Existence, uniqueness, and regularity of solutions to nonlinear and non-smooth parabolic obstacle problems*

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Abstract

We establish the existence, uniqueness, and $W^{1,2,p}$-regularity of solutions to fully-nonlinear, parabolic obstacle problems when the obstacle is the pointwise supremum of functions in $W^{1,2,p}$ and the nonlinear operator is required only to be measurable in the state and time variables. In particular, the results hold for all convex obstacles. Applied to stopping problems, they provide general conditions under which a decision maker never stops at a convex kink of the stopping payoff. The proof relies on new $W^{1,2,p}$-estimates for obstacle problems when the obstacle is the maximum of finitely many functions in $W^{1,2,p}$.

1 Introduction

We study a fully nonlinear parabolic obstacle problem with Dirichlet boundary data on the domain $\mathcal{Y} = [0,T) \times \mathcal{X}$ where T is finite and $\mathcal{X}$ is a bounded, open subset of $\mathbb{R}^d$ for some $d \in \mathbb{N}$:

$$\begin{cases} \max\left\{u_t + F\left(t,x,u,u_x,u_{xx}\right),\, g-u\right\} = 0 \text{ on } \mathcal{Y}, \\ u = b \text{ on } \partial\mathcal{Y}. \end{cases} \tag{1}$$

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Here, $\partial\mathcal{Y} = \{T\} \times \mathcal{X} \cup [0,T] \times \partial\mathcal{X}$ is the boundary of $\mathcal{Y}$ and F is a measurable nonlinear uniformly elliptic operator defined on $\mathcal{Y} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$.[1]

Parabolic obstacle problems consist of finding the smallest function that (i) exceeds a given obstacle function and (ii) is a supersolution of a given parabolic equation (Petrosyan and Shahgholian, 2007). Such a function solves an equation of the form (1) and is called the solution of the obstacle problem. Obstacle problems arise in physics, e.g., to study phase transitions (Lamé and Clapeyron (1831); Stefan (1889); Friedman (1982)), in biology, e.g., to study tumor growth (Greenspan (1976); Bazaliy and Friedman (2003)), in economics, e.g., to study learning and investment decisions (Wald (1947); Dixit (1993); Moscarini and Smith (2001); Décamps et al. (2024)), and in finance, e.g., to study American options (Jacka (1991); Shiryaev (1999); Villeneuve (1999)).

We are particularly interested in obstacle problems that arise in optimal stochastic control and stopping problems, whether they involve a single decision maker or multiple players interacting in a stochastic game. When the primitives of the corresponding obstacle problem are sufficiently regular, it is well-known that the value function for the optimal control and stopping of a diffusion is the solution of an obstacle problem where the operator F is the Hamilton-Jacobi-Bellman operator and the obstacle is the stopping payoff (Bensoussan and Lions, 1978; Friedman, 1982; Karatzas and Sudderth, 2001; Peskir and Shiryaev, 2006; Petrosyan et al., 2012; Strulovici and Szydlowski, 2015).

In many applications, however, the operator, the obstacle, or the domain that define the obstacle problem fail to be regular. In stochastic games, for example, the Hamilton-Jacobi-Bellman operator that defines a player's optimization problem has coefficients that depend on the strategies of other players and may for this reason fail to be continuous (Kuvalekar and Lipnowski, 2020). Indeed, such discontinuities typically arise when players choose at each instant from finite action sets.

Similarly, the obstacle g that arises in many stopping problems takes the form $g = \sup_{a\in A} g^a$, and may thus have kinks even when the individual functions g^a are smooth. This form arises when the decision maker must decide, upon stopping, on some action a. In this case, $g^a(t,x)$ is the stopping payoff when stopping at time t in state x and taking action a upon stopping. A decision maker who stops at time t would then optimally choose the action a that maximizes $g^a(t,x)$ over all possible actions $a \in A$, and the resulting stopping payoff, i.e., obstacle, is g. Such problems are pervasive in information acquisition models where "stopping" means concluding the information acquisition stage and a is the decision taken after that stage (Wald (1992), Décamps et al. (2006), Fudenberg et al. (2018) and Camboni and Durandard (2024)).

Finally, in many obstacle problems that arise in economics and other fields, the domain of the state

[1] $\mathbb{S}^d(\mathbb{R})$ denotes the set of symmetric $d \times d$ real-valued matrices equipped with the order $M \geq_{\mathbb{S}} N$ if and only if $M - N$ is positive semi-definite.

variable fails to be smooth. In information acquisition problems, for example, the state variable is a belief and the relevant domain is a probability simplex, which has kinks at its vertices (and, more generally, along the lower-dimensional faces of the probability simplex). As another example, the state variable in many finance and general equilibrium problems is a price vector whose relevant domain is a positive orthant, which again fails to be smooth.

To address the challenges created by a lack of regularity in the primitives of obstacle problems, researchers have often used techniques tailored to their specific application. For instance, Décamps et al. (2006) consider the elliptic obstacle equation associated with an optimal stopping problem when the payoff upon stopping (i.e., the obstacle) is the maximum of two smooth convex functions, and use a probabilistic local time argument to show that the decision-maker never stops at the convex kink. The results for regular obstacles then apply, and the authors can show that the value function is a smooth solution of (1).

This paper aims to address these challenges all at once. Our main result, Theorem 1, establishes the existence, uniqueness, and $W^{1,2,p}$-regularity of a solution to fully nonlinear obstacle problems when (i) the operator is required only to be *measurable* in (t, x), (ii) the obstacle is the *supremum* of functions in $W^{1,2,p}$—and may thus have kinks—and (iii) the domain is required only to be Lipschitz—and may thus have corners and other kinks. Theorem 1 covers various applications considered in economics. For instance, it allows us to recover Décamps et al. (2006)'s result when combined with a Sobolev embedding theorem.[2] More generally, it guarantees that the solution of (1) is continuously differentiable in space and, thus, that the decision-maker never stops at a point of nondifferentiability of the stopping payoff g. Since the results apply to any convex obstacle, they imply that any point of nondifferentiability of the stopping payoff must be in the continuation region.

Results similar to Theorem 1 have appeared under stronger assumptions. A precursor is Friedman (1982), which shows that when the primitives are smooth and the operator F is linear, the obstacle problem has a unique solution in $W^{1,2,\infty}(\mathcal{Y})$. Petrosyan and Shahgholian (2007); Audrito and Kukuljan (2023) extend this result to nonlinear operators when the obstacle and operator are smooth and the operator satisfies strong convexity and growth conditions. When the operator is not smooth, solvability and regularity results for elliptic and parabolic equations in the absence of obstacles appear in Caffarelli (1989), Escauriaza (1993), Caffarelli et al. (1996), Winter (2009), and Krylov (2010) for the elliptic case, and in Crandall et al. (1998), Crandall et al. (1999), Crandall et al. (2000), Dong et al. (2013), and Krylov (2017) for the parabolic case, among others. The recent monograph Krylov (2018) offers an up-to-date general treatment. Finally, Byun et al. (2018,

[2] Corollary 1 makes this point formally.

2022) study the regularity of viscosity solutions for obstacle problems when the obstacle is smooth and one of the following conditions hold: the operator is linear or the problem is elliptic.

We now describe more specifically how our main theorem generalizes existing results. Crandall et al. (2000) and Krylov (2017, 2018) prove the existence, uniqueness, and $W^{1,2,p}$-regularity of solutions of fully nonlinear parabolic problems *in the absence of an obstacle* (let alone an irregular one). In addition, Krylov (2017, 2018) assume that the domain is smooth and Crandall et al. (2000) assume that the operator is Lipschitz continuous in the derivatives of the value function, while the theorem of the present paper requires only continuity.[3] Byun et al. (2018) proposes a new method to prove that parabolic obstacle problems on smooth domains are solvable in $W^{1,2,p}$ when the operator is linear and the obstacle is in $W^{1,2,p}$, a condition that is not satisfied by the kinked obstacles that arise in various stopping problems. Finally, Byun et al. (2022) adapt the technique in Byun et al. (2018) to study fully nonlinear *elliptic* obstacle problems in smooth domains when the obstacle is in $W^{2,p}$.

Theorem 1 builds on the results and ideas in these earlier papers to obtain the existence, uniqueness, and $W^{1,2,p}$-regularity of the solution to fully nonlinear parabolic obstacle problems when the operator is *measurable* in (t, x), the obstacle is *irregular*, and the domain is not required to be smooth. The proof of Theorem 1 relies on PDE methods and generalizes existing results derived by probabilistic methods in the context of optimal stopping. Specifically, we extend the approximation argument developed in Byun et al. (2018) and Byun et al. (2022) to the fully nonlinear parabolic case and allow non-smooth obstacles. To do so, we first obtain a $W^{1,2,p}$-estimate when the obstacle is in $W^{1,2,p}$ and the domain is smooth using known estimates on the regularity of solutions of the Dirichlet problem, e.g., Theorem 12.1.7 and 15.1.3 in Krylov (2018). We then show by induction that the existence and regularity results for a single $W^{1,2,p}$-obstacle carry to the pointwise maximum of a finite number of $W^{1,2,p}$-obstacles (Lemma 2). Finally, we extend the result to Lipschitz domains and obstacles that are the supremum of a separable family of functions in $W^{1,2,p}$ by a limit argument.

1.1 Outline of the Paper

Section 2 introduces notations and definitions. Section 3 formally states our assumptions and main result, Theorem 1. Section 4 contains two results needed for the proof of Theorem 1: (i) a comparison principle that applies to fully nonlinear parabolic obstacle problems with measurable ingredients

[3]Krylov (2017, 2018) use the same concept of solutions as we do: L^p-solutions, while Crandall et al. (2000) work with the weaker concept of L^p-*viscosity* solutions. However, these concepts coincide under our assumptions on the nonlinear operator.

(Proposition 1), and (ii) a stability theorem for solutions to obstacle problems in $W^{1,2,p}(\mathcal{Y})$ (Theorem 2). Section 5 establishes our main result in a simpler setting in which the obstacle is the maximum of finitely many functions in $W^{1,2,p}$ and the domain $\mathcal{X}$ is smooth. Finally, Section 6 proves Theorem 1 in full generality by first extending the result of Section 5 to general separable sets A and then extending this result further to non-smooth domains.

2 Preliminaries

2.1 Notation

- For $D > 0$, $B_D(x)$ denotes the open ball of radius D centered around x and $C_D(t,x) = [t, t+D) \times B_D(x)$.
- $\bar{\mathcal{O}}$ denotes the closure of $\mathcal{O}$ for the relevant topology.
- $d(\cdot,\cdot)$ is the Euclidean distance. For any sets $\mathcal{Y}, \mathcal{Y}' \subset \mathbb{R}^{d+1}$, define
  $diam\,(\mathcal{Y}) = \sup\{d\left((t,x),(t',x')\right) \,:\, (t,x),\,(t',x') \in \mathcal{Y}\}$, and
  $dist\,(\mathcal{Y},\mathcal{Y}') = \inf\{d\left((t,x),(t',x')\right) \,:\, (t,x) \in \mathcal{Y} \text{ and } (t',x') \in \mathcal{Y}'\}$.
- $\mathcal{C}^k(\mathcal{Z})$ is the space of continuous functions on $\mathcal{Z}$ if $k=0$ and of k-times continuously differentiable functions on $\mathcal{Z}$ if $k \geq 1$.
- $W^{1,2,p}(\mathcal{Z})$ denotes the Sobolev space of functions defined on the set $\mathcal{Z}$ whose first weak time derivative and second weak space derivatives are L^p-integrable. $W^{1,2,p}_{loc}(\mathcal{Y})$ is the space of functions that belong to $W^{1,2,p}(\mathcal{Y}')$ for all compact subsets $\mathcal{Y}'$ of $\mathcal{Y}$.
- For a function $u \in W^{1,2,p}_{loc}(\mathcal{Z})$, u_t, u_x, and u_{xx} stand for the first weak derivatives of u with respect to t, the first weak derivatives of u with respect to x, and the second weak derivatives of u with respect to x, respectively.
- $\rightharpoonup$ denotes weak convergence.
- For $k \in \mathbb{N}$, an open bounded subset $\mathcal{Z}$ of $\mathbb{R}^d$ is $\mathcal{C}^{k,Lip}$ if, for every $x \in \partial\mathcal{Z}$, there exists a neighborhood V of x, $\alpha > 0, \beta > 0$, an affine map $T : \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R}^d$, and a map $\phi : B_\alpha(0) \subseteq \mathbb{R}^{d-1} \to \mathbb{R}$ that is Lipschitz continuous if $k=0$ or k-times continuously differentiable with Lipschitz derivatives if $k \geq 1$, such that

$$\partial\mathcal{Z} \cap V = T\left(\left\{(\xi,\eta) \in \mathbb{R}^{d-1} \times \mathbb{R} \,:\, \xi \in B_\alpha(0) \text{ and } \eta = \phi(\xi)\right\}\right),$$

and

$$T\left(\left\{(\xi,\eta)\in\mathbb{R}^{d-1}\times\mathbb{R}\,:\,\xi\in B_\alpha(0)\text{ and }\phi(\xi)<\eta<\phi(\xi)+\beta\right\}\right)\subseteq\mathcal{Z}\cap V$$
$$\text{and } T\left(\left\{(\xi,\eta)\in\mathbb{R}^{d-1}\times\mathbb{R}\,:\,\xi\in B_\alpha(0)\text{ and }\phi(\xi)-\beta<\eta<\phi(\xi)\right\}\right)\subseteq\mathbb{R}^d\setminus\left(\bar{\mathcal{Z}}\cap\bar{V}\right).$$

The first part of the definition requires that there locally exist a coordinate system such that the boundary coincides locally with the graph of a function whose k^{th} derivative (or the function itself if $k=0$) is Lipschitz. The two inclusions in the second part of the definition guarantee that the interior and the exterior of the domain are nonempty and locally contain a cone around any point of the boundary. They rule out, for example, the domain $\{(x,y)\,:\, x>0,\,|y|<x^2\}\subset\mathbb{R}^2$. Informally, a domain $\mathcal{Z}$ is Lipschitz if its boundary $\partial\mathcal{Z}$ can be viewed locally as the graph of a Lipschitz function for some coordinate system.

