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## DECENTRALIZED TRADING WITH PRIVATE INFORMATION

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## DECENTRALIZED TRADING WITH PRIVATE INFORMATION

BY MIKHAIL GOLOSOV, GUIDO LORENZONI, AND ALEH TSYVINSKI<sup>1</sup>

The paper studies how asset prices are determined in a decentralized market with asymmetric information about asset values. We consider an economy in which a large number of agents trade two assets in bilateral meetings. A fraction of the agents has private information about the asset values. We show that, over time, uninformed agents can elicit information from their trading partners by making small offers. This form of experimentation allows the uninformed agents to acquire information as long as there are potential gains from trade in the economy. As a consequence, the economy converges to a Pareto efficient allocation.

KEYWORDS: Information revelation, bilateral trading, over-the-counter markets.

### 1. INTRODUCTION

THIS PAPER STUDIES TRADING AND INFORMATION DIFFUSION in a decentralized market with private information. We consider an economy with two key frictions: trading takes place through bilateral meetings and some agents have private information about the value of the assets traded. In financial markets, a large number of transactions take place not in centralized exchanges, but in decentralized, over-the-counter markets. Duffie, Garleanu, and Pedersen (2005) started a literature that uses random matching and bilateral trading to model over-the-counter markets.<sup>2</sup> In this paper, we study a model with random matching and bilateral trading in which some market participants have private information about the value of the assets traded and analyze how information gradually spreads through the economy. In particular, we ask whether all relevant information is revealed over time and whether the allocation converges to a Pareto efficient allocation.

Our environment is as follows. Agents start with different endowments of two risky assets, match randomly, and trade in bilateral meetings. In each bilateral meeting, one of the agents makes a take-it-or-leave-it offer to the other, who can accept or reject. Therefore, apart from the presence of private information, we have a trading game in the tradition of Gale (1986a, 1986b,

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<sup>2</sup>See, among others, Duffie, Gârleanu, and Pedersen (2007), Lagos (2007), Lagos and Rocheteau (2009), Lagos, Rocheteau, and Weill (2007), Vayanos (1998), Vayanos and Weill (2008), and Weill (2008).

1987).<sup>3</sup> Agents are risk averse so there is the potential for mutually beneficial trades of the two assets. However, before trading begins, a fraction of agents—the informed agents—receive some information about the value of the assets. Namely, they observe a binary signal that determines which one of the two assets is more valuable. The game ends at a random time, at which point the asset payoffs are revealed, and the agents consume. Uninformed agents form beliefs about the value of the two assets based on their individual trading history, which is the only information they receive during trading.

Our objective is to characterize the efficiency properties of the allocation and the value of information in the long run. Our main result is that, in the long run, the equilibrium converges to an ex post Pareto efficient allocation and the value of information goes to zero. Our argument is as follows. First, we focus on the informed agents and prove that their marginal rates of substitution converge. The intuition for this result is similar to the proof of Pareto efficiency in decentralized environments with common information: if two informed agents have different marginal rates of substitution, they can always find a trade that improves the utility of both. We then show that the marginal rates of substitution of uninformed agents also converge. Our argument is based on finding strategies that allow the uninformed agents to learn the signal received by the informed agents at an arbitrarily small cost. The existence of such strategies implies that two cases are possible: either uninformed agents eventually learn the signal, or the benefit of learning the signal goes to zero. Both cases imply that the marginal rates of substitution of all agents are equalized. We can then show that equilibrium allocations converge to ex post Pareto efficient allocations in the long run.

Our work is related to Wolinsky's (1990) seminal article on information revelation in pairwise matching environments. Wolinsky (1990) considered a game with decentralized, bilateral trading in which agents have the option to trade an indivisible good of uncertain quality, at given prices. In his game, a fraction of traders exits in each period and is replaced by new traders. He showed that steady state equilibria are possible in which some trades that would be Pareto improving under symmetric information do not take place. That is, he obtained an inefficiency result. Blouin and Serrano (2001) showed that this inefficiency result survives in a version of Wolinsky's model with a fixed population of traders, which is thus closer to our environment. The crucial differences between our setup and the models in Wolinsky (1990) and Blouin and Serrano (2001) are that in our setting the good is perfectly divisible and that agents can choose at what price to trade. These assumptions lead to different implications in terms of efficiency, resulting in equilibria in which information is fully revealed and allocations are ex post efficient in the long run. Intuitively, divisibility allows uninformed agents to strategically experiment by making small, potentially unprofitable trades to learn valuable information. In this strand of

<sup>3</sup>See Gale (2000) for a general treatment of matching and bargaining games with symmetric information and Lauerma (2013) for a recent characterization.

literature, an early paper that explored the potential for uninformed agents to learn through trading is [Green \(1992\)](#). Green's (1992) objective was to find sufficient conditions on equilibrium strategies and on the span of traded assets that ensure that uninformed agents can perfectly elicit the information of their trading partners in equilibrium. Although our goal here is different—to prove long-run efficiency—we share his interest in characterizing the learning strategies of uninformed agents.

