robustness, however, the two instruments deliver the same worst-case payoff guarantee but differ in their expected performance under the designer's model. Quantity regulation dominates when optimal markups are high, whereas price regulation performs better when demand uncertainty is substantial. The intuition is as follows. When markups are high, the designer seeks to procure little from most sellers, so fixing quantities effectively limits the buyer's exposure to the designer's demand mis-specification. By contrast, price regulation, by capping the seller's price to protect the buyer against unexpectedly high costs, forces the designer to over-procure from high-cost sellers relative to quantity regulation, thereby reducing welfare.

Organization. We conclude the introduction with a brief discussion of the relevant literature. Section 2 presents the environment and the designer's problem. Section 3 characterizes robustly optimal mechanisms. Section 4 extends the analysis to settings with a downstream market, characterizes robustly optimal price regulations, and identifies conditions under which quantity regulation dominates (or is dominated by) price regulation. Section 5 concludes. Proofs not in the main text are in the Appendix at the end of the document. Supplement I contains extensions of our results to more permissive robustness

requirements and alternative forms of cost uncertainty. Supplement II contains extra material, namely, proof of existence and undomination of robustly optimal mechanisms, and additional comparative statics of robustly optimal mechanisms.

Related literature. A vast literature in information economics studies optimal mechanisms when agents hold private information and the designer optimizes over all mechanisms that are incentive compatible and individually rational under a conjectured model (the Bayesian approach). The closest work to ours is Baron and Myerson (1982).

Recent work relaxes the key assumptions underlying the Bayesian framework to develop robust approaches to contract design; Carroll (2019) provides an excellent survey. Closely related contributions include Garrett (2014), Bergemann et al. (2025), Guo and Shmaya (2025), and Kambhampati (2026). Garrett (2014) characterizes optimal contracts when the designer lacks information about a manager's disutility from effort in the Laffont and Tirole (1986) model. Bergemann et al. (2025) study robust procurement contracts evaluated by worst-case competitive ratios. Guo and Shmaya (2025) analyze minimax-regret design and identify conditions for the optimality of price-cap regulation under private information about both demand and costs.[2] Kambhampati (2026) studies lexicographic worst-case optimality across multiple beliefs. Our analysis is also lexicographic—restricting the designer's choice to worst-case optimal mechanisms—but differs in that second-order beliefs are given by the designer's conjectured model, a primitive of the environment. This distinction is important because it implies, among other things, that the tension between efficiency and rent extraction persists under robustness concerns. None of the above papers identify conditions under which quantity regulation outperforms price regulation under robustness, which is one of our main contributions. Related are also Börgers et al. (2025) and Mishra and Patil (2025), who study undominated mechanisms in auctions and regulation.[3] In these papers, the designer treats all admissible models symmetrically, whereas in our analysis the designer uses the conjectured model to select among mechanisms that maximize the payoff guarantee. This feature aligns our approach with Dworczak and Pavan (2022), who study robust information design without transfers or screening.

Our paper is also related to the literature on model misspecification and robust control (see Cerreia-Vioglio et al. (2026) for an excellent overview). In that literature, alternative

[2]See also Bergemann and Schlag (2008) for an earlier analysis of minimax-regret design in monopoly pricing and Segal (2003) for the design of multi-unit auctions when demand is unknown.

[3]See Supplement II for a discussion of the connection between robust optimality (as defined in this paper) and undomination.

models are weighted by their distance from the conjectured one.[4] In contrast, our designer treats all admissible models symmetrically when computing the payoff guarantee, but then uses the conjectured model to select among the worst-case optimal ones. In this respect, our approach also differs from Madarász and Prat (2017), who study misspecification in monopolistic screening and show that small errors can generate large losses.

Finally, in environments with downstream markets, our analysis connects to Valenzuela-Stookey and Poggi (2025), who study markets as instruments for implementation. Kang (2026) compares the worst-case performance of price versus quantity regulation when the correlation between willingness to pay and marginal externalities is unknown. By contrast, our comparison hinges on the equivalence of price and quantity regulation under worst-case optimality and their divergence under the designer's conjectured model. See also Weitzman (1974) for the classic analysis of price versus quantity regulation without private information or robustness concerns.

2 Model

2.1 Environment

A buyer (a government agency or a private organization) delegates to an expert the design of a contract for the procurement of a product or service. The good is supplied by a monopolistic seller who incurs a cost of θq for supplying q ≥ 0 units of the good, where $\theta \geq 0$ is private information to the seller.[5]

The designer has a model $(V^\star, F^\star)$, where the function $V^\star : \mathbb{R}_+ \to \mathbb{R}_+$ represents the designer's estimate for the value from procuring the good. The function $F^\star$ represents the estimate for the cost technology, i.e., the distribution from which the seller's marginal cost θ is drawn; $F^\star$ is absolutely continuous over $\mathbb{R}$ with density $f^\star$ strictly positive over $\Theta \equiv [\underline{\theta}, \overline{\theta}] \subset \mathbb{R}_{++}$ and $F^\star(\underline{\theta}) = 0$, $F^\star(\overline{\theta}) = 1$, and the virtual cost $z^\star(\theta) \equiv \theta + F^\star(\theta)/f^\star(\theta)$ is continuous and increasing over Θ.[6] The designer is an expected-utility maximizer and

[4] See also Ball and Kattwinkel (2026) for an alternative approach to robustness where payoff guarantees are required to be continuous with respect to the set of admissible models.

[5] The results extend to a setting with a fixed cost $c \in (0, \overline{c})$, where the bound $\overline{c} \in \mathbb{R}_{++}$ ensures that it is optimal for the buyer to procure a positive quantity even when the marginal cost is the largest and the value of the good is the lowest.

[6] Throughout, a function g is said to be increasing (alternatively, weakly increasing) if $g(x) > g(x')$

evaluates the performance of any mechanism under the conjectured model $(V^\star, F^\star)$.

The buyer does not trust the designer's confidence in the model. She is concerned that the true value of the good may be given by a function V different from $V^\star$ and the true technology describing the seller's cost may be given by a distribution F different from $F^\star$. Specifically, the buyer has a set of admissible models $\mathcal{A} \equiv \mathcal{V} \times \mathcal{F}$, where $\mathcal{V}$ is a set of (gross) value functions and $\mathcal{F}$ is a set of distributions over Θ. The set $\mathcal{F}$ is the set of all cdfs whose support is contained in Θ; that is, each $F \in \mathcal{F}$ is a weakly increasing, right-continuous function $F : \mathbb{R} \to [0,1]$ such that $F(\theta) = 0$ for all $\theta < \underline{\theta}$, and $F(\theta) = 1$ for all $\theta \geq \overline{\theta}$. Supplement I considers the case in which $\mathcal{F}$ does not coincide with the set of all cdfs whose support is contained in Θ and discusses how the results depend on the lowest element of such a set (in the main text, the lowest element is a Dirac assigning probability one to $\overline{\theta}$; in the supplement, we allow the lowest element of $\mathcal{F}$ to be a non-degenerate cdf).

We assume that, for each $V \in \mathcal{V}$, there is an integrable function $P : \mathbb{R}_+ \to \mathbb{R}_+$, interpreted as the inverse demand function corresponding to V, such that, for all $\mathrm{q} \geq 0$,

$$V(\mathrm{q}) = \int_0^{\mathrm{q}} P(s)\mathrm{d}s. \tag{1}$$

We let $\mathcal{P}$ represent the set of inverse demand functions and $\mathcal{D}$ the set of direct demand functions corresponding to the elements of the set $\mathcal{V}$ of value functions. For each $P \in \mathcal{P}$, there exists a $\mathrm{q}_P \in \mathbb{R}_{++} \cup \{+\infty\}$ such that P is decreasing and continuous over $[0, \mathrm{q}_P)$ and $P(\mathrm{q}) = 0$ if $\mathrm{q} > \mathrm{q}_P$. For each $P \in \mathcal{P}$, let D denote the corresponding direct demand. For all $\mathrm{p} \in (0, P(0)]$, $D(\mathrm{p}) \equiv P^{-1}(\mathrm{p})$; for all $\mathrm{p} > P(0)$, $D(\mathrm{p}) = 0$; and $\lim_{\mathrm{p}\to 0} D(\mathrm{p}) = \mathrm{q}_P$.

Representing each V as the integral of its inverse demand function P facilitates the application to monopoly regulation in Section 4.1. We assume that $V^\star \in \mathcal{V}$ and then denote by $P^\star \in \mathcal{P}$ and $D^\star \in \mathcal{D}$, respectively, the inverse and direct demands associated with $V^\star$.

The seller can provide any quantity $\mathrm{q} \in [0, \bar{\mathrm{q}}]$, where $\bar{\mathrm{q}} \in \mathbb{R}_{++}$ is such that $D(\underline{\theta}) < \bar{\mathrm{q}}$ for all $D \in \mathcal{D}$. This assumption ensures it is never optimal for the buyer to procure more than $\bar{\mathrm{q}}$ and that the seller's equilibrium payoff satisfies a familiar envelope-theorem representation. We also assume that $D^\star(z^\star(\theta)) > 0$ for all $\theta \in \Theta$; the assumption simplifies the exposition by guaranteeing that the quantity traded under the Bayesian optimal mechanism is strictly positive for all types.

(alternatively, $g(x) \geq g(x')$) whenever $x > x'$. The terms *decreasing* and *weakly decreasing* are analogously defined. Similarly, when we say that a function is concave, convex, or quasi-concave, we mean strictly.

We assume that there exists a "smallest" inverse demand $\underline{P} \in \mathcal{P}$ such that $P(\mathrm{q}) \geq \underline{P}(\mathrm{q})$ for all $\mathrm{q} \geq 0$ and all $P \in \mathcal{P}$. The function $\underline{P}$ provides a lower bound on marginal value and is decreasing and continuous. The smallest inverse demand function $\underline{P}$ induces the smallest value function $\underline{V}$. We also assume that $\overline{\theta} < \lim_{\mathrm{q}\downarrow 0} \underline{P}(\mathrm{q})$. Together with the continuity of $\underline{P}$, this assumption ensures that there are gains from trade no matter the seller's cost and the inverse demand.

Lastly, we assume that, once the buyer learns V, ex-post adjustments to output are either infeasible or not worthwhile. Such adjustments may be prohibitively costly, or the marginal value of additional output may be too low to justify the extra production cost. In the absence of these frictions, uncertainty over V is inconsequential.

In summary, $(V^\star, F^\star)$ represents the designer's conjectured model, with $V^\star$ capturing the value of procuring output and $F^\star$ capturing the cost technology. The conjectured model belongs to the admissible set, i.e., $(V^\star, F^\star) \in \mathcal{A}$ (this also means that $P^\star \in \mathcal{P}$ and $D^\star \in \mathcal{D}$). The buyer fears, however, that the true model may be some alternative $(V, F) \in \mathcal{A}$.

The designer and the buyer share the same objective of maximizing the buyer's value from the procured good net of the transfer to the firm. However, while the designer evaluates any mechanism (described below) under the conjectured model $(V^\star, F^\star)$, the buyer evaluates it under the worst-case scenario computed over $\mathcal{A}$. The designer seeks the buyer's approval. She understands that any mechanism for which the payoff guarantee is not the highest will not be approved. She thus selects a mechanism that maximizes her expected payoff (under the model $(V^\star, F^\star)$) among those for which the buyer's payoff guarantee is the highest.

2.2 Procurement mechanisms

We start by considering environments where there are no downstream markets that can be used to generate information about the value of the good procured, as in the case of private procurement or the public procurement of certain types of military equipment not traded in downstream markets. In these environments, the only viable mechanisms are quantity mechanisms. In this subsection, we formally define these mechanisms and then analyze their properties until we introduce price mechanisms in Section 4.1.

To elicit the seller's private information and discipline the procurement, the designer offers a (direct) mechanism $M = (q, t)$. The *quantity schedule* $q : \Theta \to [0, \bar{\mathrm{q}}]$ specifies the

procured quantity as a function of the reported cost, whereas the *transfer schedule* $t : \Theta \to \mathbb{R}$ specifies the total payment.[7]

The mechanism $M = (q, t)$ is *incentive compatible* (IC) if, for all $\theta, \theta' \in \Theta$,

$$u(\theta) \equiv t(\theta) - \theta q(\theta) \geq t(\theta') - \theta q(\theta') = u(\theta') + (\theta' - \theta)q(\theta').$$

It is *individually rational* (IR) if $u(\theta) \geq 0$ for all $\theta \in \Theta$. Because, given the quantity schedule q, there is a bijection between the transfer schedule t and the utility/rent schedule u, we will often refer to a mechanism by (q, u) instead of (q, t).

As is standard, $M = (q, u)$ is IC and IR if and only if (a) q is weakly decreasing, and (b) for all $\theta \in \Theta$, $u(\theta) = u(\overline{\theta}) + \int_{\theta}^{\overline{\theta}} q(y)\mathrm{d}y$, with $u(\overline{\theta}) \geq 0$.

Let $\mathcal{M}$ be the set of all IC and IR mechanisms. If the buyer's gross value is $V \in \mathcal{V}$ and the technology is $F \in \mathcal{F}$, the buyer's *ex-ante welfare* under the mechanism $M \equiv (q, u) \in \mathcal{M}$ is

$$W(M; V, F) \equiv \int \left(V(q(\theta)) - \theta q(\theta) - u(\theta)\right) F(\mathrm{d}\theta),$$

where $V(q(\theta)) - \theta q(\theta) - u(\theta) = V(q(\theta)) - t(\theta)$ is the buyer's net value when the cost is θ.

2.3 Designer's problem

The designer faces a two-step optimization problem.

Step 1 (guarantee maximization). The designer first computes the shortlist of mechanisms for which the guarantee is the highest. The **welfare guarantee** of any IC and IR mechanism $M \in \mathcal{M}$ is given by

$$G(M) \equiv \inf_{V \in \mathcal{V}, F \in \mathcal{F}} W(M; V, F).$$

[7] The focus on deterministic mechanisms is without loss of optimality in our setting. This is because, for any given θ, the buyer's welfare is concave in q. We also adopt a conservative approach by assuming that Nature's choice of (V, F) can be made contingent on the designer's choice of the mechanism. As it will become clear from the analysis below, this assumption, however, is not important. The mechanisms that solve our design problem remain optimal even if Nature is expected to choose (V, F) simultaneously with the designer's choice of M. This is because the solution to the designer's problem is a saddle point.

The **shortlist** of mechanisms delivering the highest guarantee is given by

$$\mathcal{M}^{\mathrm{SL}} \equiv \arg\max_{M \in \mathcal{M}} G(M).$$

Any mechanism $M \in \mathcal{M}^{\mathrm{SL}}$ in the shortlist is thus **worst-case optimal**. In Supplement I, we consider a more permissive definition of shortlist comprising all mechanisms for which the guarantee is no smaller than a fraction $\gamma \in [0,1]$ of the maximal attainable guarantee (formally, $\mathcal{M}^{\mathrm{SL}}(\gamma) \equiv \{M \in \mathcal{M} : G(M) \geq \gamma G(M') \ \forall \ M' \in \mathcal{M}\}$).[8]

Step 2 (selection under conjectured model). The designer then selects the mechanism from the shortlist that maximizes expected welfare under the conjectured model $(V^\star, F^\star)$.

Definition 1. *A mechanism* M^{OPT} *is* ***robustly optimal*** *if*

$$M^{\mathrm{OPT}} \in \arg\max_{M \in \mathcal{M}^{\mathrm{SL}}} W(M; V^\star, F^\star).$$

As anticipated in the Introduction, this two-step procedure may either reflect the need to secure approval from a max-min authority (a regulator, a policy maker, or a supervisor) facing uncertainty over the set $\mathcal{A} = \mathcal{V} \times \mathcal{F}$ of admissible models, or the designer's response to her own uncertainty over $\mathcal{A}$, whereby the designer first seeks robustness by evaluating any mechanism under the worst-case scenario, and then uses the conjectured model $(V^\star, F^\star)$ as a tool to select from the shortlist of worst-case optimal mechanisms.

3 Robustly optimal mechanisms

We begin by deriving the maximal guarantee of an arbitrary IC and IR mechanism and show that the worst-case welfare need not occur under the technology that places all probability mass at $\overline{\theta}$, but always arises under the lowest possible value function $\underline{V}$.[9]

[8] See also Andrews and Chen (2025) for a different problem (without screening) featuring selection by an expected-utility maximizer over a shortlist of alternatives defined by a worst-case threshold.

[9] The optimal mechanisms in this section are also the solution to an alternative problem in which the designer and the buyer disagree about the value of the good and their values are equal to $V^\star$ and $\underline{V}$ respectively. The designer is Bayesian with subjective belief $F^\star$ over Θ, whereas the buyer faces uncertainty over Θ and has max-min preferences.

Let

$$\mathrm{q}_\ell \equiv \arg\max_{\mathrm{q}\in[0,\bar{\mathrm{q}}]} \left\{\underline{V}(\mathrm{q}) - \overline{\theta}\mathrm{q}\right\} = \underline{D}(\overline{\theta})$$

be the unique quantity that maximizes total surplus when $V = \underline{V}$ and $\theta = \overline{\theta}$; i.e., q_ℓ is the efficient quantity at the lowest demand and highest cost. Define

$$G^* \equiv \underline{V}(\mathrm{q}_\ell) - \overline{\theta}\mathrm{q}_\ell,$$

the total surplus when gross value is lowest, cost is highest, and the buyer procures q_ℓ.

Lemma 1 (welfare guarantee)**.** *For any IC and IR mechanism $M \equiv (q, u) \in \mathcal{M}$,*

$$G(M) = \inf_{\theta\in\Theta} \{\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta)\} \qquad \textit{and} \qquad G(M) \leq G^*. \tag{2}$$

The first part of Lemma 1 highlights that Nature can reduce the buyer's welfare more effectively by selecting a cost $\theta < \overline{\theta}$. Since q is weakly decreasing, the buyer procures more from lower-cost types; thus, when Nature selects an inverse demand below $P^\star$, the welfare loss from over-procurement can be larger at low θ.

The second part states that the welfare guarantee of any IC and IR mechanism cannot exceed the total surplus obtained from procuring the efficient output q_ℓ when demand is lowest and cost is highest. This follows because Nature can always choose the lowest demand and the Dirac distribution that puts probability mass one at $\overline{\theta}$, in which case the buyer obtains the maximal welfare by procuring the quantity q_ℓ.

The next lemma shows that the upper bound on the maximal welfare guarantee is tight and fully characterizes the shortlist $\mathcal{M}^{\mathrm{SL}}$.

Lemma 2 (shortlist characterization)**.** *Take any IC and IR mechanism $M \equiv (q, u) \in \mathcal{M}$. Then, $M \in \mathcal{M}^{\mathrm{SL}}$ if and only if (a) $u(\overline{\theta}) = 0$, and (b), for all $\theta \in \Theta$,*

$$\underline{V}(q(\theta)) - \theta q(\theta) - \int_\theta^{\overline{\theta}} q(y)\mathrm{d}y \geq G^*. \tag{3}$$

Worst-case optimality therefore imposes two additional constraints beyond IC and IR. First, the highest-cost type $\overline{\theta}$ must earn zero rent: otherwise, Nature can select $\underline{V}$ and the Dirac distribution that puts probability mass one at $\overline{\theta}$, reducing welfare strictly below G^*

regardless of $q(\overline{\theta})$. Consequently, for any $M \in \mathcal{M}^{\text{SL}}$, the rent schedule u is pinned down by the output schedule q. Second, ex-post welfare under the lowest possible demand function and zero rent for $\overline{\theta}$ must be weakly above the maximal guarantee G^*, *for any cost* θ. Sufficiency follows from Lemma 1; necessity is shown by constructing a simple constant mechanism that attains G^*, namely one that procures q_ℓ from all types. This constant mechanism, however, is only one element of a continuum in the shortlist. To see this, it is useful to reformulate the robustness constraints in terms of majorization inequalities:

Lemma 3 (majorizations)**.** *Take any weakly decreasing function* $q : \Theta \to \mathbb{R}_+$*. (1) For every* $\theta \in \Theta$*, constraint (3) is equivalent to*

$$\int_{\theta}^{\overline{\theta}} q(y)\mathrm{d}y \leq \int_{\theta}^{\overline{\theta}} \underline{D}(y)\mathrm{d}y - \underbrace{\int_{\theta}^{\underline{P}(q(\theta))} \left(\underline{D}(y) - q(\theta)\right) \mathrm{d}y}_{\underline{\mathrm{DWL}}(\theta, q(\theta)) \geq 0}, \tag{4}$$

where $\underline{\mathrm{DWL}}(\theta, q(\theta))$ *is the deadweight loss incurred under* $\underline{V}$ *when output* $q(\theta)$ *is procured instead of* $\underline{D}(\theta)$*. (2) The following statements are equivalent: (a) the inequality in (4) holds for all* $\theta \in \Theta$*; (b) the inequality in (4) holds for* $\theta \in \{\underline{\theta}, \overline{\theta}\}$ *and, for all* $\theta \in (\underline{\theta}, \overline{\theta})$*,*

$$\int_{\theta}^{\overline{\theta}} q(y)\mathrm{d}y \leq \int_{\theta}^{\overline{\theta}} \underline{D}(y)\mathrm{d}y. \tag{5}$$

Because $\underline{\mathrm{DWL}}(\theta, q(\theta)) \geq 0$, the weaker majorization in (5) is implied by the stronger one in (4). However, when (4) holds for the lowest and the highest type, and, in addition, the weaker constraints are satisfied for all the remaining types, all the robustness constraints are met. The intuition is the following. Because rents $u(\theta)$ are decreasing in type, when the robustness constraint is met at the extremes of the type distribution and the quantity procured from intermediate types is small relative to $\underline{D}$ (in the sense of (4)), welfare under the lowest demand $\underline{V}(q(\theta)) - \theta q(\theta) - \int_{\theta}^{\overline{\theta}} q(y)\mathrm{d}y$ exceeds the guarantee G^* for all types, even if it is not necessarily monotone in θ.

It is easy to see that the schedule $q = \underline{D}$ satisfies the robustness constraint in (4). The following observation is also useful:

Observation 1 (quantity bound)**.** *If* $(q, u) \in \mathcal{M}^{\text{SL}}$*, then* $q(\theta) \geq \mathrm{q}_\ell$ *for all* $\theta \in \Theta$*, with*

equality at $\overline{\theta}$.

Clearly, for the robustness constraint (3) to hold at $\overline{\theta}$, it must be that $q(\overline{\theta}) = \mathrm{q}_\ell$. The observation then follows from this property along with the fact that q must be weakly decreasing.

Thus, any mechanism $(q, u) \in \mathcal{M}^{\mathrm{SL}}$ procures no less than q_ℓ from any type. For any weakly decreasing schedule q that procures $q(\theta) \in [\mathrm{q}_\ell, \underline{D}(\theta)]$ from each type θ, the robustness constraint in (4) reduces to[10]

$$\int_{\theta}^{\overline{\theta}} q(y)\mathrm{d}y \leq \int_{\theta}^{\overline{\theta}} \min\{\underline{D}(y), q(\theta)\}\mathrm{d}y,$$

which is always satisfied, and, consequently, such a q corresponds to a mechanism in the shortlist. Moreover, $\mathcal{M}^{\mathrm{SL}}$ also includes mechanisms that procure more than the efficient output $\underline{D}(\theta)$ for some types—a feature that plays a key role in the analysis below.

Using Lemma 2 and the standard "virtual surplus" representation of total welfare under the conjectured model $(V^\star, F^\star)$ (see Baron and Myerson (1982)), the designer's problem reduces to[11]

$$\max_{q} \int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q(\theta)) - z^\star(\theta)q(\theta)\Big] F^\star(\mathrm{d}\theta) \qquad \textbf{(ROPT)}$$

$$\text{subject to } q \text{ weakly decreasing, and}$$

$$\underline{V}(q(\theta)) - \theta q(\theta) - \int_{\theta}^{\overline{\theta}} q(y)\mathrm{d}y \geq G^* \qquad \forall\ \theta \in \Theta. \tag{6}$$

Hereafter, we will often use Lemma 3 to express the robustness constraints in (6) as majorization constraints. Relative to Baron and Myerson (1982), this program incorporates the additional robustness constraint requiring that, for each $\theta \in \Theta$, ex-post welfare under the

[10] To see this, note that the RHS in (4) reduces to $\int_{\theta}^{\underline{P}(q(\theta))} q(\theta)\mathrm{d}y$ over the interval $(\theta, \underline{P}(q(\theta)))$ and to $\int_{\underline{P}(q(\theta))}^{\overline{\theta}} \underline{D}(y)\mathrm{d}y$ over the interval $(\underline{P}(q(\theta)), \overline{\theta})$.

[11] Existence of a solution to the program in (**ROPT**) follows from the extreme-value theorem after establishing that the set of quantity schedules q satisfying the constraints is sequentially compact and the objective function in (**ROPT**) is sequentially continuous under the point-wise convergence topology. See Section E.1 in Supplement II.

lowest possible demand exceeds the guarantee G^*. This constraint is not standard because it combines the quantity procured from type θ with the integral of the quantity procured from all higher types.

We denote a quantity schedule solving (**ROPT**) as q^{OPT} and let $M^{\text{OPT}} \equiv (q^{\text{OPT}}, u^{\text{OPT}})$ be the mechanism corresponding to it, with u^{OPT} given by $u^{\text{OPT}}(\theta) = \int_{\theta}^{\overline{\theta}} q^{\text{OPT}}(y)\mathrm{d}y$ for all θ.

3.1 Baron-Myerson-with-quantity-floor

We start with a natural candidate solution of (**ROPT**) and provide necessary and sufficient conditions under which it is robustly optimal. Let q^{BM} be the *Baron-Myerson* quantity schedule, defined, for all θ, by

$$q^{\text{BM}}(\theta) \equiv \arg\max_{\mathrm{q}\in[0,\bar{\mathrm{q}}]} \Big[V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}\Big] = D^\star(z^\star(\theta)).$$

Definition 2. *The* ***Baron-Myerson-with-quantity-floor*** *mechanism $M^\star \equiv (q^\star, u^\star)$ is such that, for all θ,*[12]

$$q^\star(\theta) = \max\{q^{\text{BM}}(\theta), \mathrm{q}_\ell\} \tag{7}$$

and $u^\star(\theta) = \int_\theta^{\overline{\theta}} q^\star(y)\mathrm{d}y$.

Proposition 1 (optimality of Baron-Myerson-with-quantity-floor)**.** *The mechanism $M^\star \equiv (q^\star, u^\star)$ is robustly optimal if and only if*

$$\int_{\underline{\theta}}^{\overline{\theta}} q^\star(y)\mathrm{d}y \le \int_{\underline{\theta}}^{\overline{\theta}} \underline{D}(y)\mathrm{d}y - \int_{\underline{\theta}}^{\underline{P}(q^\star(\underline{\theta}))} \Big[\underline{D}(y) - q^\star(\underline{\theta})\Big]\mathrm{d}y, \tag{8}$$

$$\int_{\theta}^{\overline{\theta}} q^\star(y)\mathrm{d}y \le \int_{\theta}^{\overline{\theta}} \underline{D}(y)\mathrm{d}y \qquad \forall\ \theta \in \Theta. \tag{9}$$

The two conditions in the proposition are equivalent to the requirement that the quantity schedule $q^\star$ satisfies all the robustness constraints in (3). Because $q^\star$ maximizes virtual surplus over all weakly decreasing schedules q satisfying $q(\theta) \ge \mathrm{q}_\ell$ for all θ, Baron-Myerson-

[12] Observe that $q^{\text{BM}}(\underline{\theta}) = D^\star(\underline{\theta}) \ge \underline{D}(\underline{\theta}) > \underline{D}(\overline{\theta}) = \mathrm{q}_\ell$. On the other hand, $q^{\text{BM}}(\overline{\theta}) = D^\star(z^\star(\overline{\theta}))$ can be smaller than $\underline{D}(\overline{\theta}) = \mathrm{q}_\ell$ given that $z^\star(\overline{\theta}) > \overline{\theta}$.

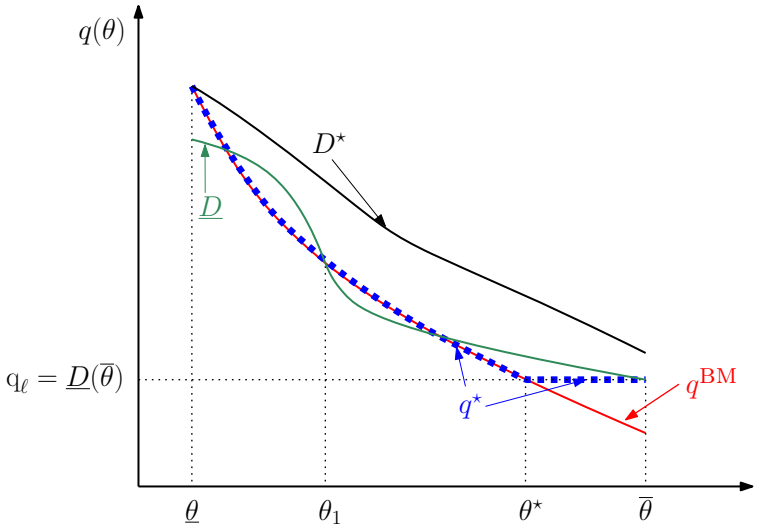


Figure 1: Baron-Myerson-with-quantity-floor.

with-quantity-floor is robustly optimal when, and only when, these conditions hold.

Note that the majorization constraints in (9) are easier to verify than the constraints in (3). In practice, it suffices to verify them at those points in which the schedule $q^\star$ crosses $\underline{D}$ from below.[13] For example, in Figure 1, it suffices to verify that $q^\star$, in addition to satisfying Condition (8), satisfies Condition (9) at θ_1.

Corollary 1 (uniqueness of output schedule)**.** *If $M^\star = (q^\star, u^\star)$ is robustly optimal, then any robustly optimal mechanism $M^{\mathrm{OPT}} = (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$ satisfies $q^{\mathrm{OPT}}(\theta) = q^\star(\theta)$ for all $\theta > \underline{\theta}$.*

Proof: If $q^\star$ solves (**ROPT**), then it also solves the relaxed program where the objective function is the same as in (**ROPT**) and the constraints in (**ROPT**) are replaced by $q(\theta) \geq \mathrm{q}_\ell$ for all $\theta \in \Theta$ (see Observation 1). Any solution to this relaxed program must coincide with $q^\star$ at almost all θ. Because $q^\star$ is continuous, it must in fact coincide at all $\theta > \underline{\theta}$. *Q.E.D.*

Proposition 1 can also be used to obtain easy-to-verify sufficient conditions for the Baron-

[13] In fact, because $q^\star$ is weakly decreasing, Lemma 5 in the Appendix implies that the function $\underline{W}(\theta, q^\star) = \underline{V}(q^\star(\theta)) - \theta q^\star(\theta) - \int_\theta^{\overline{\theta}} q^\star(y)\mathrm{d}y$ is weakly decreasing (alternatively, weakly increasing) over an interval of types $I \subset \Theta$ if $0 < q^\star(\theta) \leq \underline{D}(\theta)$ (alternatively, $q^\star(\theta) > \underline{D}(\theta)$) for all $\theta \in I$. Because, for all $\theta \in \Theta$, $q^\star(\theta) > 0$, when Condition (8) holds, to verify that Condition (9) also holds, it suffices to check that it holds at points in which the schedule $q^\star$ crosses $\underline{D}$ from below. If there are no such points, and (8) holds, then all the robustness constraints in (3) hold and $M^\star$ is robustly optimal.

Myerson-with-quantity-floor mechanism to be robustly optimal. For every $\theta \in \Theta$, let

$$d(\theta) \equiv P^\star(\underline{D}(\theta)) - \theta \tag{10}$$

denote the distance between the two *inverse demand* functions $P^\star$ and $\underline{P}$ at $\mathrm{q} = \underline{D}(\theta)$.[14]

Corollary 2 (large downward distortions)**.** *Suppose $F^\star(\theta)/f^\star(\theta) \geq d(\theta)$ for all $\theta \in \Theta$. Then, $M^\star = (q^\star, u^\star)$ is robustly optimal.*

Proof: By definition, $d(\theta) \geq 0$ for all $\theta \in \Theta$. Note that $F^\star(\underline{\theta})/f^\star(\underline{\theta}) = 0$. The condition in the corollary thus implies that $d(\underline{\theta}) = 0$. Hence, $P^\star(\underline{D}(\underline{\theta})) = \underline{\theta}$, and therefore, $\underline{D}(\underline{\theta}) = D^\star(\underline{\theta})$. Because $q^\star(\underline{\theta}) = q^{\mathrm{BM}}(\underline{\theta}) = D^\star(\underline{\theta}) = \underline{D}(\underline{\theta})$, we have that $\underline{P}(q^\star(\underline{\theta})) = \underline{\theta}$, and the two conditions in Proposition 1 reduce to the requirement that $\int_{\theta}^{\overline{\theta}} [\underline{D}(y) - q^\star(y)]\mathrm{d}y \geq 0$ for all $\theta \in \Theta$. Next, observe that, for all $\theta \in \Theta$,

$$q^{\mathrm{BM}}(\theta) = D^\star\Big(\theta + \frac{F^\star(\theta)}{f^\star(\theta)}\Big) \leq D^\star\Big(\theta + d(\theta)\Big) = \underline{D}(\theta),$$

where the inequality follows because $D^\star$ is decreasing and $F^\star(\theta)/f^\star(\theta) \geq d(\theta)$, whereas the last equality follows from the definition of $d(\theta)$. Thus, for every $\theta \in \Theta$, we have that $q^\star(\theta) = \max\{q^{\mathrm{BM}}(\theta), \mathrm{q}_\ell\} = \max\{q^{\mathrm{BM}}(\theta), \underline{D}(\overline{\theta})\} \leq \underline{D}(\theta)$. Hence, $q^\star$ satisfies the conditions in Proposition 1. *Q.E.D.*

The following is also an immediate implication of the above results:

Corollary 3 (no demand uncertainty)**.** *If $V^\star = \underline{V}$ (which is the case when there is no demand uncertainty, i.e., when $\mathcal{V} = \{V^\star\}$), then $M^\star = (q^\star, u^\star)$ is robustly optimal. In this case, the optimal mechanism features efficiency both at "the top" ($\underline{\theta}$) and "the bottom" ($\overline{\theta}$) of the type distribution.*

3.2 Beyond Baron-Myerson-with-quantity-floor

We now characterize the qualitative properties of robustly optimal mechanisms when they differ from the Baron-Myerson-with-quantity-floor (the precise characterization is in Supplement I).

[14] Note that $d(\theta) = P^\star(\underline{D}(\theta)) - \underline{P}(\underline{D}(\theta))$. Equivalently, $d(\theta)$ is the unique real solution to $D^\star(\theta + x) = \underline{D}(\theta)$.

Let $\theta^\star$ be the unique solution to $q^{\mathrm{BM}}(\theta^\star) = \mathrm{q}_\ell$ when such a solution exists (which is the case when, and only when, $q^{\mathrm{BM}}(\overline{\theta}) < \mathrm{q}_\ell$); else, let $\theta^\star \equiv \overline{\theta}$. Next, let $\underline{W}(\cdot, q^\star)$ be the function defined, for all $\theta \in \Theta$, by

$$\underline{W}(\theta, q^\star) \equiv \underline{V}(q^\star(\theta)) - \theta q^\star(\theta) - \int_\theta^{\overline{\theta}} q^\star(y)dy \tag{11}$$

and let $\theta^m \equiv \max\{\theta : \theta \in \arg\min_{y\in\Theta} \underline{W}(y, q^\star)\}$. Under regularity, θ^m is well defined and is the largest cost at which the function $\underline{W}(\cdot, q^\star)$ attains a minimum.[15] Type θ^m plays an important role in our analysis. Indeed, Lemma 7 in the Appendix shows that the Baron-Myerson-with-quantity-floor mechanism is robustly optimal if and only if $\theta^m = \overline{\theta}$. Proposition 2 below uses θ^m and $\theta^\star$ to characterize the main qualitative properties of robustly optimal mechanisms. Section S.1 in Supplement I provides a sharper characterization.

Proposition 2 (robust optimality: general case)**.** *Suppose $M^\star$ is not robustly optimal. (1) Then, $\theta^m < \theta^\star$ and*

$$\begin{aligned} q^{\mathrm{OPT}}(\theta) &= q^{\mathrm{BM}}(\theta) \qquad \forall\ \theta \in (\underline{\theta}, \theta^m), \\ q^{\mathrm{OPT}}(\theta) &\leq q^{\mathrm{BM}}(\theta) \qquad \forall\ \theta \in (\theta^m, \theta^\star), \\ q^{\mathrm{OPT}}(\theta) &= \mathrm{q}_\ell \qquad \forall\ \theta \in [\theta^\star, \overline{\theta}]. \end{aligned}$$

(2) Furthermore, if $\theta^\star < \overline{\theta}$, then $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for a subset of $(\theta^m, \theta^\star)$ of positive Lebesgue measure. If, instead, $\theta^\star = \overline{\theta}$, there exists $\theta^{\star\star} \in [\theta^m, \theta^\star]$ such that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for a subset of $(\theta^m, \theta^{\star\star})$ of positive Lebesgue measure and $q^{\mathrm{OPT}}(\theta) = \underline{D}(\theta)$ for all $\theta \in [\theta^{\star\star}, \overline{\theta}]$.

Figure 2 illustrates the structure of robustly optimal quantity schedules when they differ from the quantity schedule in Baron-Myerson-with-quantity-floor and $\theta^\star < \overline{\theta}$. In this case, robustness entails an increase in procurement from high-cost sellers and a reduction in procurement from intermediate-cost sellers, relative to the Bayesian optimum. The increase (from $q^{\mathrm{BM}}(\theta)$ to q_ℓ) for high-cost sellers prevents welfare losses that arise if Nature selects high-cost sellers with a higher probability than what is conjectured by the designer. Recall that the downward distortions in the Baron-Myerson schedule q^{BM} serve to limit the rents of low-cost types. When high-cost types are more likely, these distortions lose their

[15] Under regularity, $q^\star$ is continuous. Because Θ is compact and $\underline{W}(\cdot, q^\star)$ is continuous over Θ, the set $\{\theta : \underline{W}(\theta, q^\star) \leq \underline{W}(\theta', q^\star)\ \forall\ \theta' \in \Theta\}$ is non-empty and compact.

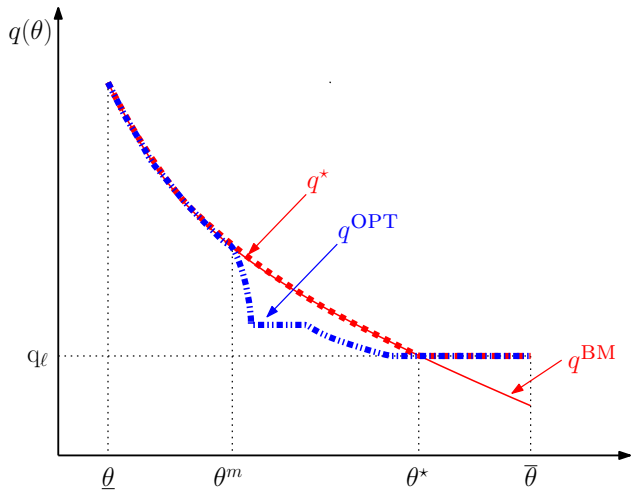


Figure 2: Illustration of Proposition 2.

rationale. In contrast, the reduction in procurement (from $q^{\text{BM}}(\theta)$ to $q^{\text{OPT}}(\theta) < q^{\text{BM}}(\theta)$) for intermediate-cost sellers limits welfare losses when actual demand is lower than conjectured. Section S.1 in Supplement I extends the result in part (2) by showing that the strict inequality $q^{\text{OPT}}(\theta) < q^{\text{BM}}(\theta)$ applies to all $\theta \in (\theta^m, \min\{\theta^\star, \theta^{\star\star}\})$. The proof uses Lagrangian methods for infinite-dimensional constraints along with ironing techniques to establish that, for $\theta \in (\theta^m, \min\{\theta^\star, \theta^{\star\star}\})$, $q^{\text{OPT}}(\theta) = D^\star(\bar{z}^M(\theta))$, where $\bar{z}^M$ is an ironed virtual cost that accounts for robustness.

