Existence, uniqueness, and regularity of solutions to nonlinear and non-smooth parabolic obstacle problems[*]

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Abstract

We establish the existence, uniqueness, and $W^{1,2,p}$-regularity of solutions to fully nonlinear parabolic obstacle problems when the obstacle is the pointwise supremum of arbitrary functions in $W^{1,2,p}$ and the operator is only assumed to be measurable in the state and time variables. The results hold for a large class of non-smooth obstacles, including all convex obstacles. Applied to stopping problems, they imply that the decision maker never stops at a convex kink of the stopping payoff. The proof relies on new $W^{1,2,p}$-estimates for obstacle problems where the obstacle is the maximum of finitely many functions in $W^{1,2,p}$.

1   Introduction and Main Result

We study the following fully nonlinear parabolic obstacle problem with Dirichlet boundary data on the domain $\mathcal{Y} = [0, T) \times \mathcal{X}$ where T is finite, $d \in \mathbb{N}$, and $\mathcal{X}$ is a bounded, open subset of $\mathbb{R}^d$:

$$\begin{cases} \max\left\{u_t + F\left(t, x, u, u_x, u_{xx}\right), \, g - u\right\} = 0 \text{ in } \mathcal{Y}, \\ u = b \text{ on } \partial\mathcal{Y} \end{cases} \tag{1}$$

where $\partial \mathcal{Y} = (\{T\} \times \mathcal{X}) \cup ([0, T] \times \partial \mathcal{X})$ is the boundary of $\mathcal{Y}$ and F is a measurable nonlinear uniformly elliptic operator defined on $\mathcal{Y} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$.[1]

We are interested in obstacles g that take the form $g = \sup_{a \in A} g^a$ for functions that lie in the Sobolev space $W^{1,2,p}(\mathcal{Y})$ for some $p \in (d+2, \infty]$. This form arises in stopping problems for which the decision maker must decide, upon stopping, on some action a. In this case, $g^a(t, x)$ is the stopping payoff when stopping at time t in state x and taking action a upon stopping. A decision maker who stops at time t would then optimally choose the action a that maximizes $g^a(t, x)$ over all possible actions $a \in A$, and the resulting stopping payoff, i.e., obstacle, is g.

Such problems are pervasive, for example, in information acquisition models where "stopping" means concluding the information acquisition stage and a is the decision taken after that stage. Examples include Wald (1992), Décamps et al. (2006), Fudenberg et al. (2018) and Camboni and Durandard (2023). In these and other instances, knowing that the Hamilton-Jacobi-Bellman equation has a solution proves instrumental in deriving the optimal policy and can lead to important insights (Dixit, 1993).

While commonly arising in applications, these problems create a challenge because the obstacle g is irregular. In particular, g will typically have kinks whenever the optimal action a changes as a function of x and A is finite.

Our main result requires the following conditions (all notation and definitions are formally introduced in Section 2).

ASSUMPTION 1 $\mathcal{X}$ is $\mathcal{C}^{0,Lip}$.

This assumption allows domains $\mathcal{X}$ with non-smooth boundaries as long as they are Lipschitz. Informally, $\mathcal{X}$ is Lipschitz if its boundary $\partial \mathcal{X}$ can be viewed locally as the graph of a Lipschitz function for some coordinate system. Allowing non-smooth domains is particularly useful for economic and financial applications, whose domains are often given by a positive orthant (e.g., in the case of price vectors), a unit cube, or a simplex (e.g., when x is a probability distribution representing the decision maker's belief about a state of the world with d possible values). All these domains fail to be smooth, but they all satisfy the Lipschitz property.

The next two assumptions concern the primitives b and g.

ASSUMPTION 2 $b : \bar{\mathcal{Y}} \to \mathbb{R}$ is in $\mathcal{C}^0(\bar{\mathcal{Y}})$.

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[1] $\mathbb{S}^d(\mathbb{R})$ denotes the set of symmetric $d \times d$ real-valued matrices equipped with the order $M \geq_{\mathbb{S}} N$ if and only if $M - N$ is positive semi-definite.

ASSUMPTION 3 $g = \sup_{a \in A} g^a$ where A is a separable topological space and (i) $g^a \in W^{1,2,p}(\mathcal{Y})$ for all a and $\sup_{a \in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} < \infty$, (ii) $g^a \leq b$ on $\partial \mathcal{Y}$, and (iii) viewed as a map from A to $C^0(\mathcal{Y})$, the map $a \mapsto g^a$ is continuous.

Part (iii) of Assumption 3 guarantees that the obstacle can be approximated by the maximum of finitely many $W^{1,2,p}$-functions, and hence the validity of the approximation argument in the proof of Theorem 1. It is a sufficient condition: The proof only requires the existence of an approximating sequence $(g^n = \max_{a \in A_n} g^a)_n$, with A_n finite for all n.

The next assumption is a **structure condition**.

ASSUMPTION 4 There exist $\lambda, \Lambda > 0$ and moduli of continuity ω_1 and ω_2 (i.e., nondecreasing continuous functions on $\mathbb{R}_+$ with $\omega_j(0) = 0$, $j = 1, 2$) such that, for almost all $(t, x) \in \mathcal{Y}$, we have

$$
\begin{cases}
\mathcal{P}^-_{\lambda,\Lambda}\left(M - \tilde{M}\right) - \omega_1\left(|r - \tilde{r}|\right) - \omega_2\left(|q - \tilde{q}|\right) \\
\qquad \leq F(t, x, r, q, M) - F\left(t, x, \tilde{r}, \tilde{q}, \tilde{M}\right) \\
\qquad \leq \mathcal{P}^+_{\lambda,\Lambda}\left(M - \tilde{M}\right) + \omega_1\left(|r - \tilde{r}|\right) + \omega_2\left(|p - \tilde{p}|\right).
\end{cases}
\tag{SC}
$$

for all $M, \tilde{M} \in \mathbb{S}^d$, $r, \tilde{r} \in \mathbb{R}$, and $q, \tilde{q} \in \mathbb{R}^d$, where $\mathcal{P}^-_{\lambda,\Lambda}$ and $\mathcal{P}^+_{\lambda,\Lambda}$ are the Pucci extremal operators. Moreover, we assume that there exists $R > 0$ such that $\omega_1(r) + \omega_2(q) \leq R(1 + r + q)$ for all $r, q \in \mathbb{R}_+$.

This assumption implies in particular the following properties: (i) F is uniformly elliptic, (ii) $F(t, x, r, p, M)$ is continuous in r, p, and M uniformly in (t, x), and (iii) F grows at most linearly in (t, x).

Assumption 4 is weaker than the conditions typically imposed on F in the literature: First, F is not required to be Lipschitz in (r, q). Second, F need not be continuous in (t, x). Third, F need not be monotonic in r.[2]

ASSUMPTION 5 $F(t, x, 0, 0, M)$ is convex in M for all (t, x).[3]

The following Vanishing Mean Oscillation (VMO) assumption is also imposed. Define

$$
\Theta\left((t, x), (\tilde{t}, \tilde{x})\right) = \sup_{M \in \mathbb{S}^d \setminus \{0\}} \frac{F(t, x, 0, 0, M) - F\left(\tilde{t}, \tilde{x}, 0, 0, M\right)}{\|M\|}
$$

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[2] We do impose a weaker monotonicity condition (Assumption 8), for our uniqueness result.

[3] The concave case follows (upon modifying Assumption 8 appropriately), since u solves $\max\{g - v, F(t, x, v, v_x, v_{xx}) + v_t\} = 0$ if and only if it solves $\min\{v - g, -F(t, x, v, v_x, v_{xx}) - v_t\} = 0$.

ASSUMPTION 6 *For almost all $(t, x) \in \mathcal{Y}$.*

$$\lim_{|Q| \to 0 \,:\, (t,x) \in Q \subset \mathcal{Y}} \frac{1}{|Q|} \int_Q \Theta\left((t,x),(\tilde{t},\tilde{x})\right) d\tilde{t} d\tilde{x} \to 0,$$

As pointed out in Dong (2020), Assumption 6 is the weakest known assumption under which a solution in $W^{1,2,p}$ always exists for Dirichlet problems, even in the linear case. Moreover, as the counter-examples in Meyers (1963) or Ural'ceva (1967) suggest, it may be close to optimal.

The next assumption is standard. It is required to derive a bound on the $W^{1,2,p}$-norm of solutions to (1).

ASSUMPTION 7 *There exists $G \in L^p(\mathcal{Y})$ such that, for all $(t,x) \in \mathcal{Y}$,*

$$|F(t,x,0,0,0)| \le G(t,x).$$

The final assumption is used to apply a comparison principle to (1). It is not needed for our existence results, but is essential to guarantee uniqueness of the solution to (1).

ASSUMPTION 8 *There exists $\kappa > 0$ such that, for all $(t,x) \in \mathcal{Y}$, $p \in \mathbb{R}^d$, and $M \in \mathbb{S}^d$,*

$$r \to F(t,x,r,p,M) - \kappa r$$

is strictly decreasing.

Our main result[4] establishes the existence, uniqueness, and regularity for solutions to (1).

THEOREM 1 *Let $p \in (d+2, \infty]$ and suppose that Assumptions 1–7 hold. Then (1) has an L^p-solution u.*

Moreover, there exists $C = C\left(d, p, \lambda, \Lambda, R, diam\left(\mathcal{X}\right), T, dist(\mathcal{Y}', \mathcal{Y})\right) \in \mathbb{R}_+$ such that, for all compact $\mathcal{Y}' \subset \mathcal{Y}$,

$$\|u\|_{W^{1,2,p}(\mathcal{Y}')} \le C \left(1 + \sup_{a \in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})} + \|b\|_{L^\infty(\mathcal{Y})} \right), \tag{2}$$

and a modulus of continuity $\bar{\omega}$ depending only on $p, d, \lambda, \Lambda, R, T, diam\left(\mathcal{X}\right), \sup_{a \in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})}, \|G\|_{L^p(\mathcal{Y})}, \|b\|_{L^\infty(\partial \mathcal{Y})}$, the modulus of continuity of b, and the parameters of the cone condition of $\mathcal{X}$, such that, for all $(t,x), (t',x') \in \bar{\mathcal{Y}}$,

$$\left|u(t,x) - u(t',x')\right| \le \bar{\omega}\left(\left|(t,x) - (t',x')\right|\right).$$

If, in addition, Assumption 8 holds, u is the unique L^p-solution of (1).

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[4]The structure condition (**SC**) guarantees that any solution of $u_t + F[u] = 0$ with $u = b$ on $\partial \mathcal{Y}$ is bounded below. Therefore, by choosing g small enough, Theorem 1 also guarantees the existence, uniqueness, and regularity of the solution without obstacles.