In the literature on asset pricing in decentralized markets, a few recent papers deal with information transmission through bilateral meetings: [Duffie and Manso \(2007\)](#), [Duffie, Giroux, and Manso \(2010\)](#), [Duffie, Malamud, and Manso \(2009, 2010\)](#). These papers characterize in closed form the dynamics of beliefs in models in which agents perfectly share the information of the agents they meet. The main difference with our work is that these papers made assumptions which ensured that information is perfectly transmitted in each bilateral meeting. The setup in our paper instead is such that agents may prefer a trading behavior which is not perfectly revealing. For example, an informed agent trying to sell the less valuable asset may decide to mimic an uninformed agent with a large endowment of the same asset to trade at more favorable price. Therefore, in our environment, the speed at which information is transmitted in bilateral meetings is endogenous and the ability of uninformed traders to elicit this information is at the center of our analysis. Finally, a recent related paper is [Ostrovsky \(2012\)](#), who studied the incentives of large, strategic traders in dynamic, centralized markets, and showed that information gets aggregated in equilibrium.

The paper is structured as follows. Section 2 describes the environment. Section 3 contains our main result on long-run efficiency. Section 4 concludes. The [Appendix](#) contains details of the proofs omitted in the paper.

## 2. SETUP AND TRADING GAME

In this section, we introduce the model and define an equilibrium.

### 2.1. Setup

There are two states of the world  $S \in \{S_1, S_2\}$  and two assets  $j \in \{1, 2\}$ . Asset  $j$  is an Arrow security that pays one unit of consumption in state  $S_j$ . There is a continuum of agents with von Neumann–Morgenstern expected utility  $E[u(c)]$ , where  $E$  is the expectation operator. At date 0, each agent is randomly assigned a type  $i$ , which determines his initial portfolio of the two assets, denoted by the vector  $x_{i,0} \equiv (x_{i,0}^1, x_{i,0}^2)$ . There is a finite set of types  $N$  and each type  $i \in N$  is assigned to a fraction  $\nu_i$  of agents. The aggregate endowment of each asset is equal in the two states and normalized to 1:

$$(1) \quad \sum_{i \in N} \nu_i x_{i,0}^j = 1 \quad \text{for } j = 1, 2.$$

We make the following assumptions on preferences and endowments. The first assumption is symmetry of the endowments.

**ASSUMPTION 1—Symmetry:** *For each type  $i \in N$ , there exists a type  $j \in N$  of equal mass  $v_j = v_i$ , holding symmetric endowments  $x_{j,0} = (x_{i,0}^2, x_{i,0}^1)$ .*

The role of this assumption is discussed in detail in Section 4. The second assumption imposes usual properties on the utility function, as well as boundedness from above and a condition ruling out zero consumption in either state.

**ASSUMPTION 2:** *The utility function  $u(\cdot)$  is increasing, strictly concave, twice continuously differentiable on  $R_{++}^2$ , bounded above, and satisfies  $\lim_{c \rightarrow 0} u(c) = -\infty$ .*

Finally, we assume that the initial endowments are interior.

**ASSUMPTION 3:** *The initial endowment  $x_{i,0}$  is in the interior of  $R_+^2$  for all types  $i \in N$ .*

At date 0, nature draws a binary signal  $s$  that takes the values  $s_1$  and  $s_2$  with equal probabilities. The posterior probability of  $S_1$  conditional on  $s$  is denoted by  $\phi(s)$ . We assume that signal  $s_1$  is favorable to state  $S_1$  and that the signals are symmetric:  $\phi(s_1) > 1/2$  and  $\phi(s_2) = 1 - \phi(s_1)$ . After  $s$  is realized, a fraction  $\alpha$  of agents of each type privately observes the realization of  $s$ . The agents who observe  $s$  are called *informed agents*; those who do not observe it are called *uninformed agents*.

