Supplementary Appendix to

Persistent Private Information Revisited

Not for Publication

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Abstract

This Supplementary Appendix to Bloedel, Krishna, and Strulovici (2020) contains proofs and results omitted from that paper and its Online Appendix. References to equations and results follow the labeling convention in the main paper.

1. Optimal FO-IC Contracts

1.1. Formal Definition of Principal's First-Order Problem

The FOA allows us to formulate (a version of) the principal's problem as one of stochastic control. By [**FO-IC**], this is a stochastic control problem in which (s, Q) are controls and (y, q, p) are state variables. Let $J(y, q, p)$ denote the principal's value function over FO-IC contracts, starting from state (y, q, p).

Definition 1.1. The principal's *auxiliary first-order (FO) problem* is

$$J(y_0, q_0, p_0) := \inf_{(c,Q)\in\mathscr{A}_P(y_0,q_0,q_0)} \mathbf{E}_0^* \left[\int_0^\infty e^{-\rho t} (c_t - b_t) \, \mathrm{d}t \right] \quad [\mathbf{1.1}]$$

where the y-adapted process $(c, Q) \in \mathscr{A}_P(y_0, q_0, q_0)$ if and only if (i) they induce unique strong solutions to [**4.2**] and [**4.3**] (including the transversality conditions) in which $\gamma \equiv p$

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in accordance with [FO-IC] and (ii) $\mathbf{E}_0^* \left[\int_0^\infty e^{-\rho t} |c_t| \, \mathrm{d}t\right] < \infty$ so that the value function J is well-defined and finite. We refer to $J : \mathbb{R} \times D \to \mathbb{R}$ as the principal's *FO value function*. The principal's *first-stage FO problem* is

$$\inf_{p_0 < 0 \text{ s.t. } (q_0, p_0) \in D} J(y_0, q_0, p_0) \tag{1.2}$$

The principal's *FO problem* is the joint optimization consisting of his auxiliary FO problem and first-stage FO problem.

When $\lambda = 0$, notice that the first-stage FO problem is trivial by Lemma H.1(ii). In that case, let $\tilde{J}(y, q) := J(y, q, \theta q)$ denote the natural projection of J from $\mathbb{R} \times D$ onto $\mathbb{R} \times \mathbb{R}_{--}$.

1.2. Proof of Theorem 5

By Lemma H.1(ii), when $\lambda = 0$ every FO-IC contract has constant geometric sensitivity $k_t \equiv \theta$. Thus, the principal's FO value function J over FO-IC contracts may be written independently of p as the function $(y, q) \mapsto \tilde{J}(y, q)$, as described in Supplementary Appendix 1.1. By the homogeneity of the

$$\tilde{J}(y, q) := -\frac{y}{\rho} - \frac{\log(-q)}{\theta\rho} + H$$

for some constant $H \in \mathbb{R}$. Thus, $\tilde{J} \in \mathbf{C}^\infty(\mathbb{R} \times \mathbb{R}_{--})$. By adapting standard arguments to the present setting (again see Yong and Zhou 1999, Theorem 3.3 and Touzi 2018, Propositions 2.4-2.5), it follows that $\tilde{J}$ satisfies the HJB equation

$$\begin{aligned} \rho \tilde{J}(y, q) = \inf_{c \in \mathbb{R}} \Big[& c - y + \tilde{J}_y(y, q) \cdot (\mu - \lambda y) + \tilde{J}_q(y, q) \cdot (\rho q - u(c)) + \frac{\sigma^2}{2} \tilde{J}_{yy}(y, q) \\ & + \frac{\sigma^2 (\theta q)^2}{2} \tilde{J}_{qq}(y, q) + \sigma (\theta q) \tilde{J}_{yq}(y, q) \Big] \end{aligned} \tag{1.3}$$