4 Downstream markets and monopoly regulation

Now suppose there exists a downstream market for the good supplied by the seller, and interpret D as the demand curve in such a market (with inverse P and associated value V). Recall that D is continuous over $\mathbb{R}_{++}$, decreasing on $(0, P(0)]$, and satisfies $D(\text{p}) = 0$ for all $\text{p} > P(0)$. The mechanisms in the previous section can then be interpreted as *quantity regulation*, whereby the buyer (acting as a regulator maximizing consumer surplus) directly controls the quantity supplied to consumers.[16]

The regulator can alternatively influence market outcomes by regulating prices rather

[16] If the revenue $P(q(\theta))q(\theta)$ is accrued to the seller, the mechanism (q, u) is implemented through a transfer schedule $\tilde{t}(\theta, P) = t(\theta) - P(q(\theta))q(\theta)$, where t is the total transfer schedule corresponding to the mechanism (q, u), and where $P(q(\theta))$ is the observed market price. Such a transfer schedule fully insulates the seller from any uncertainty about the demand D (alternatively, the price function P).

than quantities. Price regulation has the advantage of allowing the quantity to adjust to the realized demand. However, as shown in Subsection 4.2 below, it also entails disadvantages that make its overall desirability vis-à-vis quantity regulation unclear.[17] Below, we first characterize robust price regulation, and then identify conditions under which each form of regulation dominates the other. Consistently with the analysis in the previous section, we assume that the regulator delegates to a Bayesian designer the choice of the price mechanism but requires that the welfare guarantee (over the admissible set $\mathcal{D} \times \mathcal{F}$) be the highest.

4.1 Price regulation

Price regulation consists of a price function p and a transfer function t. The price function $p : \Theta \to \mathbb{R}$ specifies the price the seller must set for each reported cost. The seller must then supply any quantity demanded by consumers at that price. Because demand is uncertain, price mechanisms expose the seller to uncertainty in profits. To ensure that the seller participates and reports truthfully regardless of its beliefs or attitude toward uncertainty, the designer conditions the total transfer t to the seller on the realized demand $D \in \mathcal{D}$, which we assume becomes known ex post.[18] We maintain that, once the quantity $D(p(\theta))$ demanded at price $p(\theta)$ is observed by the designer and the seller (which can happen either concurrently with or before learning the full demand D), it is too late to make adjustments to the price or the output produced. As explained in the previous section, in the absence of these natural frictions, demand uncertainty is inconsequential.

Definition 3. *A **price regulation** $\widetilde{M} = (p, t)$ is a pair of mappings $p : \Theta \to \mathbb{R}_+$ and $t : \Theta \times \mathcal{D} \to \mathbb{R}_+$ where $p(\theta)$ is the price the regulator asks the seller to post and $t(\theta, D)$ is the total transfer to the seller when the reported cost is θ and the realized demand is D.*[19]

The price regulation $\widetilde{M} = (p, t)$ is *ex-post incentive compatible* (EPIC) if, for all $\theta, \theta' \in \Theta$ and $D \in \mathcal{D}$, $t(\theta, D) - \theta D(p(\theta)) \geq t(\theta', D) - \theta D(p(\theta'))$. It is *ex-post individually rational*

[17] Under Bayesian analysis (i.e., in the absence of worst-case approval constraints), price regulation always dominates.

[18] In some cases, (e.g., when the set $\mathcal{P}$ of inverse demand functions is fully ordered so that for all $P, P' \in \mathcal{P}$, we have that $P(\mathrm{q}) \neq P'(\mathrm{q})$ for all $\mathrm{q} \in [0, \bar{\mathrm{q}}]$) the demand D can be learned by observing the quantity $D(p(\theta))$ traded at the posted price $p(\theta)$. The assumption that D is learned ex-post gives price regulation the "best shot" at beating quantity regulation. As we show in Subsection 4.2, even under such a favorable scenario, price regulation may be dominated by quantity regulation.

[19] Again, if the revenue $p(\theta)D(p(\theta))$ is accrued to the seller, the regulator commits to transfer an additional amount $t^\dagger(\theta, D) = t(\theta, D) - p(\theta)D(p(\theta))$ to the seller. This transfer is funded with consumers' money. The total monetary cost to the consumers is thus $t(\theta, D)$.

(EPIR) if, for all $\theta \in \Theta$ and $D \in \mathcal{D}$, $\tilde{u}(\theta, D) \equiv t(\theta, D) - \theta D(p(\theta)) \geq 0$.

Let $\widetilde{\mathcal{M}}$ be the set of all EPIC and EPIR price regulations. For any $\widetilde{M} \in \widetilde{\mathcal{M}}$, demand $D \in \mathcal{D}$, and technology $F \in \mathcal{F}$, welfare (consumer surplus) is given by

$$\widetilde{W}(\widetilde{M}; D, F) \equiv \int_{\underline{\theta}}^{\overline{\theta}} \widetilde{w}(\theta, \widetilde{M}; D) F(\mathrm{d}\theta),$$

where, for all $\theta \in \Theta$, $\widetilde{M} \in \widetilde{\mathcal{M}}$, and $D \in \mathcal{D}$,

$$\widetilde{w}(\theta, \widetilde{M}; D) \equiv \int_{0}^{D(p(\theta))} D^{-1}(y)\mathrm{d}y - \theta D(p(\theta)) - \tilde{u}(\theta, D).$$

The **welfare guarantee** of any price regulation $\widetilde{M} \in \widetilde{\mathcal{M}}$ is given by

$$G(\widetilde{M}) \equiv \inf_{D \in \mathcal{D}, F \in \mathcal{F}} \widetilde{W}(\widetilde{M}; D, F).$$

The **shortlist** of approved price regulations is given by

$$\widetilde{\mathcal{M}}^{\mathrm{SL}} \equiv \arg\max_{\widetilde{M} \in \widetilde{\mathcal{M}}} G(\widetilde{M}).$$

Recall that the maximal welfare guarantee for quantity regulations is $G^* \equiv \underline{V}(\mathrm{q}_\ell) - \overline{\theta}\mathrm{q}_\ell$, as shown in Lemma 1, where $\mathrm{q}_\ell \equiv \underline{D}(\overline{\theta})$ is the efficient quantity when $\theta = \overline{\theta}$ and $D = \underline{D}$ (equivalently, when $P = \underline{P}$ and $V = \underline{V}$). This same guarantee applies to price regulations:

Lemma 4 (shortlist of price regulations)**.** *If $\widetilde{M} \in \widetilde{\mathcal{M}}^{\mathrm{SL}}$, then $G(\widetilde{M}) = G^*$. Moreover, $\widetilde{M} \equiv (p, t) \in \widetilde{\mathcal{M}}^{\mathrm{SL}}$ if and only if (a) p is weakly increasing, (b) for all $\theta \in \Theta$ and $D \in \mathcal{D}$,*

$$\tilde{u}(\theta, D) = \tilde{u}(\overline{\theta}, D) + \int_{\theta}^{\overline{\theta}} D(p(y))\mathrm{d}y \tag{12}$$

with $\tilde{u}(\overline{\theta}, D) \geq 0$ and $\tilde{u}(\overline{\theta}, \underline{D}) = 0$, (c) $p(\overline{\theta}) = \overline{\theta}$, and (d) for all $\theta \in \Theta$ and $D \in \mathcal{D}$, $\widetilde{w}(\theta, \widetilde{M}; D) \geq G^*$.

The following is an immediate implication of the previous lemma:

Corollary 4 (equivalence under worst-case optimality)**.** *The maximal welfare guarantee over all price regulations equals that under quantity regulations.*

The result follows from the fact that Nature can always select the lowest possible demand $\underline{D}$ and highest cost $\overline{\theta}$. The maximal welfare at $(\underline{D}, \overline{\theta})$ is attained when the monopolist supplies the efficient output q_ℓ. Whether the regulator enforces q_ℓ by fixing the price at marginal cost or by directly imposing that the monopolist supply that quantity is immaterial. However, as shown in Subsection 4.2, identical guarantees do not imply that the two mechanisms yield the same welfare under the designer's conjectured model.

A simple way to attain G^* is to set $p(\theta) = \overline{\theta}$ for all $\theta \in \Theta$ and $\tilde{u}(\overline{\theta}, D) = 0$ for all $D \in \mathcal{D}$. Yet many other price regulations yield the same guarantee, allowing the designer to use her conjectured model $(D^\star, F^\star)$ to select among them. Notably, as indicated in the last lemma, any approved price regulation must set $p(\overline{\theta}) = \overline{\theta}$ and leave zero rent when $\theta = \overline{\theta}$ and $D = \underline{D}$, as only these choices achieve G^* in the worst case. Unlike quantity regulation, however, the price function does not uniquely determine the transfers, because $\tilde{u}(\overline{\theta}, D)$ may be positive for $D \neq \underline{D}$.

Definition 4. *A **Baron-Myerson-with-price-cap** regulation is a pair $\widetilde{M} = (p, t)$ such that*

$$p(\theta) = \min\{z^\star(\theta), \overline{\theta}\} \tag{13}$$

for all $\theta \in \Theta$, and the induced rent schedule $\tilde{u}(\theta, D) \equiv t(\theta, D) - \theta D(p(\theta))$ satisfies $\tilde{u}(\theta, D) = \tilde{u}(\overline{\theta}, D) + \int_\theta^{\overline{\theta}} D(p(y))dy$ for all $\theta \in \Theta$ and $D \in \mathcal{D}$, with $\tilde{u}(\overline{\theta}, \underline{D}) = \tilde{u}(\overline{\theta}, D^\star) = 0$ and, for all $D \in \mathcal{D} \setminus \{\underline{D}, D^\star\}$,

$$0 \leq \tilde{u}(\overline{\theta}, D) \leq \int_0^{D(\overline{\theta})} D^{-1}(y)dy - \overline{\theta} D(\overline{\theta}) - G^*. \tag{14}$$

All such regulations share the same price schedule (for $\theta > \underline{\theta}$) but differ in their transfer schedules.

Proposition 3 (optimality of Baron-Myerson-with-price-cap)**.** *Every Baron-Myerson-with-price-cap regulation is robustly optimal. Moreover, in every robustly optimal price regulation $\widetilde{M}^{\mathrm{OPT}} = (p^{\mathrm{OPT}}, t^{\mathrm{OPT}})$, for any $\theta > \underline{\theta}$, $p^{\mathrm{OPT}}(\theta) = \min\{z^\star(\theta), \overline{\theta}\}$, and $\widetilde{u}^{\mathrm{OPT}}(\overline{\theta}, D^\star) = 0$.*

Under the conjectured model $(D^\star, F^\star)$ with gross value function $V^\star$, the virtual surplus

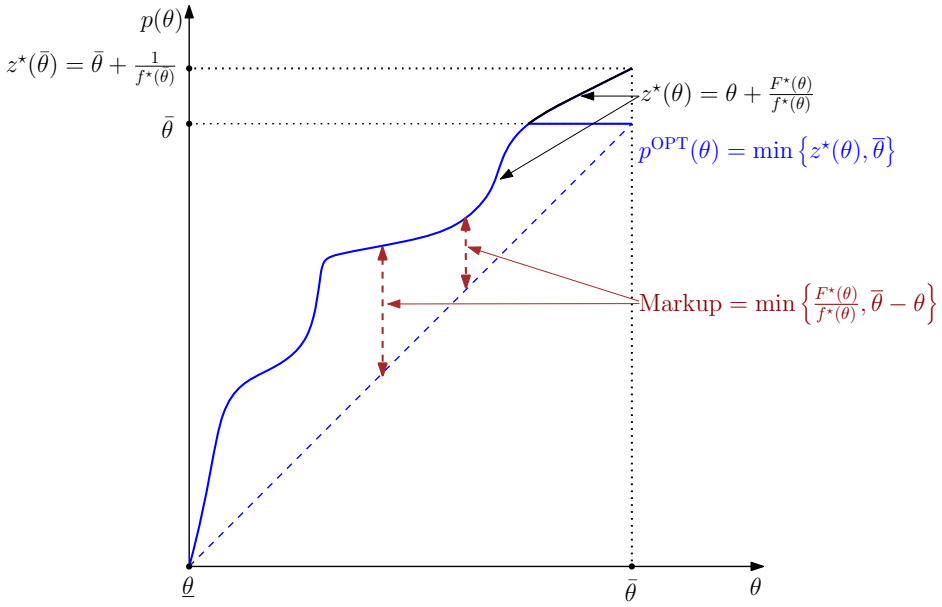


Figure 3: Robustly optimal price schedule.

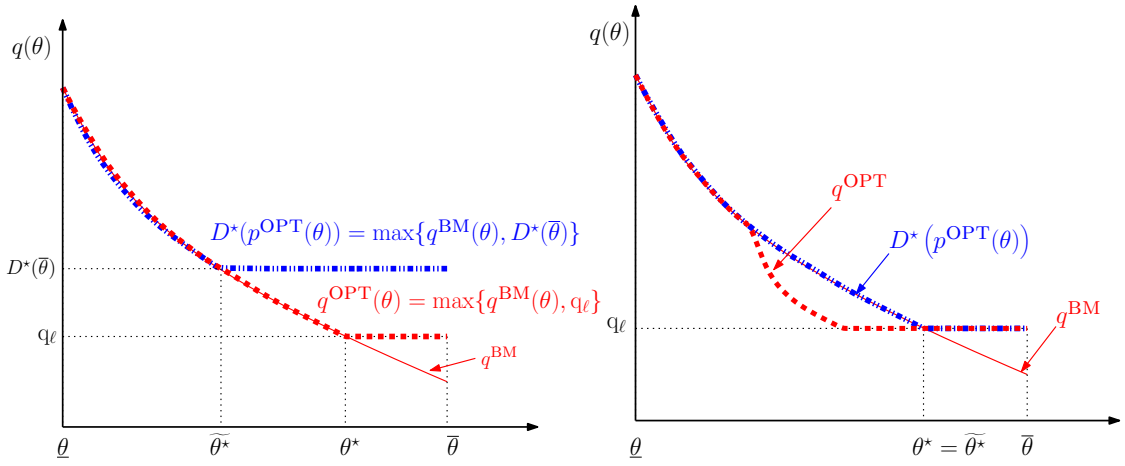
$V^\star(D^\star(\mathrm{p})) - z^\star(\theta)D^\star(\mathrm{p})$ is quasi-concave in p and maximized at $\mathrm{p} = z^\star(\theta)$ for all $\theta \in \Theta$. The constants $\{\tilde{u}^{\mathrm{OPT}}(\overline{\theta}, D)\}$ can then be chosen to satisfy all the constraints in Lemma 4 (for instance, by choosing $\widetilde{u}(\overline{\theta}, D) = 0$ for all $D \in \mathcal{D}$).

Corollary 5 (robustly optimal price schedule)**.** *In the interval $(\underline{\theta}, \overline{\theta}]$, the robustly optimal price schedule is unique and is invariant to both the designer's conjectured demand $D^\star$, and the admissible set $\mathcal{D} \times \mathcal{F}$ of demands and cost technologies the regulator considers plausible. It sets a markup of $F^\star(\theta)/f^\star(\theta)$ at each cost $\theta > \underline{\theta}$, capped at $(\overline{\theta} - \theta)$.*

Figure 3 illustrates the result. By committing to rent payments contingent on realized demand and setting a cost-dependent markup—based solely on the conjectured cost distribution $F^\star$—capped at $(\overline{\theta} - \theta)$, the designer maximizes welfare (under the conjectured model $(D^\star, F^\star)$) regardless of the regulator's uncertainty over demand or technology.

4.2 Quantity vs price regulation

Corollary 4 establishes that the maximal welfare guarantee—the highest welfare achievable under the worst-case scenario—is the same for both price and quantity regulation. However, the maximal welfare attainable under the designer's conjectured model $(D^\star, F^\star)$ over the shortlist of approved worst-case-optimal regulations may differ between the two types of



A: Part 1. B: Part 2.

Figure 4: Graphical illustration of Proposition 4.

regulation. The following definition formalizes what it means for one type of regulation to dominate the other, from the perspective of the designer:

Definition 5. *Price regulation dominates quantity regulation if*

$$\widetilde{W}(\widetilde{M}^{\mathrm{OPT}}; D^\star, F^\star) \geq W(M^{\mathrm{OPT}}; V^\star, F^\star) \tag{15}$$

(strictly if the inequality is strict). Conversely, quantity regulation dominates price regulation if

$$W(M^{\mathrm{OPT}}; V^\star, F^\star) \geq \widetilde{W}(\widetilde{M}^{\mathrm{OPT}}; D^\star, F^\star) \tag{16}$$

(strictly if the inequality is strict). Price and quantity regulation are equivalent if (15) and (16) jointly hold.

Proposition 4 (quantity vs price regulation)**.** *(1) If the Baron-Myerson-with-quantity-floor mechanism is robustly optimal, quantity regulation dominates price regulation (strictly if $D^\star(\overline{\theta}) > \underline{D}(\overline{\theta})$). (2) If $D^\star(\overline{\theta}) = \underline{D}(\overline{\theta})$, price regulation dominates quantity regulation. Furthermore, the domination is strict if the Baron-Myerson-with-quantity-floor mechanism is not robustly optimal.*

Panel A of Figure 4 illustrates the induced quantity schedules q^{OPT} and $D^\star(p^{\mathrm{OPT}})$ under the designer's conjectured demand $D^\star$ for the first part of Proposition 4, when $D^\star(\overline{\theta}) > \underline{D}(\overline{\theta}) = \mathrm{q}_\ell$. Panel B illustrates the second part, when $D^\star(\overline{\theta}) = \underline{D}(\overline{\theta}) = \mathrm{q}_\ell$.

The intuition for the first part is as follows. When the Baron-Myerson-with-quantity-floor mechanism is robustly optimal, both regulations induce the same volume of trade at low costs under the conjectured model. For high costs, however, the output traded under the optimal price regulation can exceed that under the optimal quantity regulation. This occurs because the price is capped at $\overline{\theta}$, leading the seller to supply $D^\star(\overline{\theta})$ under the conjectured demand, which exceeds q_ℓ, the quantity supplied under the optimal quantity regulation, when $D^\star(\overline{\theta}) > \underline{D}(\overline{\theta}) = \mathrm{q}_\ell$. This oversupply increases the distance from $D^\star(z^\star(\theta))$, the Bayesian optimal quantity under the conjectured model $(D^\star, F^\star)$, reducing welfare relative to quantity regulation (see Panel A of Figure 4).

For the second part, when the Baron-Myerson-with-quantity-floor mechanism is not robustly optimal, quantity regulation requires downward adjustments in output for intermediate costs to avoid excessive trading under low realized demand (see Proposition 2). These adjustments reduce welfare under the conjectured model. Price regulation, by contrast, automatically protects the regulator by adjusting output to the realized demand, sparing the designer from the need to reduce trade for intermediate-cost sellers, thus avoiding the welfare reduction. Provided the price cap $\mathrm{p} = \overline{\theta}$ does not induce over-procurement for high costs—which is the case when $D^\star(\overline{\theta}) - \underline{D}(\overline{\theta})$ is small—price regulation strictly dominates.

The following is an implication of the previous results and can be seen directly by combining Corollary 3 and Proposition 3.

Corollary 6 (equivalence in the absence of downward demand uncertainty)**.** *If the regulator faces no downward demand uncertainty (i.e., $\underline{D} = D^\star$), price and quantity regulations are equivalent.*

In the absence of downward demand uncertainty, it does not matter whether the designer induces the desired output (under the conjectured model $(D^\star, F^\star)$) by fixing the price or specifying the quantity.

Note that Corollary 2 provides a condition on primitives $(F^\star(\theta)/f^\star(\theta) \geq d(\theta)$ for all $\theta \in \Theta$, where $d(\theta) = P^\star(\underline{D}(\theta)) - \theta)$, under which the Baron-Myerson-with-quantity-floor mechanism is robustly optimal. Proposition 4 then implies that, under the same condition, quantity regulation dominates price regulation. The next result provides further primitive conditions under which each regulation dominates the other.

For every $\theta \in \Theta$, let $\delta(\theta) \equiv D^\star(\theta) - \underline{D}(\theta)$ denote the distance between the conjectured and the lowest *direct demands*, and then let $\underline{\delta} \equiv \inf\{\delta(\theta) : \theta \in \Theta\}$, $\overline{\delta} \equiv \sup\{\delta(\theta) : \theta \in \Theta\}$,

$\underline{f} \equiv \inf\{f^\star(\theta) : \theta \in \Theta\}$ and $\bar{f} \equiv \sup\{f^\star(\theta) : \theta \in \Theta\}$.

Proposition 5 (quantity vs price regulation: markup-based conditions)**.** *Suppose $D^\star$ is bi-Lipschitz continuous over $[P^\star(\bar{\mathrm{q}}), P^\star(0)]$ with constants $0 < k \leq K$.*[20]

(a) Quantity regulation dominates price regulation if, for all $\theta \in \Theta$, $F^\star(\theta)/f^\star(\theta) \geq \delta(\theta)/k$, with the domination strict if $\delta(\overline{\theta}) > 0$.

(b) When, instead, for all $\theta \in \Theta$, $F^\star(\theta)/f^\star(\theta) \leq \delta(\theta)/K[1 + \frac{\bar{\delta}}{\underline{\delta}}\sqrt{2\bar{f}/\underline{f}}]$ and $\underline{\delta} > 0$, then price regulation strictly dominates quantity regulation.

The proposition thus establishes that the type of regulation that dominates depends on the size of the markup that the designer would like to apply to each cost, relative to the difference between the two demands. Quantity regulation dominates when the markups $F^\star(\theta)/f^\star(\theta)$ are large relative to the downstream demand uncertainty $\delta(\theta)$ faced by the regulator. This is because the quantity that the designer would like to procure from each type is small and hence, in the eyes of the regulator, the danger of over-procurement in case the designer's model is wrong is also small.

The second part of the proposition, in turn, establishes that price regulation strictly dominates when the markups are small relative to the difference between the two demands. When this is the case, in the eyes of the regulator, there is significant danger of over-procurement from intermediate-cost types under quantity regulation. As a consequence, if the designer were to opt for quantity regulation, the regulator would require a cut to the quantity procured from intermediate types, as established in the previous section. The designer can avoid these cuts through price regulation (which guards the regulator by letting the quantity adjust to the realized demand). As explained above, the drawback of this regulation, in the eyes of the designer, is that it may result in over-procurement from high-cost types. However, the conditions in part (b) imply that the cost of such over-procurement to the designer is small compared to the cost of reducing the procurement from intermediate-cost sellers under quantity regulation, thus making price regulation superior.

[20] That is, for any $p, p' \in [P^\star(\bar{\mathrm{q}}), P^\star(0)]$, with $p' > p$, $D^\star(p) - D^\star(p') \in [k(p' - p), K(p' - p)]$.

5 Conclusions

We study the problem of a Bayesian designer who must obtain approval from a max-min authority that does not share the designer's confidence in the model and requires satisfactory performance across alternative models.

We conduct the analysis in a procurement setting, where approval by conservative regulators, policymakers, or supervisors is common. We show that robustness reshapes the efficiency-rent-extraction trade-off, calling for increased procurement from high-cost sellers and reduced procurement from intermediate-cost sellers relative to the Bayesian optimum. Quantity floors protect against higher-than-expected costs, while reducing procurement from intermediate-cost sellers guards against demand overestimation. When the good is sold in a downstream market, our approach also delivers a novel comparison of price and quantity regulation: the two instruments provide identical welfare guarantees but yield different payoffs under the designer's conjectured model. Which instrument dominates depends on the desired markups and the extent of demand uncertainty.

We expect the approach to be relevant in other applications. In auction markets, for example, sellers often rely on experts to design mechanisms (e.g., for spectrum allocation), yet rarely share the designers' confidence in the underlying model. This lack of trust may stem from limited understanding of the estimation methods or from concerns that the environment has changed. Similarly, in public good provision, local governments often need approval from state or federal authorities that do not share their confidence in the model used to design the relevant contracts. The same applies to taxation, where reforms of fiscal rules at the local level typically require approval by conservative central authorities. Extending the approach developed here to these settings is a promising direction for future research.

A Appendix

The Appendix is divided into two section. Sections A.1 and A.2 contain the missing proofs for the results from Sections 3 and 4 respectively.

A.1 Proofs of Section 3

Proof of Lemma 1. For any $(V, F) \in \mathcal{V} \times \mathcal{F}$,

$$\int [V(q(\theta)) - \theta q(\theta) - u(\theta)]\, F(\mathrm{d}\theta) \geq \inf_{\theta \in \Theta} \{\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta)\},$$

where the inequality follows from the definition of $\underline{V}$. Hence, $G(M) \geq \inf_{\theta\in\Theta}\{\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta)\}$. Because $\underline{V} \in \mathcal{V}$ and, for each θ, the Dirac distribution that puts probability mass one at θ is in $\mathcal{F}$, $G(M) \leq \inf_{\theta\in\Theta}\{\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta)\}$. Combining the last two inequalities, we obtain the first part of Condition (2). To see the second part, observe that $G(M) \leq \underline{V}(q(\overline{\theta})) - \overline{\theta} q(\overline{\theta}) - u(\overline{\theta}) \leq G^*$, where the first inequality holds because Nature can always choose $\underline{V} \in \mathcal{V}$ and the Dirac distribution that puts probability mass one at $\overline{\theta}$. The second inequality follows from the fact that $u(\overline{\theta}) \geq 0$ along with the definition of G^*.Q.E.D.

Proof of Lemma 2. First, we show that there exists an IC and IR mechanism that delivers the guarantee upper bound in (2). Consider the constant mechanism $M_L \equiv (q_L, u_L)$ in which $q_L(\theta) = \mathrm{q}_\ell$ and $t_L(\theta) = \overline{\theta}\mathrm{q}_\ell$, for every $\theta \in \Theta$ (yielding a rent $u_L(\theta) = (\overline{\theta} - \theta)\mathrm{q}_\ell$ to each θ). The mechanism M_L is clearly IC and IR. Furthermore, $\inf_\theta \{\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta)\} = G^*$. The first part of Condition (2) in Lemma 1 then implies that $G(M_L) = G^*$. The second part of Condition (2) in Lemma 1 in turn implies that $M_L \in \mathcal{M}^{\mathrm{SL}}$. Therefore, for any $M \in \mathcal{M}^{\mathrm{SL}}$, $G(M) = G^*$.

For a mechanism $M = (q, u)$ to be IC and IR, it must be that q is weakly decreasing and, for all θ, $u(\theta) = u(\overline{\theta}) + \int_\theta^{\overline{\theta}} q(y)dy$, with $u(\overline{\theta}) \geq 0$. Moreover, if $M \in \mathcal{M}^{\mathrm{SL}}$, then, for all θ, $\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta) \geq G^*$. This is possible only if $u(\overline{\theta}) = 0$ (else, the constraint is violated at $\overline{\theta}$) and, for any θ, constraint (3) holds. We conclude that any $M \in \mathcal{M}^{\mathrm{SL}}$ must satisfy the constraint (3).

Now, let M be any IC and IR mechanism with $u(\overline{\theta}) = 0$ satisfying constraint (3). By the first part of Condition (2) in Lemma 1, $G(M) \geq G^*$. Because every mechanism in $\mathcal{M}^{\mathrm{SL}}$ has a welfare guarantee of G^*, we conclude that $M \in \mathcal{M}^{\mathrm{SL}}$. Q.E.D.

Proof of Lemma 3. **Part (1)**. Observe that, for any $\theta \in \Theta$,

$$\underline{V}(q(\theta)) - \theta q(\theta) = \int_\theta^\infty \underline{D}(y)\mathrm{d}y - \underline{\mathrm{DWL}}(\theta, q(\theta)),$$

where, for any q, $\underline{\text{DWL}}(\theta, \text{q}) \equiv \int_{\theta}^{\underline{P}(\text{q})} \big(\underline{D}(y) - \text{q}\big)\text{d}y \geq 0$. The equivalence between the constraints in (3) and (4) follows from this observation together with the fact that $G^* \equiv \underline{V}(\text{q}_\ell) - \overline{\theta}\text{q}_\ell = \underline{V}(\underline{D}(\overline{\theta})) - \overline{\theta}\underline{D}(\overline{\theta}) = \int_{\overline{\theta}}^{\infty} \underline{D}(y)\text{d}y$.

Part (2). That (a) implies (b) is immediate given that, for all θ, $\underline{\text{DWL}}(\theta, q(\theta)) \geq 0$. To complete the proof, it thus suffices to show that, when Condition (4) holds for $\theta \in \{\underline{\theta}, \overline{\theta}\}$ and, in addition, Condition (5) holds for all $\theta \in (\underline{\theta}, \overline{\theta})$, the following inequality

$$\int_{\theta}^{\overline{\theta}} \underline{D}(y)dy - \int_{\theta}^{\overline{\theta}} q(y)dy - \int_{\theta}^{\underline{P}(q(\theta))} \big(\underline{D}(y) - q(\theta)\big)dy \geq 0 \tag{17}$$

holds for all $\theta \in (\underline{\theta}, \overline{\theta})$. We consider two cases, which are covered in Claims 1 and 2 below.

Claim 1. *Suppose that $q(\theta) \leq \underline{D}(\theta)$ and, for all $\theta' \in \Theta$, $\int_{\theta'}^{\overline{\theta}} q(y)\text{d}y \leq \int_{\theta'}^{\overline{\theta}} \underline{D}(y)\text{d}y$. Then the inequality in (17) holds for θ.*

Proof of Claim 1. Observe that $\theta \leq \underline{P}(q(\theta)) \leq \overline{\theta}$. The first inequality follows because $q(\theta) \leq \underline{D}(\theta)$. The second inequality follows from Condition (4) applied to $\theta = \overline{\theta}$ which gives $q(\overline{\theta}) = \text{q}_\ell = \underline{D}(\overline{\theta})$; because q is weakly decreasing, we then have that $q(\theta) \geq \underline{D}(\overline{\theta})$. The left-hand-side of (17) is thus equivalent to

$$\int_{\underline{P}(q(\theta))}^{\overline{\theta}} (\underline{D}(y) - q(y))\text{d}y + \int_{\theta}^{\underline{P}(q(\theta))} (q(\theta) - q(y))\text{d}y. \tag{18}$$

The first integral in (18) is non-negative; this follows from the second condition in Claim 1 applied to $\theta' = \underline{P}(q(\theta))$. The second integral is non-negative because q is weakly decreasing. This completes the proof of Claim 1. ■

Claim 2. *Suppose that $q(\theta) > \underline{D}(\theta)$, and, for all $\theta' \in \Theta$, $\int_{\theta'}^{\overline{\theta}} q(y)\text{d}y \leq \int_{\theta'}^{\overline{\theta}} \underline{D}(y)\text{d}y$. Then the inequality in (17) holds for θ.*

Proof of Claim 2. Because $q(\theta) > \underline{D}(\theta)$, $\underline{P}(q(\theta)) < \theta$, which implies that the left-hand-side of the inequality in (17) is equal to $\int_{\theta}^{\overline{\theta}} (\underline{D}(y) - q(y))\text{d}y - \int_{\underline{P}(q(\theta))}^{\theta} (q(\theta) - \underline{D}(y))\text{d}y$. Let $\theta^\sharp \equiv \inf\{y \leq \theta : q(s) \geq \underline{D}(s) \text{ for all } s \in (y, \theta]\}$ and note that $\theta^\sharp < \theta$. Suppose that $\theta^\sharp > \underline{\theta}$, which implies

that $q(\theta^\sharp) = \underline{D}(\theta^\sharp)$. Applying the second condition in the statement of Claim 2 to $\theta' = \theta^\sharp$, we have that $\int_{\theta^\sharp}^{\bar{\theta}} (\underline{D}(y) - q(y))\, dy \geq 0$. Hence, $\int_{\theta}^{\bar{\theta}} (\underline{D}(y) - q(y))\, dy \geq \int_{\theta^\sharp}^{\theta} (q(y) - \underline{D}(y))\, dy$. It follows that

$$\int_{\theta}^{\bar{\theta}} (\underline{D}(y) - q(y))\, dy - \int_{\underline{P}(q(\theta))}^{\theta} \big(q(\theta) - \underline{D}(y)\big) dy \geq \int_{\theta^\sharp}^{\theta} (q(y) - \underline{D}(y))\, dy - \int_{\underline{P}(q(\theta))}^{\theta} \big(q(\theta) - \underline{D}(y)\big) dy. \tag{19}$$

Because $q(\theta) \leq q(\theta^\sharp) = \underline{D}(\theta^\sharp)$, we have that $\underline{P}(q(\theta)) \geq \theta^\sharp$. This property, together with the fact that $q(y) > \underline{D}(y)$ for all $y \in (\theta^\sharp, \theta)$, implies that $\int_{\theta^\sharp}^{\theta} (q(y) - \underline{D}(y)) dy \geq \int_{\underline{P}(q(\theta))}^{\theta} (q(y) - \underline{D}(y)) dy$. In turn, this means that the right-hand-side of the inequality in (19) is greater than

$$\int_{\underline{P}(q(\theta))}^{\theta} (q(y) - \underline{D}(y))\, dy - \int_{\underline{P}(q(\theta))}^{\theta} \big(q(\theta) - \underline{D}(y)\big) dy = \int_{\underline{P}(q(\theta))}^{\theta} (q(y) - q(\theta))\, dy \geq 0,$$

where the last inequality follows from the fact that q is weakly decreasing. We thus conclude that the inequality in (17) holds, as claimed.

Next, suppose that $\theta^\sharp = \underline{\theta}$. That Condition (4) holds for $\theta = \underline{\theta}$ means that $\underline{W}(\underline{\theta}, q) \geq G^*$, where, for any $\theta \in \Theta$,

$$\underline{W}(\theta, q) \equiv \underline{V}(q(\theta)) - \theta q(\theta) - \int_{\theta}^{\bar{\theta}} q(y) dy. \tag{20}$$

Because $q(y) \geq \underline{D}(y)$ for all $y \in [\underline{\theta}, \theta]$, the function $\underline{W}(\cdot, q)$ is weakly increasing over $[\underline{\theta}, \theta]$ (see Lemma 5 below) and hence $\underline{W}(\theta, q) \geq \underline{W}(\underline{\theta}, q) \geq G^*$. As shown above, this means that (17) holds. This completes the proof of Claim 2. ■

Claims 1 and 2 cover all cases and jointly establish Lemma 3. Q.E.D.

Lemma 5 (monotonicity)**.** *Suppose $M \equiv (q, u)$ is an IC mechanism and $I \subseteq \Theta$ is an interval. Let $\underline{W}(\cdot, q)$ be the function defined, for all $\theta \in \Theta$, by (20).*

[A]. Suppose $0 < q(\theta) \leq \underline{D}(\theta)$ for all $\theta \in I$. Then $\underline{W}(\cdot, q)$ is weakly decreasing over I. If, in addition, q is decreasing with $q(\theta) < \underline{D}(\theta)$ for all $\theta \in I$, then $\underline{W}(\cdot, q)$ is decreasing over I.

[B]. Suppose $q(\theta) > \underline{D}(\theta)$ for all $\theta \in I$. Then, $\underline{W}(\cdot, q)$ is weakly increasing over I. If, in addition, q is decreasing over I, then $\underline{W}(\cdot, q)$ is increasing over I.

Proof of Lemma 5. Pick $\theta, \theta' \in I$, with $\theta' < \theta$. Note that

$$\underline{W}(\theta', q) - \underline{W}(\theta, q) = \int_{q(\theta)}^{q(\theta')} \underline{P}(y)dy - \theta' q(\theta') + \theta q(\theta) - \int_{\theta'}^{\theta} q(y)dy. \tag{21}$$

Proof of Part (A). We consider two cases.

Case 1: $\underline{P}(q(\theta')) \geq \theta > \theta'$. Note that the right-hand-side of (21) is equal to

$$\int_{q(\theta)}^{q(\theta')} (\underline{P}(y) - \theta)dy + \left\{ (\theta - \theta')q(\theta') - \int_{\theta'}^{\theta} q(y)dy \right\}. \tag{22}$$

The first term in (22) is non-negative because, for all $y \in (q(\theta), q(\theta'))$, $\underline{P}(y) > \underline{P}(q(\theta')) \geq \theta$, which follows from $\underline{P}$ being decreasing. Furthermore, if $q(\theta') > q(\theta)$, then the first term in (22) is positive. Next, observe that, because q is weakly decreasing, the expression in curly brackets in (22) is non-negative. Thus, $\underline{W}(\theta', q) \geq \underline{W}(\theta, q)$, i.e., $\underline{W}(\cdot, q)$ is weakly decreasing over I (decreasing when q is decreasing over I).

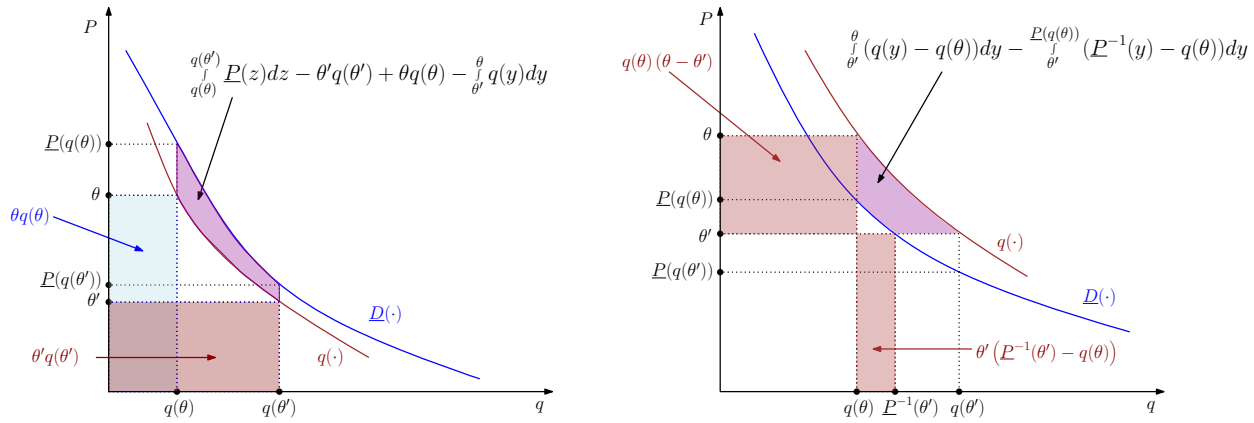
Case 2: $\theta > \underline{P}(q(\theta')) \geq \theta'$. Use Panel A in Figure 5 to observe that the sum of the first four terms in (21) is equal to

$$\int_{\theta'}^{\underline{P}(q(\theta'))} \Big(q(\theta') - q(y)\Big)dy + \int_{\underline{P}(q(\theta'))}^{\theta} \Big(\underline{D}(y) - q(y)\Big)dy + \int_{\theta}^{\underline{P}(q(\theta))} \Big(\underline{D}(y) - q(\theta)\Big)dy. \tag{23}$$

The first term in (23) is non-negative because q is weakly decreasing. Next observe that, for all $y \in (\underline{P}(q(\theta')), \theta)$, $\underline{D}(y) \geq q(y)$. Hence, the second term in (23) is also non-negative. Finally, the last term in (23) is also non-negative because, for any $y \in (\theta, \underline{P}(q(\theta)))$, $\underline{D}(y) \geq q(\theta)$, which follows from $\underline{P}$ being decreasing. We conclude that $\underline{W}(\cdot, q)$ is weakly decreasing over I (decreasing when q is decreasing and such that $q(y) < \underline{D}(y)$ for all $y \in I$).

Proof of Part (B): The difference in welfare $\underline{W}(\theta, q) - \underline{W}(\theta', q)$ across the two types is given by the (negative of) expression in (21), which can be rewritten as

$$\underline{W}(\theta, q) - \underline{W}(\theta', q) = \int_{\theta'}^{\theta} \Big(q(y) - q(\theta)\Big)dy + \int_{q(\theta)}^{q(\theta')} \Big(\theta' - \underline{P}(z)\Big)dz. \tag{24}$$



A: Case 2 in Part A.