## 2.2. Trading

After the realization of the signal  $s$ , but before the state  $S$  is revealed, all agents engage in a trading game set in discrete time. At the beginning of each period  $t \geq 1$ , the game continues with probability  $\gamma \in (0, 1)$  and ends with probability  $1 - \gamma$ . If the game ends, the state  $S$  is publicly revealed and the agents consume the asset payoffs.<sup>4</sup> If the game continues, all agents are randomly matched in pairs and a round of trading takes place. One of the two agents is selected as the *proposer* with probability  $1/2$ . The proposer makes a take-it-or-leave-it offer  $z = (z^1, z^2) \in R^2$  to the other agent, the *responder*. That is, the proposer offers to exchange  $z^1$  of asset 1 for  $-z^2$  of asset 2. The responder can accept or reject the offer. If an agent with portfolio  $x$  offers  $z$  to an agent with portfolio  $\tilde{x}$  and the offer is accepted, their end-of-period portfolios are, respectively,  $x - z$  and  $\tilde{x} + z$ . We assume that the proposer can only make feasible offers,  $x - z \geq 0$ , and the responder can only accept an offer if  $\tilde{x} + z \geq 0$ .<sup>5</sup>

<sup>4</sup>Allowing for further rounds of trading after the revelation of  $S$  would not change our results, given that at that point only one asset has positive value and no trade will occur.

<sup>5</sup>The proposer only observes if the offer is accepted or rejected. In particular, if an offer is rejected, the proposer does not know whether it was infeasible for the responder or the responder just chose to reject.

An agent does not observe the portfolio of his opponent or whether his opponent is informed or not. Moreover, an agent only observes the trades he is involved in but not those of other agents. Therefore, both trading and information revelation take place through decentralized, bilateral meetings.

### 2.3. Equilibrium Definition

We now define an equilibrium. First, let us introduce some notation for individual histories. At date 0, each agent is assigned the type  $i$  that determines his initial portfolio, with probability  $\alpha$  he is informed and observes the signal  $s$ , with probability  $1 - \alpha$  remains uninformed. The initial history  $h_0 \in N \times \{U, I_1, I_2\}$  captures this initial condition ( $U$  stands for uninformed,  $I_j$  stands for informed with signal  $s_j$ ). In each period  $t \geq 1$ , the event  $h_t = (\iota_t, z_t, r_t)$  includes the indicator variable  $\iota_t$ , equal to 1 if the agent is selected as the proposer, the offer made by the proposer  $z_t \in R^2$ , and the indicator variable  $r_t$ , equal to 1 if the offer is accepted. The sequence  $h^t = \{h_0, h_1, \dots, h_t\}$  denotes the history of play up to period  $t$  for an individual agent.  $H^t$  denotes the space of all possible histories of length  $t$  and  $H^\infty$  denotes the space of all infinite histories. Letting  $\Omega = \{s_1, s_2\} \times H^\infty$ , a point in  $\Omega$  describes an infinite history of play for an individual agent, if the game continues forever. We use  $(s, h^t)$  to denote the subset of  $\Omega$  given by all the  $\omega = (s, h^\infty)$  such that the first  $t$  elements of  $h^\infty$  are equal to  $h^t$ .

We can now describe strategies. If the agent is selected as the proposer at time  $t$ , his actions are given by the map

$$\sigma_i^p : H^{t-1} \rightarrow \mathcal{P},$$

where  $\mathcal{P}$  denotes the space of probability distributions over  $R^2$  with finite support. That is, we allow for mixed strategies and let the proposer choose the probability distribution  $\sigma_i^p(\cdot | h^{t-1})$  from which he draws the offer  $z$ .<sup>6</sup> If the agent is selected as the responder, his behavior is described by

$$\sigma_i^r : H^{t-1} \times R^2 \rightarrow [0, 1],$$

which denotes the probability that the agent accepts the offer  $z \in R^2$  for each history  $h^{t-1}$ . A strategy is fully described by the sequence  $\sigma = \{\sigma_i^p, \sigma_i^r\}_{i=1}^\infty$ .

We focus on symmetric equilibria where all agents play the same strategy  $\sigma$ . We say that the probability measure  $P$  on  $\Omega$  is *consistent with*  $\sigma$  if, for each individual agent,  $(1 - \gamma)\gamma^{t-1}P(s, h^{t-1})$  is the equilibrium probability that the signal  $s$  is selected, the game ends at  $t$ , and the agent's history is  $h^{t-1}$ . The sequence  $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}$  denotes the filtration generated by the information sets of the agent at the beginning of each period  $t$ . The measure

<sup>6</sup>We restrict agents to mix over a finite set of offers to simplify the measure-theoretic apparatus.

$P$  also characterizes the cross sectional distribution of individual histories in equilibrium: at the beginning of time  $t$ , the mass of agents with history  $h^{t-1}$  is equal to  $P(h^{t-1}|s)$ .

