Contract Negotiation and Screening with Persistent Information*

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1 Introduction

Whether it concerns financial bailouts, tax schemes, social security regulations, labor contracts, debt refinancing, or international agreements, renegotiation of social and corporate agreements is a pervasive aspect of economic activity. Modeling renegotiation is challenging, however, and many economists have focused their analysis on *renegotiation-proof contracts* which are, loosely speaking, contracts which cannot be Pareto improved following any history, and which are aimed to capture the main effects of renegotiation.[1]

While they offer a parsimonious approach to capture renegotiation, renegotiation-proof contracts raise conceptual issues, particularly when some parties hold private information. First, renegotiation-proofness is generally not an intrinsic property of contracts.[2] Instead, it depends on the negotiating parties' beliefs and on the set of contracts which the parties view as credible alternatives to the current contract. The expression "renegotiation-proof contract" is thus misleading, because the qualification really concerns pairs of contracts and beliefs. To illustrate this point, suppose that a principal, P, has signed a contract with an agent whose type, L or H, is private, and that P is given an opportunity to renegotiate the contract. Also assume that the initial contract, C, is efficient conditional on facing L, but inefficient if P is facing H. In this situation, P should renegotiate the contract only if he assigns a high enough probability to facing H. In some cases

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[1] Strulovici (2017) provides an explicit foundation for such contracts for environments, like the one of this paper, in which one of the party has private information and the other party has the bargaining power.

[2] See, e.g., Hart and Tirole (1988) and, more recently, Maestri (2015).

(see, e.g., Hart and Tirole (1988)), the principal may prefer to abstain from completely learning the agent's type to avoid subsequent commitment problems, and some inefficient contracts can be renegotiation-proof.

These observations raise another issue with renegotiation-proofness, which may be best understood with three periods. Whether a contract is renegotiation-proof in the second period depends on P's belief in that period, as was just discussed. This belief, in turn, depends on the agent's strategy in period 1. For example, suppose that L and H both choose some contract C in period 1 with positive probability. Whether C's continuation in period 2 (and, hence, C itself) is renegotiation-proof depends on P's belief in period 2 which, by Bayesian updating, depends on the probabilities with which each type of the agent have chosen C. Therefore, whether C is renegotiation-proof depends not only on P's initial belief about the agent, but also on the agent's strategy in period 1, which itself depends on the continuation of C: *renegotiation-proofness is an equilibrium concept*, even when the horizon is finite.

Offsetting this complexity, renegotiation-proofness also implies a form of interim efficiency that can significantly simplify the concept. Indeed, because contracts that are efficient for some type of the agent are typically inefficient for other types, renegotiation generates a separating force. When this force is sufficiently strong, it can entirely resolve the issues mentioned earlier. Suppose, for example, that contract efficiency in the last period guarantees that the agent's types are always separated in that period. Viewed from the penultimate period, regardless of the agent's strategy and the principal's current belief, the agent's types will be separated in the next period even, for that matter, if his type is persistent. This property reduces the set of contracts that need to be considered in the penultimate period, and may be used to show that the efficient contracts among those must also be separating in that period. In many cases, analyzed in later sections, an induction argument may be used that renegotiation-proof must always be separating. Thus, by pushing towards efficiency at all periods, renegotiation-proofness can force the principal to learn the type of the agent. This is the *separating power of renegotiation*.

With more periods, characterizing the set of renegotiation-proof contracts can be particularly challenging. To be renegotiation-proof, a contract must be interim efficient, i.e., sustain the comparison with alternative contracts whose current-period specification and continuations may both differ from the current contract and, crucially, whose continuations have to be renegotiation-proof. When assessing the interim efficiency of a given contract, the principal must thus consider perturbations of the contract that preserve renegotiation-proofness of the continuation contract. To address this difficulty, the paper introduces a concept of RP-preserving perturbations. The paper will provide a general specification in which these perturbations constitute a group, in the algebraic sense of the term. If one can characterize this group through its multiplication, as in the several examples described here, then can exploit interim efficiency to show that RP contracts must be separating.

2 Setting: Contracting with Persistent Private Information

We consider a dynamic principal-agent contracting model between a risk neutral principal and a risk-averse agent. There are $T < \infty$ periods.[3] The private type θ_t of the agent lies in some type space $\Theta \subset \mathbb{R}$ and evolves according to some Markov process.

Given a stochastic consumption stream $\{y_t\}_{t \leq T}$, the agent receives an expected utility

$$U(y) = E\left[\sum_{t=1}^{T} u(\theta_t, y_t)\right]$$

where the period utility u is defined on $\Theta \times \mathcal{Y}$ and $\mathcal{Y}$ is an interval of $\mathbb{R}$. For expositional simplicity, the discount rate is set to zero for both parties.[4]

The agent receives his consumption stream from the principal, whose objective is to minimize the expected cost

$$Q(y) = E\left[\sum_{t=1}^{T} y_t\right],$$

subject to providing the agent with a lifetime expected utility greater than or equal to

$$U(y) \geq w.$$

The principal's marginal cost is thus assumed to be constant across periods and normalized to 1.[5]

A contract is defined recursively as follows: in period T, a contract is simply a transfer $y_T \in \mathcal{Y}$. Let $\mathcal{C}_T$ denote the set of such contracts. In period $t < T$, a contract consists of a transfer y_t for that period and of a finite menu $M_{t+1} \subset \mathcal{C}_{t+1}$ of continuation contracts at period $t+1$. The set of period-t contracts is denoted $\mathcal{C}_t$.

The timing of the game is as follows:

- In period 1, the principal proposes an initial menu M_1 of contracts in $\mathcal{C}_1$.
- The agent learns his type θ_1 and picks a contract $C_1 = (y_1, M_2)$ from M_1.
- The principal gives y_1 to the agent, and the game moves to period 2.

[3] Focusing on a finite horizon gets rid of conceptual difficulties that arise with the definition of renegotiation-proof contracts.

[4] The analysis easily extends to a common positive discount rate for the principal and the agent.

[5] The analysis can be extended to the case of arbitrary deterministic time-varying cost functions. Furthermore, Atkeson and Lucas (1992) observe that the general equilibrium price process obtained to decentralize the central planner's problem is typically constant for the simple parametric specifications used in the present paper.

- The agent learns his type θ_2 for that period, and picks an item $C_2 = (y_2, M_3)$ from M_2

- y_2 is paid, etc.

The distinction between the terms “menu” and “contract” is for expositional clarity. A “menu” really corresponds to a contract at the beginning of some period, whereas the term “contract” is reserved for end-of-period contracts, after the agent has made his choice.

For simplicity, both the agent and the principal use pure strategies. This restriction is without loss of generality for the examples considered in this paper.[6]

A menu M_1 is *feasible* if there is a choice strategy for the agent that yields an expected utility greater than or equal to the promised utility w, given the principal's prior over the agent's initial type θ_1.

A menu M_1 induces a sequence of choices by the agent. Let $H_t(M_1)$ denote the set of possible choices induced M_1 until period $t-1$, and denote by h_t a typical element (“history”) of $H_t(M_1)$. Regardless of the agent's types in those periods, a given history h_t implies a menu M_t in period t, as specified by M_1.

A menu M_1 is *separating* if there exists an optimal strategy for the agent such that i) for any period t and choice history h_t, each type θ_t chooses a distinct item from the menu M_t implied by h_t, resulting in a transfer y_t that depends on the agent's type.

That is, given the menu and the agent's best response, with probability one the payment that the agent receives in any period $t < T$ always depends on his current type.[7]

Let $\Delta(\Theta)$ denote the set of probability distributions over Θ. For any $t \leq T$ and contract $C \in \mathcal{C}_t$, let $Q_\delta(C)$ and $U_\theta(C)$ denote the expected cost and expected utility corresponding to C from period t to period T, when the principal's belief about θ_t is $\delta \in \Delta(\Theta)$ and the type of the agent is θ.

Because of persistence, the expected cost of a particular contract typically depends on the current type of the agent. Indeed, the agent's current type affect the distribution of his future type, which determines which contracts he will choose from future menus, and hence how costly a given menu will turn out to be for the principal. This point is developed in detail in later sections.