Let $\beta := u(c)/q$ and invert $u(\cdot)$ so that $c = \bar{c}(q, \beta)$. After this transformation, dropping all terms on the RHS of [1.3] that do not involve β yields the equivalent minimization problem

$$\inf_{\beta > 0} \left[-\frac{\log(\beta)}{\theta} + \frac{1}{\rho\theta} \cdot \beta \right]$$

which is uniquely solved by $\beta^* = \rho$. Thus, the policy function generated by the HJB equation [1.3] induces Contract W. Plugging $\beta^* = \rho$ into Contract W shows that H is such that $\tilde{J}(y,q) \equiv \Pi(y,q,\rho)$ (as in [3.14]). Lemma 3.3 and Remark 2 (or display [3.12]) implies that Contract W indeed costs $\Pi(b,q,\rho)$ (as in [3.14] with $r = \rho$) when starting from state (b,q), from which it follows that Contract W is indeed the optimal FO-IC contract.

1.3. Transient Shocks

In this supplementary appendix, we show that, conditional on the validity of the FOA, the optimal full-commitment contract strictly dominates the optimal Self-Insurance Contract (or, equivalently, the optimal Self-Insurance Contract).

Definition 1.2. The environment is *regular* if it satisfies conditions (i) and (ii) below, and *respects the FOA* if it also satisfies condition (iii):

(i) An optimal FO-IC contract exists.

(ii) The principal's first-order value function J is a twice continuously-differentiable solution of the HJB equation [1.4] (given in Appendix 1.3.1).

(iii) The FOA is valid, meaning that (a) every IC contract is FO-IC and (b) the optimal FO-IC contract is IC.

Regularity is a technical assumption; if it were to fail, we could not solve for the optimal FO-IC contract simply by taking first-order conditions in the principal's HJB equation. By contrast, assuming that the FOA is valid is a more substantive restriction.

Proposition 1.3. Let $\lambda > 0$. If the environment is regular, then no optimal FO-IC contract is stationary. If the environment also respects the FOA, then no optimal full-commitment contract is stationary.

The proof of Proposition 1.3, which is given in Appendix 1.3.1, builds on W11's attempted derivation of the optimal contract (exposited in Appendix K). Under regularity, an optimal FO-IC contract can be stationary only if it is Contract W. But this is impossible, for Theorem 2 shows that Contract W is strictly dominated by the optimal Self-Insurance Contract, which is also FO-IC by Proposition 4.3. When the environment respects the FOA, this implies that some IC contracts strictly dominate the optimal stationary contract.

We conjecture that the present environment is indeed regular and respects the FOA for all parameter values, but have not proved this. Proposition 1.3 should be read as identifying plausible sufficient conditions under which stationary contracts are suboptimal. We present further conjectures about the optimal full-commitment contract in Section 6 below.

1.3.1. *Proof of Proposition 1.3*

Definition 1.2(ii) requires that $J \in \mathbf{C}^2(\mathbb{R} \times D)$, where J is the FO value function defined in [**1.1**]. By adapting standard arguments to the present setting (again see Yong and Zhou 1999, Theorem 3.3 and Touzi 2018, Propositions 2.4-2.5) J must satisfy the HJB equation (translated from W11, display 19, p. 1249)

$$\begin{aligned}\rho J(y,q,p) = \min_{(c,Q)\in\mathbb{R}^2, \gamma=p} \Big[& c - y + J_y(y,q)\cdot(\mu - \lambda y) \\ & + J_q(y,q,p)\cdot(\rho q - u(c)) + J_p(y,q,p)\cdot((\rho+\lambda)p + \theta u(c)) \\ & + \frac{\sigma^2}{2} J_{yy}(y,q,p) + \frac{\sigma^2\gamma^2}{2} J_{qq}(y,q,p) + \frac{\sigma^2 Q^2}{2} J_{pp}(y,q,p) \\ & + \sigma^2\gamma J_{yq}(y,q,p) + \sigma^2 Q J_{yp}(y,q,p) + \sigma^2\gamma Q J_{qp}(y,q,p)\Big]\end{aligned} \tag{1.4}$$

We wish to rewrite [**1.4**] in terms of (y, q, p). This requires two lemmas.