The probability measure  $P$  is constructed recursively in the following way. In period 0, the probability of  $(s, h^0)$  is determined by the exogenous assignment of initial portfolios and information. In any period  $t \geq 1$ , given that agents are randomly matched, the probability of receiving offer  $z \in R^2$  for the responding agent is

$$\psi_t(z|s) = \int_{h^{t-1}} \sigma_t^p(z|h^{t-1}) dP(h^{t-1}|s),$$

and the probability that offer  $z \in R^2$  is accepted for the proposing agent is

$$\chi_t(z|s) = \int_{h^{t-1}} \sigma_t^r(h^{t-1}, z) dP(h^{t-1}|s).$$

Given  $P(s, h^{t-1})$ ,  $\psi_t(\cdot|s)$ , and  $\chi_t(\cdot|s)$ , we can then construct  $P(s, h^t)$  as follows. For an agent with history  $h^{t-1}$ , the probability of reaching history  $h^t = (h^{t-1}, (0, z, 1))$  at  $t + 1$  is

$$P(s, h^t) = \frac{1}{2} \sigma^r(h^{t-1}, z) \psi_t(z|s) P(s, h^{t-1}),$$

since the probability of being selected as the responder is  $1/2$ , the probability of receiving offer  $z$  is  $\psi_t(z|s)$ , and the probability of accepting it is  $\sigma^r(h^{t-1}, z)$ . In a similar way, we have

$$P(s, h^t) = \frac{1}{2} (1 - \sigma^r(h^{t-1}, z)) \psi_t(z|s) P(s, h^{t-1}) \quad \text{if } h_t = (0, z, 0),$$

$$P(s, h^t) = \frac{1}{2} \chi_t(z|s) \sigma^p(z|h^{t-1}) P(s, h^{t-1}) \quad \text{if } h_t = (1, z, 1),$$

$$P(s, h^t) = \frac{1}{2} (1 - \chi_t(z|s)) \sigma^p(z|h^{t-1}) P(s, h^{t-1}) \quad \text{if } h_t = (1, z, 0).$$

To assess whether  $\sigma$  is individually optimal, agents have to form expectations about their opponents' behavior. Beliefs are described by two functions:

$$\delta_t : H^{t-1} \rightarrow [0, 1],$$

$$\delta_t^r : H^{t-1} \times R^2 \rightarrow [0, 1],$$

which represent, respectively, the probability assigned to signal  $s_1$  after history  $h^{t-1}$ , at the beginning of the period, and the probability assigned to signal  $s_1$  after history  $h^{t-1}$ , if the agent is the responder and receives offer  $z$ . The agent's beliefs are denoted compactly by  $\delta = \{\delta_t, \delta_t^r\}_{t=1}^\infty$ . At each history  $h^{t-1}$ , an agent

expects that in each period  $\tau \geq t$ , he will face an opponent with history  $\tilde{h}^{\tau-1}$  randomly drawn from the probability distribution  $P(\tilde{h}^{\tau-1}|s)$ , conditional on  $s$ , and he expects his opponent to play the strategy  $\sigma$ . This completely describes the agent's expectations about the current and future behavior of other players. For example, the probability distribution of offers expected at time  $\tau \geq t$  by an agent at  $h^{t-1}$  is equal to

$$\psi_\tau(z|s_1)\delta_t(h^{t-1}) + \psi_\tau(z|s_2)(1 - \delta_t(h^{t-1})).$$

The beliefs  $\delta_t$  are required to be consistent with Bayesian updating on the equilibrium path. This implies that

$$\delta_t(h^{t-1}) = \frac{P(s_1, h^{t-1})}{\sum_s P(s, h^{t-1})},$$

for all histories  $h^{t-1}$  such that  $\sum_s P(s, h^{t-1}) > 0$ . The same requirement is imposed on the beliefs  $\delta_t^r$ , which implies that

$$\delta_t^r(h^{t-1}, z) = \frac{\psi_t(z|s_1)P(s_1, h^{t-1})}{\sum_s \psi_t(z|s)P(s, h^{t-1})},$$

for all histories  $h^{t-1}$  and offers  $z$  such that  $\sum_s \psi_t(z|s)P(s, h^{t-1}) > 0$ .

This representation of the agents' beliefs embeds an important assumption: an agent who observes his opponent play an off-the-equilibrium-path action can change his beliefs about  $s$ , but maintains that the behavior of all other agents, conditional on  $s$ , is unchanged. That is, he believes that all other agents will continue to play  $\sigma$  in the future. This is a reasonable restriction on off-the-equilibrium-path beliefs in a game with atomistic agents and allows us to focus on the agent's beliefs about  $s$ , given that  $s$  is a sufficient statistic for the future behavior of the agent's opponents.