[6]In these examples, renegotiation-proof contracts are shown to be either fully separating or fully pooling. The definitions below easily extend to allow mixing by the agent. When the agent's utility is multiplicatively separable in θ and y, as in this paper, ruling out randomization by the principal is without loss because the principal can always reduce his cost by replacing lotteries by certainty equivalents without affecting incentives.

[7]This definition allows the agent to be indifferent between different items in the menu, but requires him to play a separating strategy in equilibrium. When indifferent between several strategies, the agent is assumed to choose the one that minimizes the principal's cost.

2.1 Renegotiation-Proof Contracts

We now define the concept of renegotiation-proof contracts. Whether a contract is renegotiation-proof generally depends on the principal's belief about the current type of the agent. It is intuitive, for example, that a contract that is efficient for some type θ but not for some other type $\hat{\theta}$ should be renegotiation-proof if the principal assigns probability 1 to θ, but not if he assigns probability 1 to $\hat{\theta}$. For each t, $\mathcal{R}_t \subset \mathcal{C}_t \times \Delta$ will denote the set of period-t contract-belief pairs (C, δ) such that C is renegotiation-proof given belief δ.

Because the horizon is finite, renegotiation-proof contracts can be defined by backward induction on t. In period T, all contracts $y_T \in \mathcal{Y}$ are declared renegotiation-proof, for any belief δ. Intuitively, there is nothing to renegotiate in period T, since any transfer y_T is Pareto efficient. Thus, $\mathcal{R}_T = \mathcal{C}_T \times \Delta$. Similarly, all menus $M \subset \mathcal{Y}$ in period T are declared renegotiation-proof for any belief δ.

If δ denotes the distribution of θ_t, let $\hat{\delta}$ denote the distribution of θ_{t+1}, obtained by the transition probabilities of the stochastic process $\{\theta_t\}_{t=1}^T$.

A period-t contract $C = (y_t, M_{t+1})$ is said to have an *RP-continuation given* δ if the menu M_{t+1} is renegotiation proof given $\hat{\delta}$.

The definition of renegotiation-proof contracts for periods $t < T$ involves two components. First, the continuation menu of a contract must be renegotiation-proof, given posterior beliefs. Second, the contract should not be improvable by a menu of contracts whose continuations are themselves renegotiation-proof.

DEFINITION 1 *For $t < T$, a period-t contract $C \in \mathcal{C}_t$ is* **renegotiation-proof** *given belief $\delta \in \Delta(\Theta)$ if there does not exist a partition $(\Theta_1, \ldots, \Theta_m)$ of $\Theta(\delta)$ and contracts $\{C_i \in \mathcal{C}_t\}_{i=1}^m$ such that*

i) Each $\theta \in \Theta_i$ weakly prefers C_i to C and to C_j for all j:

$$U_\theta(C_i) \geq \max\{U_\theta(C), \max_j\{U_\theta(C_j)\}\}.$$

ii) The principal's expected cost is reduced by this proposal:

$$\sum_{i=1}^{m} \delta(\Theta_i) Q_{\delta_i}(C_i) < Q_\delta(C), \tag{1}$$

where δ_i is the conditional distribution $\delta(\cdot|\Theta_i) = \delta(\cdot)/\delta(\Theta_i)$ defined over Θ_i.

iii) For each i, C_i has an RP continuation given δ_i.

To close this inductive definition of renegotiation-proofness, we need to define renegotiation-proof menus.

Definition 2 *A period-t menu* $M_t = \{C_i\}$ *is* **renegotiation-proof** *if there is an optimal choice strategy* σ^A *for the agent over* $\{t, \dots, T\}$ *given* M_t *such that the belief* δ_i *generated under that strategy by the choice* C_i *is such that* C_i *is renegotiation-proof given* δ_i*, for all* C_i*'s that are chosen with positive probability under* σ^A*.*[8]

While the formal definition of renegotiation-proofness is somewhat complicated, there are two situations in which a contract is easily shown *not* to be renegotiation-proof.

Definition 3 *Given a period* t*, contract* $C \in \mathcal{C}_t$*, and belief* δ*, say that* $\hat{C}$ *is a* **Pareto improvement** *over* C **given** δ*, if for each* $\theta \in \Theta(\delta)$*,*

$$U_\theta(\hat{C}) \geq U_\theta(C) \text{ and } Q_\delta(\hat{C}) < Q_\delta(C).$$

The following result immediately follows from the definition of renegotiation-proofness.

Proposition 1 *Given a period* t*, contract* $C \in \mathcal{C}_t$*, and belief* δ*, suppose that there exists* $\hat{C} \in \mathcal{C}_t$ *that is a Pareto improvement of* C *such that* $\hat{C}$ *is has an RP continuation given* δ*. Then* C *is not renegotiation-proof given* δ*.*

Proof. To contradict renegotiation-proofness of C, take the partition with a single element $\Theta_1 = \Theta(\delta)$ and associated contract $C_1 = \hat{C}$. Notice that the LHS of (1) corresponds to $Q_\delta(\hat{C})$. ■

Another important case is when there exist separating contracts, one for each type θ in $\Theta(\delta)$ that are incentive compatible and reduce the cost of the principal. Abusing notation, let $Q_\theta(C)$ denote the expected cost to the principal when the belief is degenerate, putting all mass on type θ.

Proposition 2 *Given a period* t*, contract* $C \in \mathcal{C}_t$*, and belief* δ*, suppose that there exist contracts* $\{C_\theta\}_{\theta \in \Theta(\delta)}$ *in* $\mathcal{C}_t$ *such that, for all* $\theta \in \Delta(\theta)$*,*

- $U_\theta(C_\theta) \geq \max\{U_\theta(C), U_\theta(C_{\theta'})\}$
- $Q_\theta(C_\theta) \leq Q_\theta(C)$
- C_θ *has renegotiation-proof continuations given* δ_θ

with at least one strict inequality among the cost inequalities. Then, C *is not renegotiation-proof given* δ*.*

Proof. Take the partition $\{\Theta_\theta = \{\theta\}\}_{\theta \in \Theta(\delta)}$. To show (1), notice that each term of the LHS is weakly smaller than the corresponding term entering the computation of Q_δ, with at least one strict inequality. ■

[8] This definition concerns only in-equilibrium choices.

Finally, a simple case in which a contract *is* renegotiation-proof is when the principal knows the type of the agent, and the contract is Pareto efficient. Precisely, given a type θ and period t. For any θ, let δ_θ denote the degenerate (Dirac) distribution on Θ such that $\delta_\theta(\theta) = 1$ and $\delta_\theta(\theta') = 0$ for all $\theta' \neq \theta$.

Say that $C \in \mathcal{C}_t$ is *θ-efficient* if it has an RP continuation given δ_θ and there there does not exist any contract $\hat{C} \in \mathcal{C}_t$ that has an RP continuation given δ_θ and is such that

- $U_\theta(\hat{C}) \geq U_\theta(C)$, and
- $Q_\theta(\hat{C}) < Q_\theta(C)$.

Proposition 3 *A period-t contract $C \in \mathcal{C}_t$ is renegotiation-proof given δ_θ if and only if it is θ-efficient.*

Proof. With $\delta = \delta_\theta$, the only feasible partition of $\Theta(\delta)$ in Definition 1 is the partition with a single element, equal to θ. Therefore, the definitions of renegotiation-proofness given δ_θ and θ-efficiency coincide. ■

3 Renegotiation Proofness of Continuation Contracts for Parametric Utility Functions

When trading off the cost of the current subsidy with future ones, the principal must take into account the renegotiation proofness constraint for continuation contracts. This section specializes the analysis to a class of parametric utility functions, for which the analysis of renegotiation-proof contracts is more tractable.

An *affine transformation group* of $\mathcal{Y}$ is a map $\upsilon : \mathcal{Y} \times I \to \mathcal{Y}$ for some open interval I of $\mathbb{R}$ such that $\upsilon(y, \rho) = \upsilon_1(\rho)y + \upsilon_2(\rho)$, where υ_1 is strictly positive, υ_1 and υ_2 are strictly increasing and differentiable, there exists $\bar{\rho}$ such that $\upsilon(y, \bar{\rho}) = y$ for all y, and for any $\rho \in I$, there exists $\tilde{\rho}$ (denoted ρ^{-1}, the "inverse" of ρ) such that $\upsilon(\upsilon(y, \rho), \tilde{\rho}) = y$.