- The $\mathcal{C}^{k,Lip}$-norm of the boundary of a $\mathcal{C}^{k,Lip}$ open bounded subset $\mathcal{Z}$ is defined by $\|\partial\mathcal{Z}\|_{\mathcal{C}^{k,Lip}}=\sup_{x\in\partial\mathcal{Z}}\|\phi\|_{\mathcal{C}^{k,1}(V)}$ where V and ϕ depend on x and are as defined above. Note that the $\mathcal{C}^{k,Lip}$-norm of the boundary of a bounded $\mathcal{C}^{k,Lip}$-set in $\mathbb{R}^d$ is finite, because in this case the boundary is compact.

- For $\Lambda\geq\lambda\geq 0$, Pucci's extremal operators $\mathcal{P}^+_{\lambda,\Lambda}$ and $\mathcal{P}^-_{\lambda,\Lambda}:\mathbb{S}^d(\mathbb{R})\to\mathbb{R}$ are defined by

$$\mathcal{P}^+_{\lambda,\Lambda}(M)=\sup_{\lambda I\leq A\leq\Lambda I}tr(AM)\quad\text{and}\quad\mathcal{P}^-_{\lambda,\Lambda}(M)=\inf_{\lambda I\leq A\leq\Lambda I}tr(AM).$$

- We will often use the letter C as a generic letter to denote different bounds appearing in various estimates, and which can be explicitly computed in terms of primitives of the problem.

2.2 Solution concepts and definitions

DEFINITION 1 *A function u is an L^p-subsolution (respectively, L^p-supersolution) of* (1) *if the following conditions hold: (i) $u\in W^{1,2,p}_{loc}(\mathcal{Y})\cap\mathcal{C}^0(\bar{\mathcal{Y}})$, (ii) $u\leq b$ (respectively, $\geq b$) on $\partial\mathcal{Y}$, and (iii)*

$$\max\left\{u_t+F\left(t,x,u,u_x,u_{xx}\right),\,g-u\right\}\geq 0\ (\textit{respectively, }\leq 0)\ \textit{a.e. on }\mathcal{Y}.$$

u is an L^p-solution if it is both an L^p-subsolution and an L^p-supersolution.

We will also use viscosity solutions in stating and applying the comparison principle of Section 4.1. There are several concepts of viscosity solutions depending on the set of test functions used in the definition. We use test functions in $W^{1,2,p}(\mathcal{Y})$, which corresponds to what is sometimes called "L^p-viscosity solutions."[4]

[4] When F is continuous, one can take the test functions to be in $\mathcal{C}^{1,2}$. In this case, the solution is called a $\mathcal{C}$-viscosity solution. In this case, the concepts of $\mathcal{C}$-viscosity solutions and of viscosity solutions as we defined them coincide (Lemma 2.9 in Crandall et al. (2000)).

Definition 2 *A function* $u : \bar{\mathcal{Y}} \to \mathbb{R}$ *is a viscosity subsolution (respectively, supersolution) of* (1) *if (i) it is continuous, (ii)* $u \leq b$ *(respectively,* $\geq b$*) on* $\partial \mathcal{Y}_T$*, and (iii) for all* $(t_0, x_0) \in \mathcal{Y}$ *and all* $\varphi \in W^{1,2,p}_{loc}(\mathcal{Y})$ *such that* $u - \varphi$ *(respectively* $\varphi - u$*) has a maximum at* (t_0, x_0)*, one has*

$$\operatorname*{ess\,lim\,sup}_{(t,x)\to(t_0,x_0)} \max\left\{ \varphi_t + F\left(t, x, u, \varphi_x, \varphi_{xx}\right), \, g - u\right\} \geq 0$$

$$\left(\textit{respectively, } \operatorname*{ess\,lim\,inf}_{(t,x)\to(t_0,x_0)} \max\left\{ \varphi_t + F\left(t, x, u, \varphi_x, \varphi_{xx}\right), \, g - u\right\} \leq 0\right).$$

u is a viscosity solution if it both a viscosity subsolution and supersolution.

Viscosity solutions are weaker than L^p-solutions, as shown, e.g., in Proposition 2.11 of Crandall et al. (2000), which easily extends to our setting.

Lemma 1 *Let u be an* L^p*-subsolution (respectively, supersolution) of* (1)*. Then u is a viscosity subsolution (respectively, supersolution) of* (1)*.*

3 Main Result

Our main result is derived under the following assumptions.

Assumption 1 $\mathcal{X}$ *is* $\mathcal{C}^{0,Lip}$*.*

This assumption allows domains $\mathcal{X}$ with non-smooth boundaries. As noted in the Introduction, allowing for such domains is particularly useful in economic and financial applications, where domains are often equal to a positive orthant (e.g., in the case of price vectors), a unit cube, or a simplex (e.g., when x is a probability distribution representing the decision maker's belief about a state of the world with d possible values). These domains all fail to be smooth, but they all satisfy the Lipschitz property.

The next two assumptions concern the exogenous boundary payoff function b and the stopping payoff function g.

Assumption 2 $b : \bar{\mathcal{Y}} \to \mathbb{R}$ *is in* $\mathcal{C}^0\left(\bar{\mathcal{Y}}\right)$*.*

Assumption 2 implies that b is bounded on $\bar{\mathcal{Y}}$, a property that we will use when computing estimates.

Assumption 3 $g = \sup_{a\in A} g^a$ *where A is a separable topological space and the functions* g^a *have the following properties: (i)* $g^a \in W^{1,2,p}(\mathcal{Y})$ *for all a and* $\sup_{a\in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} < \infty$*, (ii)* $g^a \leq b$ *on* $\partial\mathcal{Y}$*, and (iii) the map* $a \mapsto g^a$ *is continuous from A to* $C^0\left(\mathcal{Y}\right)$*.*

Part (iii) of Assumption 3 guarantees that the obstacle can be approximated by the maximum of finitely many $W^{1,2,p}$-functions, and allows us to use an approximation argument in the proof of Theorem 1.[5]

The next assumption is a **structure condition**.

ASSUMPTION 4 *There exist $\lambda, \Lambda > 0$ and moduli of continuity ω_1 and ω_2 (i.e., nondecreasing continuous functions on $\mathbb{R}_+$ with $\omega_j(0) = 0$, $j = 1, 2$) such that, for almost all $(t, x) \in \mathcal{Y}$, we have*

$$\begin{cases} \mathcal{P}^-_{\lambda,\Lambda}\left(M - \tilde{M}\right) - \omega_1\left(|\, r - \tilde{r}\,|\right) - \omega_2\left(|\, q - \tilde{q}\,|\right) \\ \qquad \leq F\left(t, x, r, q, M\right) - F\left(t, x, \tilde{r}, \tilde{q}, \tilde{M}\right) \\ \qquad\qquad \leq \mathcal{P}^+_{\lambda,\Lambda}\left(M - \tilde{M}\right) + \omega_1\left(|\, r - \tilde{r}\,|\right) + \omega_2\left(|\, q - \tilde{q}\,|\right). \end{cases} \tag{\textbf{SC}}$$

for all $M, \tilde{M} \in \mathbb{S}^d$, $r, \tilde{r} \in \mathbb{R}$, and $q, \tilde{q} \in \mathbb{R}^d$, where $\mathcal{P}^-_{\lambda,\Lambda}$ and $\mathcal{P}^+_{\lambda,\Lambda}$ are the Pucci extremal operators. Moreover, we assume that there exists $R > 0$ such that $\omega_1(r) + \omega_2(q) \leq R(1 + r + q)$ for all $r, q \in \mathbb{R}_+$.

Assumption 4 implies that F is uniformly elliptic, that $F(t, x, r, p, M)$ is continuous in r, p, and M uniformly over (t, x), and that F grows at most linearly in the value and first derivatives of the value function.

The assumption is weaker than the assumptions typically imposed on F in the literature for the following reasons: (i) F is not required to be Lipschitz in (r, q); (ii) F need not be continuous in (t, x); (iii) F need not be monotonic in r.[6]

ASSUMPTION 5 *$F(t, x, 0, 0, M)$ is convex in M for all (t, x).*[7]

The following Vanishing Mean Oscillation (VMO) assumption is also imposed. Define

$$\Theta\left((t, x), (\tilde{t}, \tilde{x})\right) = \sup_{M \in \mathbb{S}^d \setminus \{0\}} \frac{F\left(t, x, 0, 0, M\right) - F\left(\tilde{t}, \tilde{x}, 0, 0, M\right)}{\|M\|}$$

ASSUMPTION 6 *For almost all $(t, x) \in \mathcal{Y}$.*

$$\lim_{|Q| \to 0\,:\,(t,x) \in Q \subset \mathcal{Y}} \frac{1}{|Q|} \int_Q \Theta\left((t, x), (\tilde{t}, \tilde{x})\right) d\tilde{t} d\tilde{x} \to 0,$$

[5] Part (iii) is sufficient but need not be necessary. The proof requires only the existence of an approximating sequence $(g^n = \max_{a \in A_n} g^a)_n$, with A_n finite for all n.

[6] To establish the uniqueness part of our result, and only this part, we do impose a weak monotonicity condition (Assumption 8).

[7] When F is concave case, our results remain valid (upon modifying Assumption 8 appropriately), since u solves $\max\{g - v, F(t, x, v, v_x, v_{xx}) + v_t\} = 0$ if and only if it solves $\min\{v - g, -F(t, x, v, v_x, v_{xx}) - v_t\} = 0$.

Assumption 6 is needed to ensure that the weak derivatives of solutions to canonical equations related to obstacle problem exist a.e. and belong to L^p, by guaranteeing that the coefficient of the highest-order derivative in the linearized version of the operator F is regular enough in (t, x). As pointed out in Dong (2020), Assumption 6 is the weakest known assumption (even in the linear case) under which a solution in $W^{1,2,p}$ always exists for Dirichlet problems. Moreover, counter-examples in Meyers (1963) or Ural'ceva (1967) suggest that this condition is difficult to relax.

The next assumption is standard. It is required to derive a bound on the $W^{1,2,p}$-norm of solutions to (1).

Assumption 7 *There exists $G \in L^p(\mathcal{Y})$ such that, for all $(t,x) \in \mathcal{Y}$,*

$$|F(t,x,0,0,0)| \leq G(t,x).$$

The final assumption is used to apply a comparison principle to (1). It is not needed for our existence results, but it is essential to guarantee the uniqueness of a solution to (1).

Assumption 8 *There exists $\kappa > 0$ such that, for all $(t,x) \in \mathcal{Y}$, $q \in \mathbb{R}^d$, and $M \in \mathbb{S}^d$,*

$$r \to F(t,x,r,q,M) - \kappa r$$

is strictly decreasing.