Moreover, the beliefs of informed agents are required to always assign probability 1 to the signal observed at date 0:

$$\begin{aligned} \delta_t(h^{t-1}) &= \delta_t^r(h^{t-1}, z) = 1 \quad \text{if } h_0 = (i, I_1), \\ \delta_t(h^{t-1}) &= \delta_t^r(h^{t-1}, z) = 0 \quad \text{if } h_0 = (i, I_2). \end{aligned}$$

That is, informed agents do not change their beliefs on signal  $s$ , even after observing off-the-equilibrium-path behavior from their opponents. This fact plays a useful role in the analysis, since it allows us to characterize the behavior of informed agents after any possible offer.

We are now ready to define an equilibrium.



DEFINITION 1: A perfect Bayesian equilibrium is given by a strategy  $\sigma$ , beliefs  $\delta$ , and a probability space  $(\Omega, \mathcal{F}, P)$ , such that:

- (i) the strategy  $\sigma$  is individually optimal at each history  $h^{t-1}$  given the beliefs  $\delta$  and given that agents expect that at each round  $\tau \geq t$  they will face an opponent with history  $\tilde{h}^{\tau-1}$  randomly drawn from  $P(\tilde{h}^{\tau-1}|s)$  who plays  $\sigma$ ;
- (ii) the beliefs  $\delta$  are consistent with Bayes's rule whenever possible;
- (iii) the probability measure  $P$  is consistent with  $\sigma$ .

Notice that the cross sectional behavior of the economy in equilibrium is purely determined by the signal  $s$ . In other words,  $s$  is the only relevant aggregate state variable for our trading game, and, for this reason, we call it interchangeably *signal*  $s$  or *state*  $s$ .

To establish our results, we restrict attention to equilibria that satisfy two properties, which we call *symmetry across states* and *uniform market clearing*. Let us first state these two properties and then discuss their role in the analysis.

Symmetry across states means that strategies and beliefs are the same if we switch the labels of assets 1 and 2 and those of signals 1 and 2. Formally, define  $\tilde{h}^t$  as the *complement* of history  $h^t$  if the following are true: (i) if  $(x^1, x^2)$  is the initial endowment in  $h_0$ , then  $(x^2, x^1)$  is the initial endowment in  $\tilde{h}_0$ ; (ii) if the agent is informed and observes  $s_j$  in  $h_0$ , he is informed and observes  $s_{-j}$  in  $\tilde{h}_0$ ; (iii) if offer  $z = (z^1, z^2)$  is made/received in  $h_t$ , offer  $z = (z^2, z^1)$  is made/received in  $\tilde{h}_t$ ; (iv) responses are the same in  $\tilde{h}_t$  and  $h_t$ . We can then define symmetry across states.

DEFINITION 2: An equilibrium satisfies symmetry across states if the strategy and beliefs  $\sigma$  and  $\delta$  satisfy the following: (a)  $\sigma_{t+1}^p((z^1, z^2)|h^t) = \sigma_{t+1}^p((z^2, z^1)|\tilde{h}^t)$  and  $\sigma_{t+1}^r(h^t, (z^1, z^2)) = \sigma_{t+1}^r(\tilde{h}^t, (z^2, z^1))$ ; (b)  $\delta(h^t) = 1 - \delta(\tilde{h}^t)$  and  $\delta^r(h^t, (z^1, z^2)) = 1 - \delta^r(\tilde{h}^t, (z^2, z^1))$  for all  $h^t$  and  $(z^1, z^2)$ , where  $\tilde{h}^t$  is the complement of  $h^t$ .

This restriction is more stringent than the standard symmetry requirement that all agents follow the same strategy, which we also assume. Symmetry across states helps in two steps of our analysis: in the proof of Lemma 7, which is needed to prove Proposition 2, and in the proof of Theorem 1. We discuss its role in detail when we present these results.

Uniform market clearing requires that market clearing approximately holds for agents with asset holdings in an interval  $[0, M]$ , for  $M$  large enough.

DEFINITION 3: A symmetric equilibrium satisfies uniform market clearing if, for all  $\varepsilon > 0$ , there is an  $M$  such that

$$\int_{x_t^j(\omega) \leq M} x_t^j(\omega) dP(\omega|s) \geq 1 - \varepsilon,$$

for all  $t$  and for all  $j$ .

For a given  $t$ , this property is just an implication of market clearing and of the dominated convergence theorem. The additional restriction comes from imposing that the property holds uniformly over  $t$ . Notice that all equilibria in which the portfolios  $x_t$  converge almost surely satisfy uniform market clearing.<sup>7</sup>

Unfortunately, we do not have a general existence proof of equilibria that satisfy symmetry across states and uniform market clearing. In the Supplemental Material (Golosov, Lorenzoni, and Tsyvinski (2014)), we present two examples for which we can show, by construction, the existence of equilibria with these properties. In a companion paper, we take a computational approach and compute equilibria with these properties for a larger set of cases.