B: Case 2 in Part B.

Figure 5: Illustration for Lemma 5

We consider two cases.

Case 1: $\underline{P}(q(\theta)) \leq \theta'$. In this case, $\underline{P}(z) < \theta'$ for all $z > q(\theta)$. This implies that the second integral in (24) is non-negative. The first integral is non-negative because q is weakly decreasing. If q is decreasing, both integrals are positive.

Case 2: $\theta' < \underline{P}(q(\theta)) < \theta$. We then have that $q(\theta') > \underline{D}(\theta') > q(\theta)$. Hence, using (24), we have that

$$\underline{W}(\theta, q) - \underline{W}(\theta', q) \geq \int_{\theta'}^{\theta} \Big(q(y) - q(\theta)\Big)dy - \int_{q(\theta)}^{\underline{D}(\theta')} \Big(\underline{P}(z) - \theta'\Big)dz. \tag{25}$$

Changing the variable of integration, the second integral can be written as

$$\int_{q(\theta)}^{\underline{D}(\theta')} \Big(\underline{P}(z) - \theta'\Big)dz = \int_{\theta'}^{\underline{P}(q(\theta))} \Big(\underline{D}(y) - q(\theta)\Big)dy.$$

Thus, the right-hand-side of (25) reduces to (see Panel B in Figure 5 for an illustration)

$$\int_{\theta'}^{\theta} \Big(q(y) - q(\theta)\Big)dy - \int_{\theta'}^{\underline{P}(q(\theta))} \Big(\underline{D}(y) - q(\theta)\Big)dy, \tag{26}$$

which is non-negative because $\underline{P}(q(\theta)) < \theta$ and $\underline{D}(y) < q(y)$ for all $y \in I$.

Thus, $\underline{W}(\theta, q) \geq \underline{W}(\theta', q)$, i.e., $\underline{W}(\cdot, q)$ is weakly increasing over I. The inequality above also reveals that, when q is decreasing, the expression in (26) is positive, implying that

$\underline{W}(\cdot, q)$ is increasing over I. This completes the proof of the lemma. Q.E.D.

Proof of Proposition 1. By Observation 1, if $M = (q, u) \in \mathcal{M}^{\text{SL}}$, then $q(\theta) \geq \text{q}_\ell$ for all $\theta \in \Theta$. The following is thus a relaxation of problem (**ROPT**):

$$\max_{q} \int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q(\theta)) - z^\star(\theta) q(\theta) \Big] F^\star(\text{d}\theta) \qquad \textbf{(RP)}$$

$$\text{subject to } q \text{ weakly decreasing and } q(\theta) \geq \text{q}_\ell \ \forall\ \theta \in \Theta.$$

That the quantity schedule $q^\star$ satisfies the constraints in the relaxed problem (**RP**) follows from the fact that, when $F^\star$ is regular, q^{BM} is decreasing and hence $q^\star$ is weakly decreasing. That $q^\star$ also satisfies the other constraint of the relaxed problem follows directly from its definition. Next, observe that, for any θ, the function $V^\star(\text{q}) - z^\star(\theta)\text{q}$ is concave in q and attains a maximum at $q^{\text{BM}}(\theta)$. Therefore, the quantity schedule $q^\star$ maximizes the objective function in the relaxed program over all weakly decreasing functions q satisfying $q(\theta) \geq \text{q}_\ell$ for all θ.

To complete the proof, it suffices to show that $q^\star$ satisfies the robustness constraints in (3) if and only if Conditions (8) and (9) in the proposition hold. This follows from Lemma 3 along with the continuity of $q^\star$ and $\underline{D}$ which implies that, if $q^\star$ satisfies Condition (9), then $q^\star(\overline{\theta}) = \text{q}_\ell$. Q.E.D.

Proof of Proposition 2. Part 1. The result follows from Lemmas 6-9 below.

Lemma 6. *Every $M^{\text{OPT}} \equiv (q^{\text{OPT}}, u^{\text{OPT}})$ is such that $q^{\text{OPT}}(\theta) \leq q^\star(\theta)$ for all $\theta \in (\underline{\theta}, \overline{\theta}]$ and $q^{\text{OPT}}(\theta) = \text{q}_\ell$ for all $\theta \in [\theta^\star, \overline{\theta}]$.*

Proof of Lemma 6. By Observation 1, $q^{\text{OPT}}(\overline{\theta}) = \text{q}_\ell$. Assume, towards a contradiction, that $q^{\text{OPT}}(\theta) > q^\star(\theta)$ for some $\theta \in (\underline{\theta}, \overline{\theta})$. The continuity of $q^\star$ over $(\underline{\theta}, \overline{\theta})$ along with the fact that q^{OPT} is weakly decreasing, imply that there is a positive measure set of types $I \subseteq (\underline{\theta}, \overline{\theta})$ such that $q^{\text{OPT}}(y) > q^\star(y)$ for all $y \in I$. Then consider the quantity schedule $\tilde{q}(y) = \min\{q^{\text{OPT}}(y), q^\star(y)\}$ for all $y \in (\underline{\theta}, \overline{\theta})$ with $\tilde{q}(y) = q^{\text{OPT}}(y)$ for all $y \in \{\underline{\theta}, \overline{\theta}\}$. Clearly, $\tilde{q}$ is weakly decreasing. Furthermore, because q^{OPT} satisfies constraint (4) for $\theta \in \{\underline{\theta}, \overline{\theta}\}$, and constraint (5) for all $\theta \in (\underline{\theta}, \overline{\theta})$, and because $\tilde{q}(y) \leq q^{\text{OPT}}(y)$ for all y, with equality at $\underline{\theta}$ and $\overline{\theta}$, $\tilde{q}$ also satisfies constraint (4) for $\theta \in \{\underline{\theta}, \overline{\theta}\}$, and constraint (5) for all $\theta \in (\underline{\theta}, \overline{\theta})$. By Lemma 3, this means that $\tilde{q}$ satisfies the robustness constraint in (**ROPT**) for all $\theta \in \Theta$.

Next observe that (a) $q^{\mathrm{BM}}(y) \leq q^\star(y)$ for all $y \in \Theta$ and (b) q^{BM} maximizes the objective function in (**ROPT**) point-wise. The concavity of $V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}$ in q then implies that the objective function in (**ROPT**) is strictly higher under $\tilde{q}$ than under q^{OPT}, where the strict inequality follows because I has positive Lebesgue measure. This contradicts the optimality of q^{OPT}. We conclude that $q^{\mathrm{OPT}}(\theta) \leq q^\star(\theta)$ for all $\theta \in (\underline{\theta}, \overline{\theta}]$. Because $q^\star(\theta) = \mathrm{q}_\ell$ for all $\theta \in [\theta^\star, \overline{\theta}]$, we thus have that $q^{\mathrm{OPT}}(\theta) \leq \mathrm{q}_\ell$ for all $\theta \in [\theta^\star, \overline{\theta}]$. However, by Observation 1, $q^{\mathrm{OPT}}(\overline{\theta}) = \mathrm{q}_\ell$. Because q^{OPT} is weakly decreasing, it must be that $q^{\mathrm{OPT}}(\theta) = \mathrm{q}_\ell$ for all $\theta \in [\theta^\star, \overline{\theta}]$. ■

The next lemma compares θ^m and $\theta^\star$ when $M^\star$ is not robustly optimal.

Lemma 7. *The mechanism $M^\star$ is robustly optimal if and only if $\theta^m = \overline{\theta}$. Furthermore, if $M^\star$ is not robustly optimal, then $\theta^m < \theta^\star$.*

Proof of Lemma 7. Suppose $M^\star$ is robustly optimal. Then, $M^\star \in \mathcal{M}^{\mathrm{SL}}$. Observation 1 then implies that $q^\star(\overline{\theta}) = \mathrm{q}_\ell$. This implies that $\underline{W}(\overline{\theta}, q^\star) = G^*$. Lemma 2 in turn implies that $\underline{W}(\theta, q^\star) \geq G^*$ for all $\theta \in \Theta$. Hence, $\underline{W}(\theta, q^\star) \geq \underline{W}(\overline{\theta}, q^\star)$ for all θ. This implies that $\theta^m = \overline{\theta}$.

For the converse, suppose that $\theta^m = \overline{\theta}$. Then, by definition of θ^m, $\underline{W}(\theta, q^\star) \geq \underline{W}(\overline{\theta}, q^\star)$ for all $\theta \in \Theta$. This implies that $q^\star(\overline{\theta}) = \mathrm{q}_\ell$. To see this, suppose, towards a contradiction, that $q^\star(\overline{\theta}) > \mathrm{q}_\ell$. This means that $q^\star(\overline{\theta}) = q^{\mathrm{BM}}(\overline{\theta}) > \mathrm{q}_\ell = \underline{D}(\overline{\theta})$. By continuity of q^{BM} and $\underline{D}$, there exists $\epsilon > 0$ such that $q^\star(\theta) > \underline{D}(\theta)$ for all $\theta \in (\overline{\theta} - \epsilon, \overline{\theta}]$. Because q^{BM} is decreasing over $(\overline{\theta} - \epsilon, \overline{\theta}]$, Lemma 5 implies that $\underline{W}(\cdot, q^\star)$ is increasing over $(\overline{\theta} - \epsilon, \overline{\theta}]$, a contradiction to the assumption that $\theta^m = \overline{\theta}$. Thus, $q^\star(\overline{\theta}) = \mathrm{q}_\ell$. But then $\underline{W}(\overline{\theta}, q^\star) = G^*$. Together with the definition of $\theta^m = \overline{\theta}$, this implies that $\underline{W}(\theta, q^\star) \geq G^*$ for all $\theta \in \Theta$. Hence, $M^\star \in \mathcal{M}^{\mathrm{SL}}$. Proposition 1 then implies that $M^\star$ is robustly optimal.

Finally, to see that, when $M^\star$ is not robustly optimal, $\theta^m < \theta^\star$, observe that, when $\theta^\star = \overline{\theta}$, the result follows from the fact that $\theta^m < \overline{\theta}$. When, instead, $\theta^\star < \overline{\theta}$, then $\underline{W}(\theta, q^\star) = G^*$ for all $\theta \in [\theta^\star, \overline{\theta}]$. Because $M^\star$ is not robustly optimal, it must be that $\underline{W}(\theta^m, q^\star) < G^*$. Hence, $\theta^m < \theta^\star$. ■

The next Lemma compares $q^\star(\theta^m)$ with $\underline{D}(\theta^m)$.

Lemma 8. *Suppose $M^\star$ is not robustly optimal. Then, $q^\star(\theta^m) \geq \underline{D}(\theta^m)$, with the inequality holding as equality if $\theta^m > \underline{\theta}$.*

Proof of Lemma 8. By Lemma 7, $\theta^m < \theta^\star$. This means that $q^\star(\theta^m) = q^{\mathrm{BM}}(\theta^m)$. Assume,

towards a contradiction, that $q^{\mathrm{BM}}(\theta^m) < \underline{D}(\theta^m)$. Because q^{BM} and $\underline{D}$ are continuous and $\theta^m < \theta^\star$, there exists $\epsilon > 0$ such that $\theta^m + \epsilon < \theta^\star$ and $q^{\mathrm{BM}}(\theta) < \underline{D}(\theta)$ for all $\theta \in [\theta^m, \theta^m + \epsilon]$. Because q^{BM} (and thus $q^\star$) is decreasing on this interval, Lemma 5 implies that $\underline{W}(\cdot, q^\star)$ is decreasing on $[\theta^m, \theta^m + \epsilon]$. This contradicts the definition of θ^m. Hence, $q^\star(\theta^m) \geq \underline{D}(\theta^m)$.

By the same argument, if $\theta^m > \underline{\theta}$ and $q^\star(\theta^m) > \underline{D}(\theta^m)$, there exists $\epsilon > 0$ such that $\underline{W}(\cdot, q^\star)$ is increasing on $[\theta^m - \epsilon, \theta^m]$, contradicting the definition of θ^m. Hence, $q^\star(\theta^m) = \underline{D}(\theta^m)$ if $\theta^m > \underline{\theta}$. ■

The next lemma uses the property in Lemma 8 to complete the proof of Part 1 by establishing that q^{OPT} and q^{BM} agree on $(\underline{\theta}, \theta^m)$.

Lemma 9. *Suppose $M^\star$ is not robustly optimal. If $\theta^m > \underline{\theta}$, then in every robustly optimal mechanism, $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta^m)$.*

Proof of Lemma 9. Because $M^\star$ is not robustly optimal, by Lemma 7, $\theta^m < \theta^\star$. Hence, $q^\star(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta^m)$. Furthermore, by Lemma 6, $q^{\mathrm{OPT}}(\theta) \leq q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta^m)$. Now, assume, towards a contradiction, that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for some $\theta \in (\underline{\theta}, \theta^m)$. By continuity of q^{BM} and the fact that q^{OPT} is weakly decreasing, there exists a set of types $I \subseteq (\underline{\theta}, \theta^m)$ of positive Lebesgue measure such that $q^{\mathrm{OPT}}(y) < q^{\mathrm{BM}}(y)$ for all $y \in I$. Let $\widetilde{M} \equiv (\tilde{q}, \tilde{u})$ be the mechanism with quantity schedule given by

$$\tilde{q}(\theta) = \begin{cases} q^{\mathrm{BM}}(\theta) & \text{if } \theta \in [\underline{\theta}, \theta^m) \\ q^{\mathrm{OPT}}(\theta) & \text{otherwise} \end{cases}$$

and $\tilde{u}(\theta) = \int_{\theta}^{\overline{\theta}} \tilde{q}(y)\mathrm{d}y$ for all $\theta \in \Theta$. By Lemma 6, $q^{\mathrm{OPT}}(\theta^m) \leq q^{\mathrm{BM}}(\theta^m)$, and hence $\tilde{q}$ is weakly decreasing. Furthermore, because $\tilde{u}$ satisfies the above envelope formula with $\tilde{u}(\overline{\theta}) = 0$, $\widetilde{M}$ is IC and IR. Because q^{BM} maximizes the objective function of (**ROPT**) point-wise and, for any θ, $V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}$ in strictly concave in q, and $I \subseteq (\underline{\theta}, \theta^m)$ has positive Lebesgue measure, the objective function in (**ROPT**) is strictly higher under $\tilde{q}$ than under q^{OPT}. Below we show that $\tilde{q}$ satisfies the robustness constraints in (**ROPT**), thus contradicting the robust optimality of q^{OPT}. Clearly, $\tilde{q}$ satisfies the robustness constraints in (**ROPT**) for all $\theta \in [\theta^m, \overline{\theta})$ because $\tilde{q}$ agrees with q^{OPT} on $[\theta^m, \overline{\theta})$ and q^{OPT} satisfies these constraints by

its definition. Now, pick $\theta \in [\underline{\theta}, \theta^m)$. Then, $\tilde{q}(\theta) = q^{\mathrm{BM}}(\theta) = q^\star(\theta)$. Hence,

$$\begin{aligned}
\underline{W}(\theta, \tilde{q}) - \underline{W}(\theta, q^\star) &= \int_{\theta^m}^{\overline{\theta}} q^\star(y)\mathrm{d}y - \int_{\theta^m}^{\overline{\theta}} q^{\mathrm{OPT}}(y)\mathrm{d}y \\
&\geq \left[\underline{V}(q^{\mathrm{OPT}}(\theta^m)) - \theta^m q^{\mathrm{OPT}}(\theta^m)\right] - \left[\underline{V}(q^\star(\theta^m)) - \theta^m q^\star(\theta^m)\right] \\
&+ \int_{\theta^m}^{\overline{\theta}} q^\star(y)\mathrm{d}y - \int_{\theta^m}^{\overline{\theta}} q^{\mathrm{OPT}}(y)\mathrm{d}y \\
&= \underline{W}(\theta^m, q^{\mathrm{OPT}}) - \underline{W}(\theta^m, q^\star).
\end{aligned}$$

The first equality follows from the definition of $\tilde{q}$. The inequality follows because (a) $q^\star(\theta^m) = \underline{D}(\theta^m)$ (by Lemma 8) and (b) $\underline{V}(\mathrm{q}) - \theta^m \mathrm{q}$ is maximized at $\underline{D}(\theta^m)$. The last equality is by definition of the $\underline{W}$ function. As a consequence, $\underline{W}(\theta, \tilde{q}) - \underline{W}(\theta^m, q^{\mathrm{OPT}}) \geq \underline{W}(\theta, q^\star) - \underline{W}(\theta^m, q^\star) \geq 0$, where the inequality follows from the definition of θ^m. Hence, $\underline{W}(\theta, \tilde{q}) \geq \underline{W}(\theta^m, q^{\mathrm{OPT}}) \geq G^*$, where the last inequality follows from the fact that q^{OPT} is a solution to (**ROPT**) and hence satisfies the robustness constraints in (**ROPT**) for all θ. We conclude that $\underline{W}(\theta, \tilde{q}) \geq G^*$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$, implying that $\tilde{q}$ satisfies the robustness constraints in (**ROPT**) for all $\theta \in \Theta$, thus contradicting the robust optimality of q^{OPT}. ■

Part 2. Let

$$\theta^{\star\star} = \inf\{\theta \in [\theta^m, \overline{\theta}] : q^{\mathrm{OPT}}(y) = \underline{D}(y) \ \forall\ y \in [\theta, \overline{\theta}]\}.$$

By Observation 1, $q^{\mathrm{OPT}}(\overline{\theta}) = \mathrm{q}_\ell = \underline{D}(\overline{\theta})$ and therefore $\theta^{\star\star}$ is well defined. Furthermore, when $\theta^\star < \overline{\theta}$, Lemma 6 along with the assumption that $\underline{D}$ is decreasing implies that $\theta^{\star\star} = \overline{\theta}$. If $\theta^{\star\star} = \underline{\theta}$, then by the definition of $\theta^{\star\star}$, $\theta^m = \underline{\theta}$ and there is nothing to prove. Hence, we will assume that $\theta^{\star\star} > \underline{\theta}$.

Lemma 10. *If $\theta^{\star\star} > \underline{\theta}$, then $q^{\mathrm{OPT}}(\theta^{\star\star}) = \underline{D}(\theta^{\star\star})$.*

Proof of Lemma 10. If $\theta^{\star\star} = \overline{\theta}$, the result follows from the fact that $q^{\mathrm{OPT}}(\overline{\theta}) = \underline{D}(\overline{\theta})$. Thus suppose $\theta^{\star\star} < \overline{\theta}$. Because q^{OPT} is weakly decreasing and $\underline{D}$ is continuous, it cannot be that $q^{\mathrm{OPT}}(\theta^{\star\star}) < \underline{D}(\theta^{\star\star})$. The same properties imply that, if $q^{\mathrm{OPT}}(\theta^{\star\star}) > \underline{D}(\theta^{\star\star})$, there exists $\epsilon > 0$ such that $q^{\mathrm{OPT}}(\theta) > \underline{D}(\theta)$ for all $\theta \in [\theta^{\star\star} - \epsilon, \theta^{\star\star}]$. This however implies that, for any $\theta \in [\theta^{\star\star} - \epsilon, \theta^{\star\star})$, $\int_\theta^{\overline{\theta}} q^{\mathrm{OPT}}(y)\mathrm{d}y > \int_\theta^{\overline{\theta}} \underline{D}(y)\mathrm{d}y$ which, by virtue of Lemma 3, contradicts

the assumption that q^{OPT} satisfies the robustness constraints. Hence, $q^{\mathrm{OPT}}(\theta^{\star\star}) = \underline{D}(\theta^{\star\star})$. ■

If $\theta^{\star\star} = \theta^m$, the result in Part 2 thus follows directly from Lemma 10. If, instead, $\theta^{\star\star} > \theta^m$, the result follows from the next lemma.

Lemma 11. *Suppose $M^\star$ is not robustly optimal and $\theta^{\star\star} > \theta^m$. Then, there exists a set of types $I \subseteq (\theta^m, \min\{\theta^\star, \theta^{\star\star}\})$ of positive Lebesgue measure such that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in I$.*

Proof of Lemma 11. We consider two cases.

Case 1: $\theta^\star < \overline{\theta}$. As explained above, in this case, $\min\{\theta^\star, \theta^{\star\star}\} = \theta^\star$. Furthermore, by Lemma 7, $\theta^m < \theta^\star$. Lemma 6 in turn implies that, for all $\theta \in (\theta^m, \theta^\star)$, $q^{\mathrm{OPT}}(\theta) \leq q^\star(\theta) = q^{\mathrm{BM}}(\theta)$, whereas, from Lemma 9, $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta^m)$. Now suppose, towards a contradiction, that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for almost all $\theta \in (\underline{\theta}, \theta^\star)$. Because q^{BM} is continuous and q^{OPT} and q^{BM} are weakly decreasing, it must be that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta^\star)$. Lemma 6 then implies that $q^{\mathrm{OPT}}(\theta) = q^\star(\theta)$ for all $\theta > \underline{\theta}$. Furthermore, because $M^\star$ is not robustly optimal, $\underline{W}(\theta^m, q^\star) < G^*$. Because $\theta^m < \theta^\star$ and $\underline{W}(\cdot, q^\star)$ is continuous, there must exist some $\theta' > \theta^m$ such that $\underline{W}(\theta', q^\star) < G^*$. Because $q^{\mathrm{OPT}}(\theta) = q^\star(\theta)$ for all $\theta > \underline{\theta}$, $\underline{W}(\theta', q^{\mathrm{OPT}}) = \underline{W}(\theta', q^\star) < G^*$, which contradicts the assumption that q^{OPT} satisfies the robustness constraints of (**ROPT**). We thus conclude that there must exist a set of types $I \subseteq (\theta^m, \theta^\star)$ of positive Lebesgue measure such that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in I$, as claimed.

Case 2: $\theta^\star = \overline{\theta}$. In this case, $\min\{\theta^\star, \theta^{\star\star}\} = \theta^{\star\star}$. Again, assume towards a contradiction that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for almost all $\theta \in (\theta^m, \theta^{\star\star})$.

First, suppose that $\theta^m > \underline{\theta}$. Lemma 9 then implies that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta^m)$. Because q^{BM} is continuous and q^{OPT} is weakly decreasing, the assumption that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for almost all $\theta \in (\theta^m, \theta^{\star\star})$ then implies that $q^{\mathrm{OPT}}(\theta^m) = q^{\mathrm{BM}}(\theta^m)$. Next, suppose that $\theta^m = \underline{\theta}$. Again, because q^{BM} is continuous and q^{OPT} is weakly decreasing, the assumption that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for almost all $\theta \in (\theta^m, \theta^{\star\star})$ implies that $q^{\mathrm{OPT}}(\underline{\theta}) \geq q^{\mathrm{BM}}(\underline{\theta})$. Lastly, recall that the assumption that $\theta^\star = \overline{\theta}$ implies that $q^{\mathrm{BM}}(\theta) = q^\star(\theta)$ for all $\theta \in (\underline{\theta}, \theta^{\star\star})$.

The assumption that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for almost all $\theta \in (\theta^m, \theta^{\star\star})$ then implies that

$$\underline{W}(\theta^m, q^{\mathrm{OPT}}) - \underline{W}(\theta^m, q^\star) = \Big[\underline{V}(q^{\mathrm{OPT}}(\theta^m)) - \theta^m q^{\mathrm{OPT}}(\theta^m)\Big] - \Big[\underline{V}(q^{\mathrm{BM}}(\theta^m)) - \theta^m q^{\mathrm{BM}}(\theta^m)\Big] \tag{27}$$

$$\begin{aligned} &+ \int_{\theta^m}^{\overline{\theta}} \big(q^\star(y) - q^{\mathrm{OPT}}(y)\big)\mathrm{d}y \\ &= \Big[\underline{V}(q^{\mathrm{OPT}}(\theta^m)) - \theta^m q^{\mathrm{OPT}}(\theta^m)\Big] - \Big[\underline{V}(q^{\mathrm{BM}}(\theta^m)) - \theta^m q^{\mathrm{BM}}(\theta^m)\Big] \\ &+ \int_{\theta^{\star\star}}^{\overline{\theta}} \big(q^\star(y) - q^{\mathrm{OPT}}(y)\big)\mathrm{d}y. \end{aligned}$$

If $\theta^m > \underline{\theta}$, the first two bracketed terms in (27) cancel out because, in this case, $q^{\mathrm{BM}}(\theta^m) = q^{\mathrm{OPT}}(\theta^m)$, as established above. If, instead, $\theta^m = \underline{\theta}$, because $q^{\mathrm{BM}}(\underline{\theta}) = D^\star(\underline{\theta}) \geq \underline{D}(\underline{\theta})$, and because, as established above, $q^{\mathrm{OPT}}(\underline{\theta}) \geq q^{\mathrm{BM}}(\underline{\theta})$, by the concavity of $\underline{V}(\mathrm{q}) - \theta^m \mathrm{q}$ in q, the sum of the two square brackets in (27) is non-positive. Hence, no matter whether $\theta^m > \underline{\theta}$ or $\theta^m = \underline{\theta}$, in either case, we have that

$$\underline{W}(\theta^m, q^{\mathrm{OPT}}) - \underline{W}(\theta^m, q^\star) \leq \int_{\theta^{\star\star}}^{\overline{\theta}} \big(q^\star(y) - q^{\mathrm{OPT}}(y)\big)\mathrm{d}y. \tag{28}$$

By Lemma 10, $q^{\mathrm{OPT}}(\theta^{\star\star}) = \underline{D}(\theta^{\star\star})$. Hence,

$$\Big[\underline{V}(q^{\mathrm{OPT}}(\theta^{\star\star})) - \theta^{\star\star} q^{\mathrm{OPT}}(\theta^{\star\star})\Big] - \Big[\underline{V}(q^{\mathrm{BM}}(\theta^{\star\star})) - \theta^{\star\star} q^{\mathrm{BM}}(\theta^{\star\star})\Big] \geq 0, \tag{29}$$

where the inequality follows from the fact that $\underline{D}(\theta^{\star\star})$ maximizes $\underline{V}(\mathrm{q}) - \theta^{\star\star}\mathrm{q}$ over $[0, \bar{\mathrm{q}}]$. Combining (29) with (28), we thus have that

$$\begin{aligned} \underline{W}(\theta^m, q^{\mathrm{OPT}}) - \underline{W}(\theta^m, q^\star) &\leq \Big[\underline{V}(q^{\mathrm{OPT}}(\theta^{\star\star})) - \theta^{\star\star} q^{\mathrm{OPT}}(\theta^{\star\star}) - \int_{\theta^{\star\star}}^{\overline{\theta}} q^{\mathrm{OPT}}(y)\mathrm{d}y\Big] \\ &\quad - \Big[\underline{V}(q^{\mathrm{BM}}(\theta^{\star\star})) - \theta^{\star\star} q^{\mathrm{BM}}(\theta^{\star\star}) - \int_{\theta^{\star\star}}^{\overline{\theta}} q^\star(y)\mathrm{d}y\Big] \\ &= \underline{W}(\theta^{\star\star}, q^{\mathrm{OPT}}) - \underline{W}(\theta^{\star\star}, q^\star). \end{aligned}$$

Because $\theta^m < \theta^{\star\star}$, $\underline{W}(\theta^m, q^\star) < \underline{W}(\theta^{\star\star}, q^\star)$. We thus have that $\underline{W}(\theta^m, q^{\text{OPT}}) < \underline{W}(\theta^{\star\star}, q^{\text{OPT}})$. The definition of $\theta^{\star\star}$, along with Lemma 10, in turn imply that $q^{\text{OPT}}(\theta) = \underline{D}(\theta)$ for all $\theta \geq \theta^{\star\star}$. Because $\underline{W}(\overline{\theta}, q^{\text{OPT}}) = G^*$, this last property in turn implies that $\underline{W}(\theta^{\star\star}, q^{\text{OPT}}) = G^*$. Hence, $\underline{W}(\theta^m, q^{\text{OPT}}) < G^*$, a contradiction to the robust optimality of q^{OPT}. We thus conclude that there must exist a set of types $I \subseteq (\theta^m, \theta^{\star\star})$ of positive Lebesgue measure such that $q^{\text{OPT}}(\theta) < q^{\text{BM}}(\theta)$ for all $\theta \in I$. ■

This completes the proof of Proposition 2. Q.E.D.

A.2 Proofs of Section 4

Proof of Lemma 4. First, we show that $G(\widetilde{M}) \leq G^*$ for any $\widetilde{M} \equiv (p, t) \in \widetilde{\mathcal{M}}$. To establish this, we represent the mechanism as $\widetilde{M} \equiv (p, \tilde{u})$, where $\tilde{u}(\theta, D) \equiv t(\theta, D) - \theta D(p(\theta))$ for all $\theta \in \Theta$ and $D \in \mathcal{D}$. Observe that, for any $\widetilde{M} \in \widetilde{\mathcal{M}}$,

$$\begin{aligned} G(\widetilde{M}) \leq_{(i)} \underline{V}(\underline{D}(p(\overline{\theta}))) - \overline{\theta}\underline{D}(p(\overline{\theta})) - \tilde{u}(\overline{\theta}, \underline{D}) &\leq_{(ii)} \underline{V}(\underline{D}(p(\overline{\theta}))) - \overline{\theta}\underline{D}(p(\overline{\theta})) \\ &\leq_{(iii)} \underline{V}(\underline{D}(\overline{\theta})) - \overline{\theta}\underline{D}(\overline{\theta}) = G^*. \end{aligned}$$

Inequality (i) holds because the right-hand-side is just expected welfare under a Dirac distribution that puts probability one at $\overline{\theta}$, when the demand is $\underline{D}$. Inequality (ii) holds because $\tilde{u}(\overline{\theta}, \underline{D}) \geq 0$ as $\widetilde{M}$ is EPIR. Inequality (iii) holds because $\overline{\theta} = \arg\max_{\text{p}} \left\{\underline{V}(\underline{D}(\text{p})) - \overline{\theta}\underline{D}(\text{p})\right\}$.

Next, to prove that $G(\widetilde{M}) = G^*$ for any $\widetilde{M} \in \widetilde{\mathcal{M}}^{\text{SL}}$, it suffices to note that there exists a mechanism $\underline{\widetilde{M}} \in \widetilde{\mathcal{M}}$ such that $G(\underline{\widetilde{M}}) = G^*$. Let $\underline{\widetilde{M}} \equiv (p, t)$ be the price mechanism such that $p(\theta) = \overline{\theta}$ for all $\theta \in \Theta$, and where $t(\theta, D)$ satisfies Condition (12) with $\tilde{u}(\overline{\theta}, D) = 0$ for all $\theta \in \Theta$, all $D \in \mathcal{D}$. Under such a mechanism, for all $\theta \in \Theta$, all $D \in \mathcal{D}$, welfare is equal to

$$V(D(\overline{\theta})) - \theta D(\overline{\theta}) - \int_{\theta}^{\overline{\theta}} D(\overline{\theta}) dy = V(D(\overline{\theta})) - \overline{\theta} D(\overline{\theta}) \geq \underline{V}(\underline{D}(\overline{\theta})) - \overline{\theta}\underline{D}(\overline{\theta}) = G^*.$$

This means that, for all $F \in \mathcal{F}$ and $D \in \mathcal{D}$, $\widetilde{W}(\underline{\widetilde{M}}; D, F) \geq G^*$, and hence, $G(\underline{\widetilde{M}}) = G^*$.

We now prove that, if $\widetilde{M} \equiv (p, \tilde{u}) \in \widetilde{\mathcal{M}}^{\text{SL}}$, Conditions (a)-(d) in the lemma must hold. Because $\widetilde{M}$ is EPIC and EPIR, p is weakly increasing and $\tilde{u}$ satisfies Condition (12) for all $\theta \in \Theta$ and $D \in \mathcal{D}$, with $\tilde{u}(\overline{\theta}, D) \geq 0$. That $p(\overline{\theta}) = \overline{\theta}$ and $\tilde{u}(\overline{\theta}, \underline{D}) = 0$ follows from the fact that the only way welfare can be equal to G^* when Nature selects $D = \underline{D}$ and a technology

that selects $\theta = \overline{\theta}$ with probability one is by inducing efficient output by setting a price $p(\overline{\theta}) = \overline{\theta}$ and giving no rent to the monopolist, which amounts to setting $\tilde{u}(\overline{\theta}, \underline{D}) = 0$. That $\tilde{w}(\theta, \widetilde{M}; D) \geq G^*$ must hold for all $\theta \in \Theta$ and all $D \in \mathcal{D}$ follows from the fact that, if this is not the case, then $G(\widetilde{M}) < G^*$, a contradiction to $\widetilde{M} \in \widetilde{\mathcal{M}}^{\text{SL}}$.

Finally, to see that, jointly, Conditions (a)-(d) imply that $\widetilde{M} \in \widetilde{\mathcal{M}}^{\text{SL}}$, observe that Conditions (a)-(c) imply that $\widetilde{M}$ is EPIC and EPIR. Furthermore, Condition (d) implies that $G(\widetilde{M}) \geq G^*$. Since $G(\widetilde{M}) \leq G^*$, we conclude $G(\widetilde{M}) = G^*$. The first part of the lemma then implies that $\widetilde{M} \in \widetilde{\mathcal{M}}^{\text{SL}}$. Q.E.D.

Proof of Proposition 3. By standard arguments, for any EPIC and EPIR price regulation $\widetilde{M} \equiv (p, t)$, expected welfare under the conjectured model $(D^\star, F^\star)$ is equal to

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(D^\star(p(\theta))) - z^\star(\theta) D^\star(p(\theta)) \Big] F^\star(\mathrm{d}\theta) - \widetilde{u}(\overline{\theta}, D^\star).$$

Any robustly optimal price regulation thus maximizes this expression subject to Conditions (a)-(d) in Lemma 4.

Consider a relaxed program where (1) Conditions (a) and (c) in Lemma 4 are replaced by the requirement that $p(\theta) \leq \overline{\theta}$ for all $\theta \in \Theta$, (2) the constraints on $\{\tilde{u}(\theta, D)\}_{D \in \mathcal{D}}$ in Condition (b) in Lemma 4 are replaced by the requirement that $\widetilde{u}(\overline{\theta}, D^\star) \geq 0$; and (3) the constraints in Condition (d) in Lemma 4 are dropped. Any solution to this relaxed program is such that $\widetilde{u}(\overline{\theta}, D^\star) = 0$ and, for any $\theta > \underline{\theta}$, $p(\theta)$ is as in the Baron-Myerson-with-price-cap regulation. This is because, for any $\theta \in \Theta$, the expression $V^\star(D^\star(\mathrm{p})) - z^\star(\theta) D^\star(\mathrm{p})$ is quasi-concave in p. The unique maximizer of this expression over $[0, \overline{\theta}]$ is thus $p(\theta) = \min\{z^\star(\theta), \overline{\theta}\}$. Because the function $p : \Theta \to \mathbb{R}$ given by $p(\theta) = \min\{z^\star(\theta), \overline{\theta}\}$ for all $\theta \in \Theta$ is weakly increasing, any solution to the relaxed program is such that $p(\theta) = \min\{z^\star(\theta), \overline{\theta}\}$ for all $\theta > \underline{\theta}$.

Equipped with this result, we now show that any mechanism $\widetilde{M} \equiv (p, \tilde{u})$ in which (A) p is weakly increasing and such that $p(\theta) = \min\{z^\star(\theta), \overline{\theta}\}$ for all $\theta > \underline{\theta}$, and (B) $\tilde{u}$ satisfies the conditions in Definition 4 satisfies all the conditions of Lemma 4, implying that $\widetilde{M} \in \widetilde{\mathcal{M}}^{\text{SL}}$. First note that Condition (b) in Lemma 4 trivially holds. Because $z^\star$ is increasing, Condition (a) is also satisfied. Because $z^\star(\overline{\theta}) > \overline{\theta}$, Condition (c) also holds. To complete the proof, it thus suffices to show that, for any $D \in \mathcal{D}$, Condition (d) holds. That $\tilde{u}$ satisfies the conditions in Definition 4 implies that $\widetilde{w}(\overline{\theta}, \widetilde{M}; D) \geq G^*$. Now, pick any $\theta < \overline{\theta}$. If $z^\star(\theta) \geq \overline{\theta}$, then $\widetilde{w}(\theta, \widetilde{M}; D) = \widetilde{w}(\overline{\theta}, \widetilde{M}; D) \geq G^*$. If, instead, $z^\star(\theta) < \overline{\theta}$, then $p(\theta) = z^\star(\theta) < \overline{\theta}$. That

$z^\star(\theta) \geq \theta$ implies that $D(z^\star(\theta)) \leq D(\theta)$. Arguments identical to those in the proof of Lemma 5 but applied to the quantity schedule $q = D(p^\star(\cdot))$ and with welfare evaluated under the demand D instead of the demand $\underline{D}$ then imply that the welfare function $\widetilde{w}(\cdot, \widetilde{M}; D)$ is weakly decreasing in θ. Thus, $\widetilde{w}(\theta, \widetilde{M}; D) \geq \widetilde{w}(\overline{\theta}, \widetilde{M}; D) \geq G^*$. This means that Condition (d) in Lemma 4 also holds.

Jointly the above properties imply the result in the proposition. Q.E.D.

Proof of Proposition 4. The proof is in two parts, each establishing the corresponding claim in the proposition.

Part 1. Suppose $M^\star$ is robustly optimal. Corollary 1 then implies that in any robustly optimal quantity regulation M^{OPT}, $q^{\text{OPT}}(\theta) = \max\{q^{\text{BM}}(\theta), \mathrm{q}_\ell\}$ for all $\theta > \underline{\theta}$. Under any robustly optimal price regulation $\widetilde{M}^{\text{OPT}} = (p^{\text{OPT}}, \widetilde{u}^{\text{OPT}})$, when the demand is $D^\star$ and the cost is $\theta > \underline{\theta}$, the monopolist sells a quantity $D^\star(p^{\text{OPT}}(\theta)) = \max\{q^{\text{BM}}(\theta), \tilde{\mathrm{q}}_\ell\}$, where $\tilde{\mathrm{q}}_\ell \equiv D^\star(\overline{\theta}) \geq \underline{D}(\overline{\theta}) = \mathrm{q}_\ell$. Hence, there exists $\tilde{\theta}^\star \leq \theta^\star$ such that $D^\star(p^{\text{OPT}}(\theta)) = \tilde{\mathrm{q}}_\ell$ if $\theta \geq \tilde{\theta}^\star$ and $D^\star(p^{\text{OPT}}(\theta)) = q^{\text{BM}}(\theta)$ if $\theta \in (\underline{\theta}, \tilde{\theta}^\star)$. See Panel A of Figure 4 for the illustration.

For $\theta \in (\underline{\theta}, \tilde{\theta}^\star)$, we have that $D^\star(p^{\text{OPT}}(\theta)) = q^{\text{OPT}}(\theta) = q^{\text{BM}}(\theta)$. However, for $\theta \geq \tilde{\theta}^\star$, we have that $D^\star(p^{\text{OPT}}(\theta)) = \tilde{\mathrm{q}}_\ell \geq q^{\text{OPT}}(\theta) \geq q^{\text{BM}}(\theta)$. Because, for any θ, virtual surplus $V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}$ is quasi-concave in q, reaching a maximum at $q^{\text{BM}}(\theta)$, we thus have that, for any $\theta \geq \tilde{\theta}^\star$, $V^\star(D^\star(p^{\text{OPT}}(\theta))) - z^\star(\theta)D^\star(p^{\text{OPT}}(\theta)) \leq V^\star(q^{\text{OPT}}(\theta)) - z^\star(\theta)q^{\text{OPT}}(\theta)$. Because $F^\star$ does not have an atom at $\theta = \underline{\theta}$, and because $\widetilde{u}^{\text{OPT}}(\overline{\theta}, D^\star) = u^{\text{OPT}}(\overline{\theta}) = 0$, we conclude that $\widetilde{W}(\widetilde{M}^{\text{OPT}}; D^\star, F^\star) \leq W(M^{\text{OPT}}; V^\star, F^\star)$, with the inequality strict if, and only if, $D^\star(\overline{\theta}) > \underline{D}(\overline{\theta})$.