Lemma 1.4. Under any FO-IC contract and truthful reporting, the geometric sensitivity k evolves as

$$dk_t = \left[(\beta_t + \lambda)\, k_t - \theta\beta_t + \sigma^2 k_t \left(k_t^2 - \hat{Q}_t\right)\right] dt + \sigma \left[k_t^2 - \hat{Q}_t\right] dW_t, \tag{1.5}$$

where the process $\hat{Q} = (\hat{Q}_t)_{t\geq 0}$ is defined as $\hat{Q}_t := -Q_t/q_t$.

Proof of Lemma 1.4. Immediate from an application of Ito's lemma to [**4.2**] and [**4.3**] given $W^y = W$ (truthtelling) and the identify [**FO-IC**]. □

Lemma 1.5. Let $\lambda > 0$. Suppose that $J(y,q,p)$ (as defined in [**1.1**]) is well-defined and finite-valued on $\mathbb{R} \times D$. Then it satisfies $J(y,q,p) \equiv \hat{J}(y,q,p/q)$ where

$$\hat{J}(y,q,k) := -\frac{y}{\rho+\lambda} - \frac{\log(-q)}{\rho\theta} + h(k) \tag{1.6}$$

for some function $h : (0,\theta) \to \mathbb{R}$. Moreover, if $J \in \mathbf{C}^2(\mathbb{R} \times D)$, then $h \in \mathbf{C}^2((0,\theta))$.

Proof of Lemma 1.5. Given the representation [**1.6**], the claim about twice continuous-differentiability is immediate. The representation [**1.6**] follows from translation-invariance properties of the principal's feasible set in the first-order problem. Let $(q,p) \in D$, $y \in \mathbb{R}$, and $\alpha > 0$ be given for Steps 1-2 below.

Step 1: We assert that $J(y,q,p) = J(0,q,p) - y/(\rho+\lambda)$. Let $(c,Q) \in \mathcal{A}_P(y,q,p)$ be given. Define the deterministic process g by $g_t := ye^{-\lambda t}$ and the y-adapted processes $(\tilde{c}, \tilde{Q})$ by $\tilde{c}_t(y) := c_t(y+g)$ and $\tilde{Q}_t := Q_t(y+g)$. Let $\mathsf{P}^{*,(y)}$ denote the distribution over report paths starting from $y_0 = y$ and $\mathsf{P}^{*,(0)}$ denote the analogous distribution starting from $y_0 = 0$, assuming truthful reporting in both cases. It is then easy to see that the law of (c, Q) under $\mathsf{P}^{*,(y)}$ is the same as the law of $(\tilde{c}, \tilde{Q})$ under $\mathsf{P}^{*,(0)}$, for y under the former measure has the same law as $y + g$ under the latter. [1] It follows that

$$\mathsf{E}_0^{*,(y)}\left[\int_0^\infty e^{-\rho t}\left(c_t - y_t\right)dt\right] = \mathsf{E}_0^{*,(0)}\left[\int_0^\infty e^{-\rho t}\left(\tilde{c}_t - y_t\right)dt\right] - \underbrace{\int_0^\infty e^{-\rho t} g_t dt}_{= y/(\rho+\lambda)}.$$

To complete the proof of the assertion, it therefore suffices to show that $(\tilde{c}, \tilde{Q}) \in \mathcal{A}_P(0, q, p)$, which is immediate from the definitions.