### 3. LONG-RUN EFFICIENCY

In this section, we characterize the equilibrium in the long run, that is, along the path where the game does not end. Our main result is that the equilibrium allocation converges to an ex post Pareto efficient allocation. By ex post Pareto efficient, we mean Pareto efficient after  $s$  is publicly revealed but before  $S$  is revealed.<sup>8</sup>

After finitely many rounds of trading, the allocation will not be, in general, Pareto efficient, due to the matching friction. For example, with positive probability an agent could meet only agents with his same endowment and would not be able to trade. However, if the agents keep playing the game, they will eventually meet other agents with whom profitable trades are possible. Absent informational frictions, with a long enough horizon, all potential gains from trade are eventually realized and the allocation converges to efficiency. Different versions of this result under symmetric information were discussed in Gale (2000).

With asymmetric information, it is harder to show that all profitable trades will be exhausted. Now, when two agents meet, there may be a Pareto improving trade between them conditional on  $s$ , but since  $s$  is not commonly observed the agents may not be able to credibly signal to each other the presence of this trade. For example, suppose the state is  $s_1$  and an informed agent with a relatively large amount of asset 1 meets an informed agent with a relatively small amount of it. The informed agent would like to trade asset 1 for asset 2 at a

<sup>7</sup>Use Theorem 16.14 in Billingsley (1995). This assumption would not be required in a trading game with a large but *finite* number of agents, as in that case there would be a natural upper bound on the assets holdings of each agent, given by the aggregate endowment. However, to extend the model to a finite number of agents is not trivial since a law-of-large-numbers argument cannot be invoked, so the aggregate state of the game is not just  $s$ . Moreover, to derive limit theorems in trading games with a large but finite number of agents, one usually needs to impose further restrictions on strategies, as has been shown in symmetric information environments (Rubinstein and Wolinsky (1990) and Gale (2000, Chapter 3)).

<sup>8</sup>This is the standard notion of ex post efficiency as in Holmstrom and Myerson (1983). After  $S$  is revealed, all allocations are trivially efficient, as only one asset has positive value.

price that is mutually beneficial conditional on  $s_1$ . But the uninformed responder may reject the offer because he is afraid that the proposer has observed  $s_2$  and is trying to sell asset 1 because it is less valuable. Can this prevent the economy from achieving efficiency in the long run? Our main result shows that the answer is no.

### 3.1. Preliminary Considerations

We first define and characterize the stochastic process for an agent's expected utility in equilibrium. We use the martingale convergence theorem to show that expected utility converges in the long run, conditionally on the game not ending.

Take the probability space  $(\Omega, \mathcal{F}, P)$  and let  $x_t(\omega)$  and  $\delta_t(\omega)$  denote the portfolio and belief of the agent at the beginning of period  $t$ , at  $\omega$ . Since an agent's current portfolio and belief are, by construction, in his information set at time  $t$ ,  $x_t(\omega)$  and  $\delta_t(\omega)$  are  $\mathcal{F}_t$ -measurable stochastic processes on  $(\Omega, \mathcal{F}, P)$ . If the game ends, an agent with the portfolio-belief pair  $(x, \delta)$  receives the expected payoff

$$U(x, \delta) \equiv \pi(\delta)u(x^1) + (1 - \pi(\delta))u(x^2),$$

where  $\pi(\delta)$  is the probability that an agent with belief  $\delta$  assigns to state  $S_1$ ,

$$\pi(\delta) \equiv \delta\phi(s_1) + (1 - \delta)\phi(s_2).$$

Given the processes  $x_t$  and  $\delta_t$ , we obtain the stochastic process  $u_t \equiv U(x_t, \delta_t)$ , which gives the equilibrium expected utility of an agent if the trading game ends in  $t$ . Finally, we obtain the stochastic process

$$v_t \equiv (1 - \gamma)E\left[\sum_{s=t}^{\infty} \gamma^{s-t} u_s \mid \mathcal{F}_t\right],$$

which gives the expected lifetime utility at the beginning of period  $t$ . This process satisfies the recursion

$$(2) \quad v_t = (1 - \gamma)u_t + \gamma E[v_{t+1} \mid \mathcal{F}_t].$$

Notice that, since  $v_t$  is constructed using the equilibrium stochastic processes for  $x_t$  and  $\delta_t$ , it represents the expected utility from following the equilibrium strategy, which is, by definition, individually optimal. Using this fact, the next lemma establishes that  $v_t$  is a bounded martingale and converges in the long run.