Our main result establishes the existence, uniqueness, and regularity of solutions to (1).[8]

Theorem 1 *Suppose that Assumptions 1–7 hold for some $p \in (d+2, \infty)$.[9] Then (1) has an L^p-solution u.*

Moreover, for all compact $\mathcal{Y}' \subset \mathcal{Y}$, there exists $C = C(d, p, \lambda, \Lambda, R, diam(\mathcal{X}), T, dist(\mathcal{Y}', \mathcal{Y})) \in \mathbb{R}_+$ such that

$$\|u\|_{W^{1,2,p}(\mathcal{Y}')} \leq C \left(1 + \sup_{a \in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})} + \|b\|_{L^\infty(\mathcal{Y})}\right). \tag{2}$$

If, in addition, Assumption 8 holds, u is the unique L^p-solution of (1).

Remark 1 *The $W^{1,2,p}$-estimate (2) is independent of the moduli of continuity ω_1 and ω_2 introduced in Assumption 4.*

[8]The structure condition (**SC**) guarantees that any solution of $u_t + F[u] = 0$ with $u = b$ on $\partial\mathcal{Y}$ is bounded below. Therefore, by choosing g small enough, Theorem 1 also guarantees the existence, uniqueness, and regularity of the solution without obstacles.

[9]The parameter p appears in Assumptions 3 and 7.

Corollary 1 *Suppose that Assumptions 1–8 hold for some $p \in (d+2,\infty)$. Then, the unique L^p-solution u of* (1) *is continuously differentiable with respect to the space variable x on $\mathcal{Y}$.*

For stopping problems, the solution of (1) coincides with the value function. Corollary 1 then (i) implies that smooth pasting in space holds, and (ii) confirms that the decision maker never stops at a kink of g, as shown by Décamps et al. (2006) in the specific problem of irreversible investment.

Proof. By Theorem 1, the unique solution is in $W^{1,2,p}_{loc}(\mathcal{Y})$. By a Morrey-Sobolev embedding type theorem (Lemma 3.3, page 80 in Ladyženskaja et al. (1968)), $W^{1,2,p}_{loc}(\mathcal{Y})$ is embedded in $\mathcal{C}^{0,1}(\mathcal{Y})$.

4 Comparison and Stability for fully nonlinear parabolic obstacle problems with measurable ingredients

Section 4 contains two results needed for the proof of Theorem 1: (i) A comparison principle that applies to fully nonlinear parabolic obstacle problems with measurable ingredients (Proposition 1), and (ii) A stability theorem for solutions to obstacle problems in $W^{1,2,p}(\mathcal{Y})$ (Theorem 2).

4.1 Comparison Principle

Comparison principles for L^p-solutions and viscosity solutions of the Dirichlet problem are well-known in the literature. We provide here a more general result that encompasses obstacle problems and includes operators that are less restrained than those considered in the literature.[10] Our comparison principle is valid for a large class of nonlinear parabolic obstacle problems as long as F satisfies the structure condition (**SC**) and does not increase too fast with u (as required by Assumption 8).

Proposition 1 *Suppose that F satisfies Assumptions 4 and 8.*

- *Let u be an L^p-subsolution*[11] *and v be a viscosity supersolution of*

$$\max\{u_t + F(t,x,u,u_x,u_{xx}),\, g-u\} = 0. \tag{3}$$

  If $v \geq u$ on $\partial\mathcal{Y}$, then $v \geq u$ on $\bar{\mathcal{Y}}$.

[10] For instance, Proposition 2.10 in Crandall et al. (2000) gives a comparison principle when F is Lipschitz continuous in u_x and nonincreasing in u.

[11] Note that we do not impose here conditions on the functions at the boundary $\partial\mathcal{Y}$; that is, we dispense with requirement (ii) in Definitions 1 and 2 and allow u and v to take a priori any values on $\partial\mathcal{Y}$. This is because the comparison principle only depends on the condition that $v \geq u$ on $\partial\mathcal{Y}$, which can be used to satisfy Definitions 1 and 2, including requirement (ii), for functions that depend on u and v only through the difference $v-u$.

- *Let u be an L^p-supersolution*[12] *and v be a viscosity subsolution of*

$$\max\left\{ u_t + F\left(t, x, u, u_x, u_{xx}\right), \, g - u\right\} = 0. \tag{4}$$

If $v \leq u$ on $\partial\mathcal{Y}$, then $v \leq u$ on $\bar{\mathcal{Y}}$.

The proof of Proposition 1 builds on a comparison principle for viscosity solutions of fully nonlinear parabolic Dirichlet problems with continuous operators due to Giga et al. (1991).

Proof. Let u be an L^p-subsolution and v be a viscosity supersolution of

$$\max\left\{ u_t + F\left(t, x, u, u_x, u_{xx}\right), \, g - u\right\} = 0$$

such that $v \geq u$ on $\partial\mathcal{Y}$. We will prove that $v \geq u$ on $\bar{\mathcal{Y}}$.

Let $w = u - v$ and $\mathcal{E} = \left\{(t,x) \in \bar{\mathcal{Y}} \,:\, w > 0\right\}$. We need to show that $\mathcal{E}$ is empty. Suppose by way of contradiction that it is not. We note that $\mathcal{E} \subset \mathcal{Y}$ since $w \leq 0$ on $\partial\mathcal{Y}$. Moreover, w is continuous since both u and v are by construction continuous on $\bar{\mathcal{Y}}$. Therefore, $\mathcal{E}$ is an open bounded domain.

We begin by showing that w is a viscosity subsolution of the problem

$$\begin{cases} \max\left\{ w_t + H\left(t, x, w, w_x, w_{xx}\right), \, -w\right\} = 0 \text{ on } \mathcal{E} \\ w = 0 \text{ on } \partial\mathcal{E} \end{cases} \tag{5}$$

where

$$H(t, x, w, w_x, w_{xx}) = F\left(t, x, u, u_x, u_{xx}\right) - F\left(t, x, u - w, u_x - w_x, u_{xx} - w_{xx}\right).$$

To see this, we first note that $w = 0$ on $\partial\mathcal{E}$, so condition (ii) in the definition of viscosity subsolutions (which, here, amounts to $w \leq 0$) is satisfied. Next, let $\varphi \in W^{1,2,p}\left(\mathcal{E}\right)$ be such that $w - \varphi$ has a maximum at (t_0, x_0) in $\mathcal{E}$, i.e., $u - \varphi - v$ has a maximum at (t_0, x_0). By construction, $u - \varphi \in W^{1,2,p}_{loc}\left(\mathcal{E}\right)$ and v is a viscosity supersolution of (3). Therefore,

$$\operatorname*{ess\,lim\,inf}_{(t,x)\to(t_0,x_0)} \max\left\{ u_t - \varphi_t + F\left(t, x, u - w, u_x - \varphi_x, u_{xx} - \varphi_{xx}\right), \, g - (u - w)\right\} \leq 0.$$

Combining this result with the fact that u is an L^p-subsolution of (3) then yields

$$\operatorname*{ess\,lim\,sup}_{(t,x)\to(t_0,x_0)} \max\left\{ \varphi_t + H\left(t, x, w, \varphi_x, \varphi_{xx}\right), \, -w\right\} \geq 0.$$

This shows that w is a viscosity subsolution of (5).

[12] The same observation regarding the absence of a constraint on $\partial\mathcal{Y}$ applies here as in the previous footnote.

By the structure condition (**SC**), for any test function $\varphi \in W^{1,2,p}(\mathcal{E})$ and almost every $(t,x) \in \mathcal{E}$,

$$H(t,x,w,\varphi_x,\varphi_{xx}) \leq \mathcal{P}^{+}_{\lambda,\Lambda}(\varphi_{xx}) + \omega_1(|w|) + \omega_2(|\varphi_x|).$$

Therefore, w is also a viscosity subsolution of the problem

$$\begin{cases} \max\left\{-w,\ w_t + \mathcal{P}^{+}_{\lambda,\Lambda}(w_{xx}) + \omega_1(|w|) + \omega_2(|w_x|)\right\} = 0 \text{ on } \mathcal{E}, \\ w = 0 \text{ on } \partial\mathcal{E}. \end{cases}$$

By construction of the domain $\mathcal{E}$ and our earlier observation that $w = 0$ on $\partial\mathcal{E}$, this implies that w is a viscosity subsolution of

$$\begin{cases} w_t + \mathcal{P}^{+}_{\lambda,\Lambda}(w_{xx}) + \omega_1(|w|) + \omega_2(|w_x|) = 0 \text{ on } \mathcal{E}, \\ w = 0 \text{ on } \partial\mathcal{E}. \end{cases}$$

Note that 0 is a classical solution of the above equation, and thus also a (continuous) viscosity supersolution. Theorem 4.7 in Giga et al. (1991) then implies that $w \leq 0$ in $\mathcal{E}$, which yields the desired contradiction. This shows that $\mathcal{E} = \emptyset$ and, hence, that $v \geq u$ in $\bar{\mathcal{Y}}$.

The proof for the case in which u is an L^p-supersolution, v is a viscosity subsolution, and $u \geq v$ on $\partial\mathcal{Y}$ follows the same steps. ■

Remark 2 *Proposition 1 also holds when F is degenerate elliptic (i.e, $\lambda = 0$ in Assumption 4) since Theorem 4.7 in Giga et al. (1991) applies to degenerate elliptic operators. In this case, however, the existence of L^p-sub- or L^p-supersolutions is not guaranteed.*

Proposition 1 and Lemma 1 yield the following corollary, which establishes the uniqueness of L^p-solutions.

Corollary 2 *Suppose that Assumptions 4 and 8 hold and let u and v be, respectively, an L^p-subsolution*[13] *and an L^p-supersolution of*

$$\max\left\{u_t + F(t,x,u,u_x,u_{xx}),\ \sup_{a\in A} g^a - u\right\} = 0.$$

If $v \geq u$ on $\partial\mathcal{Y}$, then $v \geq u$ on $\bar{\mathcal{Y}}$.

If u and v are two L^p-solutions of (1), *then $u = v$ on $\bar{\mathcal{Y}}$.*

Remark 3 *Proposition 1 and Corollary 2 are also valid for non-obstacle problems, as is easily seen from the proof.*

[13] The same observation regarding the absence of a constraint on the functions at the frontier $\partial\mathcal{Y}$ applies as in footnote 11.

4.2 Stability Theorem

Stability theorems for both L^p-solutions and viscosity solutions of the Dirichlet problem (without obstacle) are well-known under a slight strengthening of the structure condition (**SC**).[14] We establish a more general stability theorem for our setting, which is then used to prove Theorem 1. This stability theorem is valid for a large class of nonlinear parabolic obstacle problems as long as F satisfies the structure condition (**SC**).

THEOREM 2 *Let $(\mathcal{Y}^n)_{n\in\mathbb{N}}$ be an increasing sequence of bounded, open domains such that $\bigcup_{n\in\mathbb{N}}\mathcal{Y}^n = \mathcal{Y}$.*[15] *Let $(u^n)_{n\in\mathbb{N}}$ be a sequence of functions in $W^{1,2,p}_{loc}(\mathcal{Y})\cap\mathcal{C}^0(\bar{Y})$ such that*

1. *$(u^n)_{n\in\mathbb{N}}$ converges uniformly on compact subsets and weakly in $W^{1,2,p}_{loc}(\mathcal{Y})$ to some $u\in W^{1,2,p}_{loc}(\mathcal{Y})\cap \mathcal{C}^0(\bar{\mathcal{Y}})$; and*
2. *for all $n\in\mathbb{N}$, $u^n=b^n$ on $\bar{\mathcal{Y}}\setminus\mathcal{Y}^n$, where the sequence b^n is such that $b^n\to b$ pointwise for some function b that is continuous on $\partial\mathcal{Y}$.*

Let F and F^n, $n\in\mathbb{N}$, be operators defined on $\mathcal{Y}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d$ that satisfy Assumption 4 uniformly in n and Assumption 5. Let g^n, $n\in\mathbb{N}$, be continuous functions on $\bar{\mathcal{Y}}$ such that $g^n\le b^n$ on $\bar{\mathcal{Y}}\setminus\mathcal{Y}^n$.