Step 2: We assert that $J(y, \alpha q, \alpha p) = J(y, q, p) - \log(\alpha)/(\rho\theta)$. Let $(c, Q) \in \mathcal{A}_P(y, q, p)$ be given. Define the y-adapted processes $(\tilde{c}, \tilde{Q})$ by $\tilde{c}_t := c_t - \log(\alpha)/\theta$ and $\tilde{Q}_t := \alpha Q_t$. Note that $u(\tilde{c}_t) \equiv \alpha u(c_t)$ under exponential utility. It is then immediate from [4.2], [4.3], and [FO-IC] that $(\tilde{c}, \tilde{Q}) \in \mathcal{A}_P(y, \alpha q, \alpha p)$. Because the distribution over endowment paths is the same starting from either state, the principal's cost of $\tilde{c}$ starting from $(y, \alpha q, \alpha p)$ is simply her cost of c starting from (y, q, p) plus the deterministic term $\int_0^t e^{-\rho t}\left(-\log(\alpha)/\theta\right)dt = -\log(\alpha)/(\theta\rho)$. Since $(c, Q) \in \mathcal{A}_P(y, q, p)$ was arbitrary, the assertion follows.

Step 3: Finally, fixing $(q, p) \in D$ and $y \in \mathbb{R}$, let $\alpha := -p/q$. By Lemma H.1(i), $\alpha \in (0, \theta)$. Combining Steps 1-2 then yields

$$J(y, q, p) = -\frac{y}{\rho+\lambda} - \frac{\log(-q)}{\rho\theta} + J\left(0, -1, -p/q\right).$$

Defining $h : (0, \theta) \to \mathbb{R}$ by $h(k) := J(0, -1, -k)$ and $\hat{J}$ as in [1.6] then completes the proof. □

(1) Recall that b_t is given by [2.2], and so too is y_t under truthful reporting. Thus, letting $b^{(0)}$ denote the endowment process given $b_0 := 0$ and $b^{(y)}$ denote the endowment process given $b_0 := y$, we have $b_t^{(y)} \equiv b_t^{(0)} + g_t$.

We may now equivalently rewrite the HJB equation [1.4] in terms of (y, q, k) as

$$\begin{aligned} \rho \hat{J}(y,q,k) = \min_{\beta>0, \hat{Q}\in\mathbb{R}} \Big\{ & \overline{c}(q,\beta) - y - \frac{y}{\rho+\lambda}\left[\mu - \lambda y\right] - \frac{1}{\rho\theta}\left[\rho - \beta\right] \\ & + h'(k) \cdot \left[(\beta+\lambda)\, k - \theta\beta + \sigma^2 k \left(k^2 - \hat{Q}\right) \right] \\ & + \frac{\sigma^2 k^2}{2\rho\theta} + \frac{\sigma^2 \left(k^2 - \hat{Q}\right)^2}{2} h''(k) \Big\} \end{aligned} \qquad [1.7]$$

where we have used the SDE for k [1.5], the functional form for $\hat{J}$ in [1.6] (and the implied smoothness of h), and converted the minimization over c to an equivalent minimization over β.

Lemma 1.6. Let $\lambda > 0$. If the environment is regular, then any optimal $\beta^\dagger$ satisfies $\beta_t^\dagger \equiv \hat{\beta}(k_t)$, where $\hat{\beta} : (0,\theta) \to \mathbb{R}$ is defined by

$$\hat{\beta}(k) := \frac{1}{1/\rho + \theta(k-\theta)h'(k)} \qquad [1.8]$$

Proof of Lemma 1.6. Eliminating terms on the RHS of [1.7] yields the following minimization over β:

$$\min_{\beta>0} \left[-\log(\beta) + \frac{\beta}{\rho} + \beta \cdot h'(k)(k-\theta) \right]. \qquad [1.9]$$

By arguments analogous to those given in Appendix G, there exists a unique solution to problem [1.9]. This unique solution is characterized by the FOC for [1.9], which reduces to [1.8]. It is standard that any optimal $\beta^\dagger$ must solve the problem [1.9] a.e., which completes the proof. □

Lemma 1.7. Let $\lambda > 0$. If the environment is regular, then any $k_0^\dagger \in \arg\min_{k_0 \in (0,\theta)} h(k_0)$ satisfies $\hat{\beta}(k_0^\dagger) = \rho$.