LEMMA 1: *There exists a random variable  $v^\infty(\omega)$  such that*

$$\lim_{t \rightarrow \infty} v_t(\omega) = v^\infty(\omega) \quad a.s.$$

PROOF: An agent always has the option to keep his time  $t$  portfolio  $x_t$  and wait for the end of the game, rejecting all offers and offering zero trades in all  $t' \geq t$ . His expected lifetime utility under this strategy is equal to  $u_t$ . Therefore, optimality implies  $u_t \leq E[v_{t+1} | \mathcal{F}_t]$ , which, combined with equation (2), gives  $v_t \leq E[v_{t+1} | \mathcal{F}_t]$ . This shows that  $v_t$  is a submartingale. It is bounded above because the utility function  $u(\cdot)$  is bounded above. Therefore, it converges by the martingale convergence theorem. Q.E.D.

It is useful to introduce an additional stochastic process,  $\hat{v}_t$ , which will be used as a reference point to study the behavior of agents who make and receive off-the-equilibrium-path offers. Let  $\hat{v}_t$  be the expected lifetime utility of an agent who adopts the following strategy: (i) if selected as the proposer at time  $t$ , follow the equilibrium strategy  $\sigma$ ; (ii) if selected as the responder, reject all offers at time  $t$  and follow an optimal continuation strategy from  $t + 1$  onward. The expected utility  $\hat{v}_t$  is computed at time  $t$  immediately after the agent is selected as the proposer or the responder, that is, it is measurable with respect to  $(h^{t-1}, \iota_t)$ , and, by definition, satisfies  $\hat{v}_t \leq E[v_{t+1} | h^{t-1}, \iota_t]$ .

Notice that  $u_t$  is the expected utility from holding the portfolio  $x_t$  until the end of the game. The following lemma shows that, in the long run, an agent is almost as well off keeping his time  $t$  portfolio as he is under the strategy leading to  $\hat{v}_t$ .

LEMMA 2: *Both  $u_t$  and  $\hat{v}_t$  converge almost surely to  $v^\infty$ :*

$$\lim_{t \rightarrow \infty} u_t(\omega) = \lim_{t \rightarrow \infty} \hat{v}_t(\omega) = v^\infty(\omega) \quad a.s.$$

PROOF: As argued in Lemma 1,  $v_t$  is a bounded submartingale and converges almost surely to  $v^\infty$ . Let  $y_t \equiv E[v_{t+1} | \mathcal{F}_t]$ . Since a bounded martingale is uniformly integrable (see Williams (1991)), we get  $y_t - v_t \rightarrow 0$  almost surely. Rewrite equation (2) as

$$(1 - \gamma)u_t = \gamma(v_t - E[v_{t+1} | \mathcal{F}_t]) + (1 - \gamma)v_t.$$

This gives

$$u_t - v_t = \frac{\gamma}{1 - \gamma}(v_t - E[v_{t+1} | \mathcal{F}_t]) = \frac{\gamma}{1 - \gamma}(v_t - y_t),$$

which implies  $u_t - v_t \rightarrow 0$  almost surely. The latter implies  $u_t \rightarrow v^\infty$  almost surely. Letting  $\hat{y}_t \equiv E[v_{t+1} | h^{t-1}, \iota_t]$ , notice that  $\hat{y}_t \rightarrow v^\infty$  almost surely. Since  $u_t \leq \hat{v}_t \leq \hat{y}_t$ , it follows that  $\hat{v}_t \rightarrow v^\infty$  almost surely. Q.E.D.

### 3.2. Informed Agents

We first focus on informed agents and show that their marginal rates of substitution converge in probability. In particular, we show that, conditional on  $s$ ,

the marginal rates of substitution of all informed agents converge in probability to the same sequence, which we denote  $\kappa_i(s)$ . In the following, we refer to  $\kappa_i(s)$  as the long-run marginal rate of substitution of the informed agents.

Since this result is about informed agents, the argument is similar to the one used in decentralized markets with full information. If two informed agents have different marginal rates of substitution, they can find a trade that improves the utility of both. As their utilities converge to their long-run levels, all the potential gains from bilateral trade must be exhausted, so their marginal rates of substitution must converge.

**PROPOSITION 1—Convergence of MRS for Informed Agents:** *There exist two sequences  $\kappa_i(s_1)$  and  $\kappa_i(s_2)$  such that, conditional on each  $s$ , the marginal rates of substitution of informed agents converge in probability to  $\kappa_i(s)$ :*

$$(3) \quad \lim_{t \rightarrow \infty} P\left(\left|\frac{\phi(s)u'(x_t^1)}{(1-\phi(s))u'(x_t^2)} - \kappa_i(s)\right| > \varepsilon \mid \delta_t = \delta^I(s), s\right) = 0$$

for all  $\varepsilon > 0$ .