Suppose that for any fixed $u_{xx}\in L^p(\mathcal{Y})$, $F^n(t,x,u^n,u^n_x,u_{xx})\to F(t,x,u,u_x,u_{xx})$ pointwise a.e. on $\mathcal{Y}$, and that either (i) $g^n\to g$ uniformly on compact subsets of $\mathcal{Y}$, or (ii) $g^n\uparrow g$ pointwise on $\mathcal{Y}$. If u^n, $n\in\mathbb{N}$, is an L^p-subsolution (supersolution) of

$$\begin{cases}\max\{u^n_t+F^n(t,x,u^n,u^n_x,u^n_{xx}),\, g^n-u^n\}=0 \text{ on } \mathcal{Y}^n,\\ u^n=b^n \text{ on } \partial\mathcal{Y}^n,\end{cases}\tag{Eq^n}$$

then u is an L^p-subsolution (supersolution) of

$$\begin{cases}\max\{u_t+F(t,x,u,u_x,u_{xx}),\, g-u\}=0 \text{ on } \mathcal{Y},\\ u=b \text{ on } \partial\mathcal{Y}.\end{cases}\tag{Eq}$$

Proof. We first show that if, for each n, u^n is an L^p-supersolution of (Eq^n), then u is an L^p-supersolution of (Eq).

By assumption, u satisfies the regularity condition required to be an L^p-supersolution (requirement (i) in the definition). Moreover, since $u^n\ge b^n$ on $\partial\mathcal{Y}$ for all $n\in\mathbb{N}$, taking limits shows that $u\ge b$ on $\partial\mathcal{Y}$ and, hence, that u satisfies the second requirement for being a supersolution.

[14] E.g., Theorem 6.1 in Crandall et al. (2000) gives a stability theorem when F is Lipschitz continuous in u_x and nonincreasing in u.

[15] $\mathcal{Y}$ does not need to be bounded.

We now prove the third condition, namely that

$$\max\left\{u_t + F\left(t, x, u, u_x, u_{xx}\right), \, g - u\right\} \leq 0 \text{ a.e. on } \mathcal{Y}.$$

Since u^n is an L^p-supersolution, $g^n - u^n \leq 0$ on $\mathcal{Y}^n$. Moreover, $b^n \geq g^n$ on $\bar{\mathcal{Y}} \setminus \mathcal{Y}^n$ by assumption. So, for all $n \in \mathbb{N}$, $u^n \geq g^n$ on $\bar{\mathcal{Y}}$. Passing to the limit, $u \geq g$.

There remains to show that

$$u_t + F\left(t, x, u, u_x, u_{xx}\right) \leq 0 \text{ a.e. on } \mathcal{Y},$$

which will hold if

$$u_t + F\left(t, x, u, u_x, u_{xx}\right) \leq 0 \text{ a.e. on } \mathcal{O}$$

for any strict bounded subset $\mathcal{O}$ of $\mathcal{Y}$.[16]

Thus, consider any strict bounded subset $\mathcal{O}$ of $\mathcal{Y}$. Since $\mathcal{Y} = \bigcup_{n\in\mathbb{N}} \mathcal{Y}^n$, compactness implies that there exists some integer $N \geq 1$ such that $\mathcal{Y}^N$ contains $\mathcal{O}$. Moreover, since the sequence $\{\mathcal{Y}^n\}_{n\geq 1}$ is increasing, $\mathcal{O} \subset \mathcal{Y}^n$ for all $n \geq N$. Without loss of generality, we assume that $N = 1$. For all $m \in \mathbb{N}$, define

$$\underline{H}^m\left(t, x, v_t, v_{xx}\right) = \inf_{n\geq m} \left\{v_t + F^n\left(t, x, u^n, u^n_x, v_{xx}\right)\right\}.$$

Since $u^n_t + F^n\left(t, x, u^n, u^n_x, u^n_{xx}\right) \leq 0$, we have for $n \geq m$ and almost every $(t, x) \in \mathcal{O}$,

$$\underline{H}^m\left(t, x, u^n_t, u^n_{xx}\right) \leq 0.$$

Since F^n satisfies Assumptions 4 and 5 for each n, we deduce that for all $m \in \mathbb{N}$ and all $v_t \in \mathbb{R}$, $\underline{H}^m(t, x, v_t, \cdot)$ is Lipschitz continuous in v_{xx} a.e. on $\mathcal{O}$ and

$$\lambda I_d \leq D_{v_{xx}} \underline{H}^m(t, x, v_t, \cdot) \leq \Lambda I_d.$$

Moreover, for any $v_{xx} \in \mathbb{S}^d$, the structure condition (**SC**) implies that $\underline{H}^m$ is bounded in $L^p\left(\mathcal{O}\right)$ because the functions u^n, $n \in \mathbb{N}$, are uniformly bounded in $W^{1,2,p}_{loc}\left(\mathcal{Y}\right)$ as part of a weakly convergent sequence. Finally, by assumption, $u^n \rightharpoonup u$ in $W^{1,2,p}\left(\mathcal{O}\right)$. Theorem 4.2.6 in Krylov (2018) then implies that, for all $m \in \mathbb{N}$,[17]

$$\underline{H}^m\left(t, x, u_t, u_{xx}\right) = \inf_{n\geq m} \left\{u_t + F^n\left(t, x, u^n, u^n_x, u_{xx}\right)\right\} = \lim_{n\to\infty} \underline{H}^m\left(t, x, u^n_t, u^n_{xx}\right) \leq 0 \text{ a.e. on } \mathcal{O}.$$

[16] We say that $\mathcal{O}$ is a strict subset of $\mathcal{Y}$ if the closure of $\mathcal{O}$ lies in the interior of $\mathcal{Y}$.

[17] In order to apply Theorem 4.2.6 in Krylov (2018) to the operator $\underline{H}^m$, we use the fact that it is a λ-nondegenerate $\mathcal{L}$-type operator, which follows from parts (c) and (d) of Lemma 4.2.4 in Krylov (2018): part (c) shows that F^n has this property, and part (d) shows that the supremum over n also inherits the property .

Moreover, $u_t + F^n(t, x, u^n, u_x^n, u_{xx}) \to u_t + F(t, x, u, u_x, u_{xx})$ pointwise a.e. on $\mathcal{Y}$ by assumption. Letting $m \to \infty$, we obtain

$$u_t + F(t, x, u, u_x, u_{xx}) \leq 0 \text{ a.e. on } \mathcal{O}.$$

This shows that u is an L^p-supersolution of (Eq).

Next, we prove that if, for each n, u^n is an L^p-subsolution of (Eq^n), then u is an L^p-subsolution of (Eq).

We first note that in this case, we have $u^n \leq b^n$ on $\partial\mathcal{Y}$ for all $n \in \mathbb{N}$, . Taking limits, $u \leq b$ on $\partial\mathcal{Y}$.

Next, we show that

$$\max\{u_t + F(t, x, u, u_x, u_{xx}), g - u\} \geq 0 \text{ a.e. on } \mathcal{Y},$$

which holds if and only if

$$\max\{u_t + F(t, x, u, u_x, u_{xx}), g - u\} \geq 0 \text{ a.e. on } \mathcal{O},$$

for any strict bounded subset $\mathcal{O}$ of $\mathcal{Y}$.

Let $\mathcal{O}$ be a strict bounded subset of $\mathcal{Y}$. Since $\mathcal{Y} = \bigcup_{n\in\mathbb{N}} \mathcal{Y}^n$ and the sequence of subsets $\mathcal{Y}^n$ is increasing, there exists $N \in \mathbb{N}$ such that $\mathcal{O} \subset \mathcal{Y}^n$, for all $n \geq N$. Without loss of generality, assume that $N = 1$. It is then enough to show that, on $\mathcal{E} = \{(t, x) \in \mathcal{O} : u(t, x) > g(t, x)\}$, we have

$$u_t + F(t, x, u, u_x, u_{xx}) \geq 0 \text{ a.e. on } \mathcal{O}.$$

If $\mathcal{E} = \emptyset$, there is nothing left to prove. Suppose instead that $\mathcal{E}$ is nonempty, and let k be large enough so that

$$\mathcal{E}^k = \left\{(t, x) \in \mathcal{O} : u(t, x) > g(t, x) + \frac{1}{k}\right\} \neq \emptyset.$$

Then, there exists $N^k \in \mathbb{N}$ such that $g^n < u^n$ on $\mathcal{E}^k$ for all $n \geq N^k$. To see this, choose N^k such that $|u^n - u| < \frac{1}{3k}$ on $\mathcal{E}^k$ (which exists since $u^n \to u$ uniformly). Then recall that either (i) $g^n \uparrow g$ pointwise, or (ii) $g^n \to g$ uniformly. So, either (i) $g^n \leq g < u - \frac{1}{k} < u - \frac{1}{3k} < u^n$ on $\mathcal{E}^k$ and we are done, or (ii) g^n converge uniformly to g. In that case, choose $\tilde{N}^k$ such that $|g^n - g| < \frac{1}{3k}$ for all $(t, x) \in \mathcal{O}$, and again, it follows that $g^n < g + \frac{1}{3k} < u - \frac{1}{3k} < u^n$ on $\mathcal{E}^k$ for all $n \geq \max\left\{N^k, \tilde{N}^k\right\}$.

Since $g_n < u_n$ on $\mathcal{E}^k$ for $n > N^k$, we conclude from (Eq^n) that, for all $n \geq N^k$,

$$u_t^n + F^n(t, x, u^n, u_x^n, u_{xx}^n) = 0 \text{ a.e. on } \mathcal{E}^k.$$

Similarly to the first part of the proof, define for all $m \geq N^k$

$$\bar{H}^m(t, x, v_t, v_{xx}) = \sup_{n \geq m} \{v_t + F^n(t, x, u^n, u_x^n, v_{xx})\}.$$

We note once more that, for $n \geq m$ and for almost every $(t, x) \in \mathcal{E}^k$,

$$\bar{H}^m(t, x, u_t^n, u_{xx}^n) \geq 0.$$

Moreover, as above, for all $m \in \mathbb{N}$ and all $v_t \in \mathbb{R}$, $\bar{H}^m(t, x, v_t, \cdot)$ is Lipschitz continuous in v_{xx} a.e. on $\mathcal{E}^k$ with

$$\lambda \leq D_{v_{xx}} \bar{H}^m(t, x, v_t, \cdot) \leq \Lambda I_d.$$

Also, for any $v_{xx} \in \mathbb{S}^d$, by the structure condition (**SC**), $\bar{H}^m$ is bounded in $L^p(\mathcal{O})$ because the functions u^n, $n \in \mathbb{N}$, are uniformly bounded in $W^{1,2,p}_{loc}(\mathcal{Y})$ as they form a weakly convergent sequence. Finally, by assumption, $u^n \rightharpoonup u$ in $W^{1,2,p}(\mathcal{E}^k)$. Theorem 4.2.6 in Krylov (2018) then implies that, for all $m \in \mathbb{N}$,[18]

$$\bar{H}^m(t, x, u_t, u_{xx}) = \inf_{n \geq m} \{u_t + F^n(t, x, u^n, u_x^n, u_{xx})\} = \lim_{n \to \infty} \bar{H}^m(t, x, u_t^n, u_{xx}^n) \geq 0 \text{ a.e. on } \mathcal{E}^k.$$

By assumption, $u_t + F^n(t, x, u^n, u_x^n, u_{xx}) \to u_t + F(t, x, u, u_x, u_{xx})$ pointwise a.e. on $\mathcal{Y}$. Letting $m \to \infty$, we obtain

$$u_t + F(t, x, u, u_x, u_{xx}) \geq 0 \text{ a.e. on } \mathcal{E}^k.$$

Since $\bigcup_{k \in \mathbb{N}} \mathcal{E}^k = \mathcal{E}$, it follows that

$$u_t + F(t, x, u, u_x, u_{xx}) \geq 0 \text{ a.e. on } \mathcal{E}.$$

This shows that u is an L^p-subsolution of (Eq). ■

REMARK 4 *Theorem 2 is also valid for non-obstacle problems, as can be seen from the proof. Moreover, it also holds for viscosity solutions. To see this, one can invoke Theorem 6.1 in Crandall et al. (2000) instead of Theorem 4.2.6 in Krylov (2018) in the proof.*

5 Existence, uniqueness, and $W^{1,2,p}$-estimate when $|A| < \infty$

Before considering the general case, we derive a more restrictive version of Theorem 1 when the obstacle is the maximum of finitely many $W^{1,2,p}$-obstacles and the following strengthening Assumptions 1 and 2 is imposed.