Condition 1 (Affine Transformation Group) *There exist increasing functions $a : \mathbb{R}_{++} \to \mathbb{R}_{++}$ and $b : \mathbb{R}_{++} \times \Theta \to \mathbb{R}$ and a transformation group $\upsilon : \mathcal{Y} \times I \to \mathcal{Y}$ such that, for each $\theta \in \Theta$, $y \in \mathcal{Y}$, and $\rho \in I$,*

$$u(\theta, \upsilon(y, \rho)) = a(\rho)u(\theta, y) + b(\rho, \theta).$$

Condition 1 is satisfied by the logarithmic, power, and exponential utility specifications:

- If $u(\theta, y) = \theta y^{1-\gamma}/(1-\gamma)$ for $\gamma \neq 1$, then $u(\theta, \rho y) = \rho^{1-\gamma}u(\theta, y)$,

- If $u(\theta, y) = \theta \log(y)$, then $u(\theta, \rho y) = u(\theta, y) + \theta \log(\rho)$,
- If $u(\theta, y) = -\exp(-(\theta + y))$, then $u(\theta, y + \rho) = \exp(-\rho) u(\theta, y)$.

For any period t and continuation contract C, let $C(\rho)$ denote the contract which has the same menu structure as C, but provides the subsidy $y_\tau(\rho) = \upsilon(y_\tau, \rho)$ in period $\tau \geq t$ whenever C provides the subsidy y_τ at that period. The transformation $M(\rho)$ is defined analogously.

THEOREM 1 ("CLOCKWORK" PRINCIPLE) *Suppose that u satisfies Condition 1. Then, for any t, a period t contract C is renegotiation-proof if and only if the contract $C(\rho)$ is renegotiation-proof, and a period t menu M is renegotiation-proof if and only if $M(\rho)$ is.*

Theorem 1 provides a way of building credible perturbations of a given continuation contract, and thus of analyzing meaningful tradeoffs between current and future subsidies. See Section 7.1 for a proof.

4 T Periods and Binary Types: Taste Shock Example and Cost-Weighted Euler Equation

This section focuses on the case of binary types: $\Theta = \{L, H\}$ with $L < H$. The type of the agent is persistent and mean-reverting: for any $t \in \{1, \ldots, T-1\}$,

$$0 < Pr(\theta_{t+1} = H | \theta_t = L) < Pr(\theta_{t+1} = H | \theta_t = H) < 1.$$

For tractability, the environment is further narrowed down to a taste shock model with constant relative risk aversion:

$$u(\theta, y) = \theta \frac{y^{1-\gamma}}{1-\gamma}$$

or

$$u(\theta, y) = \theta \log y$$

which corresponds to $\gamma = 1$.

The properties of renegotiation-proof contracts are summarized here. For any history h_t and type θ, let $y_t(\theta | h_t)$ denote the transfer received by type θ in period t after history h_t.

THEOREM 2 *Any renegotiation-proof menu M is separating and has the following properties after any history h_t:*

- **Ordered Allocation**

$$y_t(L, h_t) < y_t(H, h_t).$$

- **Ordered Marginal Utility** *For any hist*

$$u_y(L, y_t(L, h_t)) < u_y(H, y_t(H, h_t)).$$

- **Cost-Weighted Euler Equation** *For all $\tilde{t} > t$,*

$$u_y(\theta_t, y_t(\theta_t, h_t)) = E^{\hat{\alpha}}[u_y(\theta_{\tilde{t}}, y_{\tilde{t}})|\theta_t, h_t]$$

  for some transition probability distribution $\hat{\alpha}$ that assigns more weight, relative to the true distribution α, to type realizations that entail higher costs to the principal.[9]

- **Gradual Reward/Punishment**

$$y_{t+2}(\theta_{t+1} = \theta_{t+2} = L, h_t) > y_{t+1}(\theta_{t+1} = L, h_t)$$

$$y_{t+2}(\theta_{t+1} = \theta_{t+2} = H, h_t) < y_{t+1}(\theta_{t+1} = H, h_t)$$

4.1 Two Periods

To gain some intuition about the result, we start by considering the case of 2 periods. With $T = 2$, $\mathcal{C}_1$ reduces to $\mathcal{Y}^2$: a contract in period 1 simply consists of an immediate y_1 and of a promised transfer y_2 in the second and last period.

Renegotiation-proof menus are separating

With two periods, the cost of a contract C is simply $Q(C) = y_1 + y_2$ and is therefore independent of the agent's type. Moreover, the definition of renegotiation-proofness for that case reduces to i) and ii) of Definition 1: iii) is always satisfied. Now consider any non-degenerate belief δ on Θ.

The problem can be visualized in the contract space $\mathcal{Y}^2$ with y_1 and y_2 as coordinates. Isocost curves of the principal are straight lines, and isolutility curves of the agent, where the utility is given by $U_\theta(y_1, y_2) = u(\theta_1, y_1) + E^{\theta_1}u(\theta_2, y_2)$, define convex upper level sets. For C to be renegotiation-proof the isocost curve of the principal must be tangent to the isoutility curve of both types of the agent. Otherwise, there must be a new contract that strictly reduces the cost of the principal and is accepted by at least one type of the agent, contradicting Definition 1.

However, the single-crossing property assumed on the agent's utility guarantees that such situation can never occur. Indeed, ex post efficiency of the low type implies that

$$u_y(L, y_1) = E[u_y(\theta_2, y_2)|L]. \tag{2}$$

Because $\theta_2 \geq L$ and u_y is increasing in θ and decreasing in y, this is only possible if $y_1 < y_2$. By a similar argument, ex post efficiency of the high type implies that $y_1 > y_2$, showing the claimed impossibility.

[9] See Equations (9) and (10) for the definition of $\hat{\alpha}$.

If, for example, $y_1 > y_2$, the principal can propose a new allocation (y_1', y_2') with $y_1' < y_2'$ which provides the high type with the same expected utility at an efficient level, and reduces the cost of the principal. This shows that shows that (y_1, y_2) is not renegotiation-proof. Thus, any renegotiation-proof contract is separating.

This shows that any renegotiation-proof contract must put all weight on a single type of the agent. As a result, any renegotiation-proof menu must be separating.

Now consider the allocations $y^H = (y_1^H, y_2^H)$ and $y^L = (y_1^L, y_2^L)$ provided to each type of the agent (one may assume without loss of generality, in light of the separation, that each type picks a single contract). Renegotiation-proofness implies that these allocations must be ex post efficient, by Proposition 3. As argued above, this implies that $y_1^H > y_2^H$ and that $y_1^L < y_2^L$. Moreover, incentive compatibility requires that either $y_1^H \leq y_1^L$ and $y_2^H \geq y_2^L$ or vice versa, with both inequalities strict if one is strict. Combining these observations proves that $y_1^H > y_1^L$ and that $y_2^H < y_2^L$. To rank marginal utilities, observe that

$$\begin{aligned} u_y(y_1^H, H) &= E^H u_y(y_2^H, \theta_2) && (3)\\ &> E^H u_y(y_2^L, \theta_2) && (4)\\ &\geq E^L u_y(y_2^L, \theta_2) && (5)\\ &= u_y(y_1^L, L) && (6) \end{aligned}$$

where the equalities come from ex post efficiency, and the strict inequality comes from the fact that $y_2^H < y_2^L$ and that u is strictly concave in y.

Because the marginal utility of H is greater than for L, the IC constraint of the low type must be binding for the optimal renegotiation-proof contract. Otherwise, the principal could reduce his cost by increasing the consumption of H and reduce that of L.

This is intuitive: if he could abstract from incentive compatibility constraints, the principal would always want to give more to the high type in all periods, because a high type values the good more today and, by persistence, is more likely to value it more in the future. The only reason why such allocation is infeasible is that it violates the IC constraint of the low type.

For CRRA utility, the set of contracts C_θ that are efficient for θ form on a straight line characterized by (2), and the IC constraint for L defines another, downward sloping, curve line linking the two contracts. As the promised utility w varies, the allocation $(y_1^H, y_2^H, y_1^L, y_2^L)$ is scaled accordingly.