Proof of Lemma 1.7. Because $h \in \mathbf{C}^2\left((0,\theta)\right)$ by Definition 1.2(i) and Lemma 1.5, it follows that any such $k_0^\dagger$ satisfies the necessary first-order condition $h'(k_0^\dagger) = 0$. Plugging this into [1.8] completes the proof. □

Lemma 1.8. Let $\lambda > 0$. If the environment is regular and an optimal FO-IC contract is stationary, then that contract is Contract W.

Proof of Lemma 1.8. Consider any optimal FO-IC contract that is stationary. By Proposition 4.2(i), the induced geometric sensitivity process k must be constant. It follows from

Lemma 1.4 and the unique decomposition property for Ito processes that $k_t^2 \equiv \hat{Q}_t^\dagger$ (so that k has zero volatility) and thus that $\beta_t^\dagger(\lambda + k_t) - \theta\beta_t^\dagger \equiv 0$ (so that k has zero drift). By the optimality hypothesis, [1.2], and [1.6], it must be that $k_t \equiv k_0^\dagger \in \arg\min_{k_0 \in (0,\theta)} h(k_0)$. By Lemma 1.7, we conclude that $\hat{\beta}_t^\dagger \equiv \rho$, given which Proposition 4.3 implies that $k_t \equiv f(\rho;\lambda) = k_\circ^*$. The lemma then follows from the definition of Contract W. □

Putting the above pieces together yields a proof of the proposition.

Proof of Proposition 1.3. Suppose the environment is regular. By Lemma 1.8, any optimal FO-IC contract that is stationary is Contract W. But Theorem 2(i) implies that the optimal Self-Insurance Contract strictly dominates Contract W (under the standing hypothesis that $\lambda > 0$), while Definition 4.2 and the bijection described in Proposition 4.3 imply that it is FO-IC and thus feasible in the principal's first-order problem. This is a contradiction. Thus, under regularity, no optimal FO-IC contract can be stationary.

Suppose the environment also respects the FOA. By Definition 4.2(iii) and the above argument, it follows that no optimal full-commitment contract is stationary, for we have shown in Theorem 1 that Contract W is IC and thus feasible in the full-commitment problem. □

2. Properties of Brownian Motion and OU Process

In this supplementary appendix, we collect facts about long-run properties of Brownian motion and Ornstein-Uhlenbeck processes, which are invoked in proofs of results in the main appendix.

Lemma 2.1 (SLLN for BM)**.** Let $W = (W_t)_{t\geqslant 0}$ be a standard Brownian motion. Then

$$\lim_{t\to\infty} \frac{W_t}{g(t)} = 0$$

for any function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\to\infty} t/g(t) < \infty$.

Proof. See Problem 9.4 in Karatzas and Shreve (1998, p. 104) for the case in which $g(t) = t$. Let g satisfy the hypothesis of the lemma. Then $\lim_{t\to\infty} W_t/g(t) = \lim_{t\to\infty} (W_t/t) \cdot \lim_{t\to\infty} (t/g(t)) = 0$, which completes the proof. □

Let b be an Ornstein-Uhlenbeck (OU) process, with SDE

[2.1] $$\mathrm{d}b_t = (\mu - \lambda b_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

where $\sigma > 0$, $\lambda \geqslant 0$, and $\mu \in \mathbb{R}$. The unique strong solution to SDE [2.1], is given by

$$b_t = b_0 e^{-\lambda t} + \underbrace{\mu \left(\frac{1 - e^{-\lambda t}}{\lambda} \right)}_{= t \text{ when } \lambda = 0} + \underbrace{e^{-\lambda t} \int_0^t \sigma e^{\lambda \tau} \, \mathrm{d}W_\tau}_{=: X_t} \tag{2.2}$$

We will refer to the process $X = (X_t)_{t \geqslant 0}$ defined in [2.2] throughout the present appendix.