[18] Theorem 4.2.6 in Krylov (2018) applies to the operator $\bar{H}^m$ by the observation made in the previous footnote.

Assumption 1' $\mathcal{X}$ *is an* $\mathcal{C}^{1,Lip}$ *open bounded subset of* $\mathbb{R}^d$.

Assumption 2' $b : \bar{\mathcal{Y}} \to \mathbb{R}$ *is in* $W^{1,2,p}\left(\bar{\mathcal{Y}}\right)$.

Lemma 2 *Suppose that Assumptions 1', 2', and 3–7 hold for some* $p \in (d+2, \infty)$.[19] *If* $|A| < \infty$, *then* (1) *has an* L^p*-solution* u.

Moreover, $u \in W^{1,2,p}(\mathcal{Y})$ *and there exists* $C = C\left(d, p, \lambda, \Lambda, R, diam(\mathcal{X}), T, \|\partial\mathcal{X}\|_{\mathcal{C}^{1,Lip}}\right) \in \mathbb{R}_+$ *such that*

$$\|u\|_{W^{1,2,p}(\mathcal{Y})} \leq C\left(1 + \max_{a \in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})}\right). \tag{6}$$

If Assumption 8 also holds, the solution u *is unique.*

The proof builds on the approximation argument proposed by Byun et al. (2018) and Byun et al. (2022), extending it to fully nonlinear parabolic obstacle problems. Byun et al. (2018) approximate the obstacle problem by a Dirichlet problem similar to (D^n). They then use the Schauder fixed-point theorem to show the existence of L^p-solutions whose norms are bounded by a constant independent of ϵ. Instead, we appeal to known existence results and estimates on the regularity of solutions of the Dirichlet problem, e.g., Theorem 12.1.7, 15.1.3, and 15.1.4 in Krylov (2018) that allow for an arbitrary continuous dependence of the operator on the value of the solution u. As a result, we can sidestep the fixed-point argument in Byun et al. (2018), streamlining the proof of the existence of a solution and $W^{1,2,p}$-estimate when the obstacle is in $W^{1,2,p}$. We then show by induction that the existence and regularity results for a single $W^{1,2,p}$-obstacle carry to the pointwise maximum of a finite number of $W^{1,2,p}$-obstacles.

Proof. Without loss of generality, assume that $A = \{1, \ldots, I\}$, for some integer $I \geq 1$. The proof proceeds by induction on I. For all $I \geq 1$, define property $\mathbf{P(I)}$ as follows:

> If $g = \max_{a \in \{1,\ldots,I\}} g^a$ with $g^a \in W^{1,2,p}(\mathcal{Y})$ for all $a \in \{1, \ldots, I\}$, then (1) has a L^p-solution $u \in W^{1,2,p}(\mathcal{Y})$. Moreover, u satisfies the estimate (6).

We start by proving the base case $\mathbf{P(1)}$. Thus, let $g^1 \in W^{1,2,p}(\mathcal{Y})$ be such that $g^1 \leq b$ on $\partial\mathcal{Y}$. Let $h^1(t,x) = -g^1_t - F\left(t, x, g^1, g^1_x, g^1_{xx}\right)$. For each $\epsilon > 0$, let $\Phi_\epsilon \in \mathcal{C}^\infty(\mathbb{R})$ be a nondecreasing function such that $\Phi_\epsilon(a) = 0$ if $a \leq 0$, and $\Phi_\epsilon(a) = 1$ if $a \geq \epsilon$. In particular, for all $a \in \mathbb{R}$, $\Phi_\epsilon(a) \in [0,1]$. Finally, consider a sequence $(\epsilon_n)_{n \in \mathcal{N}} \subseteq \mathbb{R}_{++}$ such that $\epsilon_n \to 0$ as $n \to \infty$, and consider the following

[19] The parameter p appears in Assumptions 3 and 7.

auxiliary nonlinear Dirichlet problem for each $n \in \mathbb{N}$:

$$\begin{cases} F\left(t,x,u^n,u^n_t,u^n_x,u^n_{xx}\right) = h^1(t,x)^+\Phi_{\epsilon_n}\left(u^n - g^1\right) - h^1(t,x)^+ \text{ on } \mathcal{Y}, \\ u^n = b \text{ on } \partial\mathcal{Y}. \end{cases} \tag{D^n}$$

Here $h^1(t,x)^+ = \max\left\{h^1(t,x),0\right\}$ stands for the nonnegative part of h^1. By Assumptions 4 and 7, $h^1(t,x)$ is in $L^p\left(\mathcal{Y}\right)$ and

$$\left\|h^1\right\|_{L^p(\mathcal{Y})} \leq C^{h^1}\left(1 + \left\|g^1\right\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})}\right) \tag{7}$$

where $C^{h^1} = C\left(d,p,\lambda,\Lambda,R\right) \in \mathbb{R}_+$. By Theorem 15.1.4 in Krylov (2018), for each $n \in \mathbb{N}$, there exists a solution $u^n \in W^{1,2,p}\left(\mathcal{Y}\right)$ of (D^n). Moreover, by Theorem 15.1.3 in the same work,

$$\|u^n\|_{W^{1,2,p}(\mathcal{Y})} \leq C\left(1 + \left\|h^1\right\|_{L^p(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})} + \|b\|_{W^{1,2,p}(\mathcal{Y})} + \|u^n\|_{L^\infty(\mathcal{Y})}\right)$$

where $C = C\left(d,p,\lambda,\Lambda,R,diam(\mathcal{X}),T,\|\partial\mathcal{X}\|_{\mathcal{C}^{1,Lip}}\right) \in \mathbb{R}_+$. By Lemma 12.1.9 in Krylov (2018),

$$\|u^n\|_{L^\infty(\mathcal{Y})} \leq C\left(1 + \left\|h^1\right\|_{L^p(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})} + \|b\|_{L^\infty(\partial\mathcal{Y})}\right),$$

where $C = C\left(d,p,\lambda,\Lambda,R,diam(\mathcal{X}),T\right) \in \mathbb{R}_+$. Finally, $\|b\|_{L^\infty(\partial\mathcal{Y})} \leq \|b\|_{L^\infty(\mathcal{Y})} \leq C\|b\|_{W^{1,2,p}(\mathcal{Y})}$, where $C = C(d,p,diam(\mathcal{X}),T,\|\partial\mathcal{X}\|_{\mathcal{C}^{0,Lip}})$, by the Morrey-Sobolev embedding theorem. Combining the inequalities above, we obtain

$$\|u^n\|_{W^{1,2,p}(\mathcal{Y})} \leq C\left(1 + \left\|g^1\right\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})}\right)$$

where $C = C\left(d,p,\lambda,\Lambda,R,diam(\mathcal{X}),T,\|\partial\mathcal{X}\|_{\mathcal{C}^{1,Lip}}\right) \in \mathbb{R}_+$ is independent of n.

$W^{1,2,p}(\mathcal{Y})$ is separable and reflexive.[20] Therefore, its closed bounded subsets are weakly sequentially compact by Theorem 1.32 in Demengel et al. (2012). Moreover $W^{1,2,p}\left(\mathcal{Y}\right)$ is compactly embedded in $\mathcal{C}^0\left(\bar{\mathcal{Y}}\right)$ by the Rellich-Kondrachov theorem (see Theorem 2.84 in Demengel et al. (2012)). Therefore, there exists a function $u \in W^{1,2,p}\left(\mathcal{Y}\right) \cap \mathcal{C}^0\left(\bar{\mathcal{Y}}\right)$ and a subsequence $\left(u^{n_j}\right)_{j\in\mathbb{N}} \subseteq \left(u^n\right)_{n\in\mathbb{N}}$ such that

$$\begin{cases} u^{n_j} \rightharpoonup u \text{ in } W^{1,2,p}\left(\mathcal{Y}\right), \\ u^{n_j} \to u \text{ in } \mathcal{C}^0\left(\bar{\mathcal{Y}}\right), \end{cases}$$

as $j \to \infty$. Moreover, u satisfies the estimate (6):

$$\begin{aligned} \|u\|_{W^{1,2,p}(\mathcal{Y})} &\leq \liminf_{j\to\infty} \|u^{n_j}\|_{W^{1,2,p}(\mathcal{Y})} \\ &\leq C\left(1 + \left\|g^1\right\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})}\right). \end{aligned}$$

[20] The space L^p is uniformly convex for $p \in (1,\infty)$, hence so is $W^{1,2,p}\left(\mathcal{Y}\right)$. Therefore, this latter space is reflexive by Theorem 1.40 in Demengel et al. (2012).

There remains to show that $u \in W^{1,2,p}(\mathcal{Y})$ is an L^p-solution of (1).

First, for all $j \in \mathbb{N}$, $u^{n_j} = b$ on $\partial\mathcal{Y}$, and $(u^{n_j})_{j\in\mathbb{N}}$ converges uniformly to u on $\bar{\mathcal{Y}}$. So, $u = b$ on $\partial\mathcal{Y}$.

Second, we prove that

$$u_t + F(t, x, u, u_x, u_{xx}) \leq 0 \text{ a.e. on } \mathcal{Y}.$$

For all $j \in \mathbb{N}$ and a.e. $(t, x) \in \mathcal{Y}$,

$$u_t^{n_j} + F\left(t, x, u^{n_j}, u_x^{n_j}, u_{xx}^{n_j}\right) = h^1(t,x)^+ \Phi_{\epsilon_{n_j}}\left(u^{n_j} - g^1\right) - h^1(t,x)^+ \leq 0,$$

So, for all $j \in \mathbb{N}$, u^{n_j} is an L^p-supersolution of

$$u_t^{n_j} + F\left(t, x, u^{n_j}, u_x^{n_j}, u_{xx}^{n_j}\right) = 0.$$

Theorem 2 then implies that u is an L^p-supersolution of

$$u_t + F(t, x, u, u_x, u_{xx}) = 0 \text{ on } \mathcal{Y}.$$

Next, we show that $u \geq g^1$ a.e. on $\mathcal{Y}$. For each $j \in \mathbb{N}$, define

$$\mathcal{V}_{n_j} = \left\{(t,x) \in \mathcal{Y} : g^1(t,x) > u^{n_j}(t,x)\right\}.$$

We will show that $\mathcal{V}_{n_j}$ is empty for each $j \in \mathbb{N}$. Suppose by way of contradiction that $\mathcal{V}_{n_j} \neq \emptyset$ for some $j \in \mathbb{N}$. Since both u^{n_j} and g^1 are continuous on $\mathcal{Y}$, $\mathcal{V}_{n_j}$ is open. Moreover, since $b \geq g^1$, $u^{n_j} = g^1$ on $\partial\mathcal{V}_{n_j}$. On $\mathcal{V}_{n_j}$, $\Phi_{\epsilon_{n_j}}(u^{n_j}(t,x) - g(t,x)) = 0$. So $u^{n_j} \in W^{1,2,p}(\mathcal{Y})$ is an L^p-solution of

$$\begin{cases} u_t^{n_j} + F\left(t, x, u^{n_j}, u_x^{n_j}, u_{xx}^{n_j}\right) = -h^1(t,x)^+ \text{ on } \mathcal{V}_{n_j}, \\ u^{n_j} = g^1 \text{ on } \partial\mathcal{V}_{n_j}. \end{cases}$$

Since $-{h^1}^+ \leq -h^1$, u^{n_j} is an L^p-solution of

$$\begin{cases} u_t^{n_j} + F\left(t, x, u^{n_j}, u_x^{n_j}, u_{xx}^{n_j}\right) \leq -h^1(t,x) \text{ on } \mathcal{V}_{n_j} \\ u^{n_j} = g^1 \text{ on } \partial\mathcal{V}_{n_j}. \end{cases}$$

By definition of h^1, $g^1 \in W^{1,2,p}(\mathcal{Y})$ is an L^p-solution of

$$\begin{cases} g_t^1 + F\left(t, x, g^1, g_x^1, g_{xx}^1\right) = -h^1(t,x) \text{ on } \mathcal{V}_{n_j} \\ g^1 = g^1 \text{ on } \partial\mathcal{V}_{n_j}. \end{cases}$$

Therefore, by our comparison principle (Proposition 1), $g^1 \leq u^{n_j}$ on $\bar{\mathcal{V}}_{n_j}$, which yields the desired contradiction. We have thus showed that,for all $j \in \mathbb{N}$, $\mathcal{V}_{n_j} = \emptyset$, i.e., $u^{n_j} \geq g^1$ on $\bar{\mathcal{Y}}$. Taking limits, as $(u^{n_j})_{j\in\mathbb{N}}$ converges uniformly to u on $\bar{\mathcal{Y}}$, $u \geq g^1$ on $\mathcal{Y}$.