In particular, the prior δ about the agent's type, and the promised utility w do not affect the shape of the optimal renegotiation-proof menu. They only affect the scaling factor of that menu.

4.2 Induction for T Periods

The proof works by induction on number T of remaining period, for $T \geq 2$. Precisely, we consider the following induction hypothesis:

Induction Hypothesis Any renegotiation-proof menu M with T remaining periods has the following properties.

i) The menu is separating and the contracts C_H and C_L chosen by each type of the agent satisfy $y_1^H > y_1^L$ in period 1.

ii) The marginal utilities satisfy $u_y(H, y_1^H) > u_y(L, y_1^L)$ and

$$\frac{u_y(\theta, y_1^\theta)}{U_\rho^2(\theta, \rho^\theta)} = \frac{1}{C_\rho^2(\theta, \rho^\theta)}$$

for $\theta = \{H, L\}$, where $C^2(\theta, \rho)$ and $U^2(\theta, \rho)$ respectively denote the expected cost and utility from Period 2 onwards when the agent's type in Period 1 is θ, as a function of the scaling factor ρ for continuation contracts at period 2, and ρ^θ is the scaling factor for the period 2 continuation menu contained in C_θ.

iii) There exists a two-contract menu M_T such that any optimal renegotiation-proof menu M is proportional to M_T: for any prior δ about θ_1 and any promised utility w, there exists $\rho \geq 0$ such that $M = M_T(\rho)$.

The hypothesis holds for $T = 2$. Consider now the case of $T + 1 \geq 3$ periods, appending a period "0" to T periods indexed by $1, \ldots, T$ (this notation simplifies the application of the induction hypothesis). Any renegotiation-proof contract consists of a quantity y_0 for the current period and a pair of contracts $C_H, C_L \in \mathcal{C}_1$ such that C_θ is renegotiation-proof given δ_θ, and which satisfy properties i) and ii) of the hypothesis.

Ruling out Pooling Contracts

Suppose that in period 0, some contract $C = (\bar{y}_0, M_1 = \{C_H, C_L\})$ is renegotiation-proof given some non-degenerate belief δ. This implies that in period 1, the agent will, irrespective of his type in period 0, choose C_H in period 1 if $\theta_1 = H$, and C_L otherwise, by the induction hypothesis. In particular, the agent will not engage in multiple deviations: by the induction hypothesis, whatever his type turns out to be in period 1, will be revealed in that period. Moreover, Theorem 1 implies that the same will hold if the continuation menu M_1 is replaced by its scaled version $M_1(\rho)$ for any factor $\rho > 0$.[10]

The argument for ruling out pooling contracts, provided in the appendix (Section 7.2), is based on trading off small perturbations of ρ around $\bar{\rho} = 1$ with perturbations of y_0 around $\bar{y}_0$.

[10] More precisely, $\upsilon_1(\rho) = \rho$, $\upsilon_2(\rho) = 0$, $a(\rho) = \rho^{1-\gamma}$ and $b(\rho, \theta) = \theta \log \rho 1_{\gamma=1}$.

Properties of Renegotiation-Proof Contracts

Because any renegotiation-proof menu is separating in period 0, it must consist of two contracts (y_0^H, M_1^H) and (y_0^L, M_1^L), respectively chosen by $\theta_0 = H$ and $\theta_0 = L$. We now derive some properties of those contracts and verify that the induction hypothesis is satisfied with the additional period $t = 0$.

First, notice that, conditional on facing type θ in period 0, the menu M_1^θ must be the cheapest way of providing type θ with a given continuation utility (otherwise, the contract is not renegotiation proof). From iii) of the induction hypothesis, this implies that M_1^H and M_1^L are scalings of the same contract $M_T = (C_T^H, C_T^L)$, regardless of the particular continuation utilities that these contracts provide, and regardless of the probability distribution for θ_1 that each of the two types in period 0 generate. Therefore, $M_1^\theta = M_T(\rho^\theta)$ for some $\rho^H, \rho^L \geq 0$. Let $c_T^H = Q_H(C_T^H)$ and $c_T^L = Q_L(C_T^L)$ denote the expected cost implied by menu M_T conditional on facing types H and L, respectively, in period 1.

From Proposition 3, efficiency requires that the marginal utility ratio for each type of the agent be equal to corresponding marginal cost ratio of the principal, in the (y_0, ρ) space. Otherwise, the principal can strictly reduce his cost and preserve the utility of the agent by perturbing the scaling factor ρ^θ and y_0^θ of the type who violates this inequality. Replicating a computation of Section 7.2 for allocations of each type, this implies that

$$u_y(H, y_0^H) = \hat{\alpha}^H u_y(H, y_1^{HH}) + (1 - \hat{\alpha}^H) u_y(L, y_1^{HL}) \tag{7}$$

and

$$u_y(L, y_0^L) = \hat{\alpha}^L u_y(H, y_1^{LH}) + (1 - \hat{\alpha}^L) u_y(L, y_1^{LL}) \tag{8}$$

where, in the right-hand side, the first (second) superscript indicates the type of the agent in period 0 (period 1), and

$$\hat{\alpha}^H = \frac{\alpha_H c_T^H}{\alpha_H c_T^H + (1 - \alpha_H) c_T^L}. \tag{9}$$

and

$$\hat{\alpha}^L = \frac{\alpha_L c_T^H}{\alpha_L c_T^H + (1 - \alpha_L) c_T^L}. \tag{10}$$

Also notice that $\hat{\alpha}^H > \hat{\alpha}^L$ because the function $\alpha \mapsto \alpha c_1 / (\alpha c_1 + (1 - \alpha) c_2)$ is increasing in α for all $c_1, c_2 > 0$.

By the induction hypothesis, we have $y_1^{HH} > y_1^{HL}$, $y_1^{LH} > y_1^{LL}$, $u_y(H, y_1^{HH}) > u_y(L, y_1^{HL})$ and $u_y(H, y_1^{LH}) > u_y(L, y_1^{LL})$. Equation (7) then implies that $u_y(H, y_0^H) < u_y(H, y_1^{HH})$ and, by concavity of u, that

$$y_0^H > y_1^{HH} > y_1^{HL}.$$

Similarly, we have

$$y_0^L < y_1^{LL} < y_1^{LH}.$$

This, along with incentive compatibility in period 0, implies that $y_1^{LH} > y_1^{HH}$ (otherwise the high type's allocation would Pareto dominate the low type's, since continuation contracts are all scalings of one another) and that $y_0^H > y_0^L$.

Finally, notice that

$$\begin{aligned} u_y(y_0^H, H) &= \hat{\alpha}^H u_y(y_1^{HH}, H) + (1 - \hat{\alpha}^H) u_y(y_1^{HL}, L) \\ &> \hat{\alpha}^L u_y(y_1^{HH}, H) + (1 - \hat{\alpha}^L) u_y(y_1^{HL}, L) \\ &> \hat{\alpha}^L u_y(y_1^{LH}, H) + (1 - \hat{\alpha}^L) u_y(y_1^{LL}, L) \\ &= u_y(y_0^L, L), \end{aligned}$$

where the equalities come from (7) and (8), the first inequality comes from the fact that $\hat{\alpha}^H > \hat{\alpha}^L$ and $u_y(y_1^{HH}, H) > u_y(y_1^{HL}, L)$, and the second inequality comes from the fact that $y_1^{LL} > y_1^{HL}$ and $y_1^{LH} > y_1^{HH}$ (as observed earlier, these last two inequalities are equivalent, since continuation contracts are scalings of one another, and must both hold for incentive compatibility).

Renegotiation-proofness implies that the menu $M_0 = ((y_0^H, \rho^H), (y_0^L, \rho^L))$ satisfies efficiency conditions for H and L . For the menu to be optimal, the IC constraint of the low type must be binding: since H's marginal utility is higher than L's, the principal could gain by transferring resources from L to H, if it were not for L's IC constraint. Taken together, these three conditions imply that the optimal renegotiation-proof menu is pinned down in the (y_0, ρ) space, up to a single parameter, which is determined by the promised expected utility w.

This shows that the shape of M_0 is completely independent of the principal's prior δ about θ_0 and of the promised utility w. Those parameters only affect the scaling level of M_0, and concludes the verification of the induction hypothesis for $T + 1$ periods.