Lemma 2.2. Let X_t be defined as in [2.2] above. The following holds for each $t \geqslant 0$:

(i) When $\lambda = 0$,

$$\int_0^t b_\tau \, \mathrm{d}\tau = b_0 t + \frac{1}{2}\mu t^2 + \sigma \left(tW_t - \int_0^t \tau \, \mathrm{d}W_\tau \right)$$

(ii) When $\lambda > 0$,

$$\int_0^t b_\tau \, \mathrm{d}\tau = b_0 \left(\frac{1 - e^{-\lambda t}}{\lambda} \right) + \frac{\mu}{\lambda} \left(t - \frac{1 - e^{-\lambda t}}{\lambda} \right) + \frac{\sigma W_t - X_t}{\lambda}$$

Proof. In both points (i) and (ii), the deterministic terms follow from straightforward integration of the deterministic terms on [2.2]. The stochastic terms follow from stochastic integration by parts calculations, which are presented below.

Point (i): When $\lambda = 0$, we have $X_t = \sigma W_t$. Itô's lemma yields applied to tW_t yields $tW_t = \int_0^t W_\tau \, \mathrm{d}\tau + \int_0^t \tau \, \mathrm{d}W_\tau$. Thus, $\int_0^t X_\tau \, \mathrm{d}\tau = \left(tW_t - \int_0^t \tau \, \mathrm{d}W_\tau \right)$, as desired.

Point (ii): When $\lambda > 0$, we have $X_t = e^{-\lambda t} \int_0^t \sigma e^{\lambda \tau} \, \mathrm{d}W_\tau =: e^{-\lambda t} Y_t$. Itô's lemma applied to X_t yields $dX_t = -\lambda X_t dt + e^{-\lambda t} dY_t = -\lambda X_t dt + \sigma \, \mathrm{d}W_t$. Putting this in integral form and rearrangning yields $\int_0^t X_\tau \, \mathrm{d}\tau = (\sigma W_t)/\lambda - X_t/\lambda$, as desired. □

Lemma 2.3. The following hold (almost surely):

(i) If $\lambda > 0$, then $\lim_{t \to \infty} b_t/t = 0$. If $\lambda = 0$, then $\lim_{t \to \infty} b_t/t = \mu$,

(ii) For all $\lambda \geqslant 0$ and $\alpha > 0$, $\lim_{t \to \infty} e^{-\alpha t} b_t = 0$, and

(iii) For all $\lambda \geqslant 0$ and $\alpha > 0$, $\lim_{t \to \infty} e^{-\alpha t} \int_0^t b_\tau \, \mathrm{d}\tau = 0$.

Proof. We consider each point of the lemma in turn.

Point (i): When $\lambda = 0$, the result is immediate from Lemma 2.1 and display [2.2] above. We consider the $\lambda > 0$ case. We clearly have

$$\lim_{t \to \infty} \frac{1}{t} \left[b_0 e^{-\lambda t} + \mu \left(\frac{1 - e^{-\lambda t}}{\lambda} \right) \right] = 0$$

for any b_0, μ. Thus, it suffices to study the long-run behavior of the process X defined in [2.2], which solves [2.1] with $\mu = X_0 = 0$. In particular, we must show that $\lim_{t \to \infty} X_t/t = 0$.

It is well-known that X can be written as a time-changed Brownian motion. Define the *time-change* $v(t) := e^{2\lambda t} - 1$, which has inverse $t(v) := \log(v+1)/(2\lambda)$. The process $B = (B_t)_{t\geqslant 0}$ defined by $\frac{\sigma}{\sqrt{2\lambda}} e^{-\lambda t} B_{v(t)} := X_t$ is a standard Brownian motion.[2] Thus, we have

[2.3] $$\frac{X_t}{t} = \frac{\sigma\sqrt{2\lambda} B_{v(t)}}{\sqrt{v(t)}\log(v(t))}$$

Noting that $\lim_{t\to\infty} v(t) = \infty$, to prove that $\lim_{t\to\infty} X_t/t = 0$ it suffices to show that the RHS of [2.3] goes to zero as $v \to \infty$. To compute the latter limit, first recall that the Law of the Iterated Logarithm (Mörters and Peres 2010, p. 119) states that