REMARK 1 *The $W^{1,2,p}$-estimate (2) is independent of the moduli of continuity ω_1 and ω_2 introduced in Assumption 4.*

Theorem 1 establishes the regularity of solutions to fully nonlinear parabolic obstacle problems when the operator is only required to be measurable and the obstacle is may be irregular as noted above. Classic references on obstacle and free boundary problems include Bensoussan and Lions (1978), Friedman (1982) and Petrosyan et al. (2012). When the obstacle is smooth enough and the operator F is linear with smooth coefficients and smooth nonhomogeneous term (for example, if the obstacle is $\mathcal{C}^{1,2}$ and the coefficients and nonhomogeneous term are α-Hölder continuous), the obstacle problem has a unique solution in $W^{1,2,\infty}\left(\bar{\mathcal{Y}}\right)$, see, e.g., Friedman (1982). This result extends to the nonlinear case when both the operator and obstacle are smooth and the operator satisfies strong convexity and growth conditions (Petrosyan and Shahgholian, 2007; Audrito and Kukuljan, 2023). In these cases, the regularity of solutions to various types of obstacle problems has been extensively studied under appropriate regularity assumptions on the boundary of the domain, the obstacle, the coefficients of the linear operator, and the nonhomogeneous term.

For the case of irregular nonlinear operators, but without obstacles, solvability and regularity results for elliptic and parabolic equations have been obtained under either a small mean or small L^∞ oscillation condition in Caffarelli (1989), Escauriaza (1993), Caffarelli et al. (1996), Winter (2009), or Krylov (2010) for the elliptic case, and in Crandall et al. (1998), Crandall et al. (1999), Crandall et al. (2000), Dong et al. (2013), and Krylov (2017) for the parabolic case, among others. The recent monograph Krylov (2018) offers an up-to-date general treatment.

The regularity of viscosity solutions for the parabolic obstacle problem is studied by (Byun et al., 2018) in the linear case with VMO leading coefficients when the obstacle is in $W^{1,2,p}$ and by (Byun et al., 2022) in the fully nonlinear elliptic case when the operator satisfies the same VMO assumption as in our study and the obstacle is in $W^{1,2,p}$.

Our approach builds on the approximation argument developed in Byun et al. (2018) and Byun et al. (2022). We extend the argument in Byun et al. (2018) to the fully nonlinear parabolic case and allow the operator to be only measurable rather than continuous and the obstacle to be irregular. We first obtain a $W^{1,2,p}$-estimate when the obstacle is in $W^{1,2,p}$ using known estimates on the regularity of solutions of the Dirichlet problem, e.g., Theorem 12.1.7 and 15.1.3 in Krylov (2018). We then show by induction that the existence and regularity results for a single $W^{1,2,p}$-obstacle carry to the pointwise maximum of a finite number of $W^{1,2,p}$-obstacles (Lemma 2). Finally, we use a limit argument to consider the general case.

Our argument relies on PDE methods and generalizes existing results derived by probabilistic meth-

ods in the context of optimal stopping.[5] For example, Décamps et al. (2006) consider the elliptic obstacle equation associated with an optimal stopping problem when the payoff upon stopping (i.e., the obstacle) is the maximum of two smooth convex functions. The authors use a local time argument to show that the decision-maker never stops at the convex kink. This, in turn, implies that the value function, hence the solution of (1), is smooth. Theorem 1, together with a Sobolev embedding theorem, allows us to recover their result: It guarantees that the solution of (1) is continuously differentiable in space, and, thus, that the decision-maker never stops at a point of nondifferentiability of the stopping payoff.

Theorem 1 establishes the existence, uniqueness, and $W^{1,2,p}$-regularity of the solution to a fully nonlinear obstacle problem when the primitives are only *measurable* and the obstacle is the supremum of arbitrary functions in $W^{1,2,p}$. This generalization of earlier results is crucial to accommodate settings in which the operator or the payoff upon stopping may not be smooth. For example, relaxing the operator's continuity is crucial to analyzing the Hamilton-Jacobi-Bellman equation associated with Markovian equilibria in stochastic games, where best response functions may be discontinuous (Kuvalekar and Lipnowski, 2020). Such discontinuities are ubiquitous in *games with finite action sets*.

1.1 Outline of the Paper

Section 2 introduces some notation and definitions. Section 3 contains two results needed for the proof of Theorem 1: (i) A comparison principle that applies to fully nonlinear parabolic obstacle problems with measurable ingredients (Proposition 1), and (ii) A stability theorem for solutions to obstacle problems in $W^{1,2,p}(\mathcal{Y})$ (Theorem 2). Section 4 establishes our main result when A is finite and $\mathcal{X}$ is smooth. Section 5.1 proves Theorem 1, first considering smooth domains and general separable set A, then by using an approximation argument to treat Lipschitz domains.

2 Preliminaries

2.1 Notation

- For $D > 0$, $B_D(x)$ denotes the open ball of radius D centered around x, and $C_D(t,x) = [t, t+D) \times B_D(x)$.

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[5]The connection between obstacle problems and stochastic control and stopping problems and stochastic games had received considerable attention in the literature. See, e.g., (Bensoussan and Lions, 1978; Friedman, 1982; Karatzas and Sudderth, 2001; Peskir and Shiryaev, 2006; Strulovici and Szydlowski, 2015).

- $\bar{\mathcal{O}}$ denotes the closure of $\mathcal{O}$ for the relevant topology.

- $d(\cdot, \cdot)$ is the Euclidean distance. For any sets $\mathcal{Y}, \mathcal{Y}' \subset \mathbb{R}^{d+1}$, define
  $diam\,(\mathcal{Y}) = \sup\{d\,((t,x),(t',x')) \,:\, (t,x),\,(t',x') \in \mathcal{Y}\}$, and
  $dist\,(\mathcal{Y}, \mathcal{Y}') = \inf\{d\,((t,x),(t',x')) \,:\, (t,x) \in \mathcal{Y} \text{ and } (t',x') \in \mathcal{Y}'\}$.

- $\mathcal{C}^k\,(\mathcal{Y})$ is the space of k-times continuously differentiable functions on $\mathcal{Y}$.

- $W^{1,2,p}\,(\mathcal{Y})$ denotes the Sobolev space of functions whose first weak time derivative and second weak space derivative are L^p-integrable. $W^{1,2,p}_{loc}\,(\mathcal{Y})$ is the space of functions that belong to $W^{1,2,p}\,(\mathcal{Y}')$ for all compact subsets $\mathcal{Y}'$ of $\mathcal{Y}$.

- u_t, u_x, and u_{xx} stand for the first weak derivatives of u with respect to t, the first weak derivatives of u with respect to x, and the second weak derivatives of u with respect to x, respectively.

- $\rightharpoonup$ denotes weak convergence.

- For $k \in \mathbb{N}$, an open bounded subset of $\mathbb{R}^d$ is $\mathcal{C}^{k,Lip}$ if, for every $x \in \partial \mathcal{X}$, there exists a neighborhood V of x, $\alpha > 0, \beta > 0$, an affine map $T : \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R}^d$, and a map $\phi : B_\alpha(0) \subseteq \mathbb{R}^{d-1} \to \mathbb{R}$ that is either Lipschitz continuous if $k = 0$, or k-times continuously differentiable with Lipschitz derivatives for $k \geq 1$ such that
  $$\partial \mathcal{X} \cap V = T\left(\left\{(\xi, \eta) \in \mathbb{R}^{d-1} \times \mathbb{R} \,:\, \xi \in B_\alpha(0) \text{ and } \eta = \phi(\xi)\right\}\right),$$
  and
  $$T\left(\left\{(\xi, \eta) \in \mathbb{R}^{d-1} \times \mathbb{R} \,:\, \xi \in B_\alpha(0) \text{ and } \phi(\xi) < \eta < \phi(\xi) + \beta\right\}\right) \subseteq \mathcal{X} \cap V$$
  $$\text{and } T\left(\left\{(\xi, \eta) \in \mathbb{R}^{d-1} \times \mathbb{R} \,:\, \xi \in B_\alpha(0) \text{ and } \phi(\xi) - \beta < \eta < \phi(\xi)\right\}\right) \subseteq \mathbb{R}^d \setminus \left(\bar{\mathcal{X}} \cap \bar{V}\right).$$

  The first part definition requires that there locally exists a coordinate system such that the boundary coincides locally with the graph of a function whose k^{th} derivative (or the function itself if $k = 0$) is Lipschitz. The two inclusions in the second part of the definition guarantee that the interior and the exterior of the domain are nonempty and locally contain a cone around any point of the boundary. They rule out, for example, the domain $\{(x,y) \,:\, x > 0, |y| < x^2\}$.

- The $\mathcal{C}^{k,Lip}$-norm of the boundary of a $\mathcal{C}^{k,Lip}$ open bounded subset $\mathcal{X}$ is defined by $\|\partial \mathcal{X}\|_{\mathcal{C}^{k,Lip}} = \sup_{x \in \partial \mathcal{X}} \|\phi\|_{\mathcal{C}^{k,1}(V)}$ where V and ϕ are as defined above.

- For $\Lambda \geq \lambda \geq 0$, Pucci's extremal operators $\mathcal{P}^+_{\lambda, \Lambda}$ and $\mathcal{P}^-_{\lambda, \Lambda} : \mathbb{S}^d(\mathbb{R}) \to \mathbb{R}$ are defined by
  $$\mathcal{P}^+_{\lambda, \Lambda}(M) = \sup_{\lambda I \leq A \leq \Lambda I} tr(AM) \text{ and } \mathcal{P}^-_{\lambda, \Lambda}(M) = \inf_{\lambda I \leq A \leq \Lambda I} tr(AM).$$

- We will often use the letter C to denote bounds appearing in various estimates, and which can be explicitly computed in terms of primitives of the problem.

2.2 Solution concepts and definitions

DEFINITION 1 *A function u is an L^p-subsolution (respectively, L^p-supersolution) of Problem (1) if the following conditions hold: (i) $u \in W_{loc}^{1,2,p}(\mathcal{Y}) \cap \mathcal{C}^0(\bar{\mathcal{Y}})$, (ii) $u \leq b$ (respectively, $\geq b$) on $\partial \mathcal{Y}$, and (iii)*

$$\max \left\{ u_t + F\left(t, x, u, u_x, u_{xx}\right), \, g - u \right\} \geq 0 \text{ (respectively, } \leq 0) \text{ a.e. in } \mathcal{Y}_T.$$

u is an L^p-solution if it is both an L^p-subsolution and an L^p-supersolution.

We will also use viscosity solutions in stating and applying the comparison principle of Section 3.1. There are several concepts of viscosity solutions depending on the choice of test functions used in the definition. We use test functions in $W^{1,2,p}(\mathcal{Y})$, which corresponds to what is sometimes called "L^p-viscosity solutions."[6]

DEFINITION 2 *A continuous function $u : \bar{\mathcal{Y}} \to \mathbb{R}$ is a viscosity subsolution (respectively, supersolution) of (1) if (i) $u \leq b$ (respectively, $\geq b$) on $\partial \mathcal{Y}_T$, and (ii) for all (t_0, x_0) and all $\varphi \in W_{loc}^{1,2,p}(\mathcal{Y}_T)$ such that $u - \varphi$ (respectively $\varphi - u$) has a maximum at (t_0, x_0), one has*

$$\underset{(t,x) \to (t_0,x_0)}{ess \lim \sup} \, \max \left\{ \varphi_t + F\left(t, x, u, \varphi_x, \varphi_{xx}\right), \, g - u \right\} \geq 0$$

$$\left(respectively, \, \underset{(t,x) \to (t_0,x_0)}{ess \lim \inf} \, \max \left\{ \varphi_t + F\left(t, x, u, \varphi_x, \varphi_{xx}\right), \, g - u \right\} \leq 0 \right).$$

u is a viscosity solution if it both a viscosity subsolution and supersolution.