5 Income Shock with Constant Absolute Risk Aversion (Arbitrary Number of Types and Periods)

When $u(\theta, y) = -\exp(-\theta - y)$ (the CARA coefficient is set to 1 for simplicity, the analysis is identical for other values of that coefficient), the relevant transformation $\upsilon(y, \rho)$ is the translation $\upsilon(y, \rho) = y + \rho$, with $\rho \in \mathbb{R}$. To interpret θ as the income of the agent, we must reverse the convention of earlier sections that marginal utility is increasing in the type of the agent. In this section, a higher income agent has a lower marginal utility for additional consumption.

The agent's utility is still multiplicatively separable in θ and y. To fix ideas and to cover common specifications of the income process, we assume that the type space Θ is equal to $\mathbb{R}$. The analysis can be easily adapted to the case of finite or bounded intervals type spaces in $\mathbb{R}$. Mean reversion is generalized as follows:

Definition 4 (Mean Reversion) *The type of the agent is* **mean-reverting** *if* $\frac{\exp(-\theta)}{E[\exp(-\theta_2)|\theta_1=\theta]}$ *is increasing in* θ*.*

As is easily checked, when there are only two types, this definition reduces to the fact that the type of the agent can switch to the other type with positive probability.

The analysis still works by backward induction. With two periods, renegotiation-proof menus must be separating, and the agent's bundle (y_1^θ, y_2^θ) is such that y_1^θ (resp. y_2^θ) is decreasing (increasing) in the agent's first-period income θ, the marginal utility $u_y(\theta, y_1^\theta)$ still satisfies

$$u_y(\theta, y_1^\theta) = E[u_y(\theta_2, y_2^\theta)|\theta_1 = \theta],$$

and this marginal utility is decreasing in θ.

Intuitively, this means that individuals with a low income in the first period receive more from the principal in that period, but not so much that their total consumption in that period exceeds the total consumption of individuals with high incomes. This intuition continues to hold with more periods, as established by Theorem 3 below.

Moreover, the optimal renegotiation-proof contract is entirely pinned down up to a single translation parameter, which is then adjusted to match the promised utility w, given the principal's prior about the agent's type. The argument for this is proved more generally in the induction hypothesis below and is thus omitted.

The analysis for more periods is simpler than in the CRRA case, because the marginal cost $C_\rho(\theta, \rho)$ of increasing future consumption by the translation amount ρ is now independent of the type of the agent and of the contract considered, as it is simply equal to the number of remaining periods (the translation amount, ρ, being added to all y_t's, regardless of the preceding histories).

The induction hypothesis is modified as follows:

Induction Hypothesis

With T periods to go, indexed by $1, \ldots, T$, the following holds:

i) Any period-1 renegotiation-proof menu is separating and y_1^θ is decreasing in θ.

ii) The marginal utilities $u_y(\theta, y_1^\theta)$ are decreasing in θ and define martingale processes:

$$u_y(\theta, y_1^\theta) = E[u_y(\theta_t, y_t)|\theta_1 = \theta]$$

iii) There exists a menu M_T such that any optimal renegotiation-proof menu M is a translation of M_T: for any prior δ about θ_1 and any promised utility w, there exists $\rho \in \mathbb{R}$ such that $M = M_T + \rho$.

Suppose that the hypothesis holds for T, and append a period 0. The first step is to show that renegotiation proof menus are separating in period 0. We make the following assumption (Monotone Likelihood Ratio Property, "MLRP") on the transition probability.

Definition 5 (MLRP) *i) For any state $\theta_0 = \theta$, the distribution of θ_1 conditional on θ_0 has full support on Θ and has a density $\alpha^\theta(\cdot)$. ii) for any $\theta' < \theta''$, the ratio $\alpha^{\theta''}(\theta)/\alpha^{\theta'}(\theta)$ is increasing in θ.*

Now consider a contract (y_0, M_1) that is renegotiation proof given some belief δ, and suppose by contradiction that δ is non-degenerate. By the induction hypothesis, M_1 is associated with a translation parameter $\bar{\rho}$ of the reference menu M_T. For all types θ choosing the same item, the equality by the utility and cost marginal rates of substitution

$$q_U^\theta = \frac{u_y(\theta, y_0)}{V_\rho(\theta, \bar{\rho})} = \frac{1}{T} = q_C$$

becomes

$$g_\theta f_0 = \int_\Theta \alpha^\theta(\theta') g_{\theta'} f_{\theta'} d\theta',$$

where $g_\theta = \exp(-\theta)$ is decreasing in θ, $f_0 = -\exp(-y_0)$, $f_{\theta'} = -\exp(-y_1^{\theta'})$ is increasing in θ', by the induction hypothesis. This implies that

$$\beta_\theta f_0 = \int_\Theta \lambda^\theta(\theta') f(\theta') d\theta' \tag{11}$$

where $\beta_\theta = g_\theta / D_\theta$, $\lambda^\theta(\theta') = \alpha^\theta(\theta') g_{\theta'} / D_\theta$, and $D_\theta = \int_\Theta \alpha^\theta(\theta') g_{\theta'}$. Notice that $\lambda^\theta(\cdot)$ is positive and sums up to one, for each θ: it forms a family of probability distributions on Θ.

For any $\theta'' > \theta'$,

$$\frac{\lambda^{\theta''}(\theta)}{\lambda^{\theta'}(\theta)} = \frac{\alpha^{\theta''}(\theta)}{\alpha^{\theta'}(\theta)} \frac{D_{\theta'}}{D_{\theta''}}.$$

Therefore, the family $\{\lambda^\theta\}_\theta$ of density functions inherits the MLRP from the family $\{\alpha^\theta\}_\theta$.[11]

Since β_θ is decreasing in θ (by the definition of mean reversion) and $\int_\Theta \lambda^\theta(\theta') f(\theta') d\theta'$ is increasing in θ by the MLRP and the fact that f is increasing in its argument, one θ can satisfy Equation (11), showing that any renegotiation-proof contract must be separating.

This means that each type θ receives a distinct contract $(y_0^\theta, \rho^\theta)$, where ρ^θ is the translation level compared to M_T. Moreover, the fact that $q_U^\theta = 1/T$ and the induction hypothesis immediately implies that martingale property for the marginal utility and, hence, for the period-utility of the agent.

[11] It is easy to show by counter examples that the distributions λ^θ are not FOSD ordered even if α^θ are FOSD ordered. Assuming first-order stochastic dominance for the transition probability distributions therefore seems insufficient for the results of this section.

Now consider two types $\theta' < \theta''$. We will show that for any RP menu, we have $y_0^{\theta'} > y_0^{\theta''}$. For the contracts to be incentive compatible, we must have either A) $y_0^{\theta'} > y_0^{\theta''}$ and $\rho^{\theta'} < \rho^{\theta''}$, or B) $y_0^{\theta'} \leq y_0^{\theta''}$ and $\rho^{\theta'} \geq \rho^{\theta''}$.

Repeating the previous computations we get the equations

$$\beta_{\theta'} f_0^{\theta'} = \int_\Theta \lambda^{\theta'}(\theta) f_\theta^{\theta'} d\theta \tag{12}$$

and

$$\beta_{\theta''} f_0^{\theta''} = \int_\Theta \lambda^{\theta''}(\theta) f_\theta^{\theta''} d\theta \tag{13}$$

where $f_0^{\theta'} = -\exp(-y_0^{\theta'})$ and $f_\theta^{\theta'} = -\exp(-y_\theta^{\theta'})$, with similar notations for θ''.

Since continuation contracts are translations of each other, we have $f_\theta^{\theta'} = \exp(\rho^{\theta''} - \rho^{\theta'}) f_\theta^{\theta''}$ for all θ. Suppose by contradiction that we are in case B). In that case, $\exp(\rho^{\theta''} - \rho^{\theta'}) < 1$, and the RHS of (12) is less than $\int_\Theta \lambda^{\theta'}(\theta) f_\theta^{\theta''} d\theta$. Moreover, because $f_\theta^{\theta''}$ is increasing in θ and $\lambda^{\theta'} \prec_{MLRP} \lambda^{\theta''}$ we conclude that the RHS of (12) is less than the RHS of (13). However, $y_0^{\theta''} \geq y_0^{\theta'}$ and $\theta'' > \theta'$ imply that the LHS of (12) is strictly greater than the LHS of (13), showing that one the equations must be violated.