[2.4] $$\limsup_{v\to\infty} \frac{B_v}{\sqrt{2v\log(\log(v))}} = 1$$

[2.5] $$\liminf_{v\to\infty} \frac{B_v}{\sqrt{2v\log(\log(v))}} = -1$$

Also, note that L'Hôpital's rule implies that

[2.6] $$\lim_{v\to\infty} \frac{\sqrt{2v\log(\log(v))}}{\sqrt{v}\log(v)} = \lim_{v\to\infty} \frac{1}{\sqrt{2}\log(v)\sqrt{\log(\log(v))}} = 0$$

Combining [2.4] with [2.6] yields

$$\limsup_{v\to\infty} \frac{B_v}{\sqrt{v}\log(v)} = \limsup_{v\to\infty} \frac{B_v}{\sqrt{2v\log(\log(v))}} \frac{\sqrt{2v\log(\log(v))}}{\sqrt{v}\log(v)} = 0$$

and combining [2.5] with [2.6] yields

$$\liminf_{v\to\infty} \frac{B_v}{\sqrt{v}\log(v)} = \liminf_{v\to\infty} \frac{B_v}{\sqrt{2v\log(\log(v))}} \frac{\sqrt{2v\log(\log(v))}}{\sqrt{v}\log(v)} = 0$$

This completes the proof of point (i).
Point (ii): Let $\alpha > 0$ be given. Then

$$\limsup_{t\to\infty} e^{-\alpha t}|b_t| = \limsup_{t\to\infty} \frac{|b_t|}{t} \cdot \frac{t}{e^{\alpha t}} = \mathbf{1}(\lambda = 0)|\mu| \cdot \lim_{t\to\infty} \frac{t}{e^{\alpha t}} = 0$$

where the second equality follows from point (i). Point (ii) follows.
Point (iii): Consider first the $\lambda > 0$ case. Lemma 2.2(ii) yields

$$e^{-\alpha t}\int_0^t b_\tau \, \mathrm{d}\tau = e^{-\alpha t}\left[b_0\left(\frac{1-e^{-\lambda t}}{\lambda}\right) + \frac{\mu}{\lambda}\left(t - \frac{1-e^{-\lambda t}}{\lambda}\right)\right] + \frac{\sigma}{\lambda}e^{-\alpha t}W_t - \frac{e^{-\alpha t}}{\lambda}X_t$$

(2) Although B and W have the same law, they are clearly not equal pathwise because, among other things, they run at a different speed, and hence generate different filtrations.

The first (deterministic) term clearly goes to zero as $t \to 0$. The second term also goes to zero by Lemma 2.1. The third term goes to zero by point (ii) of the present lemma.

Now consider the $\lambda = 0$ case. Lemma 2.2(i) yields

$$e^{-\alpha t} \int_0^t b_\tau \, \mathrm{d}\tau = e^{-\alpha t} \left[b_0 t + \frac{1}{2}\mu t^2 \right] + \sigma e^{-\alpha t} t W_t - \sigma e^{-\alpha t} \int_0^t \tau \, \mathrm{d}W_\tau \tag{2.7}$$

The first (deterministic) term clearly goes to zero as $t \to 0$. The second term also goes to zero, for

$$\lim_{t\to\infty} e^{-\alpha t} t W_t = \lim_{t\to\infty} \frac{W_t}{e^{\alpha t/2}} \cdot \underbrace{\lim_{t\to\infty} \frac{t}{e^{\alpha t/2}}}_{=0}$$

and $\lim_{t\to\infty} W_t / e^{\alpha t/2} = 0$ by Lemma 2.1. To evaluate the final term in [2.7], define the process Z by $Z_t := \int_0^t \tau \, \mathrm{d}W_\tau$. We may write Z as a time-changed Brownian motion. Define the time-change $\phi(t) := t^3/3$, which has inverse $t(\phi) := (3\phi)^{1/3}$. The process $\hat{B}$ defined by $\hat{B}_{\phi(t)} := Z_t$ is a standard Brownian motion. Noting that $\phi(t) \to \infty$ as $t \to \infty$, we have