We have shown so far that $u = b$ on $\partial\mathcal{Y}$ and that

$$\max\left\{u_t + F\left(t, x, u, u_x, u_{xx}\right), \, g^1 - u\right\} \le 0 \text{ a.e. on } \mathcal{Y},$$

i.e., that u is an L^p-supersolution of (1), with $u = b$ on $\partial\mathcal{Y}$. To conclude, there remains to show that

$$\max\left\{u_t + F\left(t, x, u, u_x, u_{xx}\right), \, g^1 - u\right\} \ge 0 \text{ a.e. on } \mathcal{Y}.$$

We do so by showing that, on the open set $\mathcal{U} = \left\{(t,x) \in \mathcal{Y} \,:\, u(t,x) > g^1(t,x)\right\}$, we have

$$u_t + F\left(t, x, u, u_x, u_{xx}\right) \ge 0 \text{ a.e..}$$

By definition of Φ_ϵ, we have $\Phi_{\epsilon_{n_j}}\left(u^{n_j}(t,x) - g(t,x)\right) \to 1$ pointwise a.e. on $\mathcal{U}$ as $j \to \infty$. Therefore, for all $(t_0, x_0) \in \mathcal{U}$ and $D > 0$ such that $C_D(t_0, x_0) \subset \mathcal{U}$, Theorem 2 implies that u is an L^p-subsolution of

$$u_t + F\left(t, x, u, u_x, u_{xx}\right) = 0 \text{ on } C_D(t_0, x_0).$$

Since $(t_0, x_0) \in \mathcal{U}$ was arbitrary, we obtain

$$\max\left\{u_t + F\left(t, x, u, u_x, u_{xx}\right), \, g^1 - u\right\} \ge 0 \text{ a.e. on } \mathcal{U}.$$

Since $g^1 \ge u$ on the complement of $\mathcal{U}$, we condlude that

$$\max\left\{u_t + F\left(t, x, u, u_x, u_{xx}\right), \, g^1 - u\right\} \ge 0 \text{ a.e. on } \mathcal{Y}.$$

and, hence, that u is an L^p-solution of (1) with $I = 1$. This concludes the base case of our induction argument.

Next, suppose that $\mathbf{P}(\mathbf{I})$ holds for some $I \in \mathbb{N}$. We will show that it also holds for $I + 1$. Let $g^{I+1} \in W^{1,2,p}\left(\mathcal{Y}\right)$ with $g^{I+1} \le b$ on $\partial\mathcal{Y}$. Define $h^{I+1} = -g_t^{I+1} - F\left(t, x, g^{I+1}, g_x^{I+1}, g_{xx}^{I+1}\right)$; and consider the following sequence of auxiliary problems. For all $n \in \mathbb{N}$,

$$\begin{cases} \max\left\{u_t^n + F\left(t, x, u^n, u_x^n, u_{xx}^n\right) - h^{I+1}(t,x)^+ \Phi_{\epsilon_n}\left(u^n - g^{I+1}\right) + h^{I+1}(t,x)^+, \right. \\ \qquad\qquad\qquad\qquad \left. \max_{i \in \{1,\ldots,I\}} g^i - u^n\right\} = 0 \text{ on } \mathcal{Y}, \\ u^n = b \text{ on } \partial\mathcal{Y}. \end{cases} \tag{D^{I+1}}$$

By our induction hypothesis, for all $n \in \mathbb{N}$, (D^{I+1}) has an L^p-solution $u^n \in W^{1,2,p}\left(\mathcal{Y}\right)$, with

$$\|u^n\|_{W^{1,2,p}(\mathcal{Y})} \le C\left(1 + \max_{i\in\{1,\ldots,I\}} \|g\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})} + \left\|g^{I+1}\right\|_{W^{1,2,p}(\mathcal{Y})}\right),$$

where $C = C\left(d, p, \lambda, \Lambda, diam(\mathcal{X}), T, \|\partial \mathcal{X}\|_{\mathcal{C}^{1,Lip}}\right) \in \mathbb{R}_+$ is independent of ϵ_n, since

$$\left\|h^{I+1}\right\|_{L^p(\mathcal{Y})} \leq C^h \left(1 + \left\|g^{I+1}\right\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})}\right).$$

Proceeding as in the base case $I = 1$, we see that the sequence $(u^n)_{n\in\mathbb{N}}$ has a subsequence that converges weakly in $W^{1,2,p}(\mathcal{Y})$ and strongly in $C^0\left(\bar{\mathcal{Y}}\right)$ to some $u \in W^{1,2,p}(\mathcal{Y})$. Moreover, following the same steps as above, one can show that u is an L^p-solution of (1) with $I+1$ obstacles.

To conclude, there only remains to show that the bound (6) on the $W^{1,2,p}$-norm of u holds. To this end, we first show that $u_t = g_t$, $u_x = g_x$, and $u_{xx} = g_{xx}$ a.e. on $\{u = g\}$. To see this,[21] consider $\mathcal{Z} = \{(t,x) \in \mathcal{Y} : u(t,x) = g(t,x)\}$. By Lebesgue's theorem, for almost every $(t,x) \in \mathcal{Z}$,

$$\lim_{\rho \to 0^+} \frac{B_\rho(t,x) \cap Z}{B_\rho(t,x)} = 1. \tag{8}$$

Moreover, $u - g \in W^{1,2,p}_{loc}(\mathcal{Y})$. Proposition A.1 in Crandall et al. (1998) then implies that, for almost every $(t,x) \in \mathcal{Z}$ and, hence, for a.e. (t,x) such that (8) holds, as $(t',x') \to (t,x)$,

$$(u-g)(t',x') = (u-g)_t(t,x)(\tilde{t} - t) + (u-g)_x \cdot (\tilde{x} - x) + o\left(\left|(\tilde{t},\tilde{x}) - (t,x)\right|\right). \tag{9}$$

Assume that the vector $((u-g)_t(t,x), (u-g)_x')' = a \neq 0$ and set

$$S = \left\{(v^t, v^x) \in \partial B_1(0,0) \,:\, a \cdot v \geq \frac{1}{2}|a|\right\}.$$

For each $v \in S$ and $k > 0$, set $(\tilde{t},\tilde{x}) = (t,x) + kv$ in (9). Then

$$\begin{aligned}(u-g)(t + kv^t, x + kv^x) &= akv + o(|kv|) \\ &\geq \frac{1}{2}k|a| + o(k).\end{aligned}$$

Therefore, there exists $k_0 > 0$ such that

$$(u-g)(t + kv^t, x + kv^x) > 0, \qquad \forall\ 0 < k < k_0 \text{ and } v \in S,$$

a contradiction to (8). This shows that $a = 0$, and $u_t(t,x) = g_t(t,x)$ and $u_x(t,x) = g_x(t,x)$ a.e. on $\mathcal{Z}$. By a similar argument, $u_{xx} = g_{xx}$ a.e. on $\{(t,x) \in \mathcal{Y} : u_x(t,x) = g_x(t,x)\}$, hence, a.e. on $\{(t,x) \in \mathcal{Y} : u(t,x) = g(t,x)\}$.

Therefore, $u \in W^{1,2,p}(\mathcal{Y})$ solves

$$u_t + F(t,x,u,u_x,u_{xx}) = \mathbb{1}_{\left\{u > \max\limits_{i=1,\ldots,I+1} g^i(t,x)\right\}} h^{\iota(t,x)}(t,x) - h^{\iota(t,x)}(t,x) \tag{10}$$

[21] A similar argument appears in the proof of Corollary 3.1.2.1 in Evans (2018).

where $h^i(t,x) = -g^i_t - F\left(t,x,g^i,g^i_x,g^i_{xx}\right)$ and $\iota(t,x)$ is a measurable selection from $\arg\max_{i\in\{1,\dots I+1\}} g^i(t,x)$. The result then follows from Theorem 15.1.3 in Krylov (2018) (again using Lemma 12.1.9 in Krylov (2018) to bound $\|u\|_{L^\infty(\mathcal{Y})}$).

By induction, the result holds for all $I \in \mathbb{N}$. This concludes the proof of Lemma 2. ■

Applying Theorem 12.1.7 and Lemma 12.1.9 in Krylov (2018) to (10), we also obtain the following interior $W^{1,2,p}$-estimates and L^∞ bound.

COROLLARY 3 *Suppose that Assumptions 1', and 2–7 hold for some $p \in (d+2,\infty)$ and that $|A| < \infty$. If u is an L^p-solution of* (1)*, then, for all compact subsets $\mathcal{Y}'$ of $\mathcal{Y}$,*

$$\|u\|_{W^{1,2,p}(\mathcal{Y}')} \leq C\left(1 + \sup_{a\in A}\|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})} + \frac{1}{dist(\mathcal{Y}',\mathcal{Y})}\|u\|_{L^\infty(\mathcal{Y})}\right).$$

where $C = C\left(d,p,\lambda,\Lambda,R,diam\left(\mathcal{X}\right),T\right) \in \mathbb{R}_+$.

Moreover, there exists $C^\infty = C^\infty\left(d,p,\lambda,\Lambda,R,T,diam\left(\mathcal{X}\right)\right) \in \mathbb{R}_+$ such that

$$\|u\|_{L^\infty(\mathcal{Y})} \leq C^\infty\left(1 + \|G\|_{L^p(\mathcal{Y})} + \sup_{a\in A}\|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{L^\infty(\mathcal{Y})}\right).$$

6 Proof of Theorem 1

We first establish our main result for the special case of smooth domains and regular boundary functions b (Section 6.1) and then prove it for the general case (Section 6.2).

6.1 Smooth Domains and Regular Boundary Functions

LEMMA 3 *Suppose that Assumptions 1', 2', and 3–7 hold for some $p \in (d+2,\infty)$. Then,* (1) *has an L^p-solution u.*

Moreover, $u \in W^{1,2,p}\left(\mathcal{Y}\right)$ and there exists $C = C\left(d,p,\lambda,\Lambda,R,diam(\mathcal{X}),T,\|\partial\mathcal{X}\|_{\mathcal{C}^{1,Lip}}\right) \in \mathbb{R}_+$ such that

$$\|u\|_{W^{1,2,p}(\mathcal{Y})} \leq C\left(1 + \sup_{a\in A}\|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})}\right). \tag{11}$$

If, in addition, Assumption 8 holds, then u is the unique L^p-solution of (1).

The proof of this result builds on Lemma 2. We approximate the obstacle $g = \sup_{a\in A} g^a$ by a sequence of obstacles that satisfy the Assumptions of Lemma 2. We then invoke the stability result for obstacle problems derived in Section 4.2 (Theorem 2) to conclude.

Proof. Uniqueness follows from Corollary 2. So, we only need to show existence.

Since A is separable and and $g^{\cdot}$ is continuous in a, there exists a countable dense subset $A^0 \subseteq A$ such that $\sup_{a\in A} g^a = \sup_{a\in A^0} g^a$ on $\mathcal{Y}$. Moreover, there exists a sequence $\left(A^{0,n}\right)_{n\in\mathbb{N}}$ of finite subsets of A^0 such that $\sup_{a\in A^{0,n}} g^a$ converges pointwise from below to $\sup_{a\in A^0} g^a$.