Therefore, y_0^θ is decreasing in θ and, by incentive compatibility, ρ^θ is increasing in θ.

We now turn to the ranking of marginal utilities. Let $y_1(\theta)$ denote the period 1 subsidy of type θ under the reference contract M_T, of which all continuation contracts are translations. We have

$$\begin{aligned}
u_y\left(\theta'', y_0^{\theta''}\right) &= \int_\Theta \alpha^{\theta''}(\theta) u_y\left(\theta, y_1(\theta) + \rho^{\theta''}\right) d\theta \\
&= \exp(\rho^{\theta''} - \rho^{\theta'}) \int_\Theta \alpha^{\theta''}(\theta) u_y\left(\theta, y_1(\theta) + \rho^{\theta'}\right) d\theta \\
&< \int_\Theta \alpha^{\theta''}(\theta) u_y\left(\theta, y_1(\theta) + \rho^{\theta'}\right) d\theta \\
&\leq \int_\Theta \alpha^{\theta'}(\theta) u_y\left(\theta, y_1(\theta) + \rho^{\theta'}\right) d\theta \\
&= u_y\left(\theta', y_0^{\theta'}\right)
\end{aligned}$$

The first and last equalities come from the Euler equation. The second equality comes from the translation property of continuation contracts. The strict inequality comes from the fact, established earlier, that $\rho^{\theta''} > \rho^{\theta'}$, and the weak inequality comes from the MLRP property for the distributions α^θ and the fact that the induction hypothesis, which implies that that $u_y(\theta, y_1(\theta))$ is decreasing in θ.

There remains to show that optimal renegotiation-proof contracts are entirely pinned down, up to a single parameter. This is shown by exploiting the efficiency condition and incentive compatibility of the contracts to derive an ordinary differential equation for the contract. This is left for the appendix.

Theorem 3 *With exponential utility, mean reversion, and the MLRP property, the following holds*

- *Any renegotiation-proof contract is separating,*
- *The standard Euler equation is satisfied,*
- *After any common history h_t, y_t^θ and $u_y(\theta, y_t^\theta)$ are both decreasing in θ,*
- *The optimal renegotiation-proof menu is entirely pinned down up to a translation parameter.*

The mean reversion and MLRP conditions are trivially satisfied when the agent's type is i.i.d. over time, so the agent's utility is a martingale. This immediately implies the following result.

Corollary 1 *When the agent's type is i.i.d., renegotiation-proof contracts do not cause immiserization.*

More generally, the mean reversion Definition 4 is satisfied for any mean reverting $AR(1)$ process: if

$$\theta_{t+1} = \kappa\theta_t + (1-\kappa)\bar{\theta} + \varepsilon_{t+1}$$

with $\kappa \in (0,1)$ and the shocks ε_t are independently distributed, then

$$\frac{\exp(-\theta_t)}{E\exp(-\theta_{t+1})} = c_t \exp(-(1-\kappa)\theta_t),$$

where $c_t = 1/E(\exp((1-\kappa)\bar{\theta} + \varepsilon_{t+1}))$. The ratio is therefore decreasing in θ_t.

It is also easy to check that the MLRP condition is also satisfied when the shocks ε_t's are normally distributed.

Thus, Theorem 3 shows that renegotiation-proof contracts are separating for CARA utility when θ follows an $AR(1)$ process with Gaussian noise.

Corollary 2 *If ε_t's are normally distributed, RP contracts are separating.*

6 General Utility Functions and Type Space

In Progress

This section considers more general utility functions and any finite, totally ordered type space Θ, restricting the analysis to two periods. Following the income shock convention of Section 6, the utiltiy function u is assume to be submodular $u_y(\theta, y)$ is decreasing in θ. For simplicity, attention is restricted to lottery-free contracts.[12]

As before, renegotiation-proofness implies the following result.

[12]Lotteries are suboptimal for multiplicatively separable utilities. It is conjectured that they are also suboptimal in the income shock case when $\bar{u}$ exhibits decreasing absolute risk aversion.

Proposition 4 *Any outcome (δ, y) of the renegotiation stage satisfies the following property: for all $\theta \in \Theta(\delta)$,*

$$\frac{u_y(\theta_1, y_1)}{E^{\theta_1} u_y(\theta_2, y_2)} = 1.$$

The intuition for Proposition 4 is the following: if the equality were violated, the principal could always reduce his cost by propose a small perturbation of the bundle. The following single crossing property reflects the intuition, present in earlier sections, that strong enough mean reversion implies that renegotiation proof contracts are separating:

Condition 2 (Single Crossing) *The utility function u and type process $\{\theta_t\}_{t=1,2}$ satisfy the single crossing property if for any $(y_1, y_2) \in \mathcal{Y}^2$, the normalized marginal rate of substitution function*

$$\theta \mapsto \rho(\theta) = E^{\theta}[u_y(\theta_2, y_2)]/u_y(\theta, y_1) - 1$$

has the following single crossing property: there exists a threshold $\theta(y_1, y_2)$ such that $\rho(\theta) \geq (>)0$ if and only if $\theta \geq (>)\theta(y_1, y_2)$.

While Condition 2 is readily checked, it mixes properties of the type process with properties of the utility function. Simple distinct conditions on the type process and the agent's utility function implying Condition 2 are provided later in the section.

Theorem 4 *If u and θ satisfy the single crossing property, then any renegotiation-proof contract is separating. Moreover, $y_1(\theta)$ is decreasing in θ, and $y_2(\theta)$ is increasing in θ.*

Proof. Following any message m_1 and bundle (y_1, y_2), Condition 2 implies that at most one type can satisfy ex post efficiency, by Proposition 4. To show monotonicity, consider two types $\theta < \theta'$, with respective bundles (y_1, y_2) and (y_1', y_2'). Condition 2 implies that

$$u_y(\theta', y_1) < E^{\theta'} u_y(\theta_2', y_2)$$

If $y_1' \geq y_1$, incentive compatibility implies that $y_2' < y_2$. This implies, by concavity of u, that

$$u_y(\theta', y_1') \leq u_y(\theta', y_1) < E^{\theta'} u_y(\theta_2', y_2) \leq u_y(\theta', y_2')$$

which violate ex post efficiency for θ', by Proposition 4.

To obtain an ordering of marginal utilities, the following persistence condition is assumed.

Definition 6 (Persistence) *The type of the agent is* persistent *if the distribution of θ_2 given θ is increasing in the sense of first order stochastic dominance, i.e., for any $\tilde{\theta} \in \Theta$,*

$$Pr(\theta_2 \geq \tilde{\theta} | \theta_1 = \theta)$$

is increasing in θ.

Theorem 5 *If u and θ satisfy the single crossing and θ is persistent, then $u_y(y_1(\theta), \theta)$ is decreasing in θ.*

Proof. For any $\theta < \theta'$,$u_y(\theta', y_1') = E^{\theta'} u_y(\theta_2, y_2') < E^{\theta} u_y(\theta_2, y_2) = u_y(\theta, y_1)$, where the inequality comes from the fact that $y_2 < y_2'$, submodularity of u, and FOSD of the distributions for θ_2. ■

6.1 An income-shock example with an AR(1) process

Suppose that θ_t is the income of the agent at time t and that y_t is a subsidy (or a tax, depending on its sign) from the principal to the agent. Also suppose that $u(y, \theta) = \log(y + \theta)$ or that $u(y, \theta) = \frac{(y+\theta)^{1-\gamma}}{1-\gamma}$.

The agent's income follows an AR(1) process: $\theta_2 = (1-\kappa)+\kappa\theta_1+\varepsilon_2$, for $\kappa \in (0, 1)$, where ε_2 is a random variable independently distributed from θ_1, and which is bounded below (this boundedness assumption is necessary to guarantee the expected utility is finite for y high enough). The following result is proved in the appendix.