$$\lim_{t\to\infty} e^{-\alpha t} Z_t = \lim_{\phi\to\infty} e^{-\alpha t(\phi)} \hat{B}_\phi = \lim_{\phi\to\infty} \frac{\hat{B}_\phi}{\phi} \cdot \frac{\phi}{\exp(\alpha(3\phi)^{1/3})} = 0$$

where the last equality follows from Lemma 2.1, which yields $\lim_{\phi\to\infty} \hat{B}_\phi/\phi = 0$, and L'Hôpital's rule, which yields $\lim_{\phi\to\infty} \phi/\exp(\alpha(3\phi)^{1/3}) = 0$. Thus, $\lim_{t\to\infty} e^{-\alpha t} Z_t = 0$, as claimed. □

Lemma 2.4. For any $\alpha > 0$, the following holds:

$$\int_0^\infty e^{-\alpha t} b_t \, \mathrm{d}t = \frac{b_0}{\alpha+\lambda} + \frac{\mu}{\alpha(\alpha+\lambda)} + \frac{\sigma}{\alpha+\lambda} \int_0^\infty e^{-\alpha t} \, \mathrm{d}W_t$$

Proof. We integrate the expression for b_t in [2.2], discounted by $e^{-\alpha t}$. It is easy to see that

$$\int_0^\infty e^{-\alpha t} \left[b_0 e^{-\lambda t} + \mu \underbrace{\left(\frac{1 - e^{-\lambda t}}{\lambda} \right)}_{= t \text{ when } \lambda = 0} \right] dt = \frac{b_0}{\alpha+\lambda} + \frac{\mu}{\alpha(\alpha+\lambda)} \tag{2.8}$$

Thus, it suffices to compute $\int_0^\infty e^{-\alpha t} X_t dt$, where X_t is as defined in [2.2]. Applying Itô's lemma to $e^{-\alpha t} X_t$, we obtain

$$\begin{aligned} e^{-\alpha T} X_T &= -\alpha \int_0^T e^{-\alpha t} X_t dt + \int_0^T e^{-\alpha t} dX_t \\ &= -(\alpha+\lambda) \int_0^T e^{-\alpha t} X_t dt + \int_0^T e^{-\alpha t} \sigma \, \mathrm{d}W_t \end{aligned}$$

where the second line follows from the Itô's lemma applied to X_t, which yields $\mathrm{d}X_t = -\lambda X_t dt + \sigma\,\mathrm{d}W_t$. Rearranging and taking the $T \to \infty$ limit yields

$$\int_0^\infty e^{-\alpha t} X_t\,\mathrm{d}t = \frac{\sigma}{\alpha+\lambda}\int_0^\infty e^{-\alpha t}\,\mathrm{d}W_t - \frac{1}{\alpha+\lambda}\lim_{T\to\infty} e^{-\alpha T} X_T = 0, \tag{2.9}$$

where the second equality follows from Lemma 2.3(ii) (noting that X_t is an OU process with $X_0 = \mu = 0$). Combining [2.2], [2.8], and [2.9] completes the proof. □

Lemma 2.5. For any $\alpha > 0$, we have $\mathbf{E}[\int_0^\infty e^{-\alpha t}\,\mathrm{d}W_t] = 0$.

Proof. Consider the process $M = (M_t)_{t\geqslant 0}$ defined by $M_t := \int_0^t e^{-\alpha v}\,\mathrm{d}W_v$. Because $\int_0^\infty (e^{-\alpha v})^2\,\mathrm{d}v < \infty$, by Exercise 5.24 of Karatzas and Shreve (1998, p. 38), M is a uniformly integrable martingale, which implies that $\mathbf{E}[\int_0^\infty e^{-\alpha t}\,\mathrm{d}W_t] = M_0 = 0$. □

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