Part 2. If $D^\star(\overline{\theta}) = \underline{D}(\overline{\theta})$, then $\mathrm{q}_\ell = \tilde{\mathrm{q}}_\ell$. In this case, for any $\theta > \underline{\theta}$, the quantity traded under the conjectured model $(D^\star, F^\star)$ when running any robustly optimal price regulation is $D^\star(p^{\text{OPT}}(\theta)) = \max\{q^{\text{BM}}(\theta), \mathrm{q}_\ell\} = q^\star(\theta)$. Again, because $F^\star$ does not have any atom at $\underline{\theta}$, this means that, by running any robustly optimal price regulation $\widetilde{M}^{\text{OPT}}$, the regulator obtains the same welfare as by running the Baron-Myerson-with-quantity-floor regulation $M^\star$, i.e.,

$$\widetilde{W}(\widetilde{M}^{\text{OPT}}; D^\star, F^\star) = W(M^\star; V^\star, F^\star). \tag{30}$$

As shown in the proof of Proposition 1, $M^\star$ is the solution to a relaxation of the full program

yielding the robustly optimal quantity regulation, implying that

$$W(M^\star; V^\star, F^\star) \geq W(M^{\text{OPT}}; V^\star, F^\star). \tag{31}$$

When $M^\star$ is not robustly optimal, the inequality in (31) is strict.

Together, (30) and (31) imply that, when $D^\star(\overline{\theta}) = \underline{D}(\overline{\theta})$, price regulation dominates quantity regulation, strictly if $M^\star$ is not robustly optimal. Q.E.D.

Proof of Proposition 5. The proof is in two parts, each establishing the corresponding claim in the proposition.

Part (a). The bi-Lipschitz continuity of $D^\star$ implies that, for any $y \in \Theta$,

$$D^\star(y) - q^{\text{BM}}(y) = D^\star(y) - D^\star\left(y + \frac{F^\star(y)}{f^\star(y)}\right) \geq k\frac{F^\star(y)}{f^\star(y)} \geq \delta(y) = D^\star(y) - \underline{D}(y).$$

This implies that $\underline{D}(y) \geq q^{\text{BM}}(y)$ for all $y \in \Theta$. Decreasingness of $\underline{D}$ in turn implies that $\underline{D}(y) \geq \underline{D}(\overline{\theta}) = \mathrm{q}_\ell$. Hence, for all $y \in \Theta$, $q^\star(y) \leq \underline{D}(y)$. Condition (9) in Proposition 1 thus holds for all $\theta \in \Theta$. Furthermore, because, for any $\theta \in \Theta$, $\delta(\theta) \geq 0$, the assumption in part (a) of the proposition that $F^\star(\theta)/f^\star(\theta) \geq \delta(\theta)/k$ implies that $\delta(\underline{\theta}) = 0$. Hence, $q^\star(\underline{\theta}) = q^{\text{BM}}(\underline{\theta}) = D^\star(\underline{\theta}) = \underline{D}(\underline{\theta})$. In turn, this implies that Condition (8) in Proposition 1 reduces to Condition (9) evaluated at $\theta = \underline{\theta}$. We conclude that $M^\star$ is robustly optimal. The result in part (a) then follows from Proposition 4.

Part (b). The proof is in three steps.

Step 1. Let $\Delta \equiv \int_{\underline{\theta}}^{\overline{\theta}} \left(\delta(\theta) - K\frac{F^\star(\theta)}{f^\star(\theta)}\right)\mathrm{d}\theta$. The bi-Lipschitz continuity of $D^\star$ implies that, for any $\theta \in \Theta$,

$$q^{\text{BM}}(\theta) = D^\star\left(\theta + \frac{F^\star(\theta)}{f^\star(\theta)}\right) \geq D^\star(\theta) - K\frac{F^\star(\theta)}{f^\star(\theta)} = \underline{D}(\theta) + \delta(\theta) - K\frac{F^\star(\theta)}{f^\star(\theta)}. \tag{32}$$

Hence,

$$\int_{\underline{\theta}}^{\overline{\theta}} \big(q^{\text{BM}}(\theta) - q^{\text{OPT}}(\theta)\big)\mathrm{d}\theta \geq \int_{\underline{\theta}}^{\overline{\theta}} \big(\underline{D}(\theta) - q^{\text{OPT}}(\theta)\big)\mathrm{d}\theta + \Delta \geq \Delta, \tag{33}$$

where the last inequality holds because of Lemma 3.

STEP 2. We now use the lower bound in Step 1 to establish a lower bound on $W(M^{\mathrm{BM}}; V^\star, F^\star) - W(M^{\mathrm{OPT}}; V^\star, F^\star)$. Before proceeding with the proof of this Step, we establish a technical lemma that we use below.

Lemma 12 (strong concavity)**.** *If $D^\star$ is K-Lipschitz continuous on $[P^\star(\bar{\mathrm{q}}), P^\star(0)]$, then $V^\star$ is $(1/K)$-strongly concave on $[0, \bar{\mathrm{q}}]$, in the sense that, for all $\mathrm{q}_1, \mathrm{q}_2 \in [0, \bar{\mathrm{q}}]$, $V^\star(\mathrm{q}_2) - V^\star(\mathrm{q}_1) + P^\star(\mathrm{q}_2)(\mathrm{q}_1 - \mathrm{q}_2) \geq \frac{1}{2K}(\mathrm{q}_1 - \mathrm{q}_2)^2$.*

Proof of Lemma 12. By the K-Lipschitz continuity of $D^\star$, for any $\mathrm{q}_1, \mathrm{q}_2 \in [0, \bar{\mathrm{q}}]$, we have that $|\mathrm{q}_1 - \mathrm{q}_2| = |D^\star(P^\star(\mathrm{q}_1)) - D^\star(P^\star(\mathrm{q}_2))| \leq K|P^\star(\mathrm{q}_1) - P^\star(\mathrm{q}_2)|$. Because $P^\star$ is decreasing, irrespective of the sign of $(\mathrm{q}_1 - \mathrm{q}_2)$,

$$\big(P^\star(\mathrm{q}_2) - P^\star(\mathrm{q}_1)\big)(\mathrm{q}_1 - \mathrm{q}_2) \geq \frac{1}{K}(\mathrm{q}_1 - \mathrm{q}_2)^2. \tag{34}$$

Fix $\mathrm{q}_1, \mathrm{q}_2 \in [0, \bar{\mathrm{q}}]$, and let $\phi(t) \equiv V^\star(\mathrm{q}_2 + t(\mathrm{q}_1 - \mathrm{q}_2))$ for all $t \in [0, 1]$. Then, for all $t \in [0, 1]$, $\phi'(t) = P^\star(\mathrm{q}_2 + t(\mathrm{q}_1 - \mathrm{q}_2))(\mathrm{q}_1 - \mathrm{q}_2)$. Furthermore, using (34), we have that

$$\Big(P^\star(\mathrm{q}_2) - P^\star(\mathrm{q}_2 + t(\mathrm{q}_1 - \mathrm{q}_2))\Big)t(\mathrm{q}_1 - \mathrm{q}_2) \geq \frac{1}{K}t^2(\mathrm{q}_1 - \mathrm{q}_2)^2$$

which implies that

$$\phi'(t) = P^\star(\mathrm{q}_2 + t(\mathrm{q}_1 - \mathrm{q}_2))(\mathrm{q}_1 - \mathrm{q}_2) \leq P^\star(\mathrm{q}_2)(\mathrm{q}_1 - \mathrm{q}_2) - \frac{1}{K}t(\mathrm{q}_1 - \mathrm{q}_2)^2. \tag{35}$$

Using (35), we thus have that

$$\begin{aligned} V^\star(\mathrm{q}_1) - V^\star(\mathrm{q}_2) = \phi(1) - \phi(0) = \int_0^1 \phi'(t)\mathrm{d}t &\leq P^\star(\mathrm{q}_2)(\mathrm{q}_1 - \mathrm{q}_2) - \frac{1}{K}(\mathrm{q}_1 - \mathrm{q}_2)^2 \int_0^1 t\mathrm{d}t \\ &= P^\star(\mathrm{q}_2)(\mathrm{q}_1 - \mathrm{q}_2) - \frac{1}{2K}(\mathrm{q}_1 - \mathrm{q}_2)^2, \end{aligned}$$

from which we obtain that $V^\star(\mathrm{q}_2) - V^\star(\mathrm{q}_1) + P^\star(\mathrm{q}_2)(\mathrm{q}_1 - \mathrm{q}_2) \geq \frac{1}{2K}(\mathrm{q}_1 - \mathrm{q}_2)^2$. ■

Now, for each θ and each q, let

$$\psi(\mathrm{q}; \theta) \equiv V^\star(\mathrm{q}) - \left(\theta + \frac{F^\star(\theta)}{f^\star(\theta)}\right)\mathrm{q}.$$

Observe that, for any $\theta \in \Theta$, $\psi(\cdot;\theta)$ reaches a maximum at $q^{\mathrm{BM}}(\theta)$. Furthermore,

$$\psi(q^{\mathrm{BM}}(\theta);\theta) - \psi(q^{\mathrm{OPT}}(\theta);\theta) = V^\star(q^{\mathrm{BM}}(\theta)) - V^\star(q^{\mathrm{OPT}}(\theta)) + \left(\theta + \frac{F^\star(\theta)}{f^\star(\theta)}\right)\left(q^{\mathrm{OPT}}(\theta) - q^{\mathrm{BM}}(\theta)\right)$$
$$\geq \frac{1}{2K}\left(q^{\mathrm{BM}}(\theta) - q^{\mathrm{OPT}}(\theta)\right)^2, \tag{36}$$

where the inequality follows from the fact that $P^\star(q^{\mathrm{BM}}(\theta)) = \theta + \frac{F^\star(\theta)}{f^\star(\theta)}$ along with Lemma 12. Hence,

$$W(M^{\mathrm{BM}};V^\star,F^\star) - W(M^{\mathrm{OPT}};V^\star,F^\star) = \int_{\underline{\theta}}^{\overline{\theta}} \left(\psi(q^{\mathrm{BM}}(\theta);\theta) - \psi(q^{\mathrm{OPT}}(\theta);\theta)\right) f^\star(\theta)\mathrm{d}\theta$$
$$\geq \frac{\underline{f}}{2K}\int_{\underline{\theta}}^{\overline{\theta}} \left(q^{\mathrm{BM}}(\theta) - q^{\mathrm{OPT}}(\theta)\right)^2 \mathrm{d}\theta \geq \frac{\underline{f}}{2K(\overline{\theta}-\underline{\theta})}\left(\int_{\underline{\theta}}^{\overline{\theta}} \left(q^{\mathrm{BM}}(\theta) - q^{\mathrm{OPT}}(\theta)\right)\mathrm{d}\theta\right)^2 \geq \frac{\Delta^2 \underline{f}}{2K(\overline{\theta}-\underline{\theta})}, \tag{37}$$

where the first inequality follows from (36) and the definition of $\underline{f}$, the second inequality follows from Jensen's inequality, and the third inequality follows from (33).

Step 3. Finally, we compare the welfare generated by any price regulation $\widetilde{M}^{\mathrm{OPT}} \equiv (p^{\mathrm{OPT}}, \widetilde{u}^{\mathrm{OPT}})$ to that generated by the quantity regulation $M^{\mathrm{BM}} \equiv (q^{\mathrm{BM}}, u^{\mathrm{BM}})$, under the designer's model. For any $\theta > \underline{\theta}$, let $\tilde{q}(\theta) = D^\star\left(\min\{z^\star(\theta), \overline{\theta}\}\right)$ be the quantity traded under any robustly optimal price regulation $\widetilde{M}^{\mathrm{OPT}}$, when the demand is the one in the designer's model. For every $\theta > \underline{\theta}$ such that $z^\star(\theta) \leq \overline{\theta}$, $\tilde{q}(\theta) = q^{\mathrm{BM}}(\theta)$, implying that

$$\psi(q^{\mathrm{BM}}(\theta);\theta) - \psi(\tilde{q}(\theta);\theta) = 0. \tag{38}$$

On the other hand, for any $\theta > \underline{\theta}$ such that $z^\star(\theta) > \overline{\theta}$, $\tilde{q}(\theta) = D^\star(\overline{\theta}) \geq q^{\mathrm{BM}}(\theta)$ and hence

$$\psi(q^{\mathrm{BM}}(\theta);\theta) - \psi(\tilde{q}(\theta);\theta) = V^\star(q^{\mathrm{BM}}(\theta)) - V^\star(D^\star(\overline{\theta})) + z^\star(\theta)[D^\star(\overline{\theta}) - q^{\mathrm{BM}}(\theta)]. \tag{39}$$

Because $V^\star(\mathrm{q}) - \overline{\theta}\mathrm{q}$ is maximized at $D^\star(\overline{\theta})$, for any θ,

$$V^\star(D^\star(\overline{\theta})) - \overline{\theta}D^\star(\overline{\theta}) \geq V^\star(q^{\mathrm{BM}}(\theta)) - \overline{\theta}q^{\mathrm{BM}}(\theta). \tag{40}$$

Combining (39) with (40), we have that, for any $\theta > \underline{\theta}$ such that $z^\star(\theta) > \overline{\theta}$,

$$\psi(q^{\mathrm{BM}}(\theta);\theta) - \psi(\tilde{q}(\theta);\theta) \leq (z^\star(\theta) - \overline{\theta})[D^\star(\overline{\theta}) - q^{\mathrm{BM}}(\theta)] \leq (z^\star(\theta) - \overline{\theta})[D^\star(\theta) - q^{\mathrm{BM}}(\theta)]$$
$$\leq (z^\star(\theta) - \theta)\left[D^\star(\theta) - D^\star\left(\theta + \frac{F^\star(\theta)}{f^\star(\theta)}\right)\right] \leq K\Big(\frac{F^\star(\theta)}{f^\star(\theta)}\Big)^2 \leq \frac{\delta(\theta)^2}{KN^2},$$

where $N \equiv 1 + \frac{\overline{\delta}}{\underline{\delta}}\sqrt{\frac{2\overline{f}}{\underline{f}}}$. The penultimate inequality follows from the bi-Lipschitz continuity of $D^\star$ whereas the last inequality follows from the fact that, by assumption, $F^\star(\theta)/f^\star(\theta) \leq \delta(\theta)/(KN)$. By (38), this inequality trivially holds when $z^\star(\theta) \leq \overline{\theta}$. As a consequence for any $\theta > \underline{\theta}$,

$$\psi(q^{\mathrm{BM}}(\theta);\theta) - \psi(\tilde{q}(\theta);\theta) \leq \frac{\delta(\theta)^2}{KN^2}.$$

Integrating over all types, we thus have that

$$\begin{aligned} W(M^{\mathrm{BM}};V^\star,F^\star) - \widetilde{W}(\widetilde{M}^{\mathrm{OPT}};D^\star,F^\star) &= \int_{\underline{\theta}}^{\overline{\theta}} \Big(\psi(q^{\mathrm{BM}}(\theta);\theta) - \psi(\tilde{q}(\theta);\theta)\Big) f^\star(\theta)\mathrm{d}\theta \\ &\leq \tfrac{1}{KN^2}\int_{\underline{\theta}}^{\overline{\theta}} \delta(\theta)^2 f^\star(\theta)\mathrm{d}\theta \leq \tfrac{\overline{f}(\overline{\theta}-\underline{\theta})\overline{\delta}^2}{KN^2}. \end{aligned} \tag{41}$$

Combining (37) with (41), we have that

$$\widetilde{W}(\widetilde{M}^{\mathrm{OPT}};D^\star,F^\star) - W(M^{\mathrm{OPT}};V^\star,F^\star) \geq \frac{\Delta^2\underline{f}}{2K(\overline{\theta}-\underline{\theta})} - \frac{\overline{f}(\overline{\theta}-\underline{\theta})\overline{\delta}^2}{KN^2}. \tag{42}$$

To establish that price regulation strictly dominates quantity regulation, it thus suffices to show that, under the condition in part (b) of the proposition, $\Delta > \sqrt{\frac{2\overline{f}}{\underline{f}}}\left(\frac{\overline{\delta}(\overline{\theta}-\underline{\theta})}{N}\right)$. To see this, note that

$$\Delta \equiv \int_{\underline{\theta}}^{\overline{\theta}} \Big(\delta(\theta) - K\frac{F^\star(\theta)}{f^\star(\theta)}\Big)\mathrm{d}\theta > \int_{\underline{\theta}}^{\overline{\theta}} \delta(\theta)\frac{N-1}{N}\mathrm{d}\theta \geq \underline{\delta}(\overline{\theta}-\underline{\theta})\frac{N-1}{N} \geq \sqrt{\frac{2\overline{f}}{\underline{f}}}\left(\frac{\overline{\delta}(\overline{\theta}-\underline{\theta})}{N}\right). \tag{43}$$

The first inequality follows from the condition in the proposition along with the fact that $F^\star(\underline{\theta})/f^\star(\underline{\theta}) = 0 < \underline{\delta}/(KN)$ and the continuity of $F^\star/f^\star$. Jointly these properties imply that $KF^\star(\theta)/f^\star(\theta) < \delta(\theta)/N$ on a right-neighborhood of $\underline{\theta}$. The second inequality follows from

the definition of $\underline{\delta}$. The last inequality is obtained by using the fact that $N-1 = \frac{\bar{\delta}}{\underline{\delta}}\sqrt{\frac{2\bar{f}}{\underline{f}}}$. Q.E.D.

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S Supplement I

The notation in this supplement is the same as in the original article. All sections, definitions, displayed conditions, and results specific to this document have the prefix "S" to avoid confusion with the corresponding parts in the main text. Section S.1 contains the proof of a stronger version of the result in Part 2 of Proposition 2 in the main text. Section S.2 considers more permissive robustness requirements. Section S.3 considers alternative forms of cost uncertainty.

S.1 Stronger version of Part 2 of Proposition 2.

Part (2) of Proposition 2 in the main text establishes that there exists $\theta^{\star\star} \in [\theta^m, \overline{\theta}]$ such that $q^{\mathrm{OPT}}(\theta) \leq q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta^m, \min\{\theta^\star, \theta^{\star\star}\})$ with the inequality strict for a subset of $(\theta^m, \min\{\theta^\star, \theta^{\star\star}\})$ of positive Lebesgue measure. Proposition S.6 below establishes a stronger version of this result. The proof of this proposition further sharpens the characterization of the robustly optimal mechanism by showing that the optimal quantity schedule in the range $(\theta^m, \min\{\theta^\star, \theta^{\star\star}\})$ satisfies $q^{\mathrm{OPT}}(\theta) = D^\star(\bar{z}^M(\theta))$, where $\bar{z}^M$ is an ironed virtual cost that accounts for robustness.

Proposition S.6. *Suppose $M^\star$ is not robustly optimal. Then, in any robustly optimal mechanism $M^{\mathrm{OPT}} = (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$,*

$$q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta) \qquad \forall\ \theta \in (\theta^m, \min\{\theta^\star, \theta^{\star\star}\}) \tag{S.44}$$

where $\theta^{\star\star} = \inf\{\theta \in [\theta^m, \overline{\theta}] : q^{\mathrm{OPT}}(y) = \underline{D}(y)\ \forall\ y \in [\theta, \overline{\theta}]\}$.

Recall that, when $\theta^\star < \overline{\theta}$, $\theta^{\star\star} = \overline{\theta}$, and therefore $\min\{\theta^\star, \theta^{\star\star}\} = \theta^\star$. On the other hand, when $\theta^\star = \overline{\theta}$, $\min\{\theta^\star, \theta^{\star\star}\} = \theta^{\star\star}$.

Proof of Proposition S.6. The proof proceeds in five steps. Step 1 sets up a sub-program that permits us to use Lagrangian methods with infinite-dimensional constraints to identify necessary conditions for the optimal policy. Specifically, it considers an auxiliary problem in which the policy is held fixed over an interval of types $[\theta^\dagger, \bar{\theta}]$, with $\theta^\dagger \in (\theta^m, \min\{\theta^\star, \theta^{\star\star}\}]$ and the optimization is over the shape of the policy over the interval $[\underline{\theta}, \theta^\dagger)$. Step 2 identifies a few properties of the solution to the sub-program of Step 1. Step 3 uses ironing techniques to sharpen the characterization in Step 2. Step 4 shows how to select $\theta^\dagger$ to guarantee

constraint qualification thus validating the conditions identified in the previous steps, and then establishes the result in the proposition by continuity. Finally, Step 5 proves two technical lemmas used in the ironing arguments of Step 3.

Step 1. Lagrangian method.

Let $\theta^\dagger \in (\theta^m, \min\{\theta^\star, \theta^{\star\star}\}]$ and define $\mathcal{Q}^\dagger \equiv \{q : [\underline{\theta}, \overline{\theta}] \to [0, \overline{\mathrm{q}}] : q \text{ is weakly decreasing and } q(\theta) = q^{\mathrm{OPT}}(\theta) \ \forall\ \theta \in [\theta^\dagger, \overline{\theta}]\}$. Note that $\mathcal{Q}^\dagger$ is a convex set. Clearly, q^{OPT} must solve the following sub-program:[21]

$$q^{\mathrm{OPT}} \in \arg\max_{q \in \mathcal{Q}^\dagger} \int_{\underline{\theta}}^{\theta^\dagger} \Big[V^\star(q(\theta)) - z^\star(\theta) q(\theta)\Big] f^\star(\theta) \mathrm{d}\theta \tag{$\mathbf{FP}^\dagger$}$$

$$\text{subject to} \quad S(\theta; q) \equiv \int_{\theta}^{\overline{\theta}} (\underline{D}(y) - q(y)) \mathrm{d}y \geq 0 \qquad \forall\ \theta \in [\underline{\theta}, \theta^\dagger],$$

$$R(q) \equiv \underline{V}(q(\underline{\theta})) - \underline{\theta} q(\underline{\theta}) - \int_{\underline{\theta}}^{\overline{\theta}} q(y) \mathrm{d}y - G^* \geq 0.$$

Definition S.6. *The sub-program (**FP**†) satisfies the **Interiority Condition** if there exists $q^\dagger \in \mathcal{Q}^\dagger$ such that $R(q^\dagger) > 0$ and $\inf_{\theta \in [\underline{\theta}, \theta^\dagger]} S(\theta; q^\dagger) > 0$.*

As anticipated above, in Step 4 we show how to select a sequence of $\theta^\dagger$ satisfying the Interiority Condition and converging to $\min\{\theta^\star, \theta^{\star\star}\}$. Along the sequence, the properties identified in Steps 2-3 are necessary for optimality. Taking the limit for $\theta^\dagger \to \min\{\theta^\star, \theta^{\star\star}\}$ then permits us to establish (S.44). In Steps 2-4, we focus on arbitrary $\theta^\dagger$ satisfying the Interiority Condition and use Lagrangian methods with infinite-dimensional constraints to solve the sub-program ($\mathbf{FP}^\dagger$).

When the sub-program ($\mathbf{FP}^\dagger$) satisfies the Interiority Condition of Definition S.6, Theorem 1 of Luenberger (1997) (page 217) implies existence of a weakly increasing, right-continuous function Λ^{OPT}, satisfying $\Lambda^{\mathrm{OPT}}(\underline{\theta}) = 0$, and a non-negative scalar $\mu^{\mathrm{OPT}} \geq 0$ such

[21] Recall from Lemma 3 in the main text that q satisfies the robustness constraints in (3) for all $\theta \in \Theta$ if and only if (a) $R(q) \geq 0$, (b) $S(\theta; q) \geq 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$, and (c) $q(\overline{\theta}) = \mathrm{q}_\ell$. Imposing that $S(\theta; q) \geq 0$ also for $\theta = \underline{\theta}$ is inconsequential because this constraint is implied by $R(q) \geq 0$. Omitting the constraints $S(\theta; q) \geq 0$ for $\theta > \theta^\dagger$ and the constraint $q(\overline{\theta}) = \mathrm{q}_\ell$ is motivated by the fact that q is required to agree with q^{OPT} over $[\theta^\dagger, \overline{\theta}]$ along with the fact that q^{OPT} satisfies these constraints because it is robustly optimal.

that

$$q^{\mathrm{OPT}} \in \arg\max_{q\in\mathcal{Q}^\dagger} \mathcal{L}(q;\Lambda^{\mathrm{OPT}},\mu^{\mathrm{OPT}}), \tag{S.45}$$

where, for any $q \in \mathcal{Q}^\dagger$,

$$\mathcal{L}(q;\Lambda^{\mathrm{OPT}},\mu^{\mathrm{OPT}}) \equiv \int_{\underline{\theta}}^{\theta^\dagger} \Big[V^\star(q(\theta)) - z^\star(\theta)q(\theta)\Big] f^\star(\theta)\mathrm{d}\theta + \int_{\underline{\theta}}^{\theta^\dagger} \left[\int_{\theta}^{\overline{\theta}} (\underline{D}(y) - q(y))\mathrm{d}y\right] \mathrm{d}\Lambda^{\mathrm{OPT}}(\theta)$$
$$+ \mu^{\mathrm{OPT}}\Big[\underline{V}(q(\underline{\theta})) - \underline{\theta}q(\underline{\theta}) - \int_{\underline{\theta}}^{\overline{\theta}} q(y)\mathrm{d}y - G^*\Big]$$

is the Lagrangian function associated with the above sub-program ($\mathbf{FP}^\dagger$). Furthermore, complementary slackness must hold:

$$\int_{\underline{\theta}}^{\theta^\dagger} S(\theta;q^{\mathrm{OPT}})\mathrm{d}\Lambda^{\mathrm{OPT}}(\theta) = 0, \quad \text{and} \quad R(q^{\mathrm{OPT}})\mu^{\mathrm{OPT}} = 0. \tag{S.46}$$

Using integration by parts and rearranging terms, we have that

$$\mathcal{L}(q;\Lambda^{\mathrm{OPT}},\mu^{\mathrm{OPT}}) = \int_{\underline{\theta}}^{\theta^\dagger} \Big[V^\star(q(\theta)) - \Big(z^\star(\theta) + \frac{\Lambda^{\mathrm{OPT}}(\theta) + \mu^{\mathrm{OPT}}}{f^\star(\theta)}\Big)q(\theta)\Big] f^\star(\theta)\mathrm{d}\theta \tag{S.47}$$
$$+ \int_{\underline{\theta}}^{\theta^\dagger} \Lambda^{\mathrm{OPT}}(\theta)\underline{D}(\theta)\mathrm{d}\theta + \Lambda^{\mathrm{OPT}}(\theta^\dagger)\int_{\theta^\dagger}^{\overline{\theta}} [\underline{D}(\theta) - q(\theta)]\mathrm{d}\theta \tag{S.48}$$
$$+ \mu^{\mathrm{OPT}}\Big[\underline{V}(q(\underline{\theta})) - \underline{\theta}q(\underline{\theta}) - \int_{\theta^\dagger}^{\overline{\theta}} q(y)\mathrm{d}y - G^*\Big]. \tag{S.49}$$

Note that the terms in (S.48) are independent of the value that q takes on $[\underline{\theta},\theta^\dagger]$ whereas the terms in (S.49) depend on the value that q takes on $[\underline{\theta},\theta^\dagger]$ only through $q(\underline{\theta})$. We take advantage of these properties below.

Step 2. Miscellaneous properties of the optimum.

Observation S.2. *Suppose there exists* $\tilde{\theta} \in (\underline{\theta},\theta^\dagger)$ *such that* $\Lambda^{\mathrm{OPT}}(\theta) + \mu^{\mathrm{OPT}} = 0$ *for all*

$\theta \in (\underline{\theta}, \tilde{\theta})$. *Then,* $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ *for all* $\theta \in (\underline{\theta}, \tilde{\theta})$.

Proof of Observation S.2. Recall that $q^{\mathrm{OPT}}(\theta) \leq q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \tilde{\theta})$ by virtue of Lemma 6 in the main text along with the fact that $\tilde{\theta} \leq \theta^{\dagger} \leq \min\{\theta^{\star}, \theta^{\star\star}\}$. Now assume, towards a contradiction, that there exists a $\theta \in (\underline{\theta}, \tilde{\theta})$ such that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$. The continuity of q^{BM} along with the fact that q^{OPT} is weakly decreasing, then imply that there exists an interval $I \subset (\underline{\theta}, \tilde{\theta})$ of positive Lebesgue measure such that $q^{\mathrm{OPT}}(y) < q^{\mathrm{BM}}(y)$ for all $y \in I$. Then consider the quantity schedule

$$\tilde{q}(\theta) = \begin{cases} q^{\mathrm{BM}}(\theta) & \text{if } \theta \in [\underline{\theta}, \tilde{\theta}) \\ q^{\mathrm{OPT}}(\theta) & \text{if } \theta \in [\tilde{\theta}, \overline{\theta}]. \end{cases}$$

Observe that $\tilde{q}$ is weakly decreasing because q^{BM} is decreasing, q^{OPT} is weakly decreasing, and $q^{\mathrm{OPT}}(\tilde{\theta}) \leq q^{\mathrm{BM}}(\tilde{\theta})$ by Lemma 6 in the main text. Moreover, $\mathcal{L}(\tilde{q}; \Lambda^{\mathrm{OPT}}, \mu^{\mathrm{OPT}}) > \mathcal{L}(q^{\mathrm{OPT}}; \Lambda^{\mathrm{OPT}}, \mu^{\mathrm{OPT}})$, where the inequality follows because $q^{\mathrm{BM}}(\theta)$ is the unique maximizer of $V^{\star}(\mathrm{q}) - z^{\star}(\theta)\mathrm{q}$ and $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ on the interval I of positive Lebesgue measure. Because $\tilde{q} \in Q^{\dagger}$, this strict inequality contradicts Condition (S.45) and the robust optimality of q^{OPT}. We therefore conclude that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \tilde{\theta})$. ■

Next, let

$$\theta_{\min} \equiv \inf\{\theta \in (\underline{\theta}, \theta^{\dagger}) : \Lambda^{\mathrm{OPT}}(\theta) + \mu^{\mathrm{OPT}} > 0\}.$$

Step 4 will establish existence of $\theta^{\dagger}$ for which $\{\theta \in (\underline{\theta}, \theta^{\dagger}) : \Lambda^{\mathrm{OPT}}(\theta) + \mu^{\mathrm{OPT}} > 0\} \neq \emptyset$ and hence $\theta_{\min}$ is well-defined. Until then, we will assume that $\theta_{\min}$ exists and derive a few key properties of $\theta_{\min}$ instrumental to the result in the proposition.

Lemma S.13. $\theta_{\min} \leq \theta^{m}$.

Proof of Lemma S.13. If $\theta_{\min} = \underline{\theta}$, the claim is trivially true. Therefore, suppose that $\theta_{\min} > \underline{\theta}$. We first establish two important properties of $q^{\mathrm{OPT}}(\theta_{\min})$: (a) $\underline{W}(\theta_{\min}, q^{\mathrm{OPT}}) = G^{*}$, and (b) $q^{\mathrm{OPT}}(\theta_{\min}) = \underline{D}(\theta_{\min})$. To see this, note that, because $S(\theta; q^{\mathrm{OPT}}) \geq 0$ for all $\theta \in (\underline{\theta}, \theta^{\dagger})$, the first complementary slackness condition in (S.46) implies that $S(\theta; q^{\mathrm{OPT}}) = 0$ for Λ^{OPT}-almost every $\theta \in (\underline{\theta}, \theta^{\dagger})$. Hence, if Λ^{OPT} has a jump at $\theta_{\min}$, that is, if

$$\Delta\Lambda^{\mathrm{OPT}}(\theta_{\min}) \equiv \Lambda^{\mathrm{OPT}}(\theta_{\min}) - \lim_{\theta \uparrow \theta_{\min}} \Lambda^{\mathrm{OPT}}(\theta) > 0$$

then $S(\theta_{\min}, q^{\mathrm{OPT}}) = 0$. If, instead, there is no jump, i.e., if $\Delta\Lambda^{\mathrm{OPT}}(\theta_{\min}) = 0$, then by the

definition of $\theta_{\min}$ and the fact that $\theta_{\min} > \underline{\theta}$, it must be that $\mu^{\mathrm{OPT}} = 0$. Because Λ^{OPT} is weakly increasing, for every $\epsilon \in (0, \theta^{\dagger} - \theta_{\min})$, the interval $(\theta_{\min}, \theta_{\min} + \epsilon]$ has strictly positive Λ^{OPT}-measure. Because $S(\theta; q^{\mathrm{OPT}}) = 0$ for Λ^{OPT}-almost every $\theta \in (\underline{\theta}, \theta^{\dagger})$, each such interval contains some $\theta' > \theta_{\min}$ such that $S(\theta'; q^{\mathrm{OPT}}) = 0$. Then, let (θ_n) be a monotone sequence such that $\theta_n \downarrow \theta_{\min}$, and $S(\theta_n; q^{\mathrm{OPT}}) = 0$ for all n. By continuity of S, we then have that $S(\theta_{\min}; q^{\mathrm{OPT}}) = 0$. By Lemma 3 in the main text, we have that

$$\underline{W}(\theta_{\min}, q^{\mathrm{OPT}}) - G^* = S(\theta_{\min}; q^{\mathrm{OPT}}) - \underline{\mathrm{DWL}}(\theta_{\min}, q^{\mathrm{OPT}}(\theta_{\min})) \leq 0.$$

Because robust optimality implies that $\underline{W}(\theta, q^{\mathrm{OPT}}) - G^* \geq 0$, we conclude that (a) $\underline{W}(\theta_{\min}, q^{\mathrm{OPT}}) = G^*$ and $\underline{\mathrm{DWL}}(\theta_{\min}, q^{\mathrm{OPT}}(\theta_{\min})) = 0$. The last equality (deadweight loss equal to zero) implies that (b) $q^{\mathrm{OPT}}(\theta_{\min}) = \underline{D}(\theta_{\min})$.

Now, assume, towards a contradiction to what is claimed in the proposition, that $\theta_{\min} > \theta^m$. Because $\theta_{\min} < \theta^{\dagger}$, Observation S.2 implies that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta_{\min})$. This implies that $q^{\mathrm{OPT}}(\theta^m) = q^{\mathrm{BM}}(\theta^m)$ when $\theta^m > \underline{\theta}$. When, instead, $\theta^m = \underline{\theta}$, $q^{\mathrm{OPT}}(\theta^m) \geq q^{\mathrm{BM}}(\theta^m)$ because q^{OPT} is weakly decreasing and $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta^m, \theta_{\min})$. Thus, we have that $q^{\mathrm{OPT}}(\theta^m) \geq q^{\mathrm{BM}}(\theta^m)$. Also, as established earlier, $q^{\mathrm{OPT}}(\theta_{\min}) = \underline{D}(\theta_{\min})$. Now, observe that

$$\begin{aligned}
\underline{W}(\theta_{\min}, q^{\mathrm{OPT}}) - \underline{W}(\theta^m, q^{\mathrm{OPT}}) &= \left[\underline{V}(q^{\mathrm{OPT}}(\theta_{\min})) - \theta_{\min} q^{\mathrm{OPT}}(\theta_{\min})\right] \\
&\quad - \left[\underline{V}(q^{\mathrm{OPT}}(\theta^m)) - \theta^m q^{\mathrm{OPT}}(\theta^m)\right] + \int_{\theta^m}^{\theta_{\min}} q^{\mathrm{OPT}}(y)\mathrm{d}y \\
&\geq_{(i)} \left[\underline{V}(q^{\mathrm{BM}}(\theta_{\min})) - \theta_{\min} q^{\mathrm{BM}}(\theta_{\min})\right] \\
&\quad - \left[\underline{V}(q^{\mathrm{BM}}(\theta^m)) - \theta^m q^{\mathrm{BM}}(\theta^m)\right] + \int_{\theta^m}^{\theta_{\min}} q^{\mathrm{BM}}(y)\mathrm{d}y \\
&= \underline{W}(\theta_{\min}, q^{\star}) - \underline{W}(\theta^m, q^{\star}) > 0,
\end{aligned}$$

where the strict inequality follows from the fact that $\theta_{\min} > \theta^m$ and the definition of θ^m. To see why inequality (i) holds, first observe that Observation S.2 implies that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta_{\min})$. Moreover, $\underline{V}(\mathrm{q}) - \theta\mathrm{q}$ is strictly concave in q with a unique maximum at $\underline{D}(\theta)$, and (a) $q^{\mathrm{BM}}(\theta_{\min}) \geq q^{\mathrm{OPT}}(\theta_{\min}) = \underline{D}(\theta_{\min})$, and (b) $q^{\mathrm{OPT}}(\theta^m) \geq q^{\mathrm{BM}}(\theta^m) \geq \underline{D}(\theta^m)$, where the second inequality follows from Lemma 8 in the main text. Because q^{OPT} satisfies

the robustness constraints, $\underline{W}(\theta^m, q^{\mathrm{OPT}}) \geq G^*$. Hence, we have that $\underline{W}(\theta_{\min}, q^{\mathrm{OPT}}) > G^*$, a contradiction to what we established earlier. ■

Step 3. Ironing. Let

$$z^M(\theta) \equiv z^\star(\theta) + \frac{\Lambda^{\mathrm{OPT}}(\theta) + \mu^{\mathrm{OPT}}}{f^\star(\theta)}$$

be the *modified virtual cost*. Note that z^M need not be weakly increasing. Below, we thus consider an auxiliary optimization problem, defined on the interval $(\underline{\theta}, \theta^\dagger)$, which allows us to use ironing techniques similar to Baron and Myerson (1982) to identify further properties of the robustly optimal schedule q^{OPT}. Let $\mathcal{Q}^{\dagger -} \subset \mathcal{Q}^\dagger$ be the subset of $\mathcal{Q}^\dagger$ given by

$$\mathcal{Q}^{\dagger -} \equiv \{q \in \mathcal{Q}^\dagger : \ q(\underline{\theta}) = q^{\mathrm{OPT}}(\underline{\theta})\}.$$

Again, $\mathcal{Q}^{\dagger -}$ is a convex set. Taking advantage of the fact that the terms in (S.48) and (S.49) are independent of the value that q takes on $(\underline{\theta}, \theta^\dagger)$, we have that a necessary condition for $q^{\mathrm{OPT}} \in \arg\max_{q \in \mathcal{Q}^\dagger} \mathcal{L}(q; \Lambda^{\mathrm{OPT}}, \mu^{\mathrm{OPT}})$ is that

$$q^{\mathrm{OPT}} \in \arg\max_{q \in \mathcal{Q}^{\dagger -}} \int_{\underline{\theta}}^{\theta^\dagger} \Big[V^\star(q(\theta)) - z^M(\theta) q(\theta) \Big] f^\star(\theta) \mathrm{d}\theta. \tag{UNCONST}$$

The terms in the square brackets in the integrand in (**UNCONST**) are separable in $q(\theta)$ and θ in the sense of Toikka (2011). The solution can thus be obtained by ironing z^M. For any $\phi \in [0, F^\star(\theta^\dagger)]$, let

$$h(\phi) \equiv z^M((F^\star)^{-1}(\phi)) \qquad \text{and} \qquad H(\phi) \equiv \int_0^\phi h(y) \mathrm{d}y.$$

Let $\overline{H} \equiv \mathrm{conv}(H)$ be the convex hull of H, i.e., the highest convex function on $[0, F^\star(\theta^\dagger)]$ such that $\overline{H} \leq H$. Moreover, $\overline{H}(0) = H(0)$ and $\overline{H}(F^\star(\theta^\dagger)) = H(F^\star(\theta^\dagger))$. Because $\overline{H}$ is convex, it is continuously differentiable almost everywhere. At all ϕ where $\overline{H}$ is differentiable, let $\overline{h}(\phi) \equiv \mathrm{d}\overline{H}(\phi)/\mathrm{d}\phi$ and then extend $\overline{h}$ to all $[0, F^\star(\theta^\dagger)]$ by right continuity (i.e., taking the right derivative of $\overline{H}$). Then let

$$\overline{z}^M(\theta) \equiv \overline{h}(F^\star(\theta)), \qquad \forall\ \theta \in (\underline{\theta}, \theta^\dagger)$$

be the *ironed modified virtual cost*. By Theorem 3.7 in Toikka (2011), q^{OPT} solves (**UNCONST**) if and only if[22]

(a) for almost all $\theta \in (\underline{\theta}, \theta^\dagger)$,

$$q^{\mathrm{OPT}}(\theta) \in \arg \max_{\mathrm{q} \in [q^{\mathrm{OPT}}(\theta^\dagger), q^{\mathrm{OPT}}(\underline{\theta})]} \Big[V^\star(\mathrm{q}) - \overline{z}^M(\theta)\mathrm{q} \Big];$$

(b) for all open intervals $I \subseteq (\underline{\theta}, \theta^\dagger)$ such that $\overline{H}(F^\star(\theta)) < H(F^\star(\theta))$ for all $\theta \in I$, $\bar{z}^M$ is constant over I, and therefore, q^{OPT} is also constant over I (pooling property).