Proposition 5 *u and θ satisfy the single-crossing condition (Condition 2).*

6.2 Multiplicatively Separable Utility

Suppose now that

$$u(\theta, y) = g(\theta)\bar{u}(y),$$

where g is a positive and strictly decreasing in θ for the order on Θ, so that u is strictly submodular. In the taste shock application, $g(\theta) = \theta$ and the order on Θ is the opposite of the usual order on $\mathbb{R}$ (i.e, $\theta \prec_\Theta \theta' \Leftrightarrow \theta' < \theta$). In the income shock application with constant absolute risk aversion, $u(\theta, y) = -\exp(-(\theta + y))$, so $g(\theta) = \exp(-\theta)$ and $\bar{u}(y) = -\exp(-y)$. The order on Θ is the usual order on $\mathbb{R}$.

For each $\theta \in \Theta$, let $\beta(\theta) = E[g(\theta_2)|\theta_1 = \theta]$. By persistence (Definition 6), $\beta(\theta)$ is decreasing in θ if g is decreasing.

Definition 7 (Mean Reversion) *$\{\theta_t\}_{t=1,2}$ is* mean reverting *if $\beta(\theta)/g(\theta)$ is increasing in θ*

To understand this definition, recall that a higher type is associated with a lower value of g. Mean-reversion means that the expected value of $g(\theta_2)$ conditional on θ_1, while decreasing in θ_1, does not decrease as fast at $g(\theta_1)$ itself.

Proposition 6 (Single Crossing) *u and θ satisfy single crossing.*

Proof. Fix any y_1 and y_2. We have

$$E^{\theta}u_y(y_2,\theta_2)/u_y(y_1,\theta)-1=\frac{\bar{u}'(y_2)}{\bar{u}'(y_1)}\frac{\beta(\theta)}{\theta}-1.$$

This ratio is strictly increasing in θ, by mean reversion, and the single-crossing property is satisfied.

7 Appendix

7.1 Proof of Theorem 1

It suffices to show that, when Condition 1 holds, comparisons between any two continuation contracts C, C' from the agent and the principal's perspective are equivalent to comparisons between their transformations $C(\rho)$ and $C'(\rho)$, and strategy comparisons for the agent and the principal under contract C are the same as under contract $C(\rho)$. The expected utility of an agent under contract C following a given strategy σ_A for the entire game, and taking the principal's renegotiation strategy as given, is equal to

$$U_{\theta}^{\sigma_A}(C)=E^{\theta,\sigma_A}\left[\sum_{\tau=t}^{T}u(y_\tau,\theta_\tau)\right],$$

while it is equal to

$$U_{\theta}^{\sigma_A}(C(\rho))=E^{\theta,\sigma_A}\left[\sum_{\tau=t}^{T}u(\upsilon(y_\tau,\rho)\theta_\tau)\right]=a(\rho)U_{\theta}^{\sigma_A}(C)+E^{\theta}\left[\sum_{\tau=t}^{T}b(\rho,\theta_\tau)\right],$$

under the contract $C(\rho)$ and the same strategies. Since the second term of the right-hand side is independent of the contract, and $a(\rho)>0$, this shows that any strategy of the agent that is optimal for C is optimal for $C(\rho)$, and that any comparison of contracts C, C' is equivalent to the comparison of contracts $C(\rho)$ and $C'(\rho)$. Similarly, the principal expected cost under C and given a renegotiation strategy σ_P and belief δ, taking the agent's strategy as given, is equal to

$$Q_{\delta}^{\sigma_P}(C)=E^{\delta,\sigma_P}\left[\sum_{\tau=t}^{T}y_\tau\right],$$

while it is equal to

$$Q_{\delta}^{\sigma_P}(C(\rho))=E^{\delta,\sigma_P}\left[\sum_{\tau=t}^{T}\upsilon(y_\tau,\rho)\right]=\upsilon_1(\rho)Q_{\delta}^{\sigma_P}(C)-(T-t+1)\upsilon_2(\rho)$$

under the contract $C(\rho)$ and the same strategies, which shows the result.

7.2 Proof of Theorem 2

Let q_U^θ denote the marginal utility ratio for type θ between changes in y_0 and changes in ρ, evaluated at the initial contract. Similarly, let q_C^θ denote the marginal cost ratio for the principal between similar changes. The proof that pooling contracts are not renegotiation-proof works by comparing these different ratios.

Step 1 We start by showing that the inequality $q_U^L < q_U^H$ holds, which is the single crossing property in the (y_0, ρ) space. Equivalently,

$$\frac{u_y(L,\bar{y}_0)}{V_\rho(L,\bar{\rho})} < \frac{u_y(H,\bar{y}_0)}{V_\rho(H,\bar{\rho})}, \tag{14}$$

where $V(\theta,\rho) = E[\sum_{t=1}^T u(\theta_t, \rho y_t)|\theta_0 = \theta]$ is the expected continuation utility of the agent when his period-0 type is θ. We have

$$V_\rho(\theta,\bar{\rho}) = E^{\theta_0=\theta}\sum_{t=1}^T y_t u_y(\theta_t, y_t).$$

Since $u_y(\theta, y) = \theta y^{-\gamma}$, (14) is equivalent to

$$E^L\left[\sum_{t=1}^T \frac{\theta_t}{L} y_t^{1-\gamma}\right] > E^H\left[\sum_{t=1}^T \frac{\theta_t}{H} y_t^{1-\gamma}\right] \tag{15}$$

Suppose first that $\gamma = 1$. We have $L \le \theta_t \le H$ for all t, and these inequalities are strict with positive probability. Therefore, the left-hand side of (15) is strictly greater than 1, while its right-hand side is strictly less than 1, proving the inequality.

For $\gamma < 1$, (15) is equivalent, dividing by $(1-\gamma) > 0$, to

$$\frac{W(L)}{L} > \frac{W(H)}{H},$$

where $W(\theta) = V(\theta, 1)$ is the expected utility of type θ from period 1 onwards under his optimal strategy. One possible strategy for L is to mimic the strategy of H. In that case his expected utility is equal to

$$W(L;z) = E^L \sum_{t=1}^T \theta_t \frac{z_t^{1-\gamma}}{1-\gamma}$$

where $\{z_t\}$ denotes the subsidy stream generated by mimicking the strategy of H. Starting from $\theta_0 = L$, the type process $\{\theta_t\}_t$ satisfies $\theta_t H/L \ge H$ for all t, and with a strict inequality with positive probability. Because $y_t^{1-\gamma}/(1-\gamma) > 0$, we obtain

$$\frac{H}{L} W(\theta L; z) > E^H \sum_{t=1}^T \theta_t \frac{z_t^{1-\gamma}}{1-\gamma} = W(H).$$

This shows that $H/LW(L) > W(H)$.

For $\gamma > 1$, dividing (15) by $(1-\gamma) < 0$ yields

$$\frac{W(L)}{L} < \frac{W(H)}{H}.$$

Repeating the previous argument, but this time having H mimic L, yields the desired inequality.

Step 2 We now show that if $q_C^H < q_U^H$ or $q_C^L > q_U^L$, there is a Pareto improvement to the initial contract. Consider the first case, $q_C^H < q_U^H$. Then, the principal can reduce his cost and improve H's utility by increasing y_0 and reducing ρ, towards a contract that is strictly suboptimal for the L. In the second case, the principal can improve L's utility and strictly reduce his cost by reducing y_0 and increasing ρ, without attracting the high type. In either case, the principal strictly reduces his cost, preserving incentive compatibility and keeping renegotiation-proof continuations.

Step 3 The remaining case is that $q_C^H \geq q_U^H$ and $q_C^L \leq q_U^L$. We will show that this case is ruled out by the induction hypothesis. Let w^θ denotes the expected utility of an agent with type θ in period 1 under the menu M_1.

For $\gamma \neq 1$, the continuation utility in period 1 of an agent of type $\theta_1 = H$ is given as a function of ρ by

$$w^H \rho^{1-\gamma} = u(H, y_1^H \rho) + U^2(H, \rho),$$

where we recall from the statement of the induction hypothesis that $U^2(\theta, \rho)$ denotes the expected utility from Period 2 onwards when the agent's type in Period 1 is θ, as a function of the scaling factor ρ for continuation contracts at period 2. Similarly, if $\theta_1 = L$,

$$w^L \rho^{1-\gamma} = u(L, y_1^L \rho) + U^2(L, \rho).$$

For $\gamma > 1$, w^H, w^L are strictly negative (since $u(\theta, y) = \theta y^{1-\gamma}/(1-\gamma) < 0$).