Viscosity solutions are weaker than L^p-solutions, as shown, e.g., in Proposition 2.11 of Crandall et al. (2000), which easily extends to our setting.

LEMMA 1 *Let u be an L^p-subsolution (supersolution, respectively) of (1). Then u is a viscosity subsolution (supersolution, respectively) of (1).*

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[6]When F is continuous, one can take the test functions to be in $\mathcal{C}^{1,2}$. In this case, the solution is called a $\mathcal{C}$-viscosity solution, and $\mathcal{C}$- and viscosity solutions coincide (Lemma 2.9 in Crandall et al. (2000)).

3 Comparison and Stability for fully nonlinear parabolic obstacle problems with measurable ingredients

3.1 Comparison Principle

Comparison principles for L^p-solutions and viscosity solutions of the Dirichlet problem are well-known in the literature[7] under a slight strengthening of the structure condition (**SC**).[8] We extend these results to fit the assumptions of Theorem 3. Our comparison principle is valid for a large class of nonlinear parabolic obstacle problems as long as F satisfies the structure condition (**SC**) and does not increase too fast with u (as required by Assumption 8).

PROPOSITION 1 *Suppose that F satisfies Assumptions 4 and 8.*

- *Let u be an L^p-subsolution and v be an L^p-**viscosity** supersolution of*

$$\max\left\{\, u_t + F\left(t, x, u, u_x, u_{xx}\right),\, g - u \right\} = 0. \tag{3}$$

  If $v \geq u$ on $\partial\mathcal{Y}$, then $v \geq u$ in $\bar{\mathcal{Y}}$.

- *Let u be an L^p-supersolution and v be an L^p-**viscosity** subsolution of*

$$\max\left\{\, u_t + F\left(t, x, u, u_x, u_{xx}\right),\, g - u \right\} = 0. \tag{4}$$

  If $v \leq u$ on $\partial\mathcal{Y}$, then $v \leq u$ in $\bar{\mathcal{Y}}$.

The proof of Proposition 1 builds on a comparison principle for viscosity solutions of fully nonlinear parabolic Dirichlet problems with continuous operators due to Giga et al. (1991).

Proof. Let u be an L^p-subsolution and v be a viscosity supersolution of

$$\max\left\{\, u_t + F\left(t, x, u, u_x, u_{xx}\right),\, g - u \right\} = 0,$$

such that $v \geq u$ on $\partial\mathcal{Y}$. We will prove that $v \geq u$ in $\bar{\mathcal{Y}}$.

Let $w = u - v$. Let $\mathcal{E} = \left\{ (t, x) \in \bar{\mathcal{Y}} : w > 0 \right\}$ and note that $\mathcal{E} \subset \mathcal{Y}$, since $w \leq 0$ on $\partial\mathcal{Y}$. So, the conclusion follows if $\mathcal{E}$ is empty.

Suppose for a contradiction that $\mathcal{E} \neq \emptyset$. Note that w is continuous since both u and v are continuous on $\bar{\mathcal{Y}}$ by definition. Therefore, $\mathcal{E}$ is an open bounded domain.

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[7]See, e.g., Proposition 2.10 in Crandall et al. (2000)

[8]In particular, previous results assume that F is Lipschitz continuous in u_x and nonincreasing in u.

We first show that w is a viscosity subsolution of

$$\begin{cases} \max\left\{ w_t + H\left(t, x, w, w_x, w_{xx}\right), -w \right\} = 0 \text{ in } \mathcal{Y} \\ w = 0 \text{ on } \partial\mathcal{Y}, \end{cases} \tag{5}$$

where

$$H(t, x, w, w_x, w_{xx}) = F\left(t, x, u, u_x, u_{xx}\right) - F\left(t, x, u - w, u_x - w_x, u_{xx} - w_{xx}\right).$$

To see this, we first note that $w = 0$ on $\partial\mathcal{E}$, so condition (i) in the definition of viscosity subsolutions (here, $w \le 0$) is satisfied. Next, let $\varphi \in W^{1,2,p}\left(\mathcal{Y}\right)$ be such that $w - \varphi$ has a maximum at (t_0, x_0), i.e., $u - \varphi - v$ has a maximum at (t_0, x_0). By construction, $u - \varphi \in W^{1,2,p}$ and v is a viscosity supersolution of (3). Therefore,

$$\operatorname*{ess\,lim\,inf}_{(t,x)\to(t_0,x_0)} \max\left\{ u_t - \varphi_t + F\left(t, x, u - w, u_x - \varphi_x, u_{xx} - \varphi_{xx}\right), g - (u - w) \right\} \le 0.$$

It follows that

$$\operatorname*{ess\,lim\,sup}_{(t,x)\to(t_0,x_0)} \max\left\{ \varphi_t + H\left(t, x, w, \varphi_x, \varphi_{xx}\right), -w \right\} \ge 0.$$

That is, w is a viscosity subsolution of (5).

By the structure condition (**SC**), for any test function $\varphi \in W^{1,2,p}\left(\mathcal{Y}\right)$ and almost every $(t, x) \in \mathcal{Y}$,

$$H\left(t, x, w, \varphi_x, \varphi_{xx}\right) \le \mathcal{P}^+_{\lambda,\Lambda}\left(\varphi_{xx}\right) + \omega_1\left(|w|\right) + \omega_2\left(|\varphi_x|\right).$$

Therefore, w is also a viscosity subsolution of

$$\begin{cases} \max\left\{ -w, \, w_t + \mathcal{P}^+_{\lambda,\Lambda}\left(w_{xx}\right) + \omega_1\left(|w|\right) + \omega_2\left(|w_x|\right) \right\} = 0 \text{ in } \mathcal{Y}, \\ w = 0 \text{ on } \partial\mathcal{Y}. \end{cases}$$

Therefore, w is a viscosity subsolution of

$$\begin{cases} w_t + \mathcal{P}^+_{\lambda,\Lambda}\left(w_{xx}\right) + \omega_1\left(|w|\right) + \omega_2\left(|w_x|\right) = 0 \text{ in } \mathcal{E}, \\ w = 0 \text{ on } \partial\mathcal{E}. \end{cases}$$

Note that 0 is a classical solution of the above equation, hence a (continuous) viscosity supersolution. Theorem 4.7 in Giga et al. (1991) then implies that $w \le 0$ in $\mathcal{E}$: a contradiction. So $\mathcal{E} = \emptyset$, and $v \ge u$ in $\bar{\mathcal{Y}}$.

The proof when u is an L^p-supersolution, v is a viscosity subsolution, and $u \ge v$ on $\partial\mathcal{Y}$ follows the same steps. ∎

REMARK 2 *Proposition 1 also holds when F is degenerate elliptic (i.e, $\lambda = 0$ in Assumption 4) since Theorem 4.7 in Giga et al. (1991) applies to degenerate elliptic operators. However, the existence of L^p-sub- or L^p-supersolutions is not guaranteed in that case.*

Proposition 1 combined with Lemma 1 yields the following corollary, which establishes the uniqueness of L^p-solutions.

COROLLARY 1 *Suppose that Assumptions 4 and 8 hold. Let u and v be, respectively, an L^p-subsolution and an L^p-supersolution of*

$$\max\left\{ u_t + F\left(t, x, u, u_x, u_{xx}\right), \sup_{a \in A} g^a - u \right\} = 0.$$

If $v \geq u$ on $\partial \mathcal{Y}$, then $v \geq u$ in $\bar{\mathcal{Y}}$.

If u and v are two L^p-solutions of (1), then $u = v$ on $\bar{\mathcal{Y}}$.

REMARK 3 *Proposition 1 and Corollary 1 are also valid for non-obstacle problems, as is easily seen from the proof.*

3.2 Stability Theorem

Stability theorems for both L^p-solutions and viscosity solutions of the Dirichlet problem without obstacle are well-known under a slight strengthening of the structure condition $(\mathbf{SC})$[9] , see, e.g., Theorem 6.1 in Crandall et al. (2000). We prove such a stability theorem for our setting, as it is essential to prove 1. This stability theorem is valid for a large class of nonlinear parabolic obstacle problems as long as F satisfies the structure condition $(\mathbf{SC})$.

THEOREM 2 *Let $(\mathcal{Y}^n)_{n \in \mathbb{N}}$ be an increasing sequence of bounded domains such that $\bigcup_{n \in \mathbb{N}} \mathcal{Y}^n = \mathcal{Y}$.[10] Let $(u^n)_{n \in \mathbb{N}}$ be a sequence of functions in $W_{loc}^{1,2,p}(\mathcal{Y}) \cap \mathcal{C}^0(\bar{Y})$ such that*

1. *$(u^n)_{n \in \mathbb{N}}$ converges uniformly on compact subsets and weakly in $W_{loc}^{1,2,p}(\mathcal{Y})$ to some $u \in W_{loc}^{1,2,p}(\mathcal{Y}) \cap \mathcal{C}^0(\bar{\mathcal{Y}})$; and*

2. *for all $n \in \mathbb{N}$, $u^n = b^n$ on $\bar{\mathcal{Y}} \setminus \mathcal{Y}^n$, and $b^n \to b$ pointwise, with b continuous on $\partial \mathcal{Y}$.*

Let F, F^n, $n \in \mathbb{N}$, be operators defined on $\mathcal{Y} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ that satisfy Assumption 4 uniformly in n. Let g^n, $n \in \mathbb{N}$, be continuous functions on $\bar{\mathcal{Y}}$, such that $g^n \leq b^n$ on $\bar{\mathcal{Y}} \setminus \mathcal{Y}^n$.

---

[9]In particular, F has to be Lipschitz continuous in u_x and nonincreasing in u.

[10]$\mathcal{Y}$ does not need to be bounded.

Suppose that, for any fixed $u_{xx} \in L^p(\mathcal{Y})$, $F^n(t, x, u^n, u_x^n, u_{xx}) \to F(t, x, u, u_x, u_{xx})$ pointwise a.e. in $\mathcal{Y}$, and, either (i) $g^n \to g$ uniformly on compact subset of $\mathcal{Y}$, or (ii) $g^n \uparrow g$ pointwise on $\mathcal{Y}$. If u^n, $n \in \mathbb{N}$, is an L^p-subsolution (supersolution) of

$$\begin{cases} \max\{u_t^n + F^n(t, x, u^n, u_x^n, u_{xx}^n), g^n - u^n\} = 0 \ \text{in } \mathcal{Y}^n, \\ u^n = b^n \ \text{on } \partial \mathcal{Y}^n, \end{cases} \tag{Eq^n}$$

then u is an L^p-subsolution (supersolution) of

$$\begin{cases} \max\{u_t + F(t, x, u, u_x, u_{xx}), g - u\} = 0 \ \text{in } \mathcal{Y}, \\ u = b \ \text{on } \partial \mathcal{Y}. \end{cases} \tag{Eq}$$

Proof. We first show that if, for each n, u^n is an L^p-supersolution of (Eq^n), then u is an L^p-supersolution of (Eq).