For all $n \in \mathbb{N}$, Lemma 2 guarantees that there exists a solution $u^n \in W^{1,2,p}(\mathcal{Y})$ of

$$\begin{cases} \max\left\{u_t + F\left(t,x,u,u_x,u_{xx}\right), \sup_{a\in A^{0,n}} g^a - u\right\} = 0 \text{ on } \mathcal{Y}, \\ u = b \text{ on } \partial\mathcal{Y}. \end{cases}$$

Moreover, for all $n \in \mathbb{N}$,

$$\|u^n\|_{W^{1,2,p}(\mathcal{Y})} \leq C\left(1 + \sup_{a\in A}\|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})}\right).$$

Since $W^{1,2,p}(\mathcal{Y})$ is separable and reflexive, its closed bounded subsets are weakly sequentially compact by Theorem 1.32 in Demengel et al. (2012). Moreover $W^{1,2,p}(\mathcal{Y})$ is compactly embedded in $\mathcal{C}^0\left(\bar{\mathcal{Y}}\right)$ by the Rellich-Kondrachov theorem (Theorem 2.84 in Demengel et al. (2012)). Therefore, there exists a function $u \in W^{1,2,p}(\mathcal{Y}) \cap \mathcal{C}^0\left(\bar{\mathcal{Y}}\right)$ and a subsequence $\left(u^{n_j}\right)_{j\in\mathbb{N}} \subseteq \left(u^n\right)_{n\in\mathbb{N}}$ such that

$$\begin{cases} u^{n_j} \rightharpoonup u \text{ in } W^{1,2,p}(\mathcal{Y}), \\ u^{n_j} \to u \text{ in } \mathcal{C}^0\left(\bar{\mathcal{Y}}\right), \end{cases}$$

as $j \to \infty$. Furthermore, u satisfies the estimate (11):

$$\begin{aligned} \|u\|_{W^{1,2,p}(\mathcal{Y})} &\leq \liminf_{j\to\infty} \|u^{n_j}\|_{W^{1,2,p}(\mathcal{Y})} \\ &\leq C\left(1 + \sup_{a\in A}\|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})}\right). \end{aligned}$$

To conclude, there only remains to show that u solves (1), which follows from Theorem 2. ■

From Corollary 3 and the proof of Lemma 3, we obtain the following interior $W^{1,2,p}$-estimates and L^∞ bound.

Corollary 4 *Suppose that Assumptions 1', and 2–8 hold for some $p \in (d+2, \infty)$. If u is an L^p-solution of (1), then, for all compact subset $\mathcal{Y}'$ of $\mathcal{Y}$,*

$$\|u\|_{W^{1,2,p}(\mathcal{Y}')} \leq C\left(1 + \sup_{a\in A}\|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})} + \frac{1}{dist(\mathcal{Y}',\mathcal{Y})}\|u\|_{L^\infty(\mathcal{Y})}\right).$$

where $C = C\left(d,p,\lambda,\Lambda,R,diam\left(\mathcal{X}\right),T\right) \in \mathbb{R}_+$.

Moreover, there exists $C^\infty = C^\infty\left(d,p,\lambda,\Lambda,R,T,diam\left(\mathcal{X}\right)\right) \in \mathbb{R}_+$ such that

$$\|u\|_{L^\infty(\mathcal{Y})} \leq C^\infty\left(1 + \|G\|_{L^p(\mathcal{Y})} + \sup_{a\in A}\|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{L^\infty(\mathcal{Y})}\right).$$

Proof. By Corollary 2, u is the unique solution of (1). Following the proof of Lemma 3, we note that u is the limit of a sequence of $W^{1,2,p}$ functions, $(u^n)_{n\in\mathbb{N}}$, such that, for all $n\in\mathbb{N}$, u^n solves

$$\begin{cases} \max\left\{u^n_t + F\left(t,x,u^n,u^n_x,u^n_{xx}\right),\ \sup_{a\in A^n} g^a - u\right\} = 0 \text{ on } \mathcal{Y}, \\ u^n = b^n \text{ on } \partial\mathcal{Y}, \end{cases}$$

where A^n is finite and the $W^{1,2,p}$ functions b^n, $n\in\mathbb{N}$, converge to b. Corollary 3 then implies that, for all $n\in\mathbb{N}$,

$$\|u^n\|_{L^\infty(\mathcal{Y})} \le C^\infty\left(1 + \|G\|_{L^p(\mathcal{Y})} + \sup_{a\in A}\|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{L^\infty(\mathcal{Y})}\right)$$

for some $C^\infty = C^\infty\left(d,p,\lambda,\Lambda,R,T,diam\left(\mathcal{X}\right)\right)\in\mathbb{R}_+$. Moreover, for all $n\in\mathbb{N}$ and all compact subset $\mathcal{Y}'$ of $\mathcal{Y}$,

$$\|u^n\|_{W^{1,2,p}(\mathcal{Y}')} \le C\left(1 + \sup_{a\in A}\|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|G\|_{L^p(\mathcal{Y})} + \frac{1}{dist(\mathcal{Y}',\mathcal{Y})}\|u\|_{L^\infty(\mathcal{Y})}\right),$$

for some $C = C\left(d,p,\lambda,\Lambda,R,diam\left(\mathcal{X}\right),T\right)\in\mathbb{R}_+$.

The desired results then follow, since

$$\|u\|_{L^\infty(\mathcal{Y})} \le \liminf_{n\to\infty}\|u^n\|_{L^\infty(\mathcal{Y})} \text{ and } \|u\|_{W^{1,2,p}(\mathcal{Y}')} \le \liminf_{n\to\infty}\|u^n\|_{W^{1,2,p}(\mathcal{Y}')}.$$

6.2 Proof of Theorem 1

Lemma 3 guarantees that the obstacle problem has an L^p-solution when $\mathcal{X}$ and b satisfy Assumption 1' and 2'. To generalize the result to the weaker Assumptions 1 and 2, we study a sequence of equations, each satisfying the assumptions of Lemma 3, that converges to the equation (1). In particular, we approximate (i) $\mathcal{X}$ by a sequence of smooth domains whose cone parameters are uniformly controlled, and (ii) b by a sequence of equicontinuous functions in $W^{1,2,p}\left(\mathcal{Y}\right)$. The L^p solutions of the equations in the approximating sequence form an equicontinuous (by Lemma 4, below) and weakly compact (in $W^{1,2,p}_{loc}\left(\mathcal{Y}\right)$) family. The Arzelà-Ascoli theorem then guarantees that a subsequence converges uniformly on the compact subset of $\bar{\mathcal{Y}}$ to some function in $W^{1,2,p}_{loc}\left(\mathcal{Y}\right)\cap\mathcal{C}^0\left(\bar{\mathcal{Y}}\right)$. Finally, we invoke the stability result for obstacle problems (Theorem 2) derived in Section 4.2 to conclude.

Lemma 4 *Suppose that Assumptions 1', 2', and 3–8 hold for some $p\in(d+2,\infty)$. Let u be an L^p-solution of (1). For all $D>0$, there exists a family of modulus of continuity $\bar{\omega}_D$ such that, for all $(t,x),(t',x')\in\bar{\mathcal{Y}}\cap C_D(0,0)$,*

$$\left|u(t,x)-u(t',x')\right| \le \bar{\omega}_D\left(\left|(t,x)-(t',x')\right|\right), \tag{12}$$

where $\bar{\omega}_D$ *depends only on* $p, d, \lambda, \Lambda, R, D$, $\sup_{a\in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y}\cap C_{D+1}(0,0))}$, $\|G\|_{L^p(\mathcal{Y}\cap C_{D+1}(0,0))}$, $\|u\|_{L^\infty(\mathcal{Y}\cap C_{D+1}(0,0))}$, *the modulus of continuity of* b *on* $\partial\mathcal{Y}\cap C_{D+1}(0,0)$, *and the parameters of the cone condition of* $\mathcal{X}$ *on* $\mathcal{X}\cap B_{D+1}(0)$.

Remark 5 *The modulus of continuity* $\bar{\omega}_D$ *for the solution of* (1) *on* $\bar{\mathcal{Y}}\cap C_D(0,0)$ *given in Lemma 4 only depends on the characteristics of the primitive inside the cylinder* $C_{D+1}(0,0)$. *It is independent of* $\sup_{a\in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y}\setminus C_{D+1}(0,0))}$, $\|G\|_{L^p(\mathcal{Y}\setminus C_{D+1}(0,0))}$, $\|u\|_{L^\infty(\mathcal{Y}\setminus C_{D+1}(0,0))}$, *the modulus of continuity of* b *on* $\partial\mathcal{Y}\setminus C_{D+1}(0,0)$, *and of the parameters of the cone condition of* $\mathcal{X}$ *on* $\mathcal{X}\setminus B_{D+1}(0)$.

Proof. By the structure condition (**SC**), for any L^p-solution u of (1) and almost all $(t,x)\in \mathcal{Y}\cap C_{D+1}(0,0)$, we have

$$\begin{aligned}&\max\{u_t + F(t,x,u,u_x,u_{xx}),\, g-u\}\\ &\quad \le u_t + \mathcal{P}^+_{\lambda,\Lambda}(u_{xx}) + R\,|u_x| + \bar{C}\left(1 + \|u\|_{L^\infty(\mathcal{Y}\cap C_{D+1}(0,0))} + \sup_{a\in A}(|g^a| + |g^a_t| + |g^a_x| + |g^a_{xx}|) + G(t,x)\right),\end{aligned}$$

for some $\bar{C}>0$ depending only on $p, d, \lambda, \Lambda, R$, and D. Similarly,

$$\begin{aligned}&\max\{u_t + F(t,x,u,u_x,u_{xx}),\, g-u\}\\ &\quad \ge u_t + \mathcal{P}^-_{\lambda,\Lambda}(u_{xx}) - R\,|u_x| - \bar{C}\left(1 + \|u\|_{L^\infty(\mathcal{Y})} + \sup_{a\in A}(|g^a| + |g^a_t| + |g^a_x| + |g^a_{xx}|) + G(t,x)\right).\end{aligned}$$

Define $\bar{u}$ on $\partial(\mathcal{Y}\cap C_{D+1}(0,0))$ by

$$\bar{u}(t,x) = \begin{cases} b(t,x) \text{ if } (t,x)\in(\partial\mathcal{Y})\cap C_D(0,0),\\ dist(\{(t,x)\}, C_{D+1})\, b(t,x)\\ \qquad + (1 - dist(\{(t,x)\}, C_{D+1}))\, \|u|_{L^\infty(\mathcal{Y}\cap C_{D+1}(0,0))} \text{ if } (t,x)\in\partial\mathcal{Y}\cap(C_{D+1}\setminus C_D(0,0))\\ \|u|_{L^\infty(\mathcal{Y}\cap C_{D+1}(0,0))} \text{ if } (t,x)\in\partial(\mathcal{Y}\cap C_{D+1}(0,0))\setminus\partial\mathcal{Y}.\end{cases}$$

Similarly, define $\underline{u}$ on $\partial(\mathcal{Y}\cap C_{D+1}(0,0))$ by

$$\underline{u}(t,x) = \begin{cases} b(t,x) \text{ if } (t,x)\in(\partial\mathcal{Y})\cap C_D(0,0),\\ dist(\{(t,x)\}, C_{D+1})\, b(t,x)\\ \qquad - (1 - dist(\{(t,x)\}, C_{D+1}))\, \|u|_{L^\infty(\mathcal{Y}\cap C_{D+1}(0,0))} \text{ if } (t,x)\in\partial\mathcal{Y}\cap(C_{D+1}\setminus C_D(0,0))\\ -\|u|_{L^\infty(\mathcal{Y}\cap C_{D+1}(0,0))} \text{ if } (t,x)\in\partial(\mathcal{Y}\cap C_{D+1}(0,0))\setminus\partial\mathcal{Y}.\end{cases}$$

We observe that the moduli of continuity of $\bar{u}$ and $\underline{u}$ are controlled by $\|u\|_{L^\infty(\mathcal{Y}\cap C_{D+1}(0,0))}$ and the modulus of continuity of b on $\partial Y\cap C_{D+1}(0,0)$, and that $\bar{u}\ge u\ge \underline{u}$ on $\partial(\mathcal{Y}\cap C_{D+1}(0,0))$.