The following technical lemma, whose proof is in Step 5, establishes a relationship between $\bar{z}^M$ and q^{OPT} which we use below:

Lemma S.14. *If $\bar{z}^M$ is discontinuous at $\theta \in (\underline{\theta}, \theta^\dagger)$, then q^{OPT} is continuous at θ.*

For any $\theta \in (\underline{\theta}, \theta^\dagger)$, the function $V^\star(\mathrm{q}) - \overline{z}^M(\theta)\mathrm{q}$ is strictly concave in q, with a unique maximum at $D^\star(\overline{z}^M(\theta))$. Because $\overline{z}^M$ is weakly increasing, the function $D^\star(\overline{z}^M(\cdot))$ is weakly decreasing. We divide the interval $(\underline{\theta}, \theta^\dagger)$ into three parts. Let

$$\theta_1 \equiv \sup\{\theta \in (\underline{\theta}, \theta^\dagger) : D^\star(\overline{z}^M(\theta)) > q^{\mathrm{OPT}}(\underline{\theta})\}$$

if $\{\theta \in (\underline{\theta}, \theta^\dagger) : D^\star(\overline{z}^M(\theta)) > q^{\mathrm{OPT}}(\underline{\theta})\} \neq \emptyset$ and otherwise let $\theta_1 \equiv \underline{\theta}$. Similarly, let

$$\theta_2 \equiv \inf\{\theta \in (\underline{\theta}, \theta^\dagger) : D^\star(\overline{z}^M(\theta)) < q^{\mathrm{OPT}}(\theta^\dagger)\}$$

if $\{\theta \in (\underline{\theta}, \theta^\dagger) : D^\star(\overline{z}^M(\theta)) < q^{\mathrm{OPT}}(\theta^\dagger)\} \neq \emptyset$ and otherwise let $\theta_2 \equiv \theta^\dagger$.

Claim S.3. *$\theta_1 \leq \theta_2$. If $\theta_1 = \theta_2 = \hat{\theta}$, then $\hat{\theta} \in \{\underline{\theta}, \theta^\dagger\}$.*

Proof of Claim S.3. Assume, towards a contradiction to what is claimed, that $\theta_1 > \theta_2$. Then choose $\theta \in (\theta_2, \theta_1)$. By the definition of θ_1 and θ_2, $q^{\mathrm{OPT}}(\underline{\theta}) < D^\star(\overline{z}^M(\theta)) < q^{\mathrm{OPT}}(\theta^\dagger)$, a contradiction to the fact that q^{OPT} is weakly decreasing. Thus $\theta_1 \leq \theta_2$, as claimed.

[22] If $q^{\mathrm{OPT}}(\underline{\theta}) = q^{\mathrm{OPT}}(\theta^\dagger)$, then for any $\underline{\theta} < \theta < \theta' < \theta^\dagger$, we have $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{OPT}}(\theta') \leq q^{\mathrm{BM}}(\theta') < q^{\mathrm{BM}}(\theta)$, where the weak inequality follows from Lemma 6 in the main text whereas the strict inequality follows from the fact that q^{BM} is decreasing. Hence, $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta^\dagger)$, and, by Lemma 9 in the main text, $\theta^m = \underline{\theta}$. We conclude that, in this case, $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta^m, \theta^\dagger)$. The rest of the proof then follows from the arguments in Step 4. In the rest of Step 3 below, we thus assume that $q^{\mathrm{OPT}}(\theta^\dagger) < q^{\mathrm{OPT}}(\underline{\theta})$.

Next, consider the second part of the claim. That is, suppose that $\theta_1 = \theta_2 = \hat{\theta}$. We want to establish that $\hat{\theta} \in \{\underline{\theta}, \theta^\dagger\}$. The proof is again by contradiction. Suppose that $\hat{\theta} \in (\underline{\theta}, \theta^\dagger)$. Then, for any $\theta \in [\underline{\theta}, \hat{\theta})$ and $\theta' \in (\hat{\theta}, \theta^\dagger]$, $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{OPT}}(\underline{\theta}) > q^{\mathrm{OPT}}(\theta^\dagger) = q^{\mathrm{OPT}}(\theta')$, where the inequality follows from Footnote 22, i.e., from the fact that we are considering the case in which

$$\Delta \equiv q^{\mathrm{OPT}}(\underline{\theta}) - q^{\mathrm{OPT}}(\theta^\dagger) > 0.$$

Hence, q^{OPT} is discontinuous at $\hat{\theta}$. By Lemma S.14, $\bar{z}^M$ must thus be continuous at $\hat{\theta}$. Now consider a decreasing sequence of types (θ_n) converging to $\hat{\theta} = \theta_2$. Using the definition of θ_2, we have that $D^\star(\bar{z}^M(\theta_n)) \leq q^{\mathrm{OPT}}(\theta^\dagger) = q^{\mathrm{OPT}}(\underline{\theta}) - \Delta$. The continuity of $D^\star \circ \bar{z}^M$ at $\hat{\theta}$ implies that $D^\star(\bar{z}^M(\hat{\theta})) = \lim_{n\to\infty} D^\star(\bar{z}^M(\theta_n)) \leq q^{\mathrm{OPT}}(\underline{\theta}) - \Delta < q^{\mathrm{OPT}}(\underline{\theta})$. The same continuity in turn implies existence of $\epsilon > 0$ such that $D^\star(\bar{z}^M(\hat{\theta} - \epsilon)) < q^{\mathrm{OPT}}(\underline{\theta})$, which contradicts the definition of θ_1 because $\hat{\theta} = \theta_1$. We conclude that, if $\theta_1 = \theta_2 = \hat{\theta}$, then $\hat{\theta} \in \{\underline{\theta}, \theta^\dagger\}$, as claimed. ■

The optimal schedule q^{OPT} must thus satisfy the following properties:

$$q^{\mathrm{OPT}}(\theta) = q^{\mathrm{OPT}}(\underline{\theta}) \qquad \forall\ \theta \in (\underline{\theta}, \theta_1), \tag{S.50}$$

$$q^{\mathrm{OPT}}(\theta) = D^\star(\bar{z}^M(\theta)) \qquad \text{for almost all } \theta \in (\theta_1, \theta_2), \tag{S.51}$$

$$q^{\mathrm{OPT}}(\theta) = q^{\mathrm{OPT}}(\theta^\dagger) \qquad \forall\ \theta \in (\theta_2, \theta^\dagger). \tag{S.52}$$

Observation S.3. *(1) Suppose $\theta_1 > \underline{\theta}$. Then, $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta_1)$ and $\theta^m = \underline{\theta}$; (2) Suppose $\theta_2 < \theta^\dagger$. Then, $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta_2, \theta^\dagger)$ and $\theta^m \leq \theta_2$.*

Proof of Observation S.3. Because $\theta^\dagger \leq \theta^\star$, by Lemma 6 in the main text, $q^{\mathrm{OPT}}(\theta) \leq q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta^\dagger)$.

Part (1). By (S.50), for any $\underline{\theta} < \theta < \theta' < \theta_1$, $q^{\mathrm{OPT}}(\underline{\theta}) = q^{\mathrm{OPT}}(\theta) = q^{\mathrm{OPT}}(\theta') \leq q^{\mathrm{BM}}(\theta') < q^{\mathrm{BM}}(\theta)$, where the first inequality follows from Lemma 6 in the main text, whereas the second one follos from the fact that q^{BM} is decreasing. Hence, $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta_1)$. Lemma 9 in the main text then implies that $\theta^m = \underline{\theta}$.

Part (2). The proof is identical to that of Part (1): Pick $\theta_2 < \theta < \theta' < \theta^\dagger$. By (S.52), $q^{\mathrm{OPT}}(\theta^\dagger) = q^{\mathrm{OPT}}(\theta) = q^{\mathrm{OPT}}(\theta') \leq q^{\mathrm{BM}}(\theta') < q^{\mathrm{BM}}(\theta)$. Hence, $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta_2, \theta^\dagger)$. Lemma 9 in the main text then implies that $\theta^m \leq \theta_2$. ■

Now recall that we want to show that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta^m, \theta^\dagger)$. By virtue of Claim S.3 and Observation S.3, it only remains to show that, when $\theta^m < \theta_2$ and $\theta_1 < \theta_2$,

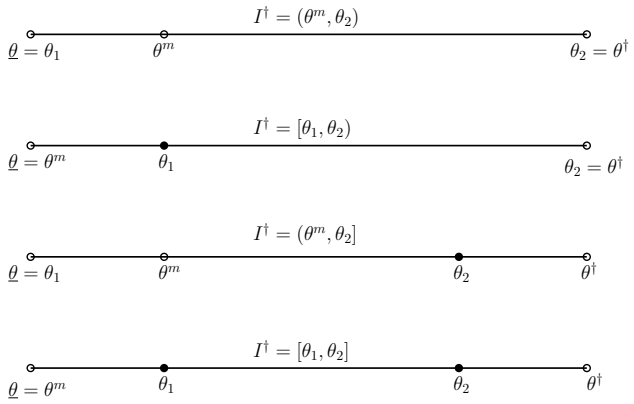


Figure S.6: Interval $I^\dagger$ where it remains to show that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$.

this inequality holds for all $\theta \in I^\dagger$, where[23]

$$I^\dagger \equiv ((\theta^m, \theta_2) \cap [\theta_1, \theta_2)) \cup \left(\{\theta_2\} \cap (\underline{\theta}, \theta^\dagger)\right).$$

Note that $I^\dagger$ is the union of the set $(\theta^m, \theta_2) \cap [\theta_1, \theta_2)$ with the singleton $\{\theta_2\}$, when $\theta_2 < \theta^\dagger$. See Figure S.6 for an illustration.

We make use of this additional technical lemma whose proof is relegated to Step 5 below.

Lemma S.15. *For all $\theta \in I^\dagger$, $\bar{z}^M(\theta) > z^\star(\theta)$.*

Now recall that we are considering the case $\theta_1 < \theta_2$. Condition (S.51) then implies that the set $E \equiv \{\theta \in (\theta_1, \theta_2) : q^{\mathrm{OPT}}(\theta) = D^\star(\bar{z}^M(\theta))\}$ has strictly positive Lebesgue measure. To show that, when $I^\dagger \neq \emptyset$, $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in I^\dagger$, we consider two cases.

Case A: $\bar{z}^M$ is discontinuous at $\theta \in I^\dagger$.

By Lemma S.14, q^{OPT} is then continuous at θ. First suppose that $\theta \in (\theta^m, \theta_2) \cap [\theta_1, \theta_2)$. Let (θ_n) be a decreasing sequence such that $\theta_n \in E$ for all n and $\lim_{n\to\infty} \theta_n = \theta$. By the right-continuity of $\bar{z}^M$ and the continuity of $D^\star$, we have that $\lim_{n\to\infty} D^\star(\bar{z}^M(\theta_n)) = D^\star(\bar{z}^M(\theta))$. The continuity of q^{OPT} at θ in turn implies that $\lim_{n\to\infty} q^{\mathrm{OPT}}(\theta_n) = q^{\mathrm{OPT}}(\theta)$. Because, for every n, $\theta_n \in E$, Condition (S.51) implies that $q^{\mathrm{OPT}}(\theta_n) = D^\star(\bar{z}^M(\theta_n))$. As a result, $D^\star(\bar{z}^M(\theta)) = q^{\mathrm{OPT}}(\theta)$. By Lemma S.15, $\bar{z}^M(\theta) > z^\star(\theta)$, and hence, $q^{\mathrm{OPT}}(\theta) = D^\star(\bar{z}^M(\theta)) <$

[23] To see this, note that, when $\theta_1 = \theta_2 = \theta^\dagger$, Part (1) of Observation S.3 implies that $\theta^m = \underline{\theta}$ and $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta^m, \theta^\dagger)$. Similarly, when $\theta_1 = \theta_2 = \underline{\theta}$, Part (2) of Observation S.3 implies that $\theta^m = \underline{\theta}$, and $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta^m, \theta^\dagger)$. Claim S.3 in turn rules out the possibility that $\theta_1 = \theta_2 = \hat{\theta} \in (\underline{\theta}, \theta^\dagger)$. Hence, it only remains to show that, when $\theta^m < \theta_2$ and $\theta_1 < \theta_2$, the same inequality holds for all $\theta \in I^\dagger$.

$D^\star(z^\star(\theta)) = q^{\mathrm{BM}}(\theta)$. Next suppose that $\theta = \theta_2 < \theta^\dagger$. By Condition (S.52), $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{OPT}}(\theta^\dagger)$ for all $\theta \in (\theta_2, \theta^\dagger)$. Because q^{OPT} is continuous at θ_2, $q^{\mathrm{OPT}}(\theta_2) = q^{\mathrm{OPT}}(\theta^\dagger)$. This means that there exists $\epsilon > 0$ such that $q^{\mathrm{OPT}}(\theta_2) = q^{\mathrm{OPT}}(\theta_2 + \epsilon) < q^{\mathrm{BM}}(\theta_2 + \epsilon) < q^{\mathrm{BM}}(\theta_2)$, where the first inequality follows from Observation S.3 and the second inequality follows from the fact that q^{BM} is decreasing. We conclude that, when $\bar{z}^M$ is discontinuous at $\theta \in I^\dagger$, no matter whether $\theta \in (\theta^m, \theta_2) \cap [\theta_1, \theta_2)$ or $\theta = \theta_2 < \theta^\dagger$, $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$.

Case B: $\bar{z}^M$ is continuous at $\theta \in I^\dagger$.

First suppose that $\theta \in I^\dagger$ with $\theta \neq \theta_1$. Then let (θ_n) be an increasing sequence such that $\theta_n \in E$ for all n and $\lim_{n\to\infty} \theta_n = \theta$. Thus, for every n, $q^{\mathrm{OPT}}(\theta_n) = D^\star(\bar{z}^M(\theta_n))$. Next, suppose that $\theta \in I^\dagger$ with $\theta = \theta_1$. Note that this means that $\theta_1 > \underline{\theta}$. Then let (θ_n) be any sequence such that $\theta_n \in (\underline{\theta}, \theta_1)$ for all n and $\lim_{n\to\infty} \theta_n = \theta_1$. Condition (S.50) then implies that, along such a sequence $q^{\mathrm{OPT}}(\theta_n) = q^{\mathrm{OPT}}(\underline{\theta}) \leq D^\star(\bar{z}^M(\theta_n))$ for every n. No matter which of the two sequences one considers, the continuity of $\bar{z}^M$ along with the continuity of $D^\star$, imply that $\lim_{n\to\infty} D^\star(\bar{z}^M(\theta_n)) = D^\star(\bar{z}^M(\theta))$. Furthermore, because q^{OPT} is weakly decreasing, $q^{\mathrm{OPT}}(\theta) \leq \lim_{n\to\infty} q^{\mathrm{OPT}}(\theta_n) \leq \lim_{n\to\infty} D^\star(\bar{z}^M(\theta_n)) = D^\star(\bar{z}^M(\theta)) < D^\star(z^\star(\theta)) = q^{\mathrm{BM}}(\theta)$, where the strict inequality follows from Lemma S.15 (which establishes that $\bar{z}^M(\theta) > z^\star(\theta)$).

Step 4. Choosing $\theta^\dagger$ to satisfy the Interiority Condition, the existence of $\theta_{\min}$, and the property that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta^m, \min\{\theta^\star, \theta^{\star\star}\})$.

The steps above have established that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta^m, \theta^\dagger)$. To establish the result in the proposition it thus suffices to construct a sequence of cut-offs $\theta^\dagger$, denoted by $(\theta^\dagger_n)$, converging to $\min\{\theta^\star, \theta^{\star\star}\}$, satisfying the Interiority Condition of Definition S.6, and such that $\{\theta \in (\underline{\theta}, \theta^\dagger) : \Lambda^{\mathrm{OPT}}(\theta) + \mu^{\mathrm{OPT}} > 0\} \neq \emptyset$, thus implying that the threshold $\theta_{\min}$ in the previous step is well defined.

Case 1. $\theta^\star < \overline{\theta}$. In this case, the sequence $(\theta^\dagger_n)$ can be taken to be degenerate with $\theta^\dagger_n = \theta^\dagger = \theta^\star$ for all n. Because $\underline{D}(\theta^\star) > \mathrm{q}_\ell = q^{\mathrm{OPT}}(\theta^\star)$, to establish that such a $\theta^\dagger$ satisfies the Interiority Condition of Definition S.6, consider the schedule $q^\dagger$ given by $q^\dagger(\theta) \equiv \underline{D}(\theta)$ for all $\theta \in [\underline{\theta}, \theta^\dagger)$ and $q^\dagger(\theta) \equiv \mathrm{q}_\ell = q^{\mathrm{OPT}}(\theta)$ for all $\theta \in [\theta^\dagger, \overline{\theta}]$. Clearly, $S(\theta; q^\dagger) = \int_{\theta^\dagger}^{\overline{\theta}} (\underline{D}(y) - \mathrm{q}_\ell)\mathrm{d}y > 0$ for all $\theta \in [\underline{\theta}, \theta^\dagger]$. Furthermore, because $q^\dagger(\underline{\theta}) = \underline{D}(\underline{\theta})$, $\underline{\mathrm{DWL}}(\underline{\theta}, q^\dagger(\underline{\theta})) = 0$. Lemma 3 in the main text then implies that $R(q^\dagger) = S(\underline{\theta}; q^\dagger) > 0$. Hence, the sub-program ($\mathbf{FP}^\dagger$) with $\theta^\dagger = \theta^\star$ satisfies the Interiority Condition of Definition S.6.

Setting $\theta^\dagger \equiv \theta^\star$ also ensures that $\{\theta \in (\underline{\theta}, \theta^\dagger) : \Lambda^{\mathrm{OPT}}(\theta) + \mu^{\mathrm{OPT}} > 0\} \neq \emptyset$ and hence that $\theta_{\min}$ is well-defined. To see why, suppose, towards a contradiction, that $\Lambda^{\mathrm{OPT}}(\theta) + \mu^{\mathrm{OPT}} = 0$

for all $\theta \in (\underline{\theta}, \theta^\star)$. The optimality condition in (S.45) then implies that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta^\star)$. Along with the fact that $q^{\mathrm{OPT}}(\theta) = \mathrm{q}_\ell$ for all $\theta \in [\theta^\star, \overline{\theta}]$, this means that $q^{\mathrm{OPT}}(\theta) = q^\star(\theta)$ for all $\theta \in (\underline{\theta}, \overline{\theta}]$. This contradicts the fact that Baron-Myerson-with-quantity-floor is not robustly optimal. We thus conclude that $\{\theta \in (\underline{\theta}, \theta^\dagger) : \Lambda^{\mathrm{OPT}}(\theta) + \mu^{\mathrm{OPT}} > 0\} \neq \emptyset$ and hence that $\theta_{\min}$ is well-defined. Because $\min\{\theta^\star, \theta^{\star\star}\} = \theta^\star$, the results in Step 3 then imply that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta^m, \min\{\theta^\star, \theta^{\star\star}\})$, as claimed.

Case 2. $\theta^\star = \overline{\theta}$. In this case, $\min\{\theta^\star, \theta^{\star\star}\} = \theta^{\star\star}$. Recall that we are considering the case in which $\theta^{\star\star} > \theta^m$. The definition of $\theta^{\star\star}$ along with the fact that q^{OPT} satisfies the constraint $S(\theta; q^{\mathrm{OPT}}) \geq 0$ for all θ, implies that for every $\delta \in (0, \theta^{\star\star} - \theta^m)$, there exists $\theta^\dagger \in (\theta^{\star\star} - \delta, \theta^{\star\star})$ such that $q^{\mathrm{OPT}}(\theta^\dagger) < \underline{D}(\theta^\dagger)$ and $S(\theta^\dagger; q^{\mathrm{OPT}}) > 0$. Now, for any such $\theta^\dagger$, let $q^\dagger \in \mathcal{Q}^\dagger$ be the schedule defined by

$$q^\dagger(\theta) = \begin{cases} \underline{D}(\theta) & \text{for all } \theta < \theta^\dagger \\ q^{\mathrm{OPT}}(\theta) & \text{otherwise.} \end{cases}$$

Clearly, for any $\theta < \theta^\dagger$, $S(\theta; q^\dagger) = S(\theta^\dagger; q^{\mathrm{OPT}}) > 0$. Moreover, because $q^\dagger(\underline{\theta}) = \underline{D}(\underline{\theta})$, $\underline{\mathrm{DWL}}(\underline{\theta}, q^\dagger(\underline{\theta})) = 0$. Lemma 3 in the main text then implies that $R(q^\dagger) = S(\underline{\theta}; q^\dagger) > 0$. Hence, the sub-program ($\mathbf{FP}^\dagger$) with such a chosen $\theta^\dagger$ satisfies the Interiority Condition of Definition S.6.

Next, observe that Lemma 11 in the main text implies that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ on a set $I \subset (\theta^m, \theta^{\star\star})$ of positive Lebesgue measure. When combined with Observation S.2 above and the fact that Λ^{OPT} is weakly increasing, this property implies that there exists $\tilde{\delta} \in (0, \theta^{\star\star} - \theta^m)$ such that, for any $\theta^\dagger \in \left(\theta^{\star\star} - \tilde{\delta}, \theta^{\star\star}\right)$, the set $\{\theta \in (\underline{\theta}, \theta^\dagger) : \Lambda^{\mathrm{OPT}}(\theta) + \mu^{\mathrm{OPT}} > 0\} \neq \emptyset$.

The above results imply existence of a monotone sequence $(\theta_n^\dagger)$, with $\theta_n^\dagger \uparrow \theta^{\star\star}$ as $n \to \infty$, such that, for any n, (a) the sub-program ($\mathbf{FP}^\dagger$) satisfies the Interiority Condition of Definition S.6, and (b) $\{\theta \in (\underline{\theta}, \theta_n^\dagger) : \Lambda^{\mathrm{OPT}}(\theta) + \mu^{\mathrm{OPT}} > 0\} \neq \emptyset$. The result that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta^m, \min\{\theta^\star, \theta^{\star\star}\})$ then follows from these properties along with what we established in Step 3.

Step 5: Proofs of technical lemmas in Step 3.

We conclude the proof of (S.44) by proving the two technical lemmas in Step 3.

Proof of Lemma S.14. We use the following notation in the proof. For any function $g : (\underline{\theta}, \theta^\dagger) \to \mathbb{R}$, and any $\theta \in (\underline{\theta}, \theta^\dagger)$, $g_+(\theta) \equiv \lim_{y \downarrow \theta} g(y)$ and $g_-(\theta) \equiv \lim_{y \uparrow \theta} g(y)$. The proof is in three steps.

Step A. We show that if $\bar{z}^M$ is discontinuous at $\theta \in (\underline{\theta}, \theta^\dagger)$, then z^M is also discontinuous at θ. Let $\varphi \equiv F^\star(\theta)$. Because $F^\star$ is continuous and increasing, it suffices to show that if $\bar{h} \equiv \bar{z}^M \circ (F^\star)^{-1}$ is discontinuous at φ, then $h \equiv z^M \circ (F^\star)^{-1}$ is also discontinuous at φ. To see this, suppose, towards a contradiction, that h is continuous at φ. Then H is differentiable at φ. On the other hand, the assumption that $\bar{h}$ is discontinuous at φ along with the fact that $\bar{h}$ is right-continuous and weakly increasing implies that $\bar{h}_-(\varphi) < \bar{h}_+(\varphi) = \bar{h}(\varphi)$. Note that this means that $\overline{H}$ is not differentiable at φ.

Next observe that $\overline{H}(\varphi) = H(\varphi)$. To see this, recall that, by definition, $\overline{H}(\varphi) \leq H(\varphi)$. Now suppose that $\overline{H}(\varphi) < H(\varphi)$. The continuity of H and $\overline{H}$ then implies that there is an open interval I containing φ such that, for all $\tilde{\varphi} \in I$, $\overline{H}(\tilde{\varphi}) < H(\tilde{\varphi})$. The pooling property then implies that $\overline{H}$ is affine on I, contradicting its non-differentiability at φ. Hence, $\overline{H}(\varphi) = H(\varphi)$.

Now observe that, because $\overline{H} \leq H$ and $\overline{H}(\varphi) = H(\varphi)$, for every $x > \varphi$,

$$\frac{\overline{H}(x) - \overline{H}(\varphi)}{x - \varphi} \leq \frac{H(x) - H(\varphi)}{x - \varphi}.$$

Taking the limit as $x \downarrow \varphi$, we have that the left-hand side converges to $\bar{h}_+(\varphi) = \bar{h}(\varphi)$. The right-hand side, instead, converges to $h(\varphi)$, as H is differentiable at φ. Thus, $\bar{h}(\varphi) \leq h(\varphi)$. Similarly, for every $x < \varphi$,

$$\frac{\overline{H}(\varphi) - \overline{H}(x)}{\varphi - x} \geq \frac{H(\varphi) - H(x)}{\varphi - x}.$$

Taking the limit as $x \uparrow \varphi$, the left-hand side converges to $\bar{h}_-(\varphi)$ whereas the right-hand side converges to $h(\varphi)$. Hence $\bar{h}_-(\varphi) \geq h(\varphi)$. Therefore, $\bar{h}_-(\varphi) \geq h(\varphi) \geq \bar{h}(\varphi)$. On the other hand, the convexity of $\overline{H}$ implies that $\bar{h}_-(\varphi) \leq \bar{h}(\varphi)$. Hence $\bar{h}_-(\varphi) = \bar{h}(\varphi) = \bar{h}_+(\varphi)$. But this contradicts the fact that, as argued above, $\bar{h}_-(\varphi) < \bar{h}_+(\varphi) = \bar{h}(\varphi)$. We thus conclude that h must be discontinuous at φ, which implies discontinuity of z^M at θ, as claimed.

Step B. We now argue that the discontinuity of z^M at θ implies that $S(\theta; q^{\mathrm{OPT}}) = 0$. To see this, note that, because $z^\star$ and $f^\star$ are continuous, the discontinuity must be due to an upward jump of Λ^{OPT} at θ. The complementary slackness conditions in (S.46) then imply that $S(\theta; q^{\mathrm{OPT}}) = 0$.

Step C. We now use the fact that $S(\cdot; q^{\mathrm{OPT}})$ is continuous at θ to establish that q^{OPT} is continuous at θ. We first argue that $q_+^{\mathrm{OPT}}(\theta) \geq \underline{D}(\theta)$. To see why, observe that the

right-derivative of $S(\cdot; q^{\text{OPT}})$ at θ is equal to

$$\lim_{x \downarrow \theta} \frac{S(x; q^{\text{OPT}}) - S(\theta; q^{\text{OPT}})}{x - \theta} = -\lim_{x \downarrow \theta} \frac{\int_\theta^x \left(\underline{D}(y) - q^{\text{OPT}}(y)\right) \mathrm{d}y}{x - \theta} = q_+^{\text{OPT}}(\theta) - \underline{D}(\theta).$$

That q^{OPT} satisfies the robustness constraints implies that $S(y; q^{\text{OPT}}) \geq 0$ for any y in a neighborhood of θ. Because $S(\theta; q^{\text{OPT}}) = 0$, it must be that $S(\cdot; q^{\text{OPT}})$ has a minimum at θ. In turn, this means that the right derivative of $S(\cdot; q^{\text{OPT}})$ is nonnegative at θ, which implies that $q_+^{\text{OPT}}(\theta) \geq \underline{D}(\theta)$. Similarly, by considering the left derivative of $S(\cdot; q^{\text{OPT}})$ at θ, we obtain that $\underline{D}(\theta) \geq q_-^{\text{OPT}}(\theta)$. Combining the last two inequalities, we have that $q_+^{\text{OPT}}(\theta) \geq \underline{D}(\theta) \geq q_-^{\text{OPT}}(\theta)$. Because q^{OPT} is weakly decreasing, $q_-^{\text{OPT}}(\theta) \geq q_+^{\text{OPT}}(\theta)$. This implies that $q_+^{\text{OPT}}(\theta) = q_-^{\text{OPT}}(\theta) = \underline{D}(\theta)$. This last result, together with the fact that q^{OPT} is weakly decreasing, implies that q^{OPT} is continuous at θ, as claimed. ■

Proof of Lemma S.15. First, we show that $\theta_{\min} \leq \theta_2$. Next, we show that, when $\theta_{\min} < \theta_2$, $\bar{z}^M(\theta) > z^\star(\theta)$ for all $\theta \in (\theta_{\min}, \theta_2) \cap [\theta_1, \theta_2)$. We then show that $\theta_{\min} = \theta^m$. Finally, we show that, when $\theta_2 < \theta^\dagger$, $\bar{z}^M(\theta_2) > z^\star(\theta_2)$. We proceed in five steps.

Step A. In this step, we establish that $\theta_{\min} \leq \theta_2$. The result follows from Lemma S.13 (establishing that $\theta_{\min} \leq \theta^m$) along with Observation S.3 (establishing that $\theta^m \leq \theta_2$ if $\theta_2 < \theta^\dagger$) and the maintained assumption that $\theta^m \leq \theta^\dagger$.

Step B. Next, we establish that the weaker inequality $\bar{z}^M(\theta) \geq z^\star(\theta)$ holds for almost all $\theta \in (\theta_1, \theta_2)$. To do so, observe that, because $\theta_2 \leq \theta^\dagger \leq \min\{\theta^\star, \theta^{\star\star}\}$, Lemma 6 in the main text implies that $q^{\text{OPT}}(\theta) \leq q^{\text{BM}}(\theta)$ for any $\theta \in (\underline{\theta}, \theta_2)$. By (S.51), $q^{\text{OPT}}(\theta) = D^\star(\bar{z}^M(\theta))$ for almost all $\theta \in (\theta_1, \theta_2)$. Combining the two properties, we have that $D^\star(\bar{z}^M(\theta)) \leq D^\star(z^\star(\theta))$ for almost all $\theta \in (\theta_1, \theta_2)$. Because $D^\star$ is decreasing, this means that $\bar{z}^M(\theta) \geq z^\star(\theta)$ for almost all $\theta \in (\theta_1, \theta_2)$.

Step C. Equipped with the results in the previous two steps, we now establish that, when $\theta_{\min} < \theta_2$, the inequality $\bar{z}^M(\theta) \geq z^\star(\theta)$ is strict for all $\theta \in (\theta_{\min}, \theta_2) \cap [\theta_1, \theta_2)$. Let $\varphi_{\min} \equiv F^\star(\theta_{\min})$, $\varphi_1 \equiv F^\star(\theta_1)$ and $\varphi_2 \equiv F^\star(\theta_2)$. Next take any $\varphi \in (\varphi_{\min}, \varphi_2) \cap [\varphi_1, \varphi_2)$, and let $\theta \equiv F^{\star -1}(\varphi)$. The proof considers two cases.

Case C.1: There exists $\delta > 0$ such that $\overline{H}(\varphi') < H(\varphi')$ for all $\varphi' \in \tilde{I} \equiv (\varphi, \varphi + \delta)$.

By the pooling property, $\bar{h}$ is constant on $\tilde{I}$. Because $\bar{h}$ is right-continuous, this constant value is equal to $\bar{h}(\varphi)$. Using $\bar{h}(\varphi) = \bar{z}^M((F^\star)^{-1}(\varphi)) = \bar{z}^M(\theta)$, we thus have that, for all $\varphi' \in \tilde{I}$, $\bar{z}^M((F^\star)^{-1}(\varphi')) = \bar{z}^M(\theta)$. Now, choose $\epsilon > 0$ such that $F^\star(\theta + \epsilon) \in \tilde{I}$ and

$\bar{z}^M(\theta+\epsilon) \geq z^\star(\theta+\epsilon)$. Note that, because $\bar{z}^M(\theta') \geq z^\star(\theta')$ for almost all $\theta' \in (\theta_1, \theta_2)$ (by Step B), such an ϵ exists. Then, $\bar{z}^M(\theta) = \bar{z}^M(\theta+\epsilon) \geq z^\star(\theta+\epsilon) > z^\star(\theta)$, where the last inequality follows from the fact that $z^\star$ is increasing.

Case C.2. For every $\delta > 0$, there exists $\varphi' \in (\varphi, \varphi+\delta) \cap (\varphi, \varphi_2)$ such that $\overline{H}(\varphi') = H(\varphi')$.

Note that, in this case, $\overline{H}(\varphi) = H(\varphi)$. If this was not true, the continuity of $\overline{H}$ and H (which follows from their definitions), would imply existence of a right neighborhood I' of φ such that $\overline{H}(\varphi') < H(\varphi')$ for all $\varphi' \in I'$, contradicting the assumption defining Case C.2. Hence, there exists a non-increasing sequence (φ_n) such that $\overline{H}(\varphi_n) = H(\varphi_n)$ for all n and with $\varphi_n \to \varphi$ as $n \to \infty$. By taking the right-derivatives of $\overline{H}$ and H along this sequence and using the fact that $\bar{h}$ and h are right-continuous, we have that

$$\bar{h}(\varphi) = \lim_{n\to\infty} \frac{\overline{H}(\varphi_n) - \overline{H}(\varphi)}{\varphi_n - \varphi} = \lim_{n\to\infty} \frac{H(\varphi_n) - H(\varphi)}{\varphi_n - \varphi} = h(\varphi).$$

Hence, $\bar{z}^M(\theta) = \bar{h}(\varphi) = h(\varphi) = z^M(\theta) > z^\star(\theta)$, where the last inequality follows because $\theta > \theta_{\min}$, and therefore, $\Lambda^{\text{OPT}}(\theta) + \mu^{\text{OPT}} > 0$.

Combining Case C.1 with Case C.2, we thus conclude that, for any $\theta \in (\theta_{\min}, \theta_2) \cap [\theta_1, \theta_2)$, $\bar{z}^M(\theta) > z^\star(\theta)$, as claimed.

Step D. Next, we establish that $\theta^m = \theta_{\min}$. Assume, towards a contradiction, that $\theta^m \neq \theta_{\min}$. Lemma S.13 (establishing that $\theta_{\min} \leq \theta^m$) then implies that $\underline{\theta} \leq \theta_{\min} < \theta^m$. By Lemma 9 in the main text,

$$q^{\text{OPT}}(\theta) = q^{\text{BM}}(\theta) \qquad \forall\ \theta \in (\underline{\theta}, \theta^m). \tag{S.53}$$

Hence, Observation S.3 (establishing that $q^{\text{OPT}}(\theta) < q^{\text{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta_1)$ if $\theta_1 > \underline{\theta}$) implies that $\theta_1 = \underline{\theta}$. The arguments in Step A above establish that $\theta^m \leq \theta_2$. Hence, $(\theta_{\min}, \theta^m) \subset (\theta_1, \theta_2)$. Step C above then implies $\bar{z}^M(\theta) > z^\star(\theta)$ for all $\theta \in (\theta_{\min}, \theta^m)$. Condition (S.51) in turn implies that there exists $\theta' \in (\theta_{\min}, \theta^m)$ such that $q^{\text{OPT}}(\theta') = D^\star(\bar{z}^M(\theta'))$. Combining the two properties, we conclude that $q^{\text{OPT}}(\theta') = D^\star(\bar{z}^M(\theta')) < D^\star(z^\star(\theta')) = q^{\text{BM}}(\theta')$, which contradicts (S.53). Hence, $\theta^m = \theta_{\min}$.

Step E. We conclude by showing that, when $\theta_2 < \theta^\dagger$, $\bar{z}^M(\theta_2) > z^\star(\theta_2)$. First, we argue that q^{OPT} is continuous at θ_2. If $\bar{z}^M$ is discontinuous at θ_2, this follows from Lemma S.14. If, instead, $\bar{z}^M$ is continuous at θ_2, the continuity of q^{OPT} at θ_2 follows from the following arguments. Because $D^\star$ is continuous, $D^\star(\bar{z}^M(\cdot))$ is also continuous at θ_2. Then consider

an increasing sequence (θ_n) such that $\theta_n \in E$ for all n and $\lim_{n\to\infty} \theta_n = \theta_2$. Along such a sequence $q^{\mathrm{OPT}}(\theta_n) = D^\star(\bar{z}^M(\theta_n))$ and hence

$$\lim_{n\to\infty} q^{\mathrm{OPT}}(\theta_n) = \lim_{n\to\infty} D^\star(\bar{z}^M(\theta_n)) = D^\star(\bar{z}^M(\theta_2)),$$

where the last equality follows from continuity of $D^\star(\bar{z}^M(\cdot))$. Furthermore, because $D^\star(\bar{z}^M(\cdot))$ is continuous, the definition of θ_2 along with (S.52) implies that $D^\star(\bar{z}^M(\theta_2)) = q^{\mathrm{OPT}}(\theta^\dagger)$. The last two properties jointly imply that $\lim_{n\to\infty} q^{\mathrm{OPT}}(\theta_n) = q^{\mathrm{OPT}}(\theta^\dagger)$. Because q^{OPT} is weakly decreasing and $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{OPT}}(\theta^\dagger)$ for all $\theta \in (\theta_2, \theta^\dagger)$, we have that $q^{\mathrm{OPT}}(\theta_2) = q^{\mathrm{OPT}}(\theta^\dagger)$ and hence q^{OPT} is continuous at θ_2.

Next, observe that there exists $\epsilon > 0$ such that

$$D^\star(\bar{z}^M(\theta_2)) \leq q^{\mathrm{OPT}}(\theta^\dagger) = q^{\mathrm{OPT}}(\theta_2) = q^{\mathrm{OPT}}(\theta_2 + \epsilon) < q^{\mathrm{BM}}(\theta_2 + \epsilon) < q^{\mathrm{BM}}(\theta_2) = D^\star(z^\star(\theta_2)).$$

The first inequality follows from the definition of θ_2, along with the property that $\bar{z}^M$ is right-continuous. The first and second equalities follow from the continuity of q^{OPT} at θ_2, along with Condition (S.52). The second inequality follows from Observation S.3. The third inequality follows from the fact that q^{BM} is decreasing. The last equality follows from the definition of the q^{BM} schedule. Because $D^\star$ is decreasing, we conclude that $\bar{z}^M(\theta_2) > z^\star(\theta_2)$, as claimed ■

This completes the proof of Proposition S.6. *Q.E.D.*

S.2 More permissive robustness constraints

We consider more permissive robustness constraints whereby the designer can choose over mechanisms whose guarantee is not too small relative to the maximal one. Formally, the shortlist of approved mechanisms is $\mathcal{M}^{\mathrm{SL}}_\gamma \equiv \{M \in \mathcal{M} : G(M) \geq \gamma G(M') \ \forall\ M' \in \mathcal{M}\}$, where $\gamma \in [0, 1]$. The analysis in the main text corresponds to the case $\gamma = 1$. Here, we consider the case $\gamma \in [0, 1)$. The proofs of all results in this section are omitted as they build on arguments similar to those for the corresponding results in the main text and are available upon request. Lemma 1 in the main text remains unchanged. Lemma 2 instead is generalized as follows:

Lemma S.16 (Shortlist characterization)**.** *Take any IC and IR mechanism $M \equiv (q, u) \in \mathcal{M}$.*

Then $M \in \mathcal{M}_\gamma^{\mathrm{SL}}$ *if and only if*

$$\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta) \geq \gamma G^* \qquad \forall\ \theta \in \Theta. \tag{S.54}$$

There are two important distinctions from the case $\gamma = 1$ in the main text. First, when $\gamma < 1$ the rent $u(\overline{\theta})$ for the least efficient type is not necessarily zero. Second, $q(\overline{\theta})$ is not restricted to be equal to q_ℓ. An immediate implication of the last lemma is that the shortlist for $\gamma' > \gamma$ is a subset of the shortlist for γ. Below, we focus on mechanisms in $\mathcal{M}_\gamma^{\mathrm{SL}}$ for which $u(\overline{\theta}) = 0$. The reason for focusing on this subset of $\mathcal{M}_\gamma^{\mathrm{SL}}$ is that robustly optimal mechanisms always feature no rent for the highest-cost type. The next lemma generalizes Lemma 3 in the main text.