Note that, for all $n \in \mathbb{N}$, $u^n \geq b^n$ on $\partial \mathcal{Y}$. Taking limits, $u \geq b$ on $\partial \mathcal{Y}$.

Next, we show that

$$\max\{u_t + F(t, x, u, u_x, u_{xx}), g - u\} \leq 0 \text{ a.e. in } \mathcal{Y}.$$

For all $n \in \mathbb{N}$, $u^n \geq g^n$ on $\bar{\mathcal{Y}}$ by the comparison principle (Proposition 1). To see this, observe that, for all $n \in \mathbb{N}$, g^n is a viscosity subsolution of (Eq^n) and u^n is an L^p-supersolution of (Eq^n) on $\mathcal{Y}^n$. Moreover, the operator F^n satisfies Assumptions 4 and 8, and, on $\partial \mathcal{Y}^n$, $u^n = b^n \geq g^n$. Thus, Proposition 1 applies, and $u^n \geq g^n$ on $\bar{\mathcal{Y}}^n$. On $\bar{\mathcal{Y}} \setminus \bar{\mathcal{Y}}^n$, $u^n = b^n$ and $b^n \geq g^n$ by assumption. So, $u^n \geq g^n$ on $\bar{\mathcal{Y}}$. Passing to the limit, $u \geq g$.

It is therefore enough to show that

$$u_t + F(t, x, u, u_x, u_{xx}) \leq 0 \text{ a.e. in } \mathcal{Y},$$

which holds if and only if

$$u_t + F(t, x, u, u_x, u_{xx}) \leq 0 \text{ a.e. in } \mathcal{O},$$

for any strict bounded subset $\mathcal{O}$ of $\mathcal{Y}$.[11]

Let $\mathcal{O}$ be a strict bounded subset of $\mathcal{Y}$. Since $\mathcal{Y} = \bigcup_{n \in \mathbb{N}} \mathcal{Y}^n$, compactness implies that there exists some integer $N \geq 1$ that contains $\mathcal{O}$. Moreover, since the sequence $\{\mathcal{Y}^n\}_{n \geq 1}$ is increasing, $\mathcal{O} \subset \mathcal{Y}^n$ for all $n \geq N$. Without loss of generality, we assume that $N = 1$. For all $m \in \mathbb{N}$, define

$$\underline{H}^m(t, x, v_t, v_{xx}) = \inf_{n \geq m} \{v_t + F^n(t, x, u^n, u_x^n, v_{xx})\}.$$

---

[11] We say that $\mathcal{O}$ is a strict subset of $\mathcal{Y}$ if the closure of $\mathcal{O}$ lies in the interior of $\mathcal{Y}$.

As $u_t^n + F^n(t, x, u^n, u_x^n, u_{xx}^n) \leq 0$, for $n \geq m$ and for almost every $(t, x) \in \mathcal{O}$,

$$\underline{H}^m(t, x, u_t^n, u_{xx}^n) \leq 0.$$

Observe also that, since the F^n's satisfy Assumptions 4 and 5, for all $m \in \mathbb{N}$ and all $v_t \in \mathbb{R}$, $\underline{H}^m(t, x, v_t, \cdot)$ is Lipschitz continuous in v_{xx} a.e. in $\mathcal{O}$ with

$$\lambda I_d \leq D_{v_{xx}} \underline{H}^m(t, x, v_t, \cdot) \leq \Lambda I_d.$$

Moreover, for any $v_{xx} \in \mathbb{S}^d$, by the structure condition (**SC**), $\underline{H}^m$ is bounded in $L^p(\mathcal{O})$ as $(u^n)_{n \in \mathbb{N}}$ is uniformly bounded in $W_{loc}^{1,2,p}(\mathcal{Y})$ (as a weakly convergent sequence). Finally, by assumption, $u^n \rightharpoonup u$ in $W^{1,2,p}(\mathcal{O})$. So, Theorem 4.2.6 in Krylov (2018) implies that, for all $m \in \mathbb{N}$,

$$\underline{H}^m(t, x, u_t, u_{xx}) = \inf_{n \geq m}\{u_t + F^n(t, x, u^n, u_x^n, u_{xx})\} = \lim_{n \to \infty} \underline{H}^m(t, x, u_t^n, u_{xx}^n) \leq 0 \text{ a.e. in } \mathcal{O}.$$

Moreover, $u_t + F^n(t, x, u^n, u_x^n, u_{xx}) \to u_t + F(t, x, u, u_x, u_{xx})$ pointwise a.e. in $\mathcal{Y}$ by assumption. Letting $m \to \infty$, we obtain

$$u_t + F(t, x, u, u_x, u_{xx}) \leq 0 \text{ a.e. in } \mathcal{O}.$$

This shows that u is an L^p-supersolution of (Eq).

Next, we prove that if the u^n's are L^p-subsolutions of (Eq^n), then u is an L^p-subsolution of (Eq).

Note that, for all $n \in \mathbb{N}$, $u^n \leq b^n$ on $\partial \mathcal{Y}$. Taking limits, $u \leq b$ on $\partial \mathcal{Y}$.

Next, we show that

$$\max\{u_t + F(t, x, u, u_x, u_{xx}), g - u\} \geq 0 \text{ a.e. in } \mathcal{Y},$$

which holds if and only if

$$\max\{u_t + F(t, x, u, u_x, u_{xx}), g - u\} \geq 0 \text{ a.e. in } \mathcal{O},$$

for any strict bounded subset $\mathcal{O}$ of $\mathcal{Y}$.

Let $\mathcal{O}$ be a strict bounded subset of $\mathcal{Y}$. Since $\mathcal{Y} = \bigcup_{n \in \mathbb{N}} \mathcal{Y}^n$ and the $\mathcal{Y}^n$'s are an increasing sequence of subsets of $\mathcal{Y}$, there exists $N \in \mathbb{N}$ such that $\mathcal{O} \subset \mathcal{Y}^n$, for all $n \geq N$. Without loss of generality, assume that $N = 1$. It is then enough to show that, on $\mathcal{E} = \{(t, x) \in \mathcal{O} : u(t, x) > g(t, x)\}$, we have

$$u_t + F(t, x, u, u_x, u_{xx}) \geq 0 \text{ a.e. in } \mathcal{O}.$$

In particular, if $\mathcal{E} = \emptyset$, we are done. So, suppose not, and let k be large enough so

$$\mathcal{E}^k = \left\{(t, x) \in \mathcal{O} : u(t, x) > g(t, x) + \frac{1}{k}\right\} \neq \emptyset.$$

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Then, there exists $N^k \in \mathbb{N}$ such that $g^n < u^n$ on $\mathcal{E}^k$ for all $n \geq N^k$. To see this, choose N^k such that $|u^n - u| < \frac{1}{3k}$ on $\mathcal{E}^k$ (which exists since $u^n \to u$ uniformly). Then recall that either (i) $g^n \uparrow g$ pointwise, or (ii) $g^n \to g$ uniformly. So, either (i) $g^n \leq g < u - \frac{1}{k} < u - \frac{1}{3k} < u^n$ on $\mathcal{E}^k$ and we are done, or (ii) g^n converge uniformly to g. In that case, choose $\tilde{N}^k$ such that $|g^n - g| < \frac{1}{3k}$ for all $(t,x) \in \mathcal{O}$, and again, it follows that $g^n < g + \frac{1}{3k} < u - \frac{1}{3k} < u^n$ in $\mathcal{E}^k$ for all $n \geq \max\left\{N^k, \tilde{N}^k\right\}$.

Since $g_n < u_n$ on $\mathcal{E}^k$ for $n > N^k$, we conclude from (Eq^n) that, for all $n \geq N^k$,

$$u_t^n + F^n\left(t, x, u^n, u_x^n, u_{xx}^n\right) = 0 \text{ a.e. in } \mathcal{E}^k.$$

Similarly to the first part of the proof, for all $m \geq N^k$, define

$$\bar{H}^m\left(t, x, v_t, v_{xx}\right) = \sup_{n \geq m}\left\{v_t + F^n\left(t, x, u^n, u_x^n, v_{xx}\right)\right\}.$$

Again, observe that, for $n \geq m$ and for almost every $(t,x) \in \mathcal{E}^k$,

$$\bar{H}^m\left(t, x, u_t^n, u_{xx}^n\right) \geq 0.$$

Moreover, as above, for all $m \in \mathbb{N}$ and all $v_t \in \mathbb{R}$, $\bar{H}^m(t, x, v_t, \cdot)$ is Lipschitz continuous in v_{xx} a.e. in $\mathcal{E}^k$ with

$$\lambda \leq D_{v_{xx}}\bar{H}^m(t, x, v_t, \cdot) \leq \Lambda I_d.$$

Also, for any $v_{xx} \in \mathbb{S}^d$, by the structure condition (**SC**), $\bar{H}^m$ is bounded in $L^p(\mathcal{O})$ as $(u^n)_{n \in \mathbb{N}}$ is uniformly bounded in $W_{loc}^{1,2,p}(\mathcal{Y})$ (as a weakly convergent sequence). Finally, by assumption, $u^n \rightharpoonup u$ in $W^{1,2,p}\left(\mathcal{E}^k\right)$. So, Theorem 4.2.6 in Krylov (2018) implies, for all $m \in \mathbb{N}$,

$$\bar{H}^m\left(t, x, u_t, u_{xx}\right) = \inf_{n \geq m}\left\{u_t + F^n\left(t, x, u^n, u_x^n, u_{xx}\right)\right\} = \lim_{n \to \infty}\bar{H}^m\left(t, x, u_t^n, u_{xx}^n\right) \geq 0 \text{ a.e. in } \mathcal{E}^k.$$

But $u_t + F^n\left(t, x, u^n, u_x^n, u_{xx}\right) \to u_t + F\left(t, x, u, u_x, u_{xx}\right)$ pointwise a.e. in $\mathcal{Y}$ by assumption. Letting $m \to \infty$, we obtain

$$u_t + F\left(t, x, u, u_x, u_{xx}\right) \geq 0 \text{ a.e. in } \mathcal{E}^k.$$

Since $\bigcup_{k \in \mathbb{N}} \mathcal{E}^k = \mathcal{E}$, it follows that

$$u_t + F\left(t, x, u, u_x, u_{xx}\right) \geq 0 \text{ a.e. in } \mathcal{E}.$$

This concludes the second part of the proof: u is an L^p-subsolution of (Eq). ∎

REMARK 4 *Theorem 2 is also valid for non-obstacle problems and viscosity solutions. To see this, one can invoke Theorem 6.1 in Crandall et al. (2000) instead of Theorem 4.2.6 in Krylov (2018) in the proof.*

4 Existence, uniqueness, and $W^{1,2,p}$-estimate when $|A| < \infty$

Before considering the general case, we derive a more restrictive version of Theorem 1 when the obstacle is the maximum of finitely many $W^{1,2,p}$-obstacles and the following strengthening Assumptions 1 and 2 is imposed.