Differentiating these inequalities with respect to ρ and evaluating the result at $\rho = \bar{\rho} = 1$, one obtains

$$(1-\gamma)w^\theta = y_1^\theta u_y(H, y_1^\theta) + U_\rho^2(\theta, \bar{\rho}).$$

By the induction hypothesis, this implies that

$$(1-\gamma)w^\theta = (y_1^\theta + C_\rho^2(\theta, 1))u_y(\theta, y_1^\theta),$$

where we recall that $C^2(\rho, \theta)$ denotes the expected cost from period 2 onwards when the agent's type in period 1 is θ, as a function of the scaling factor ρ for continuation contracts at period 2.

Moreover, because $V(\theta, \rho) = \alpha_\theta \rho^{1-\gamma} w^H + (1-\alpha_\theta)\rho^{1-\gamma} w^L$, we obtain the following formula for q_U^θ:

$$q_U^\theta = \frac{u_y(\theta, \bar{y}_0)}{\alpha_\theta u_y(H, y_1^H)(y_1^H + C_\rho^2(H, 1)) + (1-\alpha_\theta)u_y(L, y_1^L)(y_1^L + C_\rho^2(L, 1))}$$

Similarly, the expected cost for the principal as a function of the agent's type θ in period 1 is given by $c^{\theta}\rho = \rho y_1^{\theta} + C^2(\rho, \theta)$. Differentiating with respect to ρ and evaluating the derivative at $\rho = 1$ yields $c^{\theta} = y_1^{\theta} + C_{\rho}^2(\bar{\rho}, \theta)$.

Now consider the case in which $q_U^H \leq q_C^H$ and $q_U^L \geq q_C^L$. Combining the previous results, we get

$$u_y(\bar{y}_0, \theta^L) \geq \hat{\alpha}_L u_y(y_1^L, L) + (1 - \hat{\alpha}_L) u_y(y_1^H, H) \tag{16}$$

and

$$u_y(\bar{y}_0, \theta^H) \leq \hat{\alpha}_H u_y(y_1^L, H) + (1 - \hat{\alpha}_H) u_y(y_1^H, H), \tag{17}$$

where

$$\hat{\alpha}^H = \frac{\alpha_H \left(y_1^H + C_{\rho}^2(H, \bar{\rho})\right)}{\alpha_H \left(y_1^H + C_{\rho}^2(H, \bar{\rho})\right) + (1 - \alpha_H)\left(y_1^L + C_{\rho}^2(L, \bar{\rho})\right)} = \frac{\alpha_H c^H}{\alpha_H c^H + (1 - \alpha_H) c^L}. \tag{18}$$

and

$$\hat{\alpha}^L = \frac{\alpha_L \left(y_1^H + C_{\rho}^2(H, \bar{\rho})\right)}{\alpha_L \left(y_1^H + C_{\rho}^2(H, \bar{\rho})\right) + (1 - \alpha_L)\left(y_1^L + C_{\rho}^2(L, \bar{\rho})\right)} = \frac{\alpha_L c^H}{\alpha_L c^H + (1 - \alpha_L) c^L}. \tag{19}$$

Since $u_y(y_1^L, L) < u_y(y_1^H, H)$ (by the induction hypothesis), (16) implies that $u_y(\bar{y}_0, L) > u_y(y_1^L, L)$ and, hence, that $\bar{y}_0 < y_1^L < y_1^H$. However, (17) also implies that $u_y(\bar{y}_0, H) < u_y(y_1^H, H)$ and thus that $\bar{y}_0 > y_1^H > y_1^L$, a contradiction.

It is easily checked that the same computations obtain for the logarithmic case $\gamma = 1$.

7.3 Almost Uniqueness of Renegotiation-Proof Contracts for CARA utility functions

To fully verify the induction hypothesis for Section 5, there remains to verify that optimal renegotiation-proof menus are pinned down up to a single parameter. We have already established that any renegotiation proof menu is separating, and characterized by a family $\{(y_0^{\theta}, \rho^{\theta})\}_{\theta \in \mathbb{R}}$ with y_0^{θ} (resp. ρ^{θ}) decreasing (increasing) in θ. Let $U(\theta)$ denote the continuation utility of a type $\theta_0 = \theta$ under the reference continuation menu M_T.

By incentive compatibility, we must have

$$\theta \in \arg\max_{\theta'} \left\{ -\exp(-\theta)\exp(-y_0^{\theta'}) + \exp(-\rho^{\theta'}) U(\theta) \right\}.$$

Local incentive compatibility is characterized by the first-order condition[13]

$$\exp(-\theta)\exp(-y_0^{\theta}) \frac{dy_0^{\theta'}}{d\theta'}(\theta) - \exp(-\rho^{\theta}) U(\theta) \frac{d\rho^{\theta'}}{d\theta'}(\theta) = 0 \tag{20}$$

[13]By monotonicity, y_0 and ρ are a.e. differentiable in θ. Incentive compatibility also implies that y_0 and ρ are continuous in θ.

Moreover, because the agent's isoutility curves satisfy the single crossing property in the (y_0, ρ) space (as established in the main text when showing that pooling contracts are impossible), the local IC constraint is sufficient for global incentive compatibility.

Equation (20) defines a parametric equation for $(y_0^\theta, \rho^\theta)$ in the (y_0, ρ) space.

In addition, we know that renegotiation-proof contracts must be efficient, which means that any point (y_0, ρ) of the curve covered by the renegotiation-proof contract is associated with the efficiency line of exactly one type $\theta = \tau(y_0, \rho)$.

Plugging this efficiency relation into (20), and using the parametric relation $d\rho/dy_0 = (d\rho/d\theta)/(dy_0/d\theta)$, we get the ordinary differential equation[14]

$$\frac{d\rho}{dy_0} = \frac{\exp(\rho - y_0)\exp(-\tau(y_0, \rho))}{U(\tau(y_0, \rho))}. \tag{21}$$

This pins down the renegotiation-proof contract up to a single parameter, which is a translation coefficient. Indeed, it is easy to check that if the parametric curve $(y_0(\theta), \rho(\theta))$ satisfies Equation (20), then so does the curve $(y_0(\theta) + \hat{\rho}, \rho(\theta) + \hat{\rho})$ for any $\hat{\rho} \in \mathbb{R}$, and that the set of efficient allocations (y_0, ρ) for any given type θ forms a 45 degree line. Equivalently, if any function $y_0 \mapsto \rho(y_0)$ satisfies the ODE (21), then so does the function $\rho_{\hat{\rho}} : y_0 \mapsto \rho(y_0 - \hat{\rho}) + \hat{\rho}$, for $\hat{\rho} \in \mathbb{R}$.

Proof of Corollary 2

For any θ_0, the density of θ_1's distribution is given by

$$\alpha^{\theta_0}(\theta_1) = \phi(\theta_1 - (\kappa\theta_0 + (1-\kappa)\bar{\theta})),$$

where ϕ is the density of a standard Gaussian distribution. This implies that

$$\frac{\alpha^{\theta''}(\theta)}{\alpha^{\theta'}(\theta)} = \exp\left\{\frac{1}{2}\left[(\theta - (\kappa\theta' + (1-\kappa)\bar{\theta}))^2 - (\theta - (\kappa\theta'' + (1-\kappa)\bar{\theta}))^2\right]\right\}.$$

Therefore, it suffices to show that $(\theta - (\kappa\theta' + (1-\kappa)\bar{\theta}))^2 - (\theta - (\kappa\theta'' + (1-\kappa)\bar{\theta}))^2$ is nondecreasing in θ, for $\theta' < \theta''$. That expression is equal to

$$(\theta - \kappa\theta')^2 - (\theta - \kappa\theta'')^2 + 2(1-\kappa)\kappa\bar{\theta}(\theta'' - \theta')$$

The second term is independent of θ, while the first term is equal to $2\theta\kappa(\theta'' - \theta') + \kappa^2(\theta'^2 - \theta''^2)$, and is therefore increasing in θ for $\theta' < \theta''$, proving the claim. ■

[14] The RHS is negative, since U is. The solution $y_0 \mapsto \rho(y_0)$ of the ODE is therefore downward sloping.

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