Lemma S.17. *Take any weakly decreasing function* $q : \Theta \to \mathbb{R}_+$. *(1) For any* $\theta \in \Theta$, *constraint (S.54) is equivalent to*

$$\int_\theta^{\overline{\theta}} q(y)dy \leq \int_\theta^{\overline{\theta}} \underline{D}(y)dy - \int_\theta^{\underline{P}(q(\theta))} \left(\underline{D}(y) - q(\theta)\right) dy + (1-\gamma)G^*. \tag{S.55}$$

(2) The following statements are equivalent: (a) the inequality in (S.55) holds for all $\theta \in \Theta$; *(b) the inequality in (S.55) holds for* $\theta \in \{\underline{\theta}, \overline{\theta}\}$ *and, for all* $\theta \in (\underline{\theta}, \overline{\theta})$,

$$\int_\theta^{\overline{\theta}} q(y)dy \leq \int_\theta^{\overline{\theta}} \underline{D}(y)dy + (1-\gamma)G^*.$$

We now modify the definition of Baron-Myerson-with-quantity-floor mechanism to account for the more permissive shortlist. Because the function $\underline{V}(\mathrm{q}) - \overline{\theta}\mathrm{q}$ is concave in q, there are two solutions to the equation $\underline{V}(\mathrm{q}) - \overline{\theta}\mathrm{q} = \gamma G^*$. Denote these two solutions by $\underline{\mathrm{q}}_\ell^\gamma$ and $\bar{\mathrm{q}}_\ell^\gamma$, with $\bar{\mathrm{q}}_\ell^\gamma > \underline{\mathrm{q}}_\ell^\gamma$. The generalization of Baron-Myerson-with-quantity-floor mechanism uses the lowest of the two floors, $\underline{\mathrm{q}}_\ell^\gamma$. This is because, as shown in Proposition S.8 below, the relevant constraint imposed by robustness is the one requiring that $q(\theta) \geq \underline{\mathrm{q}}_\ell^\gamma$ for all $\theta \in \Theta$.

Definition S.7. *The* ***Baron-Myerson-with-quantity-floor*** *mechanism is the mechanism* $M_\gamma^\star \equiv (q_\gamma^\star, u_\gamma^\star)$ *where* $q_\gamma^\star$ *is the quantity schedule defined, for all* θ, *by*

$$q_\gamma^\star(\theta) \equiv \max\{q^{\mathrm{BM}}(\theta), \underline{\mathrm{q}}_\ell^\gamma\}$$

and where $u_\gamma^\star$ *is given by* $u_\gamma^\star(\theta) = \int_\theta^{\overline{\theta}} q_\gamma^\star(y)dy$ *for all* θ.

Proposition S.7 (Optimality of Baron-Myerson-with-quantity-floor)**.** *The mechanism $M^\star_\gamma$ is robustly optimal if and only if the following conditions jointly hold:*

$$\int_{\underline{\theta}}^{\overline{\theta}} q^\star_\gamma(y)dy \leq \int_{\underline{\theta}}^{\overline{\theta}} \underline{D}(y)dy - \int_{\underline{\theta}}^{\underline{P}(q^\star_\gamma(\underline{\theta}))} \left[\underline{D}(y) - q^\star_\gamma(\underline{\theta})\right]dy + (1-\gamma)G^*,$$

$$\int_{\theta}^{\overline{\theta}} q^\star_\gamma(y)dy \leq \int_{\theta}^{\overline{\theta}} \underline{D}(y)dy + (1-\gamma)G^* \qquad \forall \theta > \underline{\theta}.$$

Let $\theta^\star_\gamma$ be the threshold defined as follows. If $q^{\mathrm{BM}}(\overline{\theta}) \leq \underline{\mathrm{q}}^\gamma_\ell$, by continuity of q^{BM} along with the fact that $q^{\mathrm{BM}}(\underline{\theta}) > \underline{\mathrm{q}}^\gamma_\ell$, let $\theta^\star_\gamma$ be the unique solution to $q^{\mathrm{BM}}(\theta^\star_\gamma) = \underline{\mathrm{q}}^\gamma_\ell$. If, instead, $q^{\mathrm{BM}}(\overline{\theta}) > \underline{\mathrm{q}}^\gamma_\ell$ (i.e., if q^{BM} never crosses $\underline{\mathrm{q}}^\gamma_\ell$), then let $\theta^\star_\gamma \equiv \overline{\theta}$. Next let

$$\theta^m_\gamma \equiv \max\{\theta : \theta \in \arg\min_{y\in\Theta} \underline{W}(y, q^\star_\gamma)\},$$

where, as in the main text, given any function q, for all $\theta \in \Theta$, $\underline{W}(\theta, q) \equiv \underline{V}(q(\theta)) - \theta q(\theta) - \int_\theta^{\overline{\theta}} q(y)dy$.

The following proposition establishes a result similar to the one in Proposition 2 in the main text:[24]

Proposition S.8. *If Baron-Myerson-with-quantity-floor is not robustly optimal, then $\theta^m_\gamma < \theta^\star_\gamma$, and every robustly optimal mechanism $M^{\mathrm{OPT}} = (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$ is such that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta^m_\gamma]$, $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta^m_\gamma, \theta^\star_\gamma)$, and $q^{\mathrm{OPT}}(\theta) = \underline{\mathrm{q}}^\gamma_\ell$ for all $\theta \in (\theta^\star_\gamma, \overline{\theta}]$.*

Next, consider price regulation. One can then show that every robustly optimal price regulation is such that, for all $\theta > \underline{\theta}$, the price is as in the Baron-Myerson-with-price-cap mechanism. That is, $p^{\mathrm{OPT}}_\gamma(\theta) = \min(z^\star(\theta), \underline{P}(\underline{\mathrm{q}}^\gamma_\ell))$, with $\underline{P}(\underline{\mathrm{q}}^\gamma_\ell) > \overline{\theta}$. Finally, one can show that a result analogous to Proposition 4 in the main text (which compares price and quantity regulation) holds. However, the conditions for quantity regulation to dominate price regulation are more stringent because of the larger price-cap.

[24] The difference between Proposition 2 in the main text (with the stronger version of part (2) established in Section S.1 in this supplement) and Proposition S.8 stems from the fact that, when $\gamma < 1$, constraint qualification of the Lagrangian methods used to establish that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for *all* $\theta \in (\theta^m_\gamma, \theta^\star_\gamma)$ holds irrespective of whether $\theta^\star_\gamma < \overline{\theta}$ or $\theta^\star_\gamma = \overline{\theta}$.

Proposition S.9 (quantity vs price regulation)**.** *(1) If the Baron-Myerson-with-quantity-floor mechanism $M^\star_\gamma$ is robustly optimal, quantity regulation dominates price regulation (strictly if $D^\star(\underline{P}(\mathrm{q}^\gamma_\ell)) > q^\star_\gamma(\bar{\theta})$). (2) If the Baron-Myerson-with-quantity-floor mechanism $M^\star_\gamma$ is not robustly optimal and $D^\star(\underline{P}(\mathrm{q}^\gamma_\ell)) = q^\star_\gamma(\bar{\theta})$, price regulation strictly dominates quantity regulation.*

S.3 More general forms of technological uncertainty

In this section, we consider the case in which the set of feasible technologies $\mathcal{F}$ need not coincide with the entire set $\mathrm{CDF}(\Theta)$ of cdfs supported on Θ. Namely, we assume there exists a cdf $\underline{F} \in \mathrm{CDF}(\Theta)$ such that $\mathcal{F}$ is the set of all cdfs $F \in \mathrm{CDF}(\Theta)$ with $F(\theta) \geq \underline{F}(\theta)$ for all $\theta \in \Theta$. In the main text, we assumed $\underline{F}(\theta)$ is the Dirac distribution that puts unit mass at $\bar{\theta}$. Here, instead, we allow $\underline{F}$ to have support $[\theta_s, \bar{\theta}]$, where $\theta_s \in (\underline{\theta}, \bar{\theta})$.[25]

Definition S.8. *The cdf $\underline{F}$ is regular with respect to θ_s if it is absolutely continuous over $\mathbb{R}$ with density $f(\theta) > 0$ if and only if $\theta \in [\theta_s, \bar{\theta}]$ and with $\underline{z}(\theta) \equiv \theta + \underline{F}(\theta)/\underline{f}(\theta)$ continuous and increasing over $[\theta_s, \bar{\theta}]$.*

Let $\underline{M}^{\mathrm{BM}} = (\underline{q}^{\mathrm{BM}}, \underline{u}^{\mathrm{BM}}) \in \mathcal{M}$ denote an arbitrary IC and IR mechanism that is optimal under the model $(\underline{V}, \underline{F})$, in the usual Bayesian sense. Note that such a mechanism is not unique, but in any such mechanism $\underline{q}^{\mathrm{BM}}$ is weakly decreasing, $\underline{u}^{\mathrm{BM}}(\bar{\theta}) = 0$, and $\underline{u}^{\mathrm{BM}}(\theta) = \int_\theta^{\bar{\theta}} \underline{q}^{\mathrm{BM}}(y)dy$ for all θ. Then let

$$G^*_s \equiv \int_{\theta_s}^{\bar{\theta}} \underline{W}(\theta, \underline{q}^{\mathrm{BM}}) \underline{F}(\mathrm{d}\theta),$$

where, as in the main text, for any weakly decreasing functions q and all $\theta \in \Theta$, $\underline{W}(\theta, q) \equiv \underline{V}(q(\theta)) - \theta q(\theta) - \int_\theta^{\bar{\theta}} q(y)dy$. Thus, G^*_s denotes the buyer's expected welfare under the mechanism $\underline{M}^{\mathrm{BM}}$ when the gross value function is $\underline{V}$ and the technology is $\underline{F}$. Formally, $W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F}) = G^*_s$. When $\theta_s = \bar{\theta}$, as in the main text, $G^*_s = G^*$. Finally, for any IC and IR mechanism $M = (q, u) \in \mathcal{M}$, let $\underline{w}_q \equiv \inf_{\theta \leq \theta_s} \underline{W}(\theta, q)$. The next proposition generalizes Lemma 2 in the main text by providing a complete characterization of the short list.

[25] The results below do not hinge on the fact that the largest element of the support of $\underline{F}$ coincides with that of $F^\star$, i.e., with $\bar{\theta}$. They extend to the case where the support of $\underline{F}$ is $[\theta_s, \theta^H)$, with $\theta^H \geq \bar{\theta}$ possibly equal to $\theta^H = +\infty$.

Proposition S.10. *Suppose $\underline{F}$ is regular with respect to θ_s. (1) For any $M \in \mathcal{M}^{\mathrm{SL}}$, $G(M) = G_s^*$. (2) A mechanism $M \equiv (q,u) \in \mathcal{M}^{\mathrm{SL}}$ if and only if the following conditions jointly hold: (a) q is weakly decreasing; (b) for all θ, $u(\theta) = \int_\theta^{\overline{\theta}} q(y)dy$; (c) $q(\theta) = \underline{q}^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta_s, \overline{\theta})$; (d) $\underline{V}(q(\theta)) - \theta q(\theta) - \int_\theta^{\overline{\theta}} q(y)dy \geq G_s^*$ for all $\theta \leq \theta_s$; and (e) $\underline{w}_q \underline{F}(\theta) + \int_\theta^{\overline{\theta}} \underline{W}(y, \underline{q}^{\mathrm{BM}})\underline{F}(dy) \geq G_s^*$ for all $\theta \in [\theta_s, \overline{\theta}]$.*

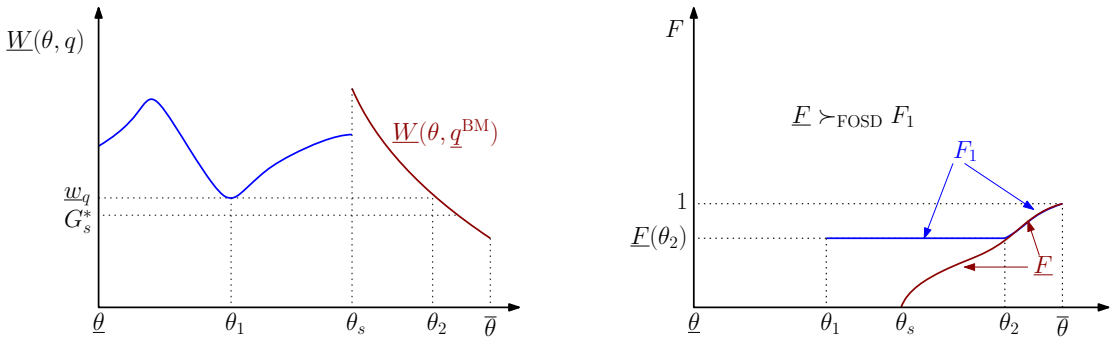
The formal proof of the proposition is below. Before getting to it, we sketch a few key arguments. Part (1) follows from the fact that Nature can always pick $(\underline{V}, \underline{F})$, which implies that the guarantee of any IC and IR mechanism is bounded above by the maximal welfare attainable under the lowest gross value function $\underline{V}$ and the worst technology $\underline{F}$. This upper bound on the guarantee can be achieved by offering a mechanism $M \equiv (q,u)$ in which $q(\theta) = \underline{q}^{\mathrm{BM}}(\theta_s)$ for all $\theta \leq \theta_s$, $q(\theta) = \underline{q}^{\mathrm{BM}}(\theta)$ for all $\theta > \theta_s$, and $u(\theta) = \int_\theta^{\overline{\theta}} q(y)dy$ for all θ. Against such a mechanism an adversarial Nature cannot do better than selecting $(V,F) = (\underline{V}, \underline{F})$, yielding the buyer a payoff of G_s^*. Conditions (a)-(d) in Part (2) are generalizations of the robustness constraints in Lemma 2 in the main text. Condition (c) follows from the fact that Nature can always choose the model $(\underline{V}, \underline{F})$ in which case worst-case optimality dictates that the output procured from types $\theta \in (\theta_s, \overline{\theta})$ be the same as under the mechanism $\underline{M}^{\mathrm{BM}} \equiv (\underline{q}^{\mathrm{BM}}, \underline{u}^{\mathrm{BM}})$. Note that Condition (d) applies only to types $\theta \leq \theta_s$ as Nature can now select a Dirac distribution at θ only when $\theta \leq \theta_s$. Condition (e) stems from the fact that Nature's best response can be a non-Dirac distribution. To understand this constraint, consider Figure S.7. Let $\theta_1 < \theta_s$ be a cost level at which the function $\underline{W}(\cdot, q)$ reaches the minimum over $[\underline{\theta}, \theta_s]$, i.e., $\underline{W}(\theta_1, q) = \underline{w}_q$, and let $\theta_2 > \theta_s$ be a cost level such that $\underline{W}(\theta_2, q) = \underline{W}(\theta_2, \underline{q}^{\mathrm{BM}}) = \underline{w}_q$. Suppose Nature picks a distribution F_1 with an atom at θ_1 equal to $\underline{F}(\theta_2)$ and which agrees with $\underline{F}$ on all $\theta \geq \theta_2$. Because $\underline{W}(\cdot, \underline{q}^{\mathrm{BM}})$ is decreasing, even if $\underline{w}_q \geq G_s^*$, in the absence of the new constraint in Condition (e), it may well be the case that

$$\underline{w}_q F_1(\theta_1) + \int_{\theta_2}^{\overline{\theta}} \underline{W}(\theta, \underline{q}^{\mathrm{BM}}) F_1(\mathrm{d}\theta) = \underline{w}_q \underline{F}(\theta_2) + \int_{\theta_2}^{\overline{\theta}} \underline{W}(\theta, \underline{q}^{\mathrm{BM}}) \underline{F}(\mathrm{d}\theta) < G_s^*,$$

in which case the mechanism's guarantee is below G_s^*. To account for such possibilities, worst-case optimality requires that the constraint in Condition (e) holds.

Proof of Proposition S.10. Part 1. For any IC and IR mechanism $M = (q,u) \in \mathcal{M}$,

$$G(M) = \inf_{(V,F) \in \mathcal{V} \times \mathcal{F}} W(M; V, F) \leq W(M; \underline{V}, \underline{F}) \leq_{(i)} W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F}) = G_s^*. \tag{S.56}$$



A: Function $\underline{W}(\cdot, q)$. B: Technology F_1 yielding welfare below G_s^*.

Figure S.7: Graphical illustration of robustness constraint in (e) of Proposition S.10.

Inequality (i) holds because $\underline{M}^{\mathrm{BM}}$ maximizes $W(\cdot; \underline{V}, \underline{F})$ over $\mathcal{M}$. We now show that there exists an IC and IR mechanism $\underline{M} \in \mathcal{M}$ such that $G(\underline{M}) = G_s^*$. Let $\underline{M} \equiv (\underline{q}, \underline{u})$ be the mechanism in which

$$\underline{q}(\theta) = \begin{cases} \underline{q}^{\mathrm{BM}}(\theta_s) & \text{if } \theta < \theta_s \\ \underline{q}^{\mathrm{BM}}(\theta) & \text{if } \theta \geq \theta_s, \end{cases}$$

and $\underline{u}(\theta) = \int_\theta^{\overline{\theta}} \underline{q}(y)dy$ for all θ. Clearly, $\underline{M} \in \mathcal{M}$. Furthermore, because $\underline{F}$ has support $[\theta_s, \overline{\theta}]$, $W(\underline{M}; \underline{V}, \underline{F}) = W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F})$. Now, recall that $\underline{M}^{\mathrm{BM}} = (\underline{q}^{\mathrm{BM}}, \underline{u}^{\mathrm{BM}})$ is the optimal mechanism for the model $(\underline{V}, \underline{F})$. When $\underline{F}$ is regular with respect to θ_s, $\underline{q}^{\mathrm{BM}}$ is such that, for all $\theta \geq \theta_s$, $\underline{q}^{\mathrm{BM}}(\theta) = \underline{D}(\underline{z}(\theta))$, where, for all $\theta \geq \theta_s$, $\underline{z}(\theta) \equiv \theta + \underline{F}(\theta)/\underline{f}(\theta)$. Thus, $\underline{q}(\theta) \leq \underline{D}(\theta)$ for all θ, with the inequality strict for $\theta \neq \theta_s$. Part A of Lemma 5 in the main text then implies that $\underline{W}(\cdot, \underline{q})$ is weakly decreasing over Θ. Furthermore, because, for all $F \in \mathcal{F}$, $\underline{F} \succ_{FOSD} F$, we have that $W(\underline{M}; \underline{V}, F) \geq W(\underline{M}; \underline{V}, \underline{F})$. Because, for any $V \in \mathcal{V}$ and any $F \in \mathcal{F}$, $W(\underline{M}; V, F) \geq W(\underline{M}; \underline{V}, F)$, we thus have that $W(\underline{M}; V, F) \geq W(\underline{M}; \underline{V}, \underline{F})$. We conclude that $G(\underline{M}) = W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F}) = G_s^*$.

Part 2: Necessity. If $M = (q, u) \in \mathcal{M}^{\mathrm{SL}}$, then M is IC and IR, and, therefore, q is weakly decreasing and $u(\theta) = u(\overline{\theta}) + \int_\theta^{\overline{\theta}} q(y)dy$ for all θ. Furthermore, by the result in Part 1, it must be that $G(M) = G_s^*$. Hence, $u(\overline{\theta}) = 0$.

Recall that, for any $\theta \geq \theta_s$, $\underline{q}^{\mathrm{BM}}(\theta) = \arg\max_{\mathrm{q} \in [0, \bar{\mathrm{q}}]} \{\underline{V}(\mathrm{q}) - \underline{z}(\theta)\mathrm{q}\}$. If $q(\theta) \neq \underline{q}^{\mathrm{BM}}(\theta)$ for a positive Lebesgue measure subset of $[\theta_s, \overline{\theta}]$, then inequality (i) in (S.56) is strict and hence $W(M; \underline{V}, \underline{F}) < W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F}) = G_s^\star$. This means that $G(M) < G_s^*$ and hence $M \notin \mathcal{M}^{\mathrm{SL}}$, a contradiction. Because $\underline{q}^{\mathrm{BM}}$ is weakly decreasing and continuous over $[\theta_s, \overline{\theta}]$, we conclude that $q(\theta) = \underline{q}^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta_s, \overline{\theta})$.

Next, observe that, for any $\theta < \theta_s$, $\mathcal{F}$ contains a distribution F corresponding to a Dirac measure at $\theta < \theta_s$ (indeed, $\underline{F} \succ_{\text{FOSD}} F$). Welfare under the lowest gross value function $\underline{V}$ and such an F is $\underline{V}(q(\theta)) - \theta q(\theta) - \int_\theta^{\overline{\theta}} q(y)dy = \underline{W}(\theta, q)$. Hence, it must be that $\underline{W}(\theta, q) \geq G_s^*$.

Finally, observe that, by Condition (c), the inequality in the constraint in Condition (e) holds with an equality for $\theta = \theta_s$. Suppose there exists $\tilde{\theta} \in (\theta_s, \overline{\theta}]$ such that

$$\underline{w}_q \underline{F}(\tilde{\theta}) + \int_{\tilde{\theta}}^{\overline{\theta}} \underline{W}(y, \underline{q}^{\text{BM}}) \underline{F}(\text{d}y) < G_s^*. \tag{S.57}$$

By definition of $\underline{w}_q$, there exists $\theta' \leq \theta_s$ such that $\underline{W}(\theta', q)$ is arbitrarily close to $\underline{w}_q$. Let $\widetilde{F}$ be the cdf given by

$$\widetilde{F}(\theta) = \begin{cases} 0 & \text{if } \theta < \theta' \\ \underline{F}(\tilde{\theta}) & \text{if } \theta \in [\theta', \tilde{\theta}) \\ \underline{F}(\theta) & \text{if } \theta \geq \tilde{\theta}. \end{cases}$$

Clearly, $\widetilde{F}(\theta) \geq \underline{F}(\theta)$ for all θ, and hence, $\widetilde{F} \in \mathcal{F}$. Welfare under the mechanism $M = (q, u)$ when Nature selects the model $(\underline{V}, \widetilde{F})$ is equal to

$$W(M; \underline{V}, \widetilde{F}) = \underline{W}(\theta', q) \underline{F}(\tilde{\theta}) + \int_{\tilde{\theta}}^{\overline{\theta}} \underline{W}(y, \underline{q}^{\text{BM}}) \underline{F}(\text{d}y) < G_s^*,$$

where the inequality follows from Condition (S.57) and the fact that $\underline{W}(\theta', q)$ is arbitrarily close to $\underline{w}_q$. This, however, is a contradiction to $M \in \mathcal{M}^{\text{SL}}$.

We conclude that properties (a)-(e) are jointly necessary for any $M \in \mathcal{M}^{\text{SL}}$.

Part 2: Sufficiency. Take any mechanism M satisfying properties (a)-(e). By virtue of (a) and (b), M is IC and IR. By virtue of the result in Part 1 of the proposition, it thus suffices to show that $W(M; V, F) \geq G_s^*$ for any model $(V, F) \in \mathcal{V} \times \mathcal{F}$. First, suppose F is a Dirac distribution on some $\theta \leq \theta_s$. Then, Condition (d) implies that

$$W(M; V, F) \geq W(M; \underline{V}, F) = \underline{V}(q(\theta)) - \theta q(\theta) - \int_\theta^{\overline{\theta}} q(y)dy \geq G_s^*.$$

Next consider any model $(V, F) \in \mathcal{V} \times \mathcal{F}$ in which F puts positive probability on $\theta < \theta_s$.

Then,[26]

$$W(M;V,F) \geq W(M;\underline{V},F) \geq \underline{w}_q F(\theta_s) + \int_{\theta_s}^{\overline{\theta}} \underline{W}(\theta,q)F(\mathrm{d}\theta) = \underline{w}_q F(\theta_s) + \int_{\theta_s}^{\overline{\theta}} \underline{W}(\theta,\underline{q}^{\mathrm{BM}})F(\mathrm{d}\theta), \tag{S.58}$$

where the second inequality follows from the definition of $\underline{w}_q$, whereas the equality follows from the fact that $q(\theta) = \underline{q}^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta_s, \bar{\theta})$. Now partition $[\theta_s, \overline{\theta}]$ into $\Theta_1 \equiv \{\theta \in [\theta_s, \overline{\theta}] : \underline{W}(\theta, \underline{q}^{\mathrm{BM}}) \leq \underline{w}_q\}$ and $\Theta_2 \equiv [\theta_s, \overline{\theta}] \setminus \Theta_1$. Note that $\underline{W}(\cdot, \underline{q}^{\mathrm{BM}})$ is decreasing over $[\theta_s, \overline{\theta}]$ and hence Θ_1 is a (possibly empty) interval. If Θ_1 is empty, let $\hat{\theta} \equiv \theta_s$. Else, let $\hat{\theta}$ be the left endpoint of Θ_1. Using (S.58), we have that

$$\begin{aligned} W(M;V,F) &\geq \underline{w}_q F(\theta_s) + \int_{\theta_s}^{\hat{\theta}} \underline{W}(\theta,\underline{q}^{\mathrm{BM}})F(\mathrm{d}\theta) + \int_{\hat{\theta}}^{\overline{\theta}} \underline{W}(\theta,\underline{q}^{\mathrm{BM}})F(\mathrm{d}\theta) \\ &\geq \underline{w}_q F(\theta_s) + \underline{w}_q \Big(F(\hat{\theta}) - F(\theta_s)\Big) + \int_{\hat{\theta}}^{\overline{\theta}} \underline{W}(\theta,\underline{q}^{\mathrm{BM}})F(\mathrm{d}\theta) \\ &= \underline{w}_q F(\hat{\theta}) + \int_{\hat{\theta}}^{\overline{\theta}} \underline{W}(\theta,\underline{q}^{\mathrm{BM}})F(\mathrm{d}\theta), \end{aligned} \tag{S.59}$$

where the second inequality follows from the fact that $\underline{W}(\cdot, \underline{q}^{\mathrm{BM}}) \geq \underline{w}_q$ on $[\theta_s, \hat{\theta}]$. Now, let $h : \Theta \to \mathbb{R}$ be the weakly decreasing function defined by

$$h(\theta) = \begin{cases} \underline{w}_q & \text{if } \theta \leq \hat{\theta} \\ \underline{W}(\theta, \underline{q}^{\mathrm{BM}}) & \text{otherwise.} \end{cases}$$

From (S.59), we then have that

$$\begin{aligned} W(M;V,F) &\geq \int_{\underline{\theta}}^{\hat{\theta}} h(\theta)F(\mathrm{d}\theta) + \int_{\hat{\theta}}^{\overline{\theta}} h(\theta)F(\mathrm{d}\theta) = \int_{\underline{\theta}}^{\overline{\theta}} h(\theta)F(\mathrm{d}\theta) \\ &\geq \int_{\underline{\theta}}^{\overline{\theta}} h(\theta)\underline{F}(\mathrm{d}\theta) = \underline{w}_q \underline{F}(\hat{\theta}) + \int_{\hat{\theta}}^{\overline{\theta}} \underline{W}(\theta,\underline{q}^{\mathrm{BM}})\underline{F}(\mathrm{d}\theta) \geq G_s^*, \end{aligned}$$

where the second inequality follows from the fact that $\underline{F} \succ_{\mathrm{FOSD}} F$, the equality follows by the definition of h, and the last inequality follows from Condition (e).

[26]The integration in any expression of the form $\underline{w}_q F(\theta) + \int_{\theta}^{\overline{\theta}} \underline{W}(\theta,\underline{q}^{\mathrm{BM}})F(\mathrm{d}\theta)$, where $\theta \geq \theta_s$, should be interpreted to be over $(\theta, \overline{\theta}]$ to avoid the double counting of any atom at θ.

Hence, $G(M) \geq G_s^*$. Part 1 establishes that, for any $M \in \mathcal{M}^{SL}$, $G(M) = G_s^*$. We conclude that any mechanism satisfying Conditions (a)-(e) is in the shortlist. This completes the proof of the proposition. Q.E.D.

The next proposition provides a characterization of properties of robustly-optimal mechanisms along the lines of Propositions 1 and 2 in the main text. To do so, we first generalize the definitions of q_ℓ, $q^\star$, $\theta^\star$ and θ^m as follows. Let $\mathrm{q}_\ell^s \equiv \underline{D}(\theta_s)$ denote the efficient output when the inverse demand is $\underline{D}$ and the cost is θ_s. Then let $q_s^\star$ be the quantity schedule defined by

$$q_s^\star(\theta) \equiv \begin{cases} \max\{q^{\mathrm{BM}}(\theta), \mathrm{q}_\ell^s\} & \theta < \theta_s \\ \underline{q}^{\mathrm{BM}}(\theta) & \theta \geq \theta_s, \end{cases} \tag{S.60}$$

where q^{BM} continues to denote the optimal quantity schedule of Baron and Myerson (1982) when the model is $(V^\star, F^\star)$, with $F^\star$ regular, whereas $\underline{q}^{\mathrm{BM}}$ is the optimal quantity schedule of Baron and Myerson (1982) when the model is $(\underline{V}, \underline{F})$. The following mechanism is a generalization of Baron-Myerson-with-quantity-floor in the main text.

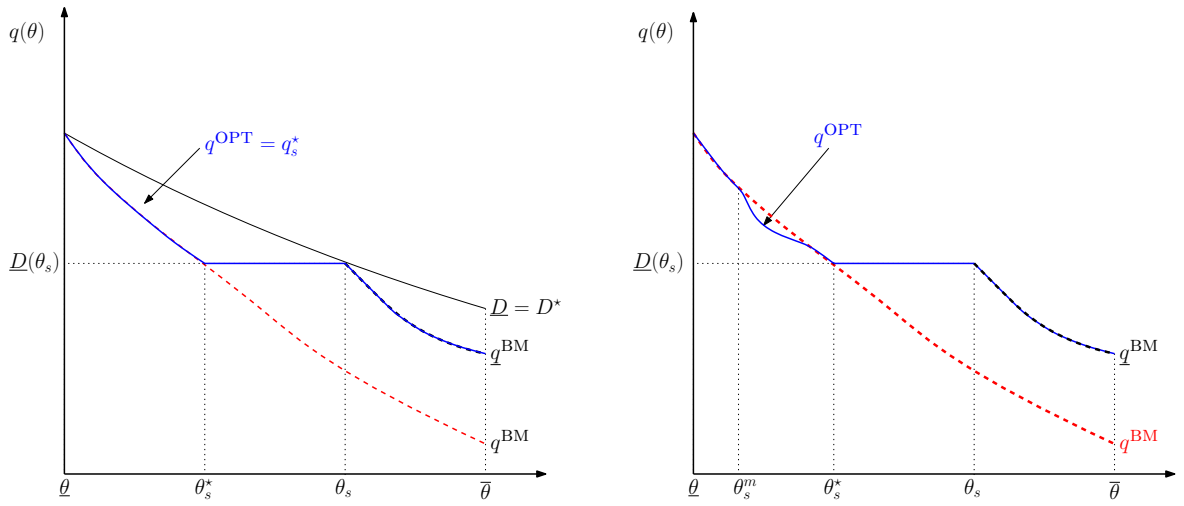
Definition S.9. *The **Baron-Myerson-with-quantity-bridge** is the mechanism $M_s^\star = (q_s^\star, u_s^\star)$ where $q_s^\star$ is given by (S.60) and $u_s^\star$ is given by $u_s^\star(\theta) = \int_\theta^{\overline{\theta}} q_s^\star(y)dy$ for all θ.*

Finally, let $\theta_s^\star$ be the threshold cost defined as follows. If $q^{\mathrm{BM}}(\theta_s) \leq \mathrm{q}_\ell^s$, by continuity of q^{BM} along with the fact that $q^{\mathrm{BM}}(\underline{\theta}) > \mathrm{q}_\ell^s$ (assured by the regularity of $F^\star$), $\theta_s^\star$ is the unique solution to $q^{\mathrm{BM}}(\theta_s^\star) = \mathrm{q}_\ell^s$. If, instead, $q^{\mathrm{BM}}(\theta_s) > \mathrm{q}_\ell^s$ (i.e., if q^{BM} never crosses q_ℓ^s over the interval $[\underline{\theta}, \theta_s]$), then $\theta_s^\star \equiv \theta_s$. In either case, $\theta_s^\star \leq \theta_s$. Similarly, let $\theta_s^m \equiv \max\{\theta : \theta \in \arg\min_{y \in [\underline{\theta}, \theta_s]} \underline{W}(y, q_s^\star)\}$. Thus, $\underline{w}_{q_s^\star} = \underline{W}(\theta_s^m, q_s^\star)$. Finally, let

$$G_s^{**} \equiv \sup_{\theta \in (\theta_s, \overline{\theta}]} \frac{1}{\underline{F}(\theta)} \int_{\theta_s}^{\theta} \underline{W}(y, \underline{q}^{\mathrm{BM}}) \underline{F}(\mathrm{d}y). \tag{S.61}$$

Proposition S.11. *Suppose $\underline{F}$ is regular with respect to θ_s.*

1. *The Baron-Myerson-with-quantity-bridge mechanism is robustly optimal if and only if $\underline{W}(\theta_s^m, q_s^\star) \geq \max\{G_s^*, G_s^{**}\}$.*
2. *If $\underline{W}(\theta_s^m, q_s^\star) < \max\{G_s^*, G_s^{**}\}$, then $\theta_s^m < \theta_s^\star$ and every robustly optimal mechanism $M^{\mathrm{OPT}} = (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$ satisfies the following properties: (a) $q^{\mathrm{OPT}}(\theta) = \underline{q}^{\mathrm{BM}}(\theta)$ for*



A: Baron-Myerson-with-quantity-bridge.

B: Robustly optimal mechanism when $V^\star \neq \underline{V}$.

Figure S.8: Graphical illustration of Proposition S.11.

all $\theta \in (\theta_s, \overline{\theta})$, *(b)* $q^{\text{OPT}}(\theta) = \text{q}_\ell^s$ *for all* $\theta \in (\theta_s^\star, \theta_s)$, *(c)* $q^{\text{OPT}}(\theta) \leq q^{\text{BM}}(\theta)$ *for all* $\theta \in (\underline{\theta}, \theta_s^\star)$, *with the inequality strict over a subset of types* $I \subseteq (\theta_s^m, \theta_s^\star)$ *of positive Lebesgue measure, and (d)* $q^{\text{OPT}}(\theta) = q^{\text{BM}}(\theta)$ *for all* $\theta \in (\underline{\theta}, \theta_s^m)$.

Before proving the result, we provide a few observations.

Remark S.1. If $V^\star = \underline{V}$ (as when the only uncertainty is over the cost technology), then $\theta_s^m = \theta_s$, and the condition in Part 1 of Proposition S.11 is satisfied. In this case, the Baron-Myerson-with-quantity-bridge mechanism, illustrated in Panel A of Figure S.8, is a robustly optimal mechanism. Panel B illustrates the features of robustly optimal mechanism when, instead, $V^\star \neq \underline{V}$. A comparison of these figures with Figure 1 and Figure 2 in the main text illustrates that the key features of the robustly optimal mechanisms continue to extend to general technological uncertainty. ■

Remark S.2. Proposition S.11 further highlights that the key forces identified in the main text continue to determine the shape of robustly optimal mechanisms to the left of θ_s. As in the main text, the quantity plateau in robustly optimal mechanisms is determined by $\underline{D}(\theta_s)$, the efficient quantity at θ_s under the lowest possible demand (recall that, in the main text, $\theta_s = \bar{\theta}$, and $\underline{D}(\theta_s) = \text{q}_\ell$). ■

Proof of Proposition S.11. The proof is in several parts, each corresponding to a part of the proposition. Before proceeding with the proof, observe that for any mechanism $M =$

$(q,u) \in \mathcal{M}^{\mathrm{SL}}$, Condition (d) in Part 2 of Proposition S.10 implies that $\underline{w}_q \geq G_s^*$. Condition (e) in the same proposition in turn implies that, for all $\theta \geq \theta_s$,

$$\underline{w}_q \underline{F}(\theta) + \int_{\theta}^{\bar{\theta}} \underline{W}(y, \underline{q}^{\mathrm{BM}}) \underline{F}(\mathrm{d}y) \geq G_s^* \equiv \int_{\theta_s}^{\bar{\theta}} \underline{W}(y, \underline{q}^{\mathrm{BM}}) \underline{F}(\mathrm{d}y),$$

which is equivalent to

$$\underline{w}_q \geq \sup_{\theta \in (\theta_s, \bar{\theta}]} \frac{1}{\underline{F}(\theta)} \int_{\theta_s}^{\theta} \underline{W}(y, \underline{q}^{\mathrm{BM}}) \underline{F}(\mathrm{d}y) \equiv G_s^{**}.$$

Thus, Condition (d) and (e) in Proposition S.10 can be equivalently written as $\underline{w}_q \geq \max\{G_s^*, G_s^{**}\}$, or alternatively, $\underline{W}(\theta, q) \geq \max\{G_s^*, G_s^{**}\}$ for all $\theta \leq \theta_s$.

Part 1. If $M_s^\star$ is robustly optimal, then $M_s^\star \in \mathcal{M}^{\mathrm{SL}}$. Observe that $\underline{w}_{q_s^\star} = \underline{W}(\theta_s^m, q_s^\star)$. Thus, as argued above, Conditions (d) and (e) in Part 2 of Proposition S.10 imply that $\underline{W}(\theta_s^m, q_s^\star) \geq \max\{G_s^*, G_s^{**}\}$.

Next, suppose that $M_s^\star$ is such that $\underline{W}(\theta_s^m, q_s^\star) \geq \max\{G_s^*, G_s^{**}\}$. We now show that $M_s^\star$ is robustly optimal. By definition, $M_s^\star$ satisfies Conditions (a)-(c) in Part 2 of Proposition S.10. As argued above, that $\underline{W}(\theta_s^m, q_s^\star) \geq \max\{G_s^*, G_s^{**}\}$ implies that Conditions (d) and (e) in Part 2 of Proposition S.10 are satisfied. This means that $M_s^\star \in \mathcal{M}^{\mathrm{SL}}$. To see that $M_s^\star$ maximizes the buyer's payoff under the conjectured model $(V^\star, F^\star)$ over $\mathcal{M}^{\mathrm{SL}}$, first observe that $q_s^\star$ is weakly decreasing because $F^\star$ is regular. Second note that every mechanism $M = (q, u)$ in the short list has a weakly decreasing quantity schedule q that agrees with $\underline{q}^{\mathrm{BM}}$ over $(\theta_s, \bar{\theta})$ (by virtue of Condition (c) in Proposition S.10). This means that, in any such mechanism, $q(\theta) \geq \mathrm{q}_\ell^s$ for all $\theta \in [\underline{\theta}, \theta_s)$. Because, for every $\theta \in [\underline{\theta}, \theta_s)$,

$$q_s^\star(\theta) = \arg \max_{\mathrm{q} \in [\mathrm{q}_\ell^s, \bar{\mathrm{q}}]} \left\{ V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q} \right\},$$

we conclude that, for any $M = (q, u) \in \mathcal{M}^{\mathrm{SL}}$,

$$\int_{\underline{\theta}}^{\bar{\theta}} \Big[V^\star(q(\theta)) - z^\star(\theta) q(\theta) \Big] F^\star(\mathrm{d}\theta) \leq \int_{\underline{\theta}}^{\bar{\theta}} \Big[V^\star(q_s^\star(\theta)) - z^\star(\theta) q_s^\star(\theta) \Big] F^\star(\mathrm{d}\theta),$$

implying that indeed $M_s^\star$ maximizes the buyer's payoff (under $(V^\star, F^\star)$) over $\mathcal{M}^{\mathrm{SL}}$.