ASSUMPTION 1' $\mathcal{X}$ is an $\mathcal{C}^{1,Lip}$ open bounded subset of $\mathbb{R}^d$.

ASSUMPTION 2' $b : \bar{\mathcal{Y}} \to \mathbb{R}$ is in $W^{1,2,p}(\mathcal{Y})$.

LEMMA 2 Let $p \in (d+2, \infty]$ and suppose that Assumptions 1', 2', and 3–7 hold. If $|A| < \infty$, then (1) has an L^p-solution u.

Moreover, $u \in W^{1,2,p}(\mathcal{Y})$ and there exists $C = C(d, p, \lambda, \Lambda, R, diam(\mathcal{X}), T, \|\partial\mathcal{X}\|_{\mathcal{C}^{1,Lip}}) \in \mathbb{R}_+$ such that

$$\|u\|_{W^{1,2,p}(\mathcal{Y})} \le C \left(1 + \max_{a \in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})} \right). \tag{6}$$

If Assumption 8 also holds, the solution u is unique.

The proof builds on the approximation argument proposed by Byun et al. (2018) and Byun et al. (2022), extending it to fully nonlinear parabolic obstacle problems. We first establish existence and obtain a $W^{1,2,p}$-estimate when the obstacle is in $W^{1,2,p}$ using known existence results and estimates on the regularity of solutions of the Dirichlet problem, e.g., Theorem 12.1.7, 15.1.3, and 15.1.4 in Krylov (2018). These estimates allow for an arbitrary continuous dependence of the operator on the value of the solution u. As a result, we can sidestep the fixed-point argument in Byun et al. (2018), streamlining the proof. We then show by induction that the existence and regularity results for a single $W^{1,2,p}$-obstacle carry to the pointwise maximum of a finite number of $W^{1,2,p}$-obstacles.

Proof. We prove Lemma 2 for $p \in (d+2, \infty)$. The case $p = \infty$ follows from the existence of an L^p-solution for some $p < \infty$ and a classic embedding theorem (see, e.g., Lemma 3.3, page 80, in Ladyženskaja et al. (1968)), which guarantees that $\max\left\{ \|u\|_{L^\infty(\mathcal{Y})}, \|u_x\|_{L^\infty(\mathcal{Y})} \right\} \le C \|u\|_{W^{1,2,p}(\mathcal{Y})}$ for some $C \in \mathbb{R}_+$ that depends only on $d, p, diam(\mathcal{X}), T$, and $\|\partial\mathcal{X}\|_{C^{0,Lip}}$. The structure condition (**SC**) and equation (8) below then implies that, for almost every $(t, x) \in \mathcal{Y}$,

$$|u_t| + |u_{xx}| \le C \left(1 + \|G\|_{L^\infty(\mathcal{Y})} + \max_{a \in A} \|g^a\|_{W^{1,2,\infty}(\mathcal{Y})} + \|u\|_{W^{1,2,p}(\mathcal{Y})} \right)$$

for some $C = C(d, \lambda, \Lambda, R, diam(\mathcal{X}), T, \|\partial\mathcal{X}\|_{C^{0,Lip}}) \in \mathbb{R}_+$. The conclusion follows from estimate (6) and the boundedness of the domain.

Accordingly, consider the case $p \in (d+2, \infty)$. Without loss of generality, assume that $A = \{1, \ldots, I\}$, with $I \in \mathbb{N}$. The proof is by induction on I. For all $I \in \mathbb{N}$, define property $P(I)$ as follows:

If $g = \max\limits_{a \in \{1, \ldots, I\}} g^a$ with $g^a \in W^{1,2,p}(\mathcal{Y})$ for all $a \in \{1, \ldots, I\}$, then (1) has a unique L^p-solution $u \in W^{1,2,p}(\mathcal{Y})$. Moreover, u satisfies the estimate (6).

We start by proving our base case: $P(I = 1)$. Let $g^1 \in W^{1,2,p}(\mathcal{Y})$ with $g^1 \leq b$ on $\partial\mathcal{Y}$. For all $\epsilon > 0$, let $\Phi_\epsilon \in \mathcal{C}^\infty(\mathbb{R})$ be a nondecreasing function such that $\Phi_\epsilon(a) = 0$ if $a \leq 0$, and $\Phi_\epsilon(a) = 1$ if $a \geq \epsilon$. In particular, for all $a \in \mathbb{R}$, $\Phi_\epsilon(a) \in [0, 1]$. Define also $h^1(t, x) = -g_t^1 - F(t, x, g^1, g_x^1, g_{xx}^1)$. Let $(\epsilon_n)_{n \in \mathcal{N}} \subseteq \mathbb{R}_{++}$ be a sequence such that $\epsilon_n \to 0$ as $n \to \infty$; and consider the following auxiliary nonlinear Dirichlet problems. For all $n \in \mathbb{N}$,

$$\begin{cases} F(t, x, u^n, u_t^n, u_x^n, u_{xx}^n) = h^1(t, x)^+ \Phi_{\epsilon_n}(u^n - g^1) - h^1(t, x)^+ \text{ in } \mathcal{Y}, \\ u^n = b \text{ on } \partial\mathcal{Y}. \end{cases} \tag{D}$$

By Assumptions 4 and 7, $h^1(t, x)$ is in $L^p(\mathcal{Y})$ with

$$\left\| h^1 \right\|_{L^p(\mathcal{Y})} \leq C^{h^1} \left(1 + \left\| g^1 \right\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})} \right), \tag{7}$$

where $C^{h^1} = C(d, p, \lambda, \Lambda, R) \in \mathbb{R}_+$. By Theorem 15.1.4 in Krylov (2018), for each $n \in \mathbb{N}$, there exists a solution $u^n \in W^{1,2,p}(\mathcal{Y})$ of (D). Moreover, by Theorem 15.1.3 (using Lemma 12.1.9 in Krylov (2018) to bound $\|u\|_{L^\infty(\mathcal{Y})}$ and Theorem 9.8.1 in Krylov (2018) to obtain the modulus of continuity of u),

$$\|u^n\|_{W^{1,2,p}(\mathcal{Y})} \leq C \left(1 + \left\| g^1 \right\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})} \right),$$

where $C = C(d, p, \lambda, \Lambda, R, diam(\mathcal{X}), T, \|\partial\mathcal{X}\|_{\mathcal{C}^{1,Lip}}) \in \mathbb{R}_+$ is independent of ϵ_n.

$W^{1,2,p}(\mathcal{Y})$ is separable and reflexive.[12] Therefore, its closed bounded subsets are weakly sequentially compact by Theorem 1.32 in Demengel et al. (2012). Moreover $W^{1,2,p}(\mathcal{Y})$ is compactly embedded in $\mathcal{C}^0(\bar{\mathcal{Y}})$ by Rellich-Kondrachov's theorem (Theorem 2.84 in Demengel et al. (2012)). So, there exists a function $u \in W^{1,2,p}(\mathcal{Y}) \cap \mathcal{C}^0(\bar{\mathcal{Y}})$ and a subsequence $(u^{n_j})_{j \in \mathbb{N}} \subseteq (u^n)_{n \in \mathbb{N}}$ such that

$$\begin{cases} u^{n_j} \rightharpoonup u \text{ in } W^{1,2,p}(\mathcal{Y}), \\ u^{n_j} \to u \text{ in } \mathcal{C}^0(\bar{\mathcal{Y}}), \end{cases}$$

---

[12]The space L^p are uniformly convex for $p \in (1, \infty)$, hence so is $W^{1,2,p}(\mathcal{Y})$ for $p \in (1, \infty)$. Therefore it is reflexive by Theorem 1.40 in Demengel et al. (2012).

as $j \to \infty$. Moreover, u satisfies the estimate (6):

$$\|u\|_{W^{1,2,p}(\mathcal{Y})} \leq \liminf_{j \to \infty} \|u^{n_j}\|_{W^{1,2,p}(\mathcal{Y})}$$

$$\leq C \left(1 + \|g^1\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})}\right).$$

There remains to show that $u \in W^{1,2,p}(\mathcal{Y})$ is an L^p-solution of (1).

First, for all $j \in \mathbb{N}$, $u^{n_j} = b$ on $\partial \mathcal{Y}$, and $(u^{n_j})_{j \in \mathbb{N}}$ converges uniformly to u on $\bar{\mathcal{Y}}$. So, $u = b$ on $\partial \mathcal{Y}$.

Second, we prove that

$$u_t + F(t, x, u, u_x, u_{xx}) \leq 0 \text{ a.e. in } \mathcal{Y}.$$

For all $j \in \mathbb{N}$ and a.e. $(t, x) \in \mathcal{Y}$,

$$u_t^{n_j} + F\left(t, x, u^{n_j}, u_x^{n_j}, u_{xx}^{n_j}\right) = h^1(t, x)^+ \Phi_{\epsilon_{n_j}}\left(u^{n_j} - g^1\right) - h^1(t, x)^+ \leq 0,$$

So, for all $j \in \mathbb{N}$, u^{n_j} is an L^p-supersolution of

$$u_t^{n_j} + F\left(t, x, u^{n_j}, u_x^{n_j}, u_{xx}^{n_j}\right) = 0.$$

Theorem 2 then implies that u is an L^p-supersolution of

$$u_t + F(t, x, u, u_x, u_{xx}) = 0 \text{ in } \mathcal{Y}.$$

Next, we show that $u \geq g^1$ a.e. on $\mathcal{Y}$. For each $j \in \mathbb{N}$, define

$$\mathcal{V}_{n_j} = \left\{(t, x) \in \mathcal{Y} : g^1(t, x) > u^{n_j}(t, x)\right\}.$$

We will show that, for all $j \in \mathbb{N}$, $\mathcal{V}_{n_j}$ is empty, by contradiction. Suppose not, i.e., $\mathcal{V}_{n_j} \neq \emptyset$ for some $j \in \mathbb{N}$. Since both u^{n_j} and g^1 are continuous on $\mathcal{Y}$, $\mathcal{V}_{n_j}$ is open. Moreover, since $b \geq g^1$, $u^{n_j} = g^1$ on $\partial \mathcal{V}_{n_j}$. On $\mathcal{V}_{n_j}$, $\Phi_{\epsilon_{n_j}}\left(u^{n_j}(t, x) - g(t, x)\right) = 0$. So $u^{n_j} \in W^{1,2,p}(\mathcal{Y})$ is an L^p-solution of

$$\begin{cases} u_t^{n_j} + F\left(t, x, u^{n_j}, u_x^{n_j}, u_{xx}^{n_j}\right) = -h^1(t, x)^+ \text{ in } \mathcal{V}_{n_j}, \\ u^n = g^1 \text{ on } \partial \mathcal{V}_{n_j}. \end{cases}$$

Since $-h^{1+} \leq -h^1$, u^{n_j} is an L^p-solution of

$$\begin{cases} u_t^{n_j} + F\left(t, x, u^{n_j}, u_x^{n_j}, u_{xx}^{n_j}\right) \leq -h^1(t, x) \text{ in } \mathcal{V}_{n_j} \\ u^{n_j} = g^1 \text{ on } \partial \mathcal{V}_{n_j}. \end{cases}$$

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But, by definition of h^1, $g^1 \in W^{1,2,p}(\mathcal{Y})$ is an L^p-solution of

$$
\begin{cases}
g_t^1 + F\left(t, x, g^1, g_x^1, g_{xx}^1\right) = -h^1(t, x) \text{ in } \mathcal{V}_{n_j} \\
g^1 = g^1 \text{ on } \partial\mathcal{V}_{n_j}.
\end{cases}
$$

Therefore, by our comparison principle (Proposition 1), $g^1 \leq u^{n_j}$ on $\bar{\mathcal{V}}_{n_j}$: a contradiction. So, for all $j \in \mathbb{N}$, $\mathcal{V}_{n_j} = \emptyset$, i.e., $u^{n_j} \geq g^1$ on $\bar{\mathcal{Y}}$. Taking limits, as $(u^{n_j})_{j\in\mathbb{N}}$ converges uniformly to u on $\bar{\mathcal{Y}}$, $u \geq g^1$ on $\mathcal{Y}$.