It follows that u is also an L^p-subsolution of

$$\begin{cases} v_t + \mathcal{P}^+_{\lambda,\Lambda}(v_{xx}) + R|v_x| \\ \quad = -\bar{C}\left(1 + \|u\|_{L^\infty(\mathcal{Y}\cap C_{D+1}(0,0))} + \sup_{a\in A}(|g^a| + |g^a_t| + |g^a_x| + |g^a_{xx}|) + G(t,x)\right) \text{ on } \mathcal{Y}\cap C_{D+1}(0,0), \\ v = \bar{u} \text{ on } \partial\mathcal{Y}\cap C_{D+1}(0,0), \end{cases} \tag{13}$$

and an L^p-supersolution of

$$\begin{cases} v_t + \mathcal{P}^-_{\lambda,\Lambda}(v_{xx}) - R|v_x| \\ \quad = \bar{C}\left(1 + \|u\|_{L^\infty(\mathcal{Y}\cap C_{D+1}(0,0))} + \sup_{a\in A}(|g^a| + |g^a_t| + |g^a_x| + |g^a_{xx}|) + G(t,x)\right) \text{ on } \mathcal{Y}\cap C_{D+1}(0,0), \\ v = \underline{u} \text{ on } \partial\mathcal{Y}\cap C_{D+1}(0,0). \end{cases} \tag{14}$$

By Theorem 4.5 in Crandall et al. (1999), there exists a viscosity solution $\bar{U}$ of (13) and a viscosity solution $\underline{U}$ of (14) whose moduli of continuity depend only on $p, d, \lambda, \Lambda, R, D, \sup_{a\in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y}\cap C_{D+1}(0,0))}$, $\|G\|_{L^p(\mathcal{Y}\cap C_{D+1}(0,0))}, \|u\|_{L^\infty(\mathcal{Y}\cap C_{D+1}(0,0))}$, the modulus of continuity of b on $\mathcal{Y}\cap C_{D+1}(0,0)$, and the parameters of the cone condition of $\mathcal{X}$ on $\mathcal{X}\cap B_{D+1}(0)$. By our comparison principle (Proposition 1), since $\underline{u} \le u \le \bar{u}$ on $\partial\mathcal{Y}\cap C_D(0,0)$,

$$\underline{U} \le u \le \bar{U} \text{ on } \bar{\mathcal{Y}}\cap C_{D+1}(0,0).$$

Therefore, for all $(t,x) \in \bar{\mathcal{Y}}\cap C_D(0,0)$ and all $(\bar{t},\bar{x})\,\partial\mathcal{Y}\cap C_D(0,0)$,

$$|u(t,x) - b(\bar{t},\bar{x})| \le \max\left\{\left|\bar{U}(t,x) - b(\bar{t},\bar{x})\right|, \left|\underline{U}(t,x) - b(\bar{t},\bar{x})\right|\right\} \le \bar{\omega}_{boundary}(|(t,x) - (\bar{t},\bar{x})|) \tag{15}$$

where $\bar{\omega}_{boundary}$ is a modulus of continuity depending only on $p, d, \lambda, \Lambda, R, D, \sup_{a\in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y}\cap C_{D+1}(0,0))}$, $\|G\|_{L^p(\mathcal{Y}\cap C_{D+1}(0,0))}, \|u\|_{L^\infty(\partial\mathcal{Y}\cap C_{D+1}(0,0))}$, the modulus of continuity of b on $\partial\mathcal{Y}\cap C_{D+1}(0,0)$, and the parameters of the cone condition of $\mathcal{X}$ on $\mathcal{X}\cap B_{D+1}(0)$.

Finally, by Corollary 4 and a Morrey-Sobolev embedding theorem (Theorem 2.84 in Demengel et al. (2012)), u is α-Hölder continuous on any compact subset $\mathcal{Y}'$ of $\mathcal{Y}\cap C_{D+1}(0,0)$ for all $\alpha < 1-(d+1/p)$, with Hölder norm bounded above by

$$C^H\left(1 + \sup_{a\in A}\|g^a\|_{W^{1,2,p}(\mathcal{Y}\cap C_{D+1}(0,0))} + \|u\|_{L^\infty(\mathcal{Y}\cap C_{D+1}(0,0))} + \|G\|_{L^p(\mathcal{Y}\cap C_{D+1}(0,0))}\right)$$

where C^H depends only, as above, on $p, d, \lambda, \Lambda, R, D$, and on the distance of the compact set to the boundary $dist(\mathcal{Y}', \partial\mathcal{Y})$.

Combining these two estimates, we obtain the desired modulus of continuity $\bar{\omega}_D$. ■

Proof of Theorem 1.

Uniqueness follows from Corollary 2. The remainder of the proof shows existence and establishes the interior estimates (2).

By Theorem 5.1 in Doktor (1976), there exists a strictly increasing (in the sense of set inclusion) sequence of bounded smooth open subsets of $\mathcal{X}$, $(\mathcal{X}_n)_{n\in\mathbb{N}}$, and a constant $M > 0$ such that, for all $n \in \mathbb{N}$,

$$\max\{\|\partial\mathcal{X}\|_{\mathcal{C}^{0,Lip}}, \|\partial\mathcal{X}_n\|_{\mathcal{C}^{0,Lip}}\} \le M.$$

By Theorem 1.2.2.2 in Grisvard (2011), $\mathcal{X}$ and $\mathcal{X}_n$, $n \in \mathbb{N}$ satisfy a uniform exterior cone condition of size (L, θ), for some $L, \theta > 0$ independent of n. In particular (L, θ) depends only on M and $diam(\mathcal{X})$.

Define $\mathcal{Y}_n = \left[0, \frac{2n-1}{2n}T\right) \times \mathcal{X}_n$, and consider the following sequence of nonlinear obstacle problems: for all $n \in \mathbb{N}$,

$$\begin{cases} \max\{u_t + F(t, x, u, u_x, u_{xx}), g - u\} = 0 \text{ on } \mathcal{Y}_n \\ u = b^n \text{ on } \partial\mathcal{Y}_n, \end{cases} \tag{Eq$_n$}$$

where $b_n \in W^{1,2,p}(\mathcal{Y})$ for all $n \in \mathbb{N}$, and $b^n \to b \in \mathcal{C}^0(\bar{Y})$.

By Lemma 3, for all $n \in \mathbb{N}$, there exists a unique L^p-solution $u^n \in W^{1,2,p}(\mathcal{Y}_n) \subset W^{1,2,p}_{loc}(\mathcal{Y}_n) \cap \mathcal{C}^0(\bar{\mathcal{Y}}_n)$ of (Eq$_n$). Extending the functions u^n, $n \in \mathbb{N}$, by letting $u^n(t,x) = b(t,x)$ on $\bar{\mathcal{Y}} \setminus \mathcal{Y}_n$, we obtain a sequence $(u^n)_{n\in\mathbb{N}} \subset W^{1,2,p}(\mathcal{Y}) \subset W^{1,2,p}_{loc}(\mathcal{Y}) \cap \mathcal{C}^0(\bar{\mathcal{Y}})$.

Next we show that the sequence $(u^n)_{n\in\mathbb{N}}$ has a subsequence that converges (i) uniformly in $\mathcal{C}^0(\bar{\mathcal{Y}})$ and (ii) weakly in $W^{1,2,p}_{loc}(\mathcal{Y})$ to some function $u \in W^{1,2,p}_{loc}(\mathcal{Y}) \cap \mathcal{C}^0(\bar{\mathcal{Y}})$.

We start with (i). By Lemma 4 with $D > T \vee diam(\mathcal{X})$ (using Lemma 12.1.9 in Krylov (2018) to uniformly control $\|u^n\|_{L^\infty(\mathcal{Y})}$, $n \in \mathbb{N}$), the sequence $(u^n)_{n\in\mathbb{N}}$ is equicontinuous. As a result, it has a convergent subsequence in $\mathcal{C}^0(\bar{\mathcal{Y}})$ by the Arzelà-Ascoli theorem (Theorem A.5 in Rudin (1973)), which, with a small abuse of notation, we relabel $(u^n)_{n\in\mathbb{N}}$.

Next, we show (ii), i.e., that the subsequence above, which converges in $\mathcal{C}^0(\bar{\mathcal{Y}})$, has a weakly convergent subsequence in $W^{1,2,p}_{loc}(\mathcal{Y})$. To do so, we prove that the $W^{1,2,p}$-norms of the functions u^n, $n \in \mathbb{N}$, restricted to any given compact subset of $\mathcal{Y}$, are eventually uniformly bounded.

It is easy to see that, for all $n \ge 3$, $u^n \in W^{1,2,p}_{loc}(\mathcal{Y}_n) \cap \mathcal{C}^0(\bar{\mathcal{Y}}_n)$ is also an L^p-solution of

$$\begin{cases} \max\{u_t + F(t, x, u, u_x, u_{xx}), g - u\} = 0 \text{ on } \mathcal{Y}_{n'} \\ u(t,x) = u^n(t,x) \text{ on } \partial\mathcal{Y}_{n'}, \end{cases}$$

for all $n' \leq n-2$. So, by Corollary 4, for all $n \geq 3$ and all $n' \leq n-2$,

$$\|u^n\|_{W^{1,2,p}(\mathcal{Y}_{n'})} \leq C^{n'} \left(1 + \|G\|_{L^p(\mathcal{Y})} + \sup_{a\in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{L^\infty(\mathcal{Y})}\right), \tag{16}$$

for some $C^{n'} = C^{n'}(d, p, \lambda, \Lambda, R, diam(\mathcal{X}), T, dist(\mathcal{Y}_{n'}, \mathcal{Y}_{n'+1})) \in \mathbb{R}_+$. Crucially, the right-hand side of (16) is independent of n.

We then show that there exists a subsequence of $(u^n)_{n\in\mathbb{N}}$ that converges to the limit $u \in \mathcal{C}^0(\bar{\mathcal{Y}})$ obtained in (i) by a diagonal argument, and, therefore, that $u \in W^{1,2,p}_{loc}(\mathcal{Y}) \cap \mathcal{C}^0(\bar{\mathcal{Y}})$. For all $n \geq 3$,(16) implies that

$$\|u^n\|_{W^{1,2,p}(\mathcal{Y}_1)} \leq C^1 \left(1 + \|G\|_{L^p(\mathcal{Y})} + \sup_{a\in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{L^\infty(\mathcal{Y})}\right).$$

$W^{1,2,p}(\mathcal{Y}_1)$ is separable and reflexive, and, hence, its closed bounded subsets are weakly sequentially compact by Theorem 1.32 in Demengel et al. (2012). Moreover $W^{1,2,p}(\mathcal{Y}_1)$ is compactly embedded in $\mathcal{C}^0(\mathcal{Y}_1)$ by the Rellich-Kondrachov theorem (Theorem 2.84 in Demengel et al. (2012)). Therefore, there exists a function $\tilde{u}$ defined on $\bar{\mathcal{Y}}_T$ which restriction on $\mathcal{Y}_1$ is continuous, and a subsequence $(u^{n_j})_{j\in\mathbb{N}} \subseteq (u^n)_{n\in\mathbb{N}}$ such that

$$\begin{cases} u^{n_j} \rightharpoonup \tilde{u} \text{ in } W^{1,2,p}(\mathcal{Y}_1) \\ u^{n_j} \to \tilde{u} \text{ in } \mathcal{C}^0(\bar{\mathcal{Y}}_1), \end{cases}$$

as $j \to \infty$. In particular, $\tilde{u}$ must coincide with u on $\mathcal{Y}_1$.

Proceeding with a diagonal argument, we see that there exists a subsequence $(u^{n_j})_{j\in\mathbb{N}} \subseteq (u_n)_{n\in\mathbb{N}}$ such that

$$\begin{cases} u^{n_j} \rightharpoonup u \text{ in } W^{1,2,p}_{loc}(\mathcal{Y}) \\ u^{n_j} \to u \text{ in } \mathcal{C}^0(\bar{\mathcal{Y}}), \end{cases}$$

as $j \to \infty$. Moreover, u satisfies the interior estimates (2) since, for all $n \in \mathbb{N}$,

$$\begin{aligned} \|u\|_{W^{1,2,p}(\mathcal{Y}_n)} &\leq \liminf_{j\to\infty} \|u^{n_j}\|_{W^{1,2,p}(\mathcal{Y}_n)} \\ &\leq C^n \left(1 + \|G\|_{L^p(\mathcal{Y})} + \sup_{a\in A} \|g^a\|_{W^{1,2,p}(\mathcal{Y})} + \|b\|_{L^\infty(\mathcal{Y})}\right). \end{aligned}$$

That is, $u \in W^{1,2,p}_{loc}(\mathcal{Y}) \cap \mathcal{C}^0(\bar{\mathcal{Y}})$ and u satisfies the interior $W^{1,2,p}$-estimates (2).

There remains to show that $u \in W^{1,2,p}_{loc}(\mathcal{Y}) \cap \mathcal{C}^0(\bar{\mathcal{Y}})$ is an L^p-solution of (1). By construction, $(u^{n_j})_{j\in\mathbb{N}} \subseteq W^{1,2,p}_{loc}(\mathcal{Y}_T) \cap \mathcal{C}^0(\bar{\mathcal{Y}}_T)$ converges to u weakly in $W^{1,2,p}_{loc}(\mathcal{Y})$ and uniformly on $\bar{\mathcal{Y}}$. The result then follows from Theorem 2. ■

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