Part 2. We start with the following lemma:

Lemma S.18. *Suppose* $\underline{w}_{q_s^\star} < \max\{G_s^*, G_s^{**}\}$. *Then the following are true: (1)* $\theta_s^m < \theta_s^\star$, *(2) if* $\theta_s^m > \underline{\theta}$, *then* $q_s^\star(\theta_s^m) = \underline{D}(\theta_s^m)$.

Proof of Lemma S.18. Part 1. We consider two cases.

Case1: $q^{\mathrm{BM}}(\theta_s) > \mathrm{q}_\ell^s = \underline{D}(\theta_s)$.

Then, $\theta_s^\star = \theta_s$. Because $\underline{D}$ and q^{BM} are decreasing and continuous, there exists a non-empty left neighborhood of θ_s where $q^{\mathrm{BM}}(\theta) > \underline{D}(\theta)$. Part B of Lemma 5 in the main text then implies that $\underline{W}(\cdot, q)$ is increasing on this interval implying that $\theta_s^m < \theta_s = \theta_s^\star$.

Case 2: $q^{\mathrm{BM}}(\theta_s) \leq \mathrm{q}_\ell^s = \underline{D}(\theta_s)$.

Then, $\theta_s^\star \leq \theta_s$, and for every $\theta \in [\theta_s^\star, \theta_s]$, $q_s^\star(\theta) = \mathrm{q}_\ell^s$ and $\underline{W}(\theta, q_s^\star) = \underline{W}(\theta_s, q_s^\star) \geq \max\{G_s^*, G_s^{**}\} > \underline{w}_{q_s^\star}$. The first inequality follows from the definitions of G_s^* and G_s^{**} along with Part A of Lemma 5 in the main text (which implies that $\underline{W}(\cdot, q)$ is weakly decreasing over $[\theta_s, \bar{\theta}]$). The second inequality follows from the assumption of Lemma S.18. This implies that $\underline{W}(\theta_s^\star, q_s^\star) > \underline{w}_{q_s^\star}$. Hence, $\theta_s^m < \theta_s^\star$.

Part 2. Because $\theta_s^m < \theta_s^\star$, we have that $q_s^\star(\theta_s^m) = q^{\mathrm{BM}}(\theta_s^m)$. This property, together with arguments similar to those in the proof of Lemma 8 in the main text, then implies the result. ■

Equipped with the last lemma, we now establish Parts 2(a) and 2(b) of the proposition.

Parts 2(a) and 2(b). Part 2(a) follows from Part 2(c) of Proposition S.10 because any robustly optimal mechanism belongs to $\mathcal{M}^{\mathrm{SL}}$. Thus consider Part 2(b). We consider two cases. First, suppose that $q^{\mathrm{BM}}(\theta_s) \geq \mathrm{q}_\ell^s$. By the definition of $\theta_s^\star$, we then have that $\theta_s^\star = \theta_s$, and hence the interval $(\theta_s^\star, \theta_s)$ is empty and the result applies vacuously. Next, suppose that $q^{\mathrm{BM}}(\theta_s) < \mathrm{q}_\ell^s$. Then, $\theta_s^\star < \theta_s$. Now, assume, toward a contradiction, that there exists $\theta' \in (\theta_s^\star, \theta_s)$ such that $q^{\mathrm{OPT}}(\theta') > \mathrm{q}_\ell^s$. Monotonicity of q^{OPT} then implies that $q^{\mathrm{OPT}}(\theta) > \mathrm{q}_\ell^s$ for all $\theta \in [\theta_s^\star, \theta']$. This means that there exists a set of types of positive Lebesgue measure such that $q^{\mathrm{OPT}}(\theta) > \mathrm{q}_\ell^s$. Then, consider the IC and IR mechanism $\widetilde{M} = (\tilde{q}, \tilde{u})$ where the quantity schedule is given by

$$\tilde{q}(\theta) = \begin{cases} q^{\mathrm{OPT}}(\theta) & \text{if } \ \theta < \theta_s^\star \\ \mathrm{q}_\ell^s & \text{if } \ \theta \in [\theta_s^\star, \theta_s] \\ \underline{q}^{\mathrm{BM}}(\theta) = q^{\mathrm{OPT}}(\theta) & \text{if } \ \theta \geq \theta_s, \end{cases}$$

and where the rents $\tilde{u}$ are given by $\tilde{u}(\theta) = \int_{\theta}^{\overline{\theta}} \tilde{q}(y)dy$ for all θ. The buyer's payoff from $\widetilde{M}$ (under the conjectured model $(V^\star, F^\star)$) is equal to $\int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(\tilde{q}(\theta)) - z^\star(\theta)\tilde{q}(\theta)\Big] F^\star(\mathrm{d}\theta)$ which is strictly higher than under M^{OPT}. This follows from the fact that, for any $\theta \in [\theta_s^\star, \theta_s]$, q_ℓ^s maximizes $V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}$ over $\mathrm{q} \geq \mathrm{q}_\ell^s$, along with the fact that $F^\star$ is absolutely continuous. Thus, to produce a contradiction to the robust optimality of M^{OPT}, it suffices to show that $\widetilde{M} \in \mathcal{M}^{\mathrm{SL}}$, which we show next.

By definition, $\widetilde{M}$ satisfies Conditions (a)-(c) in Part 2 of Proposition S.10. As for Conditions (d) and (e), recall that these conditions are equivalent to $\underline{w}_{\tilde{q}} \geq \max\{G_s^*, G_s^{**}\}$, as established above. That $\underline{w}_{\tilde{q}} \geq G_s^*$ follows from arguments analogous to those in the proof of Lemma 6 in the main text, along with the fact that $\tilde{q}(\theta) \leq q^{\mathrm{OPT}}(\theta)$ for all θ. To establish that $\underline{w}_{\tilde{q}} \geq G_s^{**}$, notice that, for all $\theta \in [\theta_s^\star, \theta_s]$, $\underline{W}(\theta, \tilde{q}) = \underline{W}(\theta_s, \underline{q}^{\mathrm{BM}}) \geq G_s^{**}$. Thus, it suffices to focus on $\theta < \theta_s^\star$. Observe that $\inf_{\theta < \theta_s^\star} \underline{W}(\theta, \tilde{q}) \geq \inf_{\theta < \theta_s^\star} \underline{W}(\theta, q^{\mathrm{OPT}}) \geq G_s^{**}$, where the first inequality follows from the fact that $\tilde{q}(\theta) = q^{\mathrm{OPT}}(\theta)$ for $\theta < \theta_s^\star$, along with the fact that $\tilde{q}(\theta) \leq q^{\mathrm{OPT}}(\theta)$ for all θ, which implies that $\tilde{u}(\theta) \leq u^{\mathrm{OPT}}(\theta)$ for all θ. The second inequality follows from the fact that $M^{\mathrm{OPT}} \in \mathcal{M}^{\mathrm{SL}}$.

Part 2(c). From Part 2(b), $q^{\mathrm{OPT}}(\theta) = \mathrm{q}_\ell^s$ for all $\theta \in (\theta_s^\star, \theta_s)$. Now suppose there is a set $S \subseteq (\underline{\theta}, \theta_s^\star)$ of positive Lebesgue measure such that $q^{\mathrm{OPT}}(\theta) > q_s^\star(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta \in S$. Consider the IC and IR mechanism $\widetilde{M} = (\tilde{q}, \tilde{u})$ where the quantity schedule is given by

$$\tilde{q}(\theta) = \min\{q_s^\star(\theta), q^{\mathrm{OPT}}(\theta)\} \qquad \forall\ \theta \in \Theta,$$

and where $\tilde{u}(\theta) = \int_\theta^{\overline{\theta}} \tilde{q}(y)\mathrm{d}y$ for all θ. Parts 2(a) and 2(b) in the proposition then imply that $\tilde{q}(\theta) = q^{\mathrm{OPT}}(\theta) = q_s^\star(\theta)$ for all $\theta \geq \theta_s^\star$.

The buyer's payoff under $\widetilde{M}$ is strictly higher than under M^{OPT} (this follows from $q_s^\star$ maximizing virtual surplus). Clearly, $\widetilde{M}$ satisfies Conditions (a)-(c) of Part 2 of Proposition S.10. The next two claims establish that $\widetilde{M}$ also satisfies Conditions (d) and (e), that is, $\underline{w}_{\tilde{q}} \geq \max\{G_s^*, G_s^{**}\}$ or, equivalently, $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$ for all $\theta \leq \theta_s$. First, observe that, for every $\theta \in [\theta_s^\star, \theta_s]$, $\tilde{q}(\theta) = q_s^\star(\theta) = \mathrm{q}_\ell^s$ and $\underline{W}(\theta, \tilde{q}) = \underline{W}(\theta_s, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$. Below we establish that $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$ also for $\theta < \theta_s^\star$. The result is a consequence of the following two claims.

Claim S.4. *Suppose $\theta < \theta_s^\star$ is such that either $\tilde{q}(\theta) = q^{\mathrm{OPT}}(\theta)$ or $\underline{D}(\theta) \leq \tilde{q}(\theta) = q_s^\star(\theta) <$*

$q^{\mathrm{OPT}}(\theta)$. *Then* $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$.

Proof of Claim S.4. Pick $\theta < \theta_s^\star$. It suffices to show that $\underline{W}(\theta, \tilde{q}) \geq \underline{W}(\theta, q^{\mathrm{OPT}})$; because $\underline{W}(\theta, q^{\mathrm{OPT}}) \geq \max\{G_s^*, G_s^{**}\}$, the claim then follows. Note that $q_s^\star(\theta) = q^{\mathrm{BM}}(\theta)$. For any θ such that $\tilde{q}(\theta) = q^{\mathrm{OPT}}(\theta)$, the fact that $\tilde{q}(y) \leq q^{\mathrm{OPT}}(y)$ for all $y \geq \theta$, implies that $\underline{W}(\theta, \tilde{q}) \geq \underline{W}(\theta, q^{\mathrm{OPT}})$. Thus, consider a θ for which $\underline{D}(\theta) \leq \tilde{q}(\theta) = q_s^\star(\theta) = q^{\mathrm{BM}}(\theta) < q^{\mathrm{OPT}}(\theta)$. The quasi-concavity of the function $\underline{V}(\mathrm{q}) - \theta \mathrm{q}$ in q implies that $\underline{V}(q_s^\star(\theta)) - \theta q_s^\star(\theta) > \underline{V}(q^{\mathrm{OPT}}(\theta)) - \theta q^{\mathrm{OPT}}(\theta)$. Because $\tilde{q}(y) \leq q^{\mathrm{OPT}}(y)$ for all $y \geq \theta$, we have that $\underline{W}(\theta, \tilde{q}) \geq \underline{W}(\theta, q^{\mathrm{OPT}})$. ■

Claim S.5. *If* $\theta < \theta_s^\star$ *and* $q_s^\star(\theta) < \min\{\underline{D}(\theta), q^{\mathrm{OPT}}(\theta)\}$, *then* $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$.

Proof of Claim S.5. The proof considers two cases to establish the existence of $\theta' > \theta$ such that $W(\cdot, \tilde{q})$ is weakly decreasing on $[\theta, \theta']$ with $W(\theta', \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$.

Case 1. Suppose $q^{\mathrm{BM}}(\theta_s) \leq \mathrm{q}_\ell^s = \underline{D}(\theta_s)$. Because $q_s^\star$ and $\underline{D}$ are both continuous, there exists $\theta < \theta' \leq \theta_s$ such that $q_s^\star(y) \leq \underline{D}(y)$ for all $y \in [\theta, \theta']$, with $q_s^\star(\theta') = \underline{D}(\theta')$. Thus, $\tilde{q}(\theta') = \min\{\underline{D}(\theta'), q^{\mathrm{OPT}}(\theta')\}$. Furthermore, for all $y \in [\theta, \theta']$, $\tilde{q}(y) = \min\{q^{\mathrm{OPT}}(y), q_s^\star(y)\} \leq \underline{D}(y)$. Part A of Lemma 5 in the main text implies that $\underline{W}(\cdot, \tilde{q})$ is weakly decreasing over $[\theta, \theta']$ whereas Claim S.4 implies that $\underline{W}(\theta', \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$. Hence, $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$.

Case 2. Now suppose $q^{\mathrm{BM}}(\theta_s) > \mathrm{q}_\ell^s = \underline{D}(\theta_s)$. Then, because $q_s^\star(\theta) < \underline{D}(\theta)$, and $\underline{D}$ and $q_s^\star$ are continuous (the latter due to the regularity of $F^\star$), there exists $\theta < \hat{\theta} < \overline{\theta}$ such that $q_s^\star(\hat{\theta}) = \underline{D}(\hat{\theta})$ and $q_s^\star(y) > \underline{D}(y)$ for all $y \in (\hat{\theta}, \theta_s)$. Again, just like we argued in Case 1, there exists $\theta < \theta' \leq \hat{\theta}$ such that $q_s^\star(y) \leq \underline{D}(y)$ for all $y \in [\theta, \theta']$ with $q_s^\star(\theta') = \underline{D}(\theta')$. The same arguments as in Case 1 then imply that $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$. ■

The above two claims establish that $\tilde{q}$ is such that $\underline{w}_{\tilde{q}} \geq \max\{G_s^*, G_s^{**}\}$. From Proposition S.10, we thus conclude that $\widetilde{M} = (\tilde{q}, \tilde{u}) \in \mathcal{M}^{\mathrm{SL}}$.

The arguments above imply that $q^{\mathrm{OPT}}(\theta) \leq q^{\mathrm{BM}}(\theta)$ for almost all $\theta \in (\underline{\theta}, \theta_s^\star)$. Because q^{OPT} is weakly decreasing and q^{BM} is continuous, the inequality must in fact hold for all $\theta \in (\underline{\theta}, \theta_s^\star)$.

Next, we show that there exists a set of types $I \subseteq (\underline{\theta}, \theta_s^\star)$ of positive Lebesgue measure such that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in I$. To see this, assume, towards a contradiction, that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for almost all $\theta \in (\underline{\theta}, \theta_s^\star)$. Because $q_s^\star$ and q^{OPT} are weakly decreasing and $q_s^\star$ is continuous, this means that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta_s^\star)$. This however implies that $M_s^\star$ is robustly optimal, a contradiction. That I is in fact a subset of $(\theta_s^m, \theta_s^\star)$ follows

from what just established along with part (d) in the proposition, which we prove next.

Part 2(d). Assume, toward a contradiction, that there exists a $\theta \in (\underline{\theta}, \theta_s^m)$ such that $q^{\mathrm{OPT}}(\theta) \neq q^{\mathrm{BM}}(\theta)$. Because $q^{\mathrm{OPT}}(y) \leq q^{\mathrm{BM}}(y)$ for all $y \in (\underline{\theta}, \theta_s^m)$ (by virtue of Part 2(c) of Proposition S.11), and because q^{BM} and q^{OPT} are weakly decreasing and q^{BM} is continuous, this means that there exists a set of types $I \subseteq (\underline{\theta}, \theta_s^m)$ of positive Lebesgue measure such that $q^{\mathrm{OPT}}(y) < q^{\mathrm{BM}}(y)$ for all $y \in I$. Then, let $\widetilde{M} = (\tilde{q}, \tilde{u})$ be the mechanism where the quantity schedule is given by

$$\tilde{q}(\theta) = \begin{cases} q^{\mathrm{BM}}(\theta) & \text{if } \ \theta \in [\underline{\theta}, \theta_s^m] \\ q^{\mathrm{OPT}}(\theta) & \text{otherwise} \end{cases}$$

and where $\tilde{u}(\theta) = \int_{\theta}^{\overline{\theta}} \tilde{q}(y)dy$ for all θ. Clearly, $\widetilde{M}$ is IC and IR. Below, we show that $\widetilde{M}$ yields a higher payoff to the buyer than M^{OPT} and $\widetilde{M} \in \mathcal{M}^{\mathrm{SL}}$, contradicting the optimality of M^{OPT}.

Because, for any θ, $q^{\mathrm{BM}}(\theta)$ is the unique maximizer of $V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}$,

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(\tilde{q}(\theta)) - z^\star(\theta)\tilde{q}(\theta)\Big] F^\star(\mathrm{d}\theta) > \int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q^{\mathrm{OPT}}(\theta)) - z^\star(\theta)q^{\mathrm{OPT}}(\theta)\Big] F^\star(\mathrm{d}\theta).$$

We now show that $\tilde{q}$ is such that $\underline{w}_{\tilde{q}} \geq \max\{G_s^*, G_s^{**}\}$. To do so, it suffices to show that $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$ for $\theta \leq \theta_s^m$ (in fact, for $\theta > \theta_s^m$, $\underline{W}(\theta, \tilde{q}) = \underline{W}(\theta, q^{\mathrm{OPT}}) \geq \max\{G_s^*, G_s^{**}\}$, where the inequality follows from the fact that $M^{\mathrm{OPT}} \in \mathcal{M}^{\mathrm{SL}}$). Thus consider $\theta \in [\underline{\theta}, \theta_s^m]$. For any such θ, $\tilde{q}(\theta) = q^{\mathrm{BM}}(\theta) = q_s^\star(\theta)$. The latter equality follows from Lemma S.18, which establishes that $\theta_s^m < \theta_s^\star$. Moreover,

$$\begin{aligned}
\underline{W}(\theta, \tilde{q}) - \underline{W}(\theta, q_s^\star) = & \int_{\theta_s^m}^{\overline{\theta}} q_s^\star(y)dy - \int_{\theta_s^m}^{\overline{\theta}} q^{\mathrm{OPT}}(y)dy \\
\geq_{(i)} & \Big[\underline{V}(q^{\mathrm{OPT}}(\theta_s^m)) - \theta_s^m q^{\mathrm{OPT}}(\theta_s^m)\Big] - \Big[\underline{V}(q^{\mathrm{BM}}(\theta_s^m)) - \theta_s^m q^{\mathrm{BM}}(\theta_s^m)\Big] \\
& + \int_{\theta_s^m}^{\overline{\theta}} q_s^\star(y)dy - \int_{\theta_s^m}^{\overline{\theta}} q^{\mathrm{OPT}}(y)dy =_{(ii)} \underline{W}(\theta_s^m, q^{\mathrm{OPT}}) - \underline{W}(\theta_s^m, q_s^\star) \\
\geq_{(iii)} & \max\{G_s^*, G_s^{**}\} - \underline{W}(\theta_s^m, q_s^\star) \geq_{(iv)} \max\{G_s^*, G_s^{**}\} - \underline{W}(\theta, q_s^\star).
\end{aligned}$$

Inequality (i) follows from the fact that $\underline{D}(\theta_s^m)$ maximizes $\underline{V}(\mathrm{q}) - \theta_s^m \mathrm{q}$ over all $[0, \bar{\mathrm{q}}]$ and $q^{\mathrm{BM}}(\theta_s^m) = \underline{D}(\theta_s^m)$ (Lemma S.18). Equality (ii) follows from the fact that $q_s^\star(\theta) = q^{\mathrm{BM}}(\theta)$. Inequality (iii) follows from the fact that $M^{\mathrm{OPT}} \in \mathcal{M}^{\mathrm{SL}}$ which implies that $q^{\mathrm{OPT}}(y)$ satisfies the robustness constraints $\underline{W}(\theta_s^m, q^{\mathrm{OPT}}) \geq \max\{G_s^*, G_s^{**}\}$. Inequality (iv) follows from the definition of θ_s^m. Hence, $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$. We conclude that $\widetilde{M} \in \mathcal{M}^{\mathrm{SL}}$ and yields a higher payoff to the buyer than M^{OPT}, contradicting the optimality of M^{OPT}. Q.E.D.

E Supplement II

The notation in this supplement is the same as in the original article. All sections, definitions, displayed conditions, and results specific to this document have the prefix "E" to avoid confusion with the corresponding parts in the main text. Section E.1 establishes existence of robustly optimal mechanisms. Section E.2 studies the connection between robustly optimal and undominated mechanisms. Section E.3 examines comparative statics of robustly optimal mechanisms with respect to the lowest distribution $\underline{F}$ in the feasible set $\mathcal{F}$ (in the main text such a distribution is a Dirac measure at $\bar{\theta}$).

E.1 Existence of robustly optimal mechanisms

Proposition E.12. *A robustly optimal mechanism exists.*[27]

Proof of Proposition E.12. Recall that a mechanism $M = (q, u)$ is robustly optimal if and only if u satisfies $u(\theta) = \int_\theta^{\bar{\theta}} q(s)\mathrm{d}s$ for all $\theta \in \Theta$, with q solving the following program (where we use the fact that $F^\star$ is absolutely continuous over $\mathbb{R}$, with density $f^\star(\theta) > 0$ if and only if $\theta \in \Theta$):

$$\max_{q:[\underline{\theta},\bar{\theta}]\to[0,\bar{\mathrm{q}}]} \int_{\underline{\theta}}^{\bar{\theta}} \left[V^\star(q(\theta)) - z^\star(\theta)q(\theta)\right] f^\star(\theta)\mathrm{d}\theta$$

$$\text{s.t. } q \text{ weakly decreasing and } \underline{V}(q(\theta)) - \theta q(\theta) - \int_\theta^{\bar{\theta}} q(y)\mathrm{d}y \geq G^* \quad \forall \theta \in \Theta.$$

Hereafter, we refer to the set of schedules q satisfying the restrictions in the above problem as the "feasible set".

Because each q in the feasible set is uniformly bounded, i.e., $0 \leq q(\theta) \leq \bar{\mathrm{q}}$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$, by Helly's selection theorem, the set of weakly decreasing schedules q is sequentially compact under the point-wise convergence topology. Furthermore, because $\underline{V}$ is continuous and q is uniformly bounded, by the dominated convergence theorem, for any $\theta \in \Theta$, the function $H(q;\theta)$ defined by $H(q;\theta) \equiv \underline{V}(q(\theta)) - \theta q(\theta) - \int_\theta^{\bar{\theta}} q(y)\mathrm{d}y$ is sequentially continuous in q. Hence, the feasible set is sequentially compact under the point-wise convergence topology. That this set is non-empty is immediate given that the function q satisfying $q(\theta) = \mathrm{q}_\ell$ for all $\theta \in \Theta$ meets all the requirements.

[27] We thank Issan Patri for providing helpful comments and suggestions on this part.

Next, observe that the objective function in the above program is sequentially continuous in q. To see this, let ϕ be the function defined, for all $(\theta, \mathrm{q}) \in \Theta \times [0, \bar{\mathrm{q}}]$ by

$$\phi(\theta, \mathrm{q}) = \Big[V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}\Big] f^\star(\theta).$$

Note that, given any function q in the feasible set, the value of the objective function is

$$\int_{\underline{\theta}}^{\overline{\theta}} \phi(\theta, q(\theta)) \mathrm{d}\theta.$$

Clearly, $\phi(\theta, \mathrm{q})$ is continuous over $\Theta \times [0, \bar{\mathrm{q}}]$. By Weierstrass' theorem, for all $\theta \in \Theta$, $\underline{\phi}(\theta) \equiv \min_{\mathrm{q} \in [0, \bar{\mathrm{q}}]} \phi(\theta, \mathrm{q})$ and $\bar{\phi}(\theta) \equiv \max_{\mathrm{q} \in [0, \bar{\mathrm{q}}]} \phi(\theta, \mathrm{q})$ exist. Furthermore, by the theorem of the maximum, $g(\theta) \equiv \max\{|\underline{\phi}(\theta)|, |\bar{\phi}(\theta)|\}$ is continuous. Because Θ is compact, g is bounded and hence integrable over Θ. Now take a sequence (q_n) of feasible schedules converging to q under the point-wise convergence topology. Then, for every $\theta \in [\underline{\theta}, \overline{\theta}]$, we have that $\lim_{n \to \infty} \phi(\theta, q_n(\theta)) = \phi(\theta, q(\theta))$ by continuity of ϕ in the second argument. Furthermore, for each q_n in the sequence, we have that $|\phi(\theta, q_n(\theta))| \leq g(\theta)$ for all $\theta \in \Theta$. Then, by the dominated convergence theorem,

$$\lim_{n \to \infty} \int_{\underline{\theta}}^{\overline{\theta}} \phi(\theta, q_n(\theta)) \mathrm{d}\theta = \int_{\underline{\theta}}^{\overline{\theta}} \phi(\theta, q(\theta)) \mathrm{d}\theta.$$

This establishes the sequential continuity of the objective function under the point-wise convergence topology. Because the range of the objective function is a subset of $\mathbb{R}$, from the extreme value theorem, we conclude that the above optimization program has a solution, i.e., a robustly optimal mechanism exists. This completes the proof of the proposition. Q.E.D.

E.2 Undomination and robustness

We formally define what it means for a mechanism to be undominated, and then establish that robustly optimal mechanisms are undominated.

Definition E.10. *For any pair of mechanisms $M = (q, u)$ and $\widehat{M} = (\hat{q}, \hat{u})$, M* ***dominates***

$\widehat{M}$ *if, for every* $(V, F) \in \mathcal{A}$,[28]

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V(q(\theta)) - \theta q(\theta) - u(\theta)\Big] F(\mathrm{d}\theta) \geq \int_{\underline{\theta}}^{\overline{\theta}} \Big[V(\hat{q}(\theta)) - \theta \hat{q}(\theta) - \hat{u}(\theta)\Big] F(\mathrm{d}\theta),$$

with the inequality strict for some (V, F)*. A mechanism* $\widehat{M} \in \mathcal{M}$ *is* ***undominated*** *if there does not exist a mechanism* $M \in \mathcal{M}$ *that* ***dominates*** *it.*

The following lemma points to an internal consistency property of the set of robustly optimal mechanisms: each robustly optimal mechanism is either undominated, or it is dominated by another robustly optimal mechanism.

Lemma E.19. *Suppose* $M^{\mathrm{OPT}} = (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$ *is a robustly optimal mechanism and* $M = (q, u) \in \mathcal{M}$ *dominates* M^{OPT}*. Then* M *is a robustly optimal mechanism.*

Proof of Lemma E.19. Because M^{OPT} is a robustly optimal mechanism, for all $\theta \in \Theta$, $\underline{V}(q^{\mathrm{OPT}}(\theta)) - \theta q^{\mathrm{OPT}}(\theta) - u^{\mathrm{OPT}}(\theta) \geq G^*$. Now pick any $\theta \in \Theta$. Because M dominates M^{OPT}, under $V = \underline{V}$ and $F = \delta_\theta$ (where δ_θ is the Dirac distribution that puts unit point mass at θ), we have that $\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta) \geq \underline{V}(q^{\mathrm{OPT}}(\theta)) - \theta q^{\mathrm{OPT}}(\theta) - u^{\mathrm{OPT}}(\theta)$. Combining the two inequalities, we have that $\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta) \geq G^*$. Because the last inequality holds for all θ, $M \in \mathcal{M}^{\mathrm{SL}}$.

Next, pick any $\theta \in \Theta$. Because M dominates M^{OPT}, under $V = V^\star$ and $F = \delta_\theta$, we have that $V^\star(q(\theta)) - \theta q(\theta) - u(\theta) \geq V^\star(q^{\mathrm{OPT}}(\theta)) - \theta q^{\mathrm{OPT}}(\theta) - u^{\mathrm{OPT}}(\theta)$, which implies that

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q(\theta)) - \theta q(\theta) - u(\theta)\Big] F^\star(\mathrm{d}\theta) \geq \int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q^{\mathrm{OPT}}(\theta)) - \theta q^{\mathrm{OPT}}(\theta) - u^{\mathrm{OPT}}(\theta)\Big] F^\star(\mathrm{d}\theta).$$

Because $M \in \mathcal{M}^{\mathrm{SL}}$ and M^{OPT} is robustly optimal, the above inequality implies that M is also robustly optimal. In turn, this implies that the above inequality is an equality. This completes the proof of the lemma. Q.E.D.

Proposition E.13. *If the Baron-Myerson-with-quantity-floor mechanism* $M^\star = (q^\star, u^\star)$ *is robustly optimal, it is undominated.*

[28] As in the main text, we assume that $\mathcal{F}$ is the collection of all cdfs supported on Θ and that $(V^\star, F^\star) \in \mathcal{A} \equiv \mathcal{V} \times \mathcal{F}$.

Proof of Proposition E.13. Suppose $M^\star$ is robustly optimal and $M = (q, u)$ dominates it. By Lemma E.19, M is also robustly optimal. By Corollary 1 in the main text, $q(\theta) = q^\star(\theta)$ for all $\theta > \underline{\theta}$. This implies that $u(\theta) = u^\star(\theta)$ for all θ.

Consider the pair $(V^\star, \delta_{\underline{\theta}})$, where $\delta_{\underline{\theta}}$ is the Dirac distribution that puts unit mass at $\underline{\theta}$. Then $V^\star(q^\star(\underline{\theta})) - \underline{\theta} q^\star(\underline{\theta}) - u^\star(\underline{\theta}) > V^\star(q(\underline{\theta})) - \underline{\theta} q(\underline{\theta}) - u(\underline{\theta})$. The inequality holds because $u(\underline{\theta}) = u^\star(\underline{\theta})$ and because $q^\star(\underline{\theta}) \equiv q^{\text{BM}}(\underline{\theta})$ uniquely maximizes surplus $V^\star(\text{q}) - \underline{\theta}\text{q}$. This inequality, however, contradicts the fact that $M = (q, u)$ dominates $(q^\star, u^\star)$. Q.E.D.

Hence, when the conditions in Proposition 1 in the main text are satisfied, the Baron-Myerson-with-quantity-floor mechanism is not only robustly optimal but also undominated. This result generalizes to other robustly optimal mechanisms, albeit under a mild condition, which is satisfied by $M^\star = (q^\star, u^\star)$. Let

$$Q^{\#}(\underline{\theta}) \equiv \cup_{V \in \mathcal{V}} \left\{ \arg\max_{\text{q} \in [0, \bar{\text{q}}]} \left(V(\text{q}) - \underline{\theta}\text{q} \right) \right\}$$

denote the set of output levels that, when $\theta = \underline{\theta}$, deliver the highest total surplus for some $V \in \mathcal{V}$.

Proposition E.14. *Let $M^{\text{OPT}} = (q^{\text{OPT}}, u^{\text{OPT}})$ be a robustly optimal mechanism in which q^{OPT} is left-continuous and, if $\theta^m > \underline{\theta}$, $q^{\text{OPT}}(\underline{\theta}) \in Q^{\#}(\underline{\theta})$. Then M^{OPT} is undominated.*

Proof of Proposition E.14. Towards a contradiction, assume that M^{OPT} is dominated by another mechanism $\widehat{M} = (\hat{q}, \hat{u})$. By Proposition E.13, we then have that $M^{\text{OPT}} \neq M^\star$. Furthermore, by Lemma E.19, $\widehat{M}$ is also robustly optimal.

Case A. Suppose first that $M^\star$ is robustly optimal, in which case Lemma 7 in the main text implies that $\theta^m = \overline{\theta} > \underline{\theta}$. Because $\widehat{M}$ is also robustly optimal, Corollary 1 in the main text then implies that $q^{\text{OPT}}(\theta) = \hat{q}(\theta)$ for all $\theta > \underline{\theta}$. This in turn implies that, for all θ, $u^{\text{OPT}}(\theta) = \hat{u}(\theta)$. Because $q^{\text{OPT}}(\underline{\theta}) \in Q^{\#}(\underline{\theta})$, there exists $V \in \mathcal{V}$ such that $V(q^{\text{OPT}}(\underline{\theta})) - \underline{\theta} q^{\text{OPT}}(\underline{\theta}) - u^{\text{OPT}}(\underline{\theta}) > V(\hat{q}(\underline{\theta})) - \underline{\theta}\hat{\text{q}}(\underline{\theta}) - \hat{u}(\underline{\theta})$, where the strict inequality follows from the concavity of $V(\text{q}) - \underline{\theta}\text{q}$ in q. Because Nature can always pair V with a Dirac distribution selecting $\underline{\theta}$ with probability one, the above inequality contradicts the assumption that $\widehat{M}$ dominates M^{OPT}.

Case B. Next, suppose that $M^\star$ is not robustly optimal. Because M^{OPT} and $\widehat{M}$ are both robustly optimal and $M^\star$ is not, Proposition 2 in the main text then implies that $\hat{q}$ and q^{OPT} can disagree only over the set $\{\underline{\theta}\} \cup [\theta^m, \theta^\star)$, with the thresholds θ^m and $\theta^\star$ as defined in the

main text.

Lemma E.20. *Suppose $M^{\text{OPT}} = (q^{\text{OPT}}, u^{\text{OPT}})$ is a robustly optimal mechanism such that q^{OPT} is left-continuous and, if $\theta^m > \underline{\theta}$, $q^{\text{OPT}}(\underline{\theta}) \in Q^{\#}(\underline{\theta})$. If M^{OPT} is dominated by $\widehat{M} = (\hat{q}, \hat{u})$, then the set*

$$\Theta_{\hat{q},q^{\text{OPT}}} \equiv \{\theta \in [\theta^m, \theta^\star) : \hat{q}(\theta) < q^{\text{OPT}}(\theta)\}$$

has positive Lebesgue measure.

Proof of Lemma E.20. Assume, toward a contradiction, that $\Theta_{\hat{q},q^{\text{OPT}}}$ has zero Lebesgue measure. Then, either (1) $\hat{q}(\theta) \geq q^{\text{OPT}}(\theta)$ for all $\theta \in [\theta^m, \theta^\star)$, or (2) $\hat{q}(\theta) < q^{\text{OPT}}(\theta)$ only on countably many $\theta \in [\theta^m, \theta^\star)$. Below, we establish a contradiction to the fact that M^{OPT} is dominated by $\widehat{M}$ in each of these two cases.

Case 1. Suppose that $\hat{q}(\theta) \geq q^{\text{OPT}}(\theta)$ for all $\theta \in [\theta^m, \theta^\star)$. If this last inequality holds as an equality for all $\theta \in [\theta^m, \theta^\star)$, then, because $M^{\text{OPT}} \neq \widehat{M}$, it must be that $\theta^m > \underline{\theta}$. Furthermore, because $\hat{q}(\theta) = q^{\text{OPT}}(\theta) = \mathrm{q}_\ell$ for all $\theta \geq \theta^\star$, $\hat{q}$ and q^{OPT} disagree only at $\theta = \underline{\theta}$, implying that $u^{\text{OPT}}(\theta) = \hat{u}(\theta)$ for all θ. The assumption that $q^{\text{OPT}}(\underline{\theta}) \in Q^{\#}(\underline{\theta})$ then implies that $\widehat{M}$ does not dominate M^{OPT}, a contradiction to the assumption it does (the arguments are identical to those of Case A above). Thus there must exist $\theta' \in [\theta^m, \theta^\star)$ such that $\hat{q}(\theta') > q^{\text{OPT}}(\theta')$.

First suppose that $\theta^m > \underline{\theta}$. Because q^{OPT} is left-continuous and $\hat{q}$ is weakly decreasing, and because $\hat{q}(\theta) = q^{\text{OPT}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta^m)$, it must be that $\theta' > \theta^m$. That $\hat{q}(\theta') > q^{\text{OPT}}(\theta')$, along with the fact that q^{OPT} is left continuous and $\hat{q}$ is weakly decreasing, implies that there exists a subset of $(\theta^m, \theta^\star)$ of positive Lebesgue measure over which $\hat{q}(y) > q^{\text{OPT}}(y)$. Together with the fact that $\hat{q}(\theta) \geq q^{\text{OPT}}(\theta)$ for all $\theta \geq \theta^m$, this last property implies that $\hat{u}(\theta^m) > u^{\text{OPT}}(\theta^m)$. Next, observe that, because q^{OPT} is left continuous, and because $q^{\text{OPT}}(\theta) = q^{\text{BM}}(\theta)$ for all $\theta \in (\underline{\theta}, \theta^m)$ (by virtue of Proposition 2 in the main text), we have that $q^{\text{OPT}}(\theta^m) = q^{\text{BM}}(\theta^m)$. Furthermore, Lemma 8 in the main text implies that $q^{\text{BM}}(\theta^m) = \underline{D}(\theta^m)$. Because $\widehat{M}$ is robustly optimal, Lemma 6 in the main text then implies that $\hat{q}(\theta^m) \leq q^{\text{BM}}(\theta^m)$. Jointly, the last three properties imply that $\hat{q}(\theta^m) \leq q^{\text{BM}}(\theta^m) = \underline{D}(\theta^m) = q^{\text{OPT}}(\theta^m)$. This last inequality, together with the fact that $\hat{u}(\theta^m) > u^{\text{OPT}}(\theta^m)$, and the fact that $\underline{D}(\theta^m)$ is the unique maximizer of $\underline{V}(\mathrm{q}) - \theta^m \mathrm{q}$, implies that

$$\underline{V}(\hat{q}(\theta^m)) - \theta^m \hat{q}(\theta^m) - \hat{u}(\theta^m) < \underline{V}(q^{\text{OPT}}(\theta^m)) - \theta^m q^{\text{OPT}}(\theta^m) - u^{\text{OPT}}(\theta^m).$$

This means that $W(\widehat{M}; \underline{V}, \delta_{\theta^m}) < W(M^{\text{OPT}}; \underline{V}, \delta_{\theta^m})$, a contradiction to the assumption that $\widehat{M}$ dominates M^{OPT}.

Next suppose that $\theta^m = \underline{\theta}$. Claim E.6 (stated and proved below at the end of the proof) establishes that $q^{\mathrm{OPT}}(\underline{\theta}) \geq \underline{D}(\underline{\theta})$. Hence, $\underline{D}(\underline{\theta}) \leq q^{\mathrm{OPT}}(\underline{\theta}) \leq \hat{q}(\underline{\theta})$, where the second inequality follows from the fact, by assumption, $\hat{q}(\theta) \geq q^{\mathrm{OPT}}(\theta)$ for all $\theta \in [\theta^m, \theta^\star)$. Now recall that we have established above that there must exist $\theta' \in [\theta^m, \theta^\star)$ such that $\hat{q}(\theta') > q^{\mathrm{OPT}}(\theta')$.

If $\theta' = \theta^m$, because $\theta^m = \underline{\theta}$ and because $\underline{D}(\underline{\theta}) \leq q^{\mathrm{OPT}}(\underline{\theta}) \leq \hat{q}(\underline{\theta})$, we then have that $\underline{D}(\underline{\theta}) \leq q^{\mathrm{OPT}}(\underline{\theta}) < \hat{q}(\underline{\theta})$. This last property, together with the fact that $\hat{u}(\underline{\theta}) \geq u^{\mathrm{OPT}}(\underline{\theta})$ (by virtue of the fact that $\hat{q}(\theta) \geq q^{\mathrm{OPT}}(\theta)$ for all $\theta \geq \theta^m = \underline{\theta}$) and the property that $\underline{V}(\mathrm{q}) - \underline{\theta}\mathrm{q}$ is concave in q with a unique maximum at $\underline{D}(\underline{\theta})$ then implies that $W(\widehat{M}; \underline{V}, \delta_{\theta^m}) < W(M^{\mathrm{OPT}}; \underline{V}, \delta_{\theta^m})$, which contradicts the assumption that $\widehat{M}$ dominates M^{OPT}.

If, instead, $\theta' > \theta^m$, then, as argued above, $\hat{u}(\theta^m) > u^{\mathrm{OPT}}(\theta^m)$. This property, together with the fact that $\underline{D}(\underline{\theta}) \leq q^{\mathrm{OPT}}(\underline{\theta}) \leq \hat{q}(\underline{\theta})$ and the concavity of $\underline{V}(\mathrm{q}) - \underline{\theta}\mathrm{q}$ in q then implies again that $W(\widehat{M}; \underline{V}, \delta_{\theta^m}) < W(M^{\mathrm{OPT}}; \underline{V}, \delta_{\theta^m})$, a contradiction to the assumption that $\widehat{M}$ dominates M^{OPT}.