So far, we have shown that $u = b$ on $\partial\mathcal{Y}$ and that

$$
\max\left\{u_t + F\left(t, x, u, u_x, u_{xx}\right), g^1 - u\right\} \leq 0 \text{ a.e. in } \mathcal{Y},
$$

i.e., u is an L^p-supersolution of (1), with $u = b$ on $\partial\mathcal{Y}$. To conclude, there remains to show that

$$
\max\left\{u_t + F\left(t, x, u, u_x, u_{xx}\right), g^1 - u\right\} \geq 0 \text{ a.e. in } \mathcal{Y}.
$$

We do so by showing that, on the open set $\mathcal{U} = \left\{(t, x) \in \mathcal{Y} : u(t, x) > g^1(t, x)\right\}$, we have

$$
u_t + F\left(t, x, u, u_x, u_{xx}\right) \geq 0 \text{ a.e..}
$$

By definition of Φ_ϵ, note that $\Phi_{\epsilon_{n_j}}\left(u^{n_j}(t, x) - g(t, x)\right) \to 1$ pointwise a.e. as $j \to \infty$ on $\mathcal{U}$. Therefore, for all $(t_0, x_0) \in \mathcal{U}$ and $D > 0$ such that $C_D(t_0, x_0) \subset \mathcal{U}$, by Theorem 2, u is an L^p-subsolution of

$$
u_t + F\left(t, x, u, u_x, u_{xx}\right) = 0 \text{ in } C_r(t_0, x_0).
$$

Since $(t_0, x_0) \in \mathcal{U}$ was arbitrary, we obtain

$$
\max\left\{u_t + F\left(t, x, u, u_x, u_{xx}\right), g^1 - u\right\} \geq 0 \text{ a.e. in } \mathcal{U}.
$$

So,

$$
\max\left\{u_t + F\left(t, x, u, u_x, u_{xx}\right), g^1 - u\right\} \geq 0 \text{ a.e. in } \mathcal{Y}.
$$

and u is an L^p-solution of (1) with $I = 1$. This concludes the base case of our induction argument.

Next, suppose that $P(I)$ holds for some $I \in \mathbb{N}$. We show that it also holds for $I + 1$. Let $g^{I+1} \in W^{1,2,p}(\mathcal{Y})$ with $g^{I+1} \leq b$ on $\partial\mathcal{Y}$. Define $h^{I+1} = -g_t^{I+1} - F\left(t, x, g^{I+1}, g_x^{I+1}, g_{xx}^{I+1}\right)$; and consider the following sequence of auxiliary problems. For all $n \in \mathbb{N}$,

$$
\begin{cases}
\max\left\{u_t^n + F\left(t, x, u^n, u_x^n, u_{xx}^n\right) - h^{I+1}(t, x)^+\Phi_{\epsilon_n}\left(u^n - g^{I+1}\right) + h^{I+1}(t, x)^+, \right. \\
\qquad\qquad \left. \max_{i\in\{1,\dots,I\}} g^i - u^n\right\} = 0 \text{ in } \mathcal{Y}, \qquad\qquad\qquad (\mathrm{D}^{I+1}) \\
u^n = b \text{ on } \partial\mathcal{Y}.
\end{cases}
$$

By our induction hypothesis, for all $n \in \mathbb{N}$, (D^{I+1}) has an L^p-solution $u^n \in W^{1,2,p}(\mathcal{Y})$, with

$$\|u^n\|_{W^{1,2,p}(\mathcal{Y})} \leq C \left(1 + \max_{i \in \{1,\dots,I\}} \|g\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})} + \|g^{I+1}\|_{W^{1,2,p}(\mathcal{Y})} \right),$$

where $C = C\left(d, p, \lambda, \Lambda, diam(\mathcal{X}), T, \|\partial \mathcal{X}\|_{\mathcal{C}^{1,Lip}}\right) \in \mathbb{R}_+$ is independent of ϵ_n, since

$$\|h^{I+1}\|_{L^p(\mathcal{Y})} \leq C^h \left(1 + \|g^{I+1}\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})} \right).$$

Proceeding exactly as in the case $I = 1$, we see that the sequence $(u^n)_{n \in \mathbb{N}}$ has a subsequence that converges weakly in $W^{1,2,p}(\mathcal{Y})$ and strongly in $C^0(\bar{\mathcal{Y}})$ to some $u \in W^{1,2,p}(\mathcal{Y})$. Moreover, following the same steps as above, one can show that u is an L^p-solution of (1) with $I + 1$ obstacles.

To conclude, there only remains to show that the bound (6) on the $W^{1,2,p}$-norm of u holds. Note that $u_t = g_t$, $u_x = g_x$, and $u_{xx} = g_{xx}$ a.e. on $\{u = g\}$. This follows from the proof of Corollary 3.1.2.1 in Evans (2018), using Proposition A.1 in Crandall et al. (1998) instead of Rademacher's Theorem. So, $u \in W^{1,2,p}(\mathcal{Y})$ solves

$$u_t + F(t, x, u, u_x, u_{xx}) = \mathbb{1}_{\left\{ u > \max_{i=1,\dots,I+1} g^i(t,x) \right\}} h^{i(t,x)}(t, x) - h^{i(t,x)}(t, x), \tag{8}$$

where $h^i(t, x) = -g_t^i - F\left(t, x, g^i, g_x^i, g_{xx}^i\right)$ and $i(t, x)$ is a measurable selection from $\arg \max_{i \in \{1,\dots I+1\}} g^i(t, x)$. The result then follows from Theorem 15.1.3 in Krylov (2018) (using Lemma 12.1.9 in Krylov (2018) to bound $\|u\|_{L^\infty(\mathcal{Y})}$ and Theorem 9.8.1 in Krylov (2018) to obtain the modulus of continuity of u).

By induction, the result holds for all $I \in \mathbb{N}$. This concludes the proof of Lemma 2. ∎

Applying Theorem 12.1.7 and Lemma 12.1.9 in Krylov (2018) to (8), we also obtain the following interior $W^{1,2,p}$-estimates and L^∞ bound.

COROLLARY 2 *Let $p \in (d+2, \infty]$ and suppose that Assumptions 1', and 2–7 hold and that $|A| < \infty$. If u is an L^p-solution of (1), then, for all compact subset $\mathcal{Y}'$ of $\mathcal{Y}$,*

$$\|u\|_{W^{1,2,p}(\mathcal{Y}')} \leq C \left(1 + \sup_{a \in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})} + \frac{1}{dist(\mathcal{Y}', \mathcal{Y})} \|u\|_{L^\infty(\mathcal{Y})} \right).$$

where $C = C\left(d, p, \lambda, \Lambda, R, diam(\mathcal{X}), T\right) \in \mathbb{R}_+$.

Moreover, there exists $C^\infty = C^\infty\left(d, p, \lambda, \Lambda, R, T, diam(\mathcal{X})\right) \in \mathbb{R}_+$ such that

$$\|u\|_{L^\infty(\mathcal{Y})} \leq C^\infty \left(1 + \|G\|_{L^p(\mathcal{Y})} + \sup_{a \in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{L^\infty(\mathcal{Y})} \right).$$

5 Proof of Theorem 1

We first prove the result for the special case of smooth domains.

LEMMA 3 *Let $p \in (d+2, \infty]$ and suppose that Assumptions 1', 2', and 3–7 hold. Then (1) has an L^p-solution u.*

Moreover, $u \in W^{1,2,p}(\mathcal{Y})$ and there exists $C = C(d, p, \lambda, \Lambda, R, diam(\mathcal{X}), T, \|\partial \mathcal{X}\|_{\mathcal{C}^{1,Lip}}) \in \mathbb{R}_+$ such that

$$\|u\|_{W^{1,2,p}(\mathcal{Y})} \leq C \left(1 + \sup_{a \in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})} \right). \tag{9}$$

If, in addition, Assumption 8 holds, then u is the unique L^p-solution of (1).

5.1 Proof of Lemma 3

Building on Lemma 2, we We approximate the obstacle $g = \sup_{a \in A} g^a$ by a sequence of obstacles that satisfy the Assumptions of Lemma 2. We then invoke the stability result for obstacle problems derived in Section 3.2 (Theorem 2) to conclude.

Proof. Uniqueness follows from Corollary 1. So, we only need to show existence.

Since A is separable and and $g^{\cdot}$ is continuous in a, there exists a countable dense subset $A^0 \subseteq A$ such that $\sup_{a \in A} g^a = \sup_{a \in A^0} g^a$ on $\mathcal{Y}$. Moreover, there exists a sequence $(A^{0,n})_{n \in \mathbb{N}}$ of finite subsets of A^0 such that $\sup_{a \in A^{0,n}} g^a$ converges pointwise from below to $\sup_{a \in A^0} g^a$.