Case 2. Now suppose that there exists $\theta' \in [\theta^m, \theta^\star)$ such that $\hat{q}(\theta') < q^{\mathrm{OPT}}(\theta')$ but the set

$$\{\theta \in [\theta^m, \theta^\star) : \hat{q}(\theta) < q^{\mathrm{OPT}}(\theta)\}$$

has zero Lebesgue measure. Because $\hat{q}(\theta) \geq q^{\mathrm{OPT}}(\theta)$ for almost all $\theta \geq \theta^m$, $\hat{u}(\theta') \geq u^{\mathrm{OPT}}(\theta')$.

Suppose $\theta' > \underline{\theta}$. Lemma 6 in the main text then implies that $\hat{q}(\theta') < q^{\mathrm{OPT}}(\theta') \leq q^{\mathrm{BM}}(\theta') \leq D^\star(\theta')$. Along with the concavity of the function $V^\star(\mathrm{q}) - \theta'\mathrm{q}$ in q (reaching a maximum at $D^\star(\theta')$), these last series of inequalities implies that

$$V^\star(\hat{q}(\theta')) - \theta'\hat{q}(\theta') - \hat{u}(\theta') < V^\star(q^{\mathrm{OPT}}(\theta')) - \theta' q^{\mathrm{OPT}}(\theta') - u^{\mathrm{OPT}}(\theta').$$

Hence, $W(\widehat{M}; V^\star, \delta_{\theta'}) < W(M^{\mathrm{OPT}}; V^\star, \delta_{\theta'})$, a contradiction to the assumption that $\widehat{M}$ dominates M^{OPT}.

Finally, suppose that $\theta' = \underline{\theta}$. Because $\theta' \in [\theta^m, \theta^\star)$, this implies that $\theta^m = \underline{\theta}$. Thus, we have that $\hat{q}(\underline{\theta}) < q^{\mathrm{OPT}}(\underline{\theta}) \leq D^\star(\underline{\theta})$, where the last inequality follows from Claim E.6. The same arguments as above then imply that $W(\widehat{M}; V^\star, \delta_{\theta'}) < W(M^{\mathrm{OPT}}; V^\star, \delta_{\theta'})$, a contradiction to the assumption that $\widehat{M}$ dominates M^{OPT}.

Combining the two cases, we thus conclude that $\Theta_{\hat{q}, q^{\mathrm{OPT}}}$ has positive Lebesgue measure. ■

We now use Lemma E.20 to establish a contradiction to the assumption that $\widehat{M}$ dominates

M^{OPT}. Let

$$\theta_h \equiv \sup\{\theta \in \Theta_{\hat{q},q^{\text{OPT}}}\}.$$

Observe that, because $\Theta_{\hat{q},q^{\text{OPT}}}$ has positive Lebesgue measure, $\theta_h > \theta^m$. Furthermore, by definition, $\theta_h \leq \theta^\star$. Proposition 2 in the main text then implies that

$$q^{\text{OPT}}(\theta_h) \leq q^{\text{BM}}(\theta_h) < D^\star(\theta_h). \tag{E.62}$$

The rest of the proof below shows that there exists θ such that $W(M^{\text{OPT}}; V^\star, \delta_\theta) > W(\widehat{M}; V^\star, \delta_\theta)$.

First, consider the case in which $\hat{q}(\theta_h) < q^{\text{OPT}}(\theta_h)$. Condition (E.62) then implies that $\hat{q}(\theta_h) < q^{\text{OPT}}(\theta_h) < D^\star(\theta_h)$. Because $V^\star(\text{q}) - \theta_h \text{q}$ is concave in q, with a maximum at $D^\star(\theta_h)$, $V^\star(\hat{q}(\theta_h)) - \theta_h(\hat{q}(\theta_h)) < V^\star(q^{\text{OPT}}(\theta_h)) - \theta_h q^{\text{OPT}}(\theta_h)$. Furthermore, because, for all $y > \theta_h$, $\hat{q}(y) \geq q^{\text{OPT}}(y)$, $\int_{\theta_h}^{\bar{\theta}} \hat{q}(y)dy \geq \int_{\theta_h}^{\bar{\theta}} q^{\text{OPT}}(y)dy$. Jointly, these properties imply that $W(\widehat{M}; V^\star, \delta_{\theta_h}) < W(M^{\text{OPT}}; V^\star, \delta_{\theta_h})$, a contradiction to the assumption that $\widehat{M}$ dominates M^{OPT}.

Next, suppose that $\hat{q}(\theta_h) = q^{\text{OPT}}(\theta_h)$. The left-continuity of q^{OPT} along with the fact that $q^{\text{OPT}}(\theta_h) < D^\star(\theta_h)$ (by Condition (E.62)) then imply that there exists $\theta_\ell \in (\theta^m, \theta_h)$ such that, for all $\theta \in \mathcal{N} \equiv [\theta_\ell, \theta_h)$, $q^{\text{OPT}}(\theta) < D^\star(\theta_h)$. Then let $\Delta \equiv \sup_{y \in \mathcal{N}} \left[q^{\text{OPT}}(y) - \hat{q}(y)\right]$. Because $\mathcal{N} \cap \Theta_{\hat{q},q^{\text{OPT}}} \neq \emptyset$, $\Delta > 0$. By the definition of Δ, there exists $\theta \in \mathcal{N}$ and $\epsilon > 0$, with

$$\epsilon < \Delta \frac{P^\star(q^{\text{OPT}}(\theta_\ell)) - \theta_h}{P^\star(q^{\text{OPT}}(\theta_\ell)) - \theta^m} < \Delta, \tag{E.63}$$

such that $q^{\text{OPT}}(\theta) - \hat{q}(\theta) > \Delta - \epsilon$. Note that the second inequality in (E.63) follows from the fact that $q^{\text{OPT}}(\theta_\ell) < D^\star(\theta_h)$ (by virtue of the fact that $\theta_\ell \in \mathcal{N}$) which in turn implies that $P^\star(q^{\text{OPT}}(\theta_\ell)) > \theta_h > \theta^m$. Moreover, because $\Delta - \epsilon > 0$, $\theta \in \Theta_{\hat{q},q^{\text{OPT}}}$, which implies that $q^{\text{OPT}}(\theta) - \hat{q}(\theta) > 0$. Next, observe that

$$\begin{aligned}
&W(M^{\text{OPT}}; V^\star, \delta_\theta) - W(\widehat{M}; V^\star, \delta_\theta) = \int_{\hat{q}(\theta)}^{q^{\text{OPT}}(\theta)} \big(P^\star(z) - \theta\big)\mathrm{d}z - \int_{\theta}^{\bar{\theta}} \big(q^{\text{OPT}}(y) - \hat{q}(y)\big)\mathrm{d}y \\
&= \int_{\hat{q}(\theta)}^{q^{\text{OPT}}(\theta)} \big(P^\star(z) - \theta\big)\mathrm{d}z - \int_{\theta}^{\theta_h} \big(q^{\text{OPT}}(y) - \hat{q}(y)\big)\mathrm{d}y - \int_{\theta_h}^{\bar{\theta}} \big(q^{\text{OPT}}(y) - \hat{q}(y)\big)\mathrm{d}y \\
&\geq \int_{\hat{q}(\theta)}^{q^{\text{OPT}}(\theta)} \big(P^\star(z) - \theta\big)\mathrm{d}z - \int_{\theta}^{\theta_h} \big(q^{\text{OPT}}(y) - \hat{q}(y)\big)\mathrm{d}y \\
&\geq \big(P^\star(q^{\text{OPT}}(\theta_\ell)) - \theta\big)(q^{\text{OPT}}(\theta) - \hat{q}(\theta)) - \Delta(\theta_h - \theta)
\end{aligned}$$

$$\geq \big(P^\star(q^{\mathrm{OPT}}(\theta_\ell)) - \theta\big)(\Delta - \epsilon) - \Delta(\theta_h - \theta) = \Delta\big(P^\star(q^{\mathrm{OPT}}(\theta_\ell)) - \theta_h\big) - \epsilon\big(P^\star(q^{\mathrm{OPT}}(\theta_\ell)) - \theta\big)$$
$$\geq \Delta\big(P^\star(q^{\mathrm{OPT}}(\theta_\ell)) - \theta_h\big) - \epsilon\big(P^\star(q^{\mathrm{OPT}}(\theta_\ell)) - \theta^m\big) > 0.$$

The first two equalities follow from the definitions of $V^\star$, $W(M^{\mathrm{OPT}}; V^\star, \delta_\theta)$, and $W(\widehat{M}; V^\star, \delta_\theta)$. The first inequality follows from the fact that, by definition of θ_h, $q^{\mathrm{OPT}}(y) \leq \hat{q}(y)$ for all $y > \theta_h$. The second inequality follows from the monotonicity of q^{OPT}, the fact that $\theta > \theta_\ell$, and the definition of Δ. The third inequality follows from the choice of θ. The third equality is immediate. The fourth inequality follows from the fact that $\theta > \theta^m$. The last inequality follows from the fact that $\epsilon < \Delta \frac{P^\star(q^{\mathrm{OPT}}(\theta_\ell)) - \theta_h}{P^\star(q^{\mathrm{OPT}}(\theta_\ell)) - \theta^m}$. We thus conclude that $W(M^{\mathrm{OPT}}; V^\star, \delta_\theta) > W(\widehat{M}; V^\star, \delta_\theta)$. This, however, contradicts the assumption that $\widehat{M}$ dominates M^{OPT}.

Because a contradiction to the assumption that $\widehat{M}$ dominates M^{OPT} is reached under all possible scenarios, we conclude that M^{OPT} is undominated. Q.E.D.

We now state and prove a claim that we made in the proof of Lemma E.20 above.

Claim E.6. *Let $M^{\mathrm{OPT}} \equiv (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$ be a robustly optimal mechanism. (A) Then $q^{\mathrm{OPT}}(\underline{\theta}) \geq \underline{D}(\underline{\theta})$. (B) Furthermore, if $\theta^m = \underline{\theta}$, then $q^{\mathrm{OPT}}(\underline{\theta}) \leq D^\star(\underline{\theta})$.*

Proof of Claim E.6 Part (A). Assume, towards a contradiction, that $q^{\mathrm{OPT}}(\underline{\theta}) < \underline{D}(\underline{\theta})$. Then $q^{\mathrm{OPT}}(\underline{\theta}) < \underline{D}(\underline{\theta}) \leq D^\star(\underline{\theta}) = q^{\mathrm{BM}}(\underline{\theta})$. Continuity of q^{BM} and $\underline{D}$, along with the monotonicity of q^{OPT}, then imply that there exists $0 < \epsilon < \theta^\star - \underline{\theta}$ such that, for all $\theta \in [\underline{\theta}, \underline{\theta} + \epsilon]$, $q^{\mathrm{OPT}}(\theta) < \min\{q^{\mathrm{BM}}(\theta), \underline{D}(\theta)\}$. Then consider the mechanism $\widetilde{M} \equiv (\tilde{q}, \tilde{u})$, where the quantity schedule is given by

$$\tilde{q}(\theta) = \begin{cases} \min\{q^{\mathrm{BM}}(\theta), \underline{D}(\theta)\} & \text{if } \theta \in [\underline{\theta}, \underline{\theta} + \epsilon] \\ q^{\mathrm{OPT}}(\theta) & \text{otherwise,} \end{cases}$$

and $\tilde{u}(\theta) = \int_\theta^{\overline{\theta}} \tilde{q}(y)\mathrm{d}y$ for all $\theta \in \Theta$. We claim that $\widetilde{M} \in \mathcal{M}^{\mathrm{SL}}$. To see this, first observe that $\tilde{q}$ is weakly decreasing because q^{OPT}, $\underline{D}$, and q^{BM} are weakly decreasing, and $\tilde{q}(\underline{\theta}+\epsilon) > q^{\mathrm{OPT}}(\underline{\theta}+\epsilon)$. Hence, $\widetilde{M}$ is IC and IR. Next, observe that, because $\underline{\theta} + \epsilon < \theta^\star$, $\tilde{q}(\overline{\theta}) = q^{\mathrm{OPT}}(\overline{\theta}) = \mathrm{q}_\ell$. Because $q^{\mathrm{BM}}(\underline{\theta}) = D^\star(\underline{\theta})$, $\min\{q^{\mathrm{BM}}(\underline{\theta}), \underline{D}(\theta)\} = \underline{D}(\theta)$ and hence $\tilde{q}(\underline{\theta}) = \underline{D}(\underline{\theta})$. This means that $\underline{\mathrm{DWL}}(\underline{\theta}, \tilde{q}(\underline{\theta})) = 0$. Lemma 3 in the main text then implies that the robustness constraint for every $\theta \in [\underline{\theta}, \overline{\theta})$ reduces to $\int_\theta^{\overline{\theta}} (\underline{D}(y) - \tilde{q}(y))\,\mathrm{d}y \geq 0$. For $\theta > \underline{\theta} + \epsilon$, the inequality holds

because $\tilde{q} = q^{\mathrm{OPT}}$ and q^{OPT} is robustly optimal. For $\theta \in [\underline{\theta}, \underline{\theta} + \epsilon]$, we have that

$$\int_{\theta}^{\overline{\theta}} (\underline{D}(y) - \tilde{q}(y))\,\mathrm{d}y = \int_{\theta}^{\underline{\theta}+\epsilon} (\underline{D}(y) - \tilde{q}(y))\,\mathrm{d}y + \int_{\underline{\theta}+\epsilon}^{\overline{\theta}} \left(\underline{D}(y) - q^{\mathrm{OPT}}(y)\right)\mathrm{d}y \geq 0,$$

where the inequality follows from the fact that the first integral is non-negative because of the definition of $\tilde{q}$ and the second integral is non-negative because of the fact that q^{OPT} is a robustly optimal schedule. Together the above properties thus imply that $\widetilde{M} \in \mathcal{M}^{\mathrm{SL}}$.

Now observe that, by the definition of $\tilde{q}$, for all $\theta \in [\underline{\theta}, \underline{\theta} + \epsilon]$, $q^{\mathrm{OPT}}(\theta) < \tilde{q}(\theta) \leq q^{\mathrm{BM}}(\theta)$. Because, for all θ, $V^{\star}(\mathrm{q}) - z^{\star}(\theta)\mathrm{q}$ is concave with a maximum at $q^{\mathrm{BM}}(\theta)$, and because $\tilde{u}(\overline{\theta}) = 0$, we conclude that $W(\widetilde{M}; V^{\star}, F^{\star}) > W(M^{\mathrm{OPT}}; V^{\star}, F^{\star})$, a contradiction to the robust optimality of M^{OPT}. Hence, $q^{\mathrm{OPT}}(\underline{\theta}) < \underline{D}(\underline{\theta})$, as claimed.

Part (B). Next we establish that, if $\theta^m = \underline{\theta}$, then $q^{\mathrm{OPT}}(\underline{\theta}) \leq D^{\star}(\underline{\theta})$. Because $\theta^m = \underline{\theta}$, Lemma 7 in the main text implies that $M^{\star}$ is not robustly optimal. Consider the following two cases: (i) $\theta^m = \theta^{\star\star}$, and (ii) $\theta^m < \theta^{\star\star}$.

First, suppose that $\theta^m = \theta^{\star\star}$. Then, $q^{\mathrm{OPT}}(\theta) = \underline{D}(\theta)$ for all $\theta > \underline{\theta}$. Thus,

$$S(\underline{\theta}; q^{\mathrm{OPT}}) \equiv \int_{\theta}^{\overline{\theta}} \left(\underline{D}(y) - q^{\mathrm{OPT}}(y)\right)\mathrm{d}y = 0,$$

and, therefore,

$$R(q^{\mathrm{OPT}}) = S(\underline{\theta}; q^{\mathrm{OPT}}) - \underline{\mathrm{DWL}}(\underline{\theta}, q^{\mathrm{OPT}}(\underline{\theta})) = -\underline{\mathrm{DWL}}(\underline{\theta}, q^{\mathrm{OPT}}(\underline{\theta})).$$

Because $\underline{\mathrm{DWL}}(\underline{\theta}, q^{\mathrm{OPT}}(\underline{\theta})) \geq 0$, and because $R(q^{\mathrm{OPT}}) \geq 0$, we conclude that $\underline{\mathrm{DWL}}(\underline{\theta}, q^{\mathrm{OPT}}(\underline{\theta})) = 0$, which implies that $q^{\mathrm{OPT}}(\underline{\theta}) = \underline{D}(\underline{\theta}) \leq D^{\star}(\underline{\theta})$, as claimed.

Next suppose that $\theta^m < \theta^{\star\star}$, and assume, towards a contradiction, that $q^{\mathrm{OPT}}(\underline{\theta}) > D^{\star}(\underline{\theta})$. Consider the mechanism $\widetilde{M} \equiv (\tilde{q}, \tilde{u})$, where

$$\tilde{q}(\theta) = \begin{cases} D^{\star}(\theta) & \text{if } \theta = \underline{\theta} \\ q^{\mathrm{OPT}}(\theta) & \text{otherwise} \end{cases} \qquad \tilde{u}(\theta) = \int_{\theta}^{\overline{\theta}} \tilde{q}(y)\mathrm{d}y \quad \forall\ \theta \in \Theta.$$

Note that $\tilde{q}$ is weakly decreasing because q^{OPT} is weakly decreasing and, for every $\theta \in (\underline{\theta}, \theta^{\star})$,

$q^{\mathrm{OPT}}(\theta) \leq q^{\mathrm{BM}}(\theta) < q^{\mathrm{BM}}(\underline{\theta}) = D^\star(\underline{\theta})$, where the first inequality follows from Lemma 6, whereas the second inequality follows from the fact that q^{BM} is decreasing. Next, observe that

$$\underline{V}(D^\star(\underline{\theta})) - \underline{\theta} D^\star(\underline{\theta}) - u^{\mathrm{OPT}}(\underline{\theta}) > \underline{V}(q^{\mathrm{OPT}}(\underline{\theta})) - \underline{\theta} q^{\mathrm{OPT}}(\underline{\theta}) - u^{\mathrm{OPT}}(\underline{\theta}) \geq G^*. \tag{E.64}$$

The first inequality follows from the fact that $q^{\mathrm{OPT}}(\underline{\theta}) > D^\star(\underline{\theta}) \geq \underline{D}(\underline{\theta})$ along with the the fact that $\underline{V}(\mathrm{q}) - \underline{\theta}\mathrm{q}$ is concave in q attaining a maximum at $\underline{D}(\underline{\theta})$. The second inequality holds because $M^{\mathrm{OPT}} \in \mathcal{M}^{\mathrm{SL}}$. Because $\tilde{u}(\underline{\theta}) = \int_{\underline{\theta}}^{\overline{\theta}} \tilde{q}(y)\mathrm{d}y = u^{\mathrm{OPT}}(\underline{\theta})$ and $\tilde{q}(\underline{\theta}) = D^\star(\underline{\theta})$, we thus have that

$$R(\tilde{q}) = \underline{V}(\tilde{q}(\underline{\theta})) - \underline{\theta}\tilde{q}(\underline{\theta}) - \int_{\underline{\theta}}^{\overline{\theta}} \tilde{q}(y)\mathrm{d}y - G^* > 0.$$

Furthermore, because for all $\theta > \underline{\theta}$, $\tilde{q}(\theta) = q^{\mathrm{OPT}}(\theta)$, and because $\mathcal{M}^{\mathrm{OPT}}$ is robustly optimal, we have that $S(\theta; \tilde{q}) = S(\theta; q^{\mathrm{OPT}}) \geq 0$ for all $\theta \in (\underline{\theta}, \overline{\theta})$ and $\tilde{q}(\overline{\theta}) = \mathrm{q}_\ell$. We thus conclude that $\widetilde{M} \in \mathcal{M}^{\mathrm{SL}}$. Finally, by definition of $\widetilde{M}$, we have that $W(\widetilde{M}; V^\star, F^\star) = W(M^{\mathrm{OPT}}; V^\star, F^\star)$. Thus, $\widetilde{M}$ is also robustly optimal.

Because $\widetilde{M}$ is robustly optimal, the analysis in Section S.1 of Supplement I implies that there exist $\widetilde{\mu} \geq 0$ and a weakly increasing, right continuous function $\widetilde{\Lambda}$, with $\widetilde{\Lambda}(\underline{\theta}) = 0$, such that $\tilde{q}$ satisfies the analog of Conditions (S.45) and (S.46) with μ^{OPT} replaced by $\widetilde{\mu}$ and Λ^{OPT} replaced by $\widetilde{\Lambda}$. The complementary slackness conditions along with the fact that $R(\tilde{q}) > 0$ implies that $\widetilde{\mu} = 0$. Moreover, because $R(\tilde{q}) = S(\underline{\theta}; \tilde{q}) - \underline{\mathrm{DWL}}(\underline{\theta}, \tilde{q}(\underline{\theta})) > 0$ and because $\underline{\mathrm{DWL}}(\underline{\theta}, \tilde{q}(\underline{\theta})) \geq 0$, we have that $S(\underline{\theta}; \tilde{q}) > 0$. Continuity of $S(\cdot; \tilde{q})$ then implies that there exists $\delta > 0$ such that $S(\theta; \tilde{q}) > 0$ for all $\theta \in [\underline{\theta}, \underline{\theta} + \delta]$. Thus, by the complementary slackness conditions, we have that $\widetilde{\Lambda}(\theta) = 0$ for all $\theta \in [\underline{\theta}, \underline{\theta} + \delta]$. We thus have that $\widetilde{\Lambda}(\theta) + \widetilde{\mu} = 0$ for all $\theta \in [\underline{\theta}, \underline{\theta} + \delta]$. Because $\underline{\theta} = \theta^m$, Observation S.2 in Supplement I then implies that $\tilde{q}(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta^m, \theta^m + \delta]$. Because $\tilde{q}(\theta) = q^{\mathrm{OPT}}(\theta)$ for all $\theta \in (\theta^m, \theta^m + \delta]$, this implies that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta^m, \theta^m + \delta]$. However, this contradicts the fact that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta^m, \min\{\theta^\star, \theta^{\star\star}\})$ as established in Section S.1 of Supplement I.[29] We thus conclude that $q^{\mathrm{OPT}}(\underline{\theta}) \leq D^\star(\underline{\theta})$, as claimed. Q.E.D.

We conclude by commenting on the assumption that, when $\theta^m > \underline{\theta}$, $q^{\mathrm{OPT}}(\underline{\theta}) \in Q^\#(\underline{\theta})$.

[29] Recall that, for any robustly optimal quantity schedule q, $\theta_q^{\star\star} \equiv \inf\{\theta \in [\theta^m, \overline{\theta}] : q(y) = \underline{D}(y)\ \forall\, y \in [\theta, \overline{\theta}]\}$. For $\tilde{q}$ and q^{OPT} the cut-off is the same, that is, $\theta_{\tilde{q}}^{\star\star} = \theta_{q^{\mathrm{OPT}}}^{\star\star} = \theta^{\star\star}$.

Proposition 2 in the main text implies that

$$\lim_{\theta \downarrow \underline{\theta}} q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\underline{\theta}) = D^{\star}(\underline{\theta}) = \arg\max_{\mathrm{q} \in [0, \bar{\mathrm{q}}]} \{V^{\star}(\mathrm{q}) - \underline{\theta}\mathrm{q}\}.$$

Because q^{OPT} is weakly decreasing, $q^{\mathrm{OPT}}(\underline{\theta}) \geq D^{\star}(\underline{\theta})$. Because $D^{\star}(\underline{\theta}) \geq \underline{D}(\underline{\theta})$ and

$$\underline{D}(\underline{\theta}) = \arg\max_{\mathrm{q} \in [0, \bar{\mathrm{q}}]} \left\{ \underline{V}(\mathrm{q}) - \underline{\theta}\mathrm{q} - \int_{\underline{\theta}}^{\bar{\theta}} q^{\mathrm{OPT}}(y)\mathrm{d}y \right\}$$

by choosing $q^{\mathrm{OPT}}(\underline{\theta}) = D^{\star}(\underline{\theta})$, the designer can then relax the robustness constraint at $\underline{\theta}$ with no effect on the robustness constraints for any $\theta > \underline{\theta}$ and on the buyer's payoff under the conjectured model. This means that, when $\theta^m > \underline{\theta}$, starting from any robustly optimal mechanism M^{OPT} in which q^{OPT} is left-continuous, one can construct another robustly optimal mechanism $\mathring{M}^{\mathrm{OPT}}$ in which $\mathring{q}^{\mathrm{OPT}}$ is left-continuous and satisfies $\mathring{q}^{\mathrm{OPT}}(\underline{\theta}) = D^{\star}(\underline{\theta})$. Because $D^{\star}(\underline{\theta}) \in Q^{\#}(\underline{\theta})$, Proposition E.14 then implies that the mechanism $\mathring{M}^{\mathrm{OPT}}$ is undominated.

E.3 Non-monotonicity of the procured quantity in $\underline{F}$

As shown in Supplement I, the quantity procured under robustly optimal quantity mechanisms depends on the set $\mathcal{F}$ of feasible technologies only through $F^{\star}$ and $\underline{F}$. In this section, we show that, as $\underline{F}$ becomes larger (in a sense made precise below), the quantity procured under robustly optimal mechanisms may change in a non-monotonic way. To simplify the exposition, we assume $V^{\star} = \underline{V}$, which implies that Baron-Myerson-with-quantity-bridge mechanisms of Definition S.9 are robustly optimal. Below, we analyze how the quantity procured under such mechanisms changes as $\underline{F}$ changes.

Fix $F^{\star}$ and consider a sequence $(\underline{F}_n)$ of cdfs (corresponding to the lowest elements of the set $\mathcal{F}$ of feasible technologies) with the following properties:

(a) for every n there exists $\underline{\theta}_n \in \Theta$ and $\delta_n \geq 0$ such that,

  (1) $\underline{F}_n$ is absolutely continuous over $(-\infty, \overline{\theta})$, with density $\underline{f}_n(\theta) > 0$ for all $\theta \in [\underline{\theta}_n, \overline{\theta})$; if there is no atom at $\overline{\theta}$, we let $\underline{f}_n(\overline{\theta}) > 0$ denote the density of $\underline{F}_n$ at $\theta = \overline{\theta}$,

  (2) $\underline{F}_n(\theta) = 0$ for all $\theta < \underline{\theta}_n$, $\underline{F}_n(\theta) = 1$ for all $\theta \geq \overline{\theta}$,

  (3) $\lim_{\theta \uparrow \bar{\theta}} \underline{F}_n(\theta) = 1 - \delta_n$,

(b) for every n, $\underline{\theta}_{n+1} \geq \underline{\theta}_n$, and for every $\theta \in (\underline{\theta}, \overline{\theta})$, there exists n such that $\theta < \underline{\theta}_n < \overline{\theta}$,

(c) for every n, $\delta_{n+1} \geq \delta_n$,

(d) there exists $\overline{n}, \overline{\overline{n}} \in \mathbb{N} \cup \{+\infty\}$ with $\overline{\overline{n}} > \overline{n}$ such that $\underline{\theta}_n = \underline{\theta}$ if, and only if, $n \leq \overline{n}$, and $\delta_n > 0$ if, and only if, $n \geq \overline{\overline{n}}$.

(e) for every n, the function $\underline{z}_n : [\underline{\theta}_n, \overline{\theta}] \to \mathbb{R}$ defined by

$$\underline{z}_n(\theta) \equiv \begin{cases} \theta + \underline{F}_n(\theta)/\underline{f}_n(\theta) & \text{if } \theta \in [\underline{\theta}_n, \overline{\theta}) \\ \overline{\theta} + 1/\underline{f}_n(\overline{\theta}) & \text{if } \theta = \overline{\theta} \text{ and } \delta_n = 0 \\ \overline{\theta} + (1-\delta_n)/\delta_n & \text{if } \theta = \overline{\theta} \text{ and } \delta_n > 0 \end{cases}$$

is increasing over $[\underline{\theta}_n, \overline{\theta}]$ and continuous over $[\underline{\theta}_n, \overline{\theta})$.

(f) for all $\theta \in [\underline{\theta}_{n+1}, \overline{\theta}]$,

$$\frac{\underline{F}_{n+1}(\theta)}{\underline{f}_{n+1}(\theta)} \leq \frac{\underline{F}_n(\theta)}{\underline{f}_n(\theta)}, \tag{E.65}$$

and for every n and every $\theta \in [\underline{\theta}_n, \overline{\theta}]$,

$$\underline{z}_n(\theta) < z^\star(\theta), \tag{E.66}$$

which is the case when $\frac{\underline{F}_n(\theta)}{\underline{f}_n(\theta)} < \frac{F^\star(\theta)}{f^\star(\theta)}$.

Figure E.9 provides an illustration of the sequence $(\underline{F}_n)$. Note that property (f) above means that the technologies are ranked in the reverse-hazard-rate order. The sequence can thus be interpreted as capturing an increase in the severity of the buyer's (downside) uncertainty over the technology that determines the seller's cost.

Let q_n^{OPT} be a robustly optimal quantity schedule when the lowest technology in $\mathcal{F}$ is $\underline{F}_n$. The following proposition establishes that the quantity procured under any robustly optimal quantity mechanism is not monotone in the lowest distribution $\underline{F}_n$. This property holds despite the fact that, as is well known, the Baron-Myerson quantity schedule $\underline{q}_n^{\mathrm{BM}}$ defined, for all $\theta \in [\underline{\theta}_n, \overline{\theta})$, by

$$\underline{q}_n^{\mathrm{BM}}(\theta) \equiv \arg\max_{\mathrm{q} \in [0, \bar{\mathrm{q}}]} \{V^\star(\mathrm{q}) - \underline{z}_n(\theta)\mathrm{q}\}$$

is increasing in the inverse-hazard rare order: for any $n, n' \in \mathbb{N}$, with $n' > n$ and any $\theta \geq \underline{\theta}_{n'}$, $\underline{q}_{n'}^{\mathrm{BM}}(\theta) \geq \underline{q}_n^{\mathrm{BM}}(\theta)$. That is, when the buyer's model over the technology governing

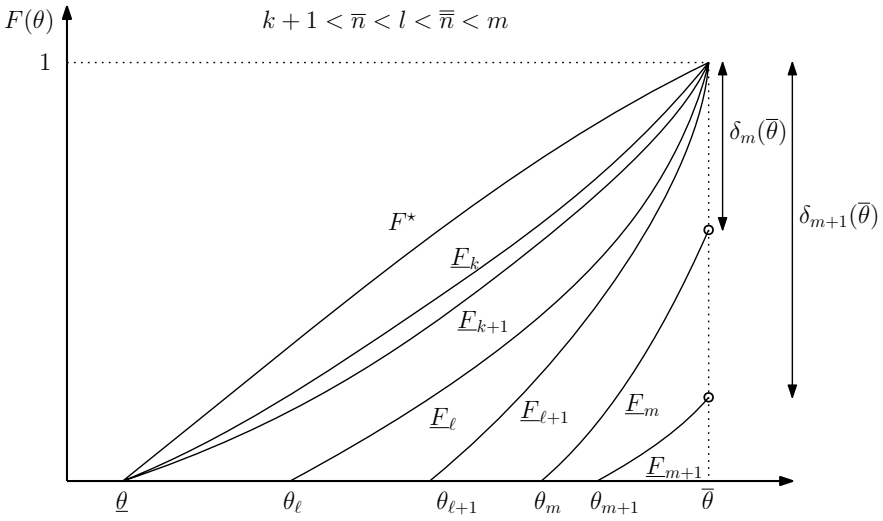


Figure E.9: Pictorial depiction of the sequence $(\underline{F}_n)$.

the seller's cost coincides with the distribution $\underline{F}_n$, an increase in the distribution (in the inverse-hazard-rate order) leads to an increase in the output procured.

Proposition E.15 (Non-mon. of output in cost uncertainty)**.** *Suppose that $V^\star = \underline{V}$. Let $(\underline{F}_n)$ be any sequence of cdfs satisfying properties (a)-(f) above and let (M_n^{OPT}) be any sequence of mechanisms such that, for each n, $M_n^{\mathrm{OPT}} \equiv (q_n^{\mathrm{OPT}}, u_n^{\mathrm{OPT}})$ is a robustly optimal mechanism when the lowest distribution in $\mathcal{F}$ is $\underline{F}_n$. Then, for every $\theta \in (\underline{\theta}, \overline{\theta})$,*

1. *there exists $n(\theta) \in \mathbb{N}$ such that $q_n^{\mathrm{OPT}}(\theta)$ is weakly increasing in n (respectively, weakly decreasing in n) over $n \leq n(\theta) - 1$ (respectively, over $n > n(\theta)$).*

2. *there exists $j, k \in \mathbb{N}$ with $j < k$ such that $q_j^{\mathrm{OPT}}(\theta) > q_k^{\mathrm{OPT}}(\theta)$.*

Figure E.10 illustrates the result in Proposition E.15. For any $\theta \in [\underline{\theta}, \theta^\dagger]$, as the lowest technology changes from $\underline{F}_1$ to $\underline{F}_2$, the quantity procured increases. In fact, the robustly optimal quantity schedule changes from the dash-dotted line to the dash-double-dotted line. Note that both $\underline{F}_1$ to $\underline{F}_2$ have support Θ; a reduction in the inverse of reverse hazard rate then implies a reduction in the value of containing the rents for the most efficient types, leading to an increase in the output procured under the optimal mechanism. When the lowest technology changes from $\underline{F}_2$ to $\underline{F}_3$, the robustly optimal quantity schedule changes from the dash-double-dotted line to the solid line and the quantity procured from types in the range $[\underline{\theta}, \theta^\dagger]$ is reduced. This is because the support of $\underline{F}_3$ no longer contains low-cost types. The buyer can then afford to procure less output from these types without jeopardizing

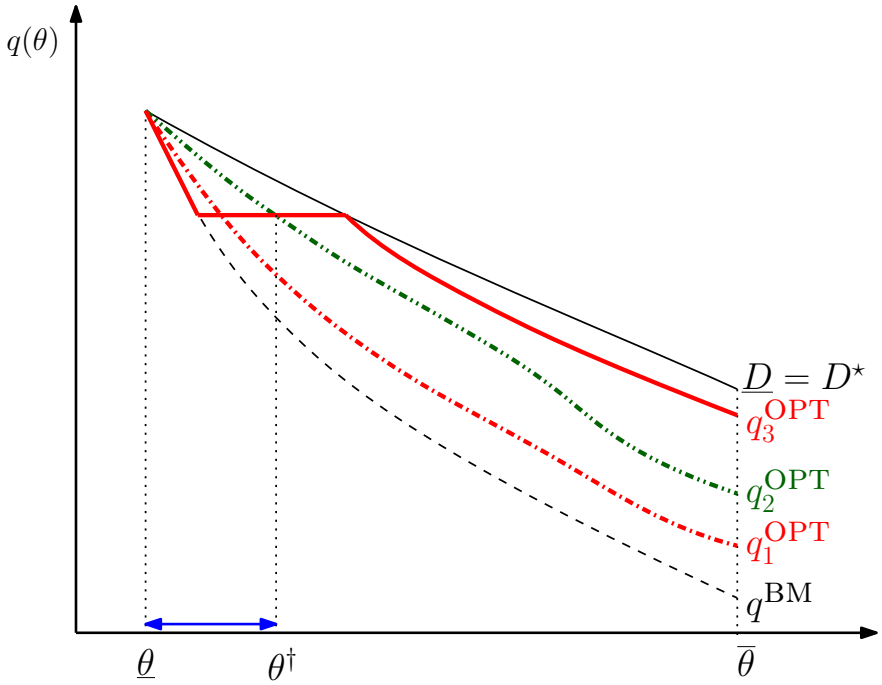


Figure E.10: Illustration of Proposition E.15.

robustness. Thus, the quantity procured from types in the range $[\underline{\theta}, \theta^\dagger]$ is not monotone in n, equivalently, in the worst possible technology.

Proof of Proposition E.15. Each part below establishes the corresponding part in the proposition.

Part 1. For any $\theta \in (\underline{\theta}, \overline{\theta})$, let $n(\theta)$ be the largest $n \geq \overline{n}$ such that $\underline{\theta}_n \leq \theta < \underline{\theta}_{n+1}$. Existence of such an $n(\theta)$ is guaranteed by Condition (b) in the definition of the sequence $(\underline{F}_n)$, which implies that $\underline{\theta}_n \leq \underline{\theta}_{n+1} < \overline{\theta}$ and $\lim_{n\to\infty} \underline{\theta}_n = \overline{\theta}$. For any $n \leq n(\theta)-1$, we have that $\theta \in [\underline{\theta}_n, \overline{\theta}]$ and $\theta \in [\underline{\theta}_{n+1}, \overline{\theta}]$. As established in Supplement I, $q_n^{\mathrm{OPT}}(\theta) = \underline{q}_n^{\mathrm{BM}}(\theta)$ and $q_{n+1}^{\mathrm{OPT}}(\theta) = \underline{q}_{n+1}^{\mathrm{BM}}(\theta)$. Condition (E.65) in turn implies that $\underline{q}_n^{\mathrm{BM}}(\theta) \leq \underline{q}_{n+1}^{\mathrm{BM}}(\theta)$, that is, $q_n^{\mathrm{OPT}}(\theta)$ is weakly increasing in n for $n \leq n(\theta) - 1$.

For any $n > n(\theta)$, $\theta < \underline{\theta}_n$, and therefore, $q_n^{\mathrm{OPT}}(\theta) = \max\{q^{\mathrm{BM}}(\theta), \underline{D}(\underline{\theta}_n)\}$. The quantity $\underline{D}(\underline{\theta}_n)$ is weakly decreasing in n because $\underline{\theta}_n \leq \underline{\theta}_{n+1} < \overline{\theta}$ for every n. Consequently, $q_n^{\mathrm{OPT}}(\theta)$ is also weakly decreasing in n.

Part 2. To establish the second part of the proposition, it suffices to exhibit a pair $j, k \in \mathbb{N}$, with $j < k$, such that $q_j^{\mathrm{OPT}}(\theta) > q_k^{\mathrm{OPT}}(\theta)$. To do so, consider the following two cases.

Case 1. Suppose $q^{\mathrm{BM}}(\theta) \geq \underline{D}(\underline{\theta}_{n(\theta)+1})$. Then let $j = n(\theta)$ and $k = n(\theta) + 1$, and observe that

$$q_j^{\mathrm{OPT}}(\theta) = \underline{q}_j^{\mathrm{BM}}(\theta) = \underline{D}\left(\theta + \frac{\underline{F}_j(\theta)}{\underline{f}_j(\theta)}\right) > \underline{D}\left(\theta + \frac{F^\star(\theta)}{f^\star(\theta)}\right) = q^{\mathrm{BM}}(\theta) = q_k^{\mathrm{OPT}}(\theta),$$

where the inequality follows from (E.66).

Case 2. Suppose $q^{\mathrm{BM}}(\theta) < \underline{D}(\underline{\theta}_{n(\theta)+1})$. Then let $j = n(\theta)+1$ and let k be such that $\underline{\theta}_k > \underline{\theta}_j$. Existence of such an k is ensured by Condition (b) in the definition of the sequence $(\underline{F}_n)$. Then

$$q_k^{\mathrm{OPT}}(\theta) = \max\{q^{\mathrm{BM}}(\theta), \underline{D}(\underline{\theta}_k)\} < \underline{D}(\underline{\theta}_j) = q_j^{\mathrm{OPT}}(\theta).$$

To see this, observe that $\underline{\theta}_k > \underline{\theta}_j$ implies that $\underline{D}(\underline{\theta}_k) < \underline{D}(\underline{\theta}_j)$. Hence, if $q_k^{\mathrm{OPT}}(\theta) = \underline{D}(\underline{\theta}_k)$, then $q_k^{\mathrm{OPT}}(\theta) = \underline{D}(\underline{\theta}_k) < \underline{D}(\underline{\theta}_j) = q_j^{\mathrm{OPT}}(\theta)$. If, instead, $q_k^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$, the result follows from the fact that, by assumption, $q^{\mathrm{BM}}(\theta) < \underline{D}(\underline{\theta}_j)$. *Q.E.D.*