For all $n \in \mathbb{N}$, Lemma 2 guarantees that there exists a solution $u^n \in W^{1,2,p}(\mathcal{Y})$ of

$$\begin{cases} \max \left\{ u_t + F(t, x, u, u_x, u_{xx}), \ \sup_{a \in A^{0,n}} g^a - u \right\} = 0 \text{ in } \mathcal{Y}, \\ u = b \text{ on } \partial \mathcal{Y}. \end{cases}$$

Moreover, for all $n \in \mathbb{N}$,

$$\|u^n\|_{W^{1,2,p}(\mathcal{Y})} \leq C \left(1 + \sup_{a \in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})} \right).$$

Since $W^{1,2,p}(\mathcal{Y})$ is separable and reflexive, its closed bounded subsets are weakly sequentially compact by Theorem 1.32 in Demengel et al. (2012). Moreover $W^{1,2,p}(\mathcal{Y})$ is compactly embedded in $\mathcal{C}^0(\bar{\mathcal{Y}})$ by Rellich-Kondrachov's theorem (Theorem 2.84 in Demengel et al. (2012)). So, there exists a function $u \in W^{1,2,p}(\mathcal{Y}) \cap \mathcal{C}^0(\bar{\mathcal{Y}})$ and a subsequence $(u^{n_j})_{j \in \mathbb{N}} \subseteq (u^n)_{n \in \mathbb{N}}$ such that

$$\begin{cases} u^{n_j} \rightharpoonup u \text{ in } W^{1,2,p}(\mathcal{Y}), \\ u^{n_j} \to u \text{ in } \mathcal{C}^0(\bar{\mathcal{Y}}), \end{cases}$$

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as $j \to \infty$. Furthermore, u satisfies the estimate (9):

$$\|u\|_{W^{1,2,p}(\mathcal{Y})} \leq \liminf_{j \to \infty} \|u^{n_j}\|_{W^{1,2,p}(\mathcal{Y})}$$

$$\leq C \left(1 + \sup_{a \in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})} \right).$$

To conclude, there only remains to show that u is the solution to (1), which follows from Theorem 2. ∎

From the proof of Lemma 3 and Corollary 2, we obtain the following interior $W^{1,2,p}$-estimates and L^∞ bound.

COROLLARY 3 *Let $p \in (d+2, \infty]$ and suppose that Assumptions 1', and 2–7 hold. If u is an L^p-solution of (1), then, for all compact subset $\mathcal{Y}'$ of $\mathcal{Y}$,*

$$\|u\|_{W^{1,2,p}(\mathcal{Y}')} \leq C \left(1 + \sup_{a \in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})} + \frac{1}{dist(\mathcal{Y}', \mathcal{Y})} \|u\|_{L^\infty(\mathcal{Y})} \right).$$

where $C = C\left(d, p, \lambda, \Lambda, R, diam\left(\mathcal{X}\right), T\right) \in \mathbb{R}_+$.

Moreover, there exists $C^\infty = C^\infty\left(d, p, \lambda, \Lambda, R, T, diam\left(\mathcal{X}\right)\right) \in \mathbb{R}_+$ such that

$$\|u\|_{L^\infty(\mathcal{Y})} \leq C^\infty \left(1 + \|G\|_{L^p(\mathcal{Y})} + \sup_{a \in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{L^\infty(\mathcal{Y})} \right).$$

5.2 Proof of Theorem 1

Lemma 3 guarantees that the obstacle problem has an L^p-solution when $\mathcal{X}$ and b satisfy Assumption 1' and 2'. To generalize the result to the weaker Assumptions 1 and 2, we study a sequence of equations, each satisfying the Assumptions of Lemma 3, that converges to the equation (1). In particular, we approximate (i) $\mathcal{X}$ by a sequence of smooth domains whose cone parameters are uniformly controlled, and (ii) b by a sequence of equicontinuous functions in $W^{1,2,p}(\mathcal{Y})$. The L^p solutions of the equations in the approximating sequence form an equicontinuous (Lemma 4) and weakly compact (in $W^{1,2,p}_{loc}(\mathcal{Y})$) family. Arzela-Ascoli's theorem then guarantees that a subsequence converges uniformly on the compact subset of $\bar{\mathcal{Y}}$ to some function in $W^{1,2,p}_{loc}(\mathcal{Y}) \cap \mathcal{C}^0(\bar{\mathcal{Y}})$. Finally, we invoke the stability result for obstacle problems (Theorem 2) derived in Section 3.2 to conclude.

LEMMA 4 *Let $p \in (d+2, \infty]$ and suppose that Assumptions 1', 2', and 3–7 hold. Let u be an L^p-solution of (1). For all $D > 0$, there exists a family of modulus of continuity $\bar{\omega}_D$ such that, for all $(t, x), (t', x') \in \bar{\mathcal{Y}} \cap C_D(0, 0)$,*

$$\left|u(t, x) - u(t', x')\right| \leq \bar{\omega}_D\left(\left|(t, x) - (t', x')\right|\right), \tag{10}$$

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where $\bar\omega_D$ depends only on $p, d, \lambda, \Lambda, R, D, \sup_{a \in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y} \cap C_{D+1}(0,0))}, \|G\|_{L^p(\mathcal{Y} \cap C_{D+1}(0,0))}, \|u\|_{L^\infty(\mathcal{Y} \cap C_{D+1}(0,0))}$, the modulus of continuity of b in $\partial\mathcal{Y} \cap C_{D+1}(0,0)$, the parameters of the cone condition of $\mathcal{X}$ on $\mathcal{X} \cap B_{D+1}(0)$, and $\|u\|_{L^\infty(\mathcal{Y} \cap C_{D+1}(0,0))}$.

Proof. By the structure condition (**SC**), for any L^p-solution u of (1) and almost all $(t, x) \in \mathcal{Y} \cap C_{D+1}(0,0)$, we have

$$\max\{u_t + F(t, x, u, u_x, u_{xx}), \, g - u\}$$
$$\leq u_t + \mathcal{P}^+_{\lambda,\Lambda}(u_{xx}) + R|u_x| + \bar{C}\left(1 + \|u\|_{L^\infty(\mathcal{Y} \cap C_{D+1}(0,0))} + \sup_{a \in A}(|g^a| + |g^a_t| + |g^a_x| + |g^a_{xx}|) + G(t, x)\right),$$

for some $\bar{C} > 0$ depending only on $p, d, \lambda, \Lambda, R$, and D. Similarly,

$$\max\{u_t + F(t, x, u, u_x, u_{xx}), \, g - u\}$$
$$\geq u_t + \mathcal{P}^-_{\lambda,\Lambda}(u_{xx}) - R|u_x| - \bar{C}\left(1 + \|u\|_{L^\infty(\mathcal{Y})} + \sup_{a \in A}(|g^a| + |g^a_t| + |g^a_x| + |g^a_{xx}|) + G(t, x)\right).$$

Define $\bar{u} : \partial\left(\mathcal{Y} \cap C_{D+1}(0,0)\right)$ as

$$\bar{u}(t,x) = \begin{cases} b(t,x) \text{ if } (t,x) \in (\partial\mathcal{Y}) \cap C_D(0,0), \\ dist\left(\{(t,x)\}, C_{D+1}\right) b(t,x) \\ \quad + (1 - dist\left(\{(t,x)\}, C_{D+1}\right)) \|u\|_{L^\infty(\mathcal{Y} \cap C_{D+1}(0,0))} \text{ if } (t,x) \in \partial\mathcal{Y} \cap (C_{D+1} \setminus C_D(0,0)) \\ \|u\|_{L^\infty(\mathcal{Y} \cap C_{D+1}(0,0))} \text{ if } (t,x) \in \partial\left(\mathcal{Y} \cap C_{D+1}(0,0)\right) \setminus \partial\mathcal{Y}. \end{cases}$$

Similarly, define $\underline{u} : \partial\left(\mathcal{Y} \cap C_{D+1}(0,0)\right)$ as

$$\underline{u}(t,x) = \begin{cases} b(t,x) \text{ if } (t,x) \in (\partial\mathcal{Y}) \cap C_D(0,0), \\ dist\left(\{(t,x)\}, C_{D+1}\right) b(t,x) \\ \quad - (1 - dist\left(\{(t,x)\}, C_{D+1}\right)) \|u\|_{L^\infty(\mathcal{Y} \cap C_{D+1}(0,0))} \text{ if } (t,x) \in \partial\mathcal{Y} \cap (C_{D+1} \setminus C_D(0,0)) \\ - \|u\|_{L^\infty(\mathcal{Y} \cap C_{D+1}(0,0))} \text{ if } (t,x) \in \partial\left(\mathcal{Y} \cap C_{D+1}(0,0)\right) \setminus \partial\mathcal{Y}. \end{cases}$$

In particular, note that $\bar{u}$'s and $\underline{u}$'s moduli of continuity are controlled by $\|u\|_{L^\infty(\mathcal{Y} \cap C_{D+1}(0,0))}$, and the modulus of continuity of b on $\partial Y \cap C_{D+1}(0,0)$, and that $\bar{u} \geq u \geq \underline{u}$ on $\partial\left(\mathcal{Y} \cap C_{D+1}(0,0)\right)$.

It follows that u is also an L^p-subsolution of

$$\begin{cases} v_t + \mathcal{P}^+_{\lambda,\Lambda}(v_{xx}) + R|v_x| \\ \quad = -\bar{C}\left(1 + \|u\|_{L^\infty(\mathcal{Y} \cap C_{D+1}(0,0))} + \sup_{a \in A}(|g^a| + |g^a_t| + |g^a_x| + |g^a_{xx}|) + G(t, x)\right) \text{ in } \mathcal{Y} \cap C_{D+1}(0,0), \\ v = \bar{u} \text{ on } \partial\mathcal{Y} \cap C_{D+1}(0,0), \end{cases}$$

$$(11)$$

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and an L^p-supersolution of

$$
\begin{cases}
v_t + \mathcal{P}^-_{\lambda,\Lambda}(v_{xx}) - R|v_x| \\
\quad = \bar{C}\left(1 + \|u\|_{L^\infty(\mathcal{Y}\cap C_{D+1}(0,0))} + \sup_{a\in A}(|g^a| + |g_t^a| + |g_x^a| + |g_{xx}^a|) + G(t,x)\right) \text{ in } \mathcal{Y}\cap C_{D+1}(0,0), \\
v = \underline{u} \text{ on } \partial\mathcal{Y}\cap C_{D+1}(0,0).
\end{cases}
$$
(12)

By Theorem 4.5 in Crandall et al. (1999), there exists a viscosity solution $\bar{U}$ of (11) and a viscosity solution $\underline{U}$ of (12) whose moduli of continuity depend only on $p, d, \lambda, \Lambda, R, D, \sup_{a\in A}\|g^a\|_{W^{1,2,p}(\mathcal{Y}\cap C_{D+1}(0,0))}$, $\|G\|_{L^p(\mathcal{Y}\cap C_{D+1}(0,0))}, \|u\|_{L^\infty(\mathcal{Y}\cap C_{D+1}(0,0))}$, the modulus of continuity of b on $\mathcal{Y}\cap C_{D+1}(0,0)$, and the parameters of the cone condition of $\mathcal{X}$ on $\mathcal{X}\cap B_{D+1}(0)$. By our comparison principle (Proposition 1), since $\underline{u} \le u \le \bar{u}$ on $\partial\mathcal{Y}\cap C_D(0,0)$,

$$\underline{U} \le u \le \bar{U} \text{ on } \bar{\mathcal{Y}}\cap C_{D+1}(0,0).$$

Therefore, for all $(t,x) \in \bar{\mathcal{Y}}\cap C_D(0,0)$ and all $(\bar{t},\bar{x})\, \partial\mathcal{Y}\cap C_D(0,0)$,

$$|u(t,x) - b(\bar{t},\bar{x})| \le \max\left\{\left|\bar{U}(t,x) - b(\bar{t},\bar{x})\right|, |\underline{U}(t,x) - b(\bar{t},\bar{x})|\right\} \le \bar{\omega}_{boundary}\left(|(t,x) - (\bar{t},\bar{x})|\right).$$
(13)

where $\bar{\omega}_{boundary}$ is a modulus of continuity depending only on $p, d, \lambda, \Lambda, R, D, \sup_{a\in A}\|g^a\|_{W^{1,2,p}(\mathcal{Y}\cap C_{D+1}(0,0))}$, $\|G\|_{L^p(\mathcal{Y}\cap C_{D+1}(0,0))}, \|u\|_{L^\infty(\partial\mathcal{Y}\cap C_{D+1}(0,0))}$, the modulus of continuity of b on $\partial\mathcal{Y}\cap C_{D+1}(0,0)$, and the parameters of the cone condition of $\mathcal{X}$ on $\mathcal{X}\cap B_{D+1}(0)$.

Finally, by Corollary 3 and a Morrey-Sobolev embedding theorem (Theorem 2.84 in Demengel et al. (2012)), u is α-Hölder continuous on any compact subset $\mathcal{Y}'$ of $\mathcal{Y}\cap C_{D+1}(0,0)$ for all $\alpha < 1-(d+1/p)$, with Hölder norm bounded above by

$$C^H\left(1 + \sup_{a\in A}\|g^a\|_{W^{1,2,p}(\mathcal{Y}\cap C_{D+1}(0,0))} + \|u\|_{L^\infty(\mathcal{Y}\cap C_{D+1}(0,0))} + \|G\|_{L^p(\mathcal{Y}\cap C_{D+1}(0,0))}\right)$$

where C^H depends only, as above, on $p, d, \lambda, \Lambda, R, D$, and on the distance of the compact set to the boundary $dist(\mathcal{Y}', \partial\mathcal{Y})$.

Combining these two estimates, we obtain the desired modulus of continuity $\bar{\omega}_D$. ∎

Proof of Theorem 1.

Uniqueness follows from Corollary 1. The remainder of the proof shows existence and establishes the interior estimates (2).

By Theorem 5.1 in Doktor (1976), there exists a strictly increasing (in the sense of set inclusion) sequence of bounded smooth open subsets of $\mathcal{X}$, $(\mathcal{X}_n)_{n\in\mathbb{N}}$, and a constant $M > 0$ such that, for all

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$n \in \mathbb{N}$,

$$\max \left\{ \|\partial \mathcal{X}\|_{\mathcal{C}^{0,Lip}}, \|\partial \mathcal{X}_n\|_{\mathcal{C}^{0,Lip}} \right\} \leq M.$$

By Theorem 1.2.2.2 in Grisvard (2011), both $\mathcal{X}$ and $\mathcal{X}_n$, $n \in \mathbb{N}$, satisfy a uniform exterior cone condition of size (L, θ), for some $L, \theta > 0$ independent of n. In particular (L, θ) depends only on M and $diam(\mathcal{X})$.

Define $\mathcal{Y}_n = \left[0, \frac{2n-1}{2n} T \right) \times \mathcal{X}_n$, and consider the following sequence of nonlinear obstacle problems: for all $n \in \mathbb{N}$,

$$\begin{cases} \max \left\{ u_t + F\left(t, x, u, u_x, u_{xx}\right), g - u \right\} = 0 \text{ in } \mathcal{Y}_n \\ u = b^n \text{ on } \partial \mathcal{Y}_n, \end{cases} \tag{Eq$_n$}$$

where $b_n \in W^{1,2,p}(\mathcal{Y})$ for all $n \in \mathbb{N}$, and $b^n \to b \in \mathcal{C}^0(\bar{Y})$.

By Lemma 3, for all $n \in \mathbb{N}$, there exists a unique L^p-solution $u^n \in W^{1,2,p}(\mathcal{Y}_n) \subset W^{1,2,p}_{loc}(\mathcal{Y}_n) \cap \mathcal{C}^0(\bar{\mathcal{Y}}_n)$ of (Eq$_n$). Extending all the the the u^n's by $u^n(x, t) = b(x, t)$ on $\bar{\mathcal{Y}} \setminus \mathcal{Y}_n$, we obtain a sequence $(u^n)_{n \in \mathbb{N}} \subset W^{1,2,p}(\mathcal{Y}) \subset W^{1,2,p}_{loc}(\mathcal{Y}) \cap \mathcal{C}^0(\bar{\mathcal{Y}})$.

Next we show that the sequence $(u^n)_{n \in \mathbb{N}}$ has a subsequence that converges (i) uniformly in $\mathcal{C}^0(\bar{\mathcal{Y}})$ and (ii) weakly in $W^{1,2,p}_{loc}(\mathcal{Y})$ to some function $u \in W^{1,2,p}_{loc}(\mathcal{Y}) \cap \mathcal{C}^0(\bar{\mathcal{Y}})$.

We start with (i). By Lemma 4 with $D > T \vee diam(\mathcal{X})$ (using Lemma 12.1.9 in Krylov (2018) to uniformly control $\|u^n\|_{L^\infty(\mathcal{Y})}$, $n \in \mathbb{N}$), the sequence $(u^n)_{n \in \mathbb{N}}$ is equicontinuous. As a result, it has a convergent subsequence in $\mathcal{C}^0(\bar{\mathcal{Y}})$ by Arzelà-Ascoli's theorem (Theorem A.5 in Rudin (1973)), which, with a small abuse of notation, we relabel $(u^n)_{n \in \mathbb{N}}$.

Next, we show (ii). I.e., the above convergent subsequence in $\mathcal{C}^0(\bar{\mathcal{Y}})$ has a weakly convergent subsequence in $W^{1,2,p}_{loc}(\mathcal{Y})$. To do so, we prove that $W^{1,2,p}$-norm of the u^n's restricted to any compact subset of $\mathcal{Y}$ is eventually uniformly bounded.

It is easy to see that, for all $n \geq 3$, $u^n \in W^{1,2,p}_{loc}(\mathcal{Y}_n) \cap \mathcal{C}^0(\bar{\mathcal{Y}}_n)$ is also an L^p-solution of

$$\begin{cases} \max \left\{ u_t + F\left(t, x, u, u_x, u_{xx}\right), g - u \right\} = 0 \text{ in } \mathcal{Y}_{n'} \\ u(t, x) = u^n(t, x) \text{ on } \partial \mathcal{Y}_{n'}, \end{cases}$$

for all $n' \leq n - 2$. So, by Corollary 3, for all $n \geq 3$ and all $n' \leq n - 2$,

$$\|u^n\|_{W^{1,2,p}(\mathcal{Y}_{n'})} \leq C^{n'} \left(1 + \|G\|_{L^p(\mathcal{Y})} + \sup_{a \in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{L^\infty(\mathcal{Y})} \right), \tag{14}$$

for some $C^{n'} = C^{n'}(d, p, \lambda, \Lambda, R, diam(\mathcal{X}), T, dist(\mathcal{Y}_{n'}, \mathcal{Y}_{n'+1})) \in \mathbb{R}_+$. Crucially, the right-hand side of (14) is independent of n.

We then show that there exists a subsequence of $(u^n)_{n \in \mathbb{N}}$ that converges to the limit $u \in \mathcal{C}^0 (\bar{\mathcal{Y}})$ obtained in (i) by a diagonal argument, and, therefore, that $u \in W_{loc}^{1,2,p} (\mathcal{Y}) \cap \mathcal{C}^0 (\bar{\mathcal{Y}})$. For all $n \geq 3$, by (14),

$$\|u^n\|_{W^{1,2,p}(\mathcal{Y}_1)} \leq C^1 \left(1 + \|G\|_{L^p(\mathcal{Y})} + \sup_{a \in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{L^\infty(\mathcal{Y})} \right).$$

$W^{1,2,p} (\mathcal{Y}_1)$ is separable and reflexive, and, hence, its closed bounded subsets are weakly sequentially compact by Theorem 1.32 in Demengel et al. (2012). Moreover $W^{1,2,p}(\mathcal{Y}_1)$ is compactly embedded in $\mathcal{C}^0 (\mathcal{Y}_1)$ by Rellich-Kondrachov's theorem (Theorem 2.84 in Demengel et al. (2012)). So, there exists a function $\tilde{u}$ defined on $\bar{\mathcal{Y}}_T$ which restriction on $\mathcal{Y}_1$ is continuous, and a subsequence $(u^{n_j})_{j \in \mathbb{N}} \subseteq (u^n)_{n \in \mathbb{N}}$ such that

$$\begin{cases} u^{n_j} \rightharpoonup \tilde{u} \text{ in } W^{1,2,p} (\mathcal{Y}_1) \\ u^{n_j} \to \tilde{u} \text{ in } \mathcal{C}^0 (\bar{\mathcal{Y}}_1), \end{cases}$$

as $j \to \infty$. In particular, $\tilde{u}$ must coincide with u on $\mathcal{Y}_1$.

Proceeding with the diagonal argument, we see that there exists a subsequence $(u^{n_j})_{j \in \mathbb{N}} \subseteq (u_n)_{n \in \mathbb{N}}$ such that

$$\begin{cases} u^{n_j} \rightharpoonup u \text{ in } W_{loc}^{1,2,p} (\mathcal{Y}) \\ u^{n_j} \to u \text{ in } \mathcal{C}^0 (\mathcal{Y}), \end{cases}$$

as $j \to \infty$. Moreover, u satisfies the interior estimates (2) since, for all $n \in \mathbb{N}$,

$$\begin{aligned} \|u\|_{W^{1,2,p}(\mathcal{Y}_n)} &\leq \liminf_{j \to \infty} \|u^{n_j}\|_{W^{1,2,p}(\mathcal{Y}_n)} \\ &\leq C^n \left(1 + \|G\|_{L^p(\mathcal{Y})} + \sup_{a \in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{L^\infty(\mathcal{Y})} \right). \end{aligned}$$

That is, $u \in W_{loc}^{1,2,p} (\mathcal{Y}) \cap \mathcal{C}^0 (\bar{\mathcal{Y}})$ and u satisfies the interior $W^{1,2,p}$-estimates (2).

There remains to show that $u \in W_{loc}^{1,2,p} (\mathcal{Y}) \cap \mathcal{C}^0 (\bar{\mathcal{Y}})$ is an L^p-solution of (1). By construction, $(u^{n_j})_{j \in \mathbb{N}} \subseteq W_{loc}^{1,2,p}(\mathcal{Y}_T) \cap \mathcal{C}^0 (\bar{\mathcal{Y}}_T)$ converges to u weakly in $W_{loc}^{1,2,p} (\mathcal{Y})$ and uniformly on $\bar{\mathcal{Y}}$. The result then follows from Theorem 2. ∎

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