**PROOF:** We provide a sketch of the proof here and leave the details to the [Appendix](#). Without loss of generality, suppose (3) is violated for  $s = s_1$ . Then, it is always possible to find a period  $T$ , arbitrarily large, in which there are two groups, of positive mass, of informed agents with marginal rates of substitution sufficiently different from each other. In particular, we can find a  $\kappa^*$  such that a positive mass of informed agents have marginal rates of substitution below  $\kappa^*$ :

$$\frac{\phi(s_1)u'(x_T^1)}{(1-\phi(s_1))u'(x_T^2)} < \kappa^*,$$

and a positive mass of informed agents have marginal rates of substitution above  $\kappa^* + \varepsilon$ :

$$\frac{\phi(s_1)u'(x_T^1)}{(1-\phi(s_1))u'(x_T^2)} > \kappa^* + \varepsilon,$$

for some positive  $\varepsilon$ . An informed agent in the first group can then offer to sell a small quantity  $\zeta^*$  of asset 1 at the price  $p^* = \kappa^* + \varepsilon/2$ , that is, he can offer the trade  $z^* = (\zeta^*, -p^*\zeta^*)$ . Suppose this offer is accepted and the proposer stops trading afterward. Then his utility can be approximated as follows:

$$\begin{aligned} & \phi(s_1)u(x_T^1 - \zeta^*) + (1-\phi(s_1))u(x_T^2 + p^*\zeta^*) \\ & \approx u_T + [-\phi(s_1)u'(x_T^1) + (1-\phi(s_1))p^*u'(x_T^2)]\zeta^* \\ & \approx \hat{v}_T + (1-\phi(s_1))u'(x_T^2)\zeta^*\varepsilon/2, \end{aligned}$$

where we use a Taylor expansion to approximate the utility gain and we use Lemma 2 to show that the continuation utility  $\hat{v}_T$  can be approximated by the current utility  $u_T$ . By choosing  $T$  sufficiently large and the size of the trade  $\zeta^*$  sufficiently small, we can make the approximation errors in the above equation small enough, so that when this trade is accepted it strictly improves the utility of the proposer. All the informed responders with marginal rate of substitution above  $\kappa^* + \varepsilon$  are also better off, by a similar argument. Therefore, they will all accept the offer. Since there is a positive mass of them, the strategy described gives strictly higher utility than the equilibrium strategy to the proposer, and we have a contradiction. *Q.E.D.*

### 3.3. Uninformed Agents

We now turn to the characterization of equilibria for uninformed agents. The main difficulty here is that uninformed agents may change their beliefs upon observing their opponent's behavior. Thus an agent who would be willing to accept a given trade *ex ante*—before updating his beliefs—might reject it *ex post*. Moreover, updated beliefs are not determined by Bayes's rule after off-the-equilibrium-path offers, and our objective is to develop a general argument, independent of how off-the-equilibrium-path beliefs are specified. For these reasons, we need a strategy of proof different from the one used for informed agents.

Our argument is based on finding strategies that allow the uninformed agents to learn the signal  $s$  at an arbitrarily small cost. This is done in Lemma 3 below. The existence of such strategies implies that either uninformed agents eventually learn the signal or the benefit of learning goes to zero.

To build our argument, we first show that, in equilibrium, the marginal rates of substitution of all agents cannot converge to the same value *independently of the state  $s$* . Since individual marginal rates of substitution determine the prices at which agents are willing to trade, this rules out equilibria in which agents, in the long run, are all willing to trade at the same price, independent of  $s$ . The fact that agents are willing to trade at different prices in the two states  $s_1$  and  $s_2$  will be key in constructing the experimentation strategies below. This fact will allow us to construct small trades that are accepted with different probability in the two states. By offering such trades, an uninformed agent will be able to extract information on  $s$  and thus acquire the information obtained by the informed agents at date 0.

Remember that  $\kappa_t(s)$  denotes the long-run marginal rates of substitution of informed agents in state  $s$ . The next proposition shows that, in the long run, two cases are possible: either the two values  $\kappa_t(s_1)$  and  $\kappa_t(s_2)$  are sufficiently far from each other, or, in each state  $s$ , there is a sufficient mass of agents with marginal rates of substitution far enough from  $\kappa_t(s)$ . That is, either the informed agents' marginal rates of substitution converge to different values or there are enough uninformed agents with marginal rates of substitution different from that of the informed.

PROPOSITION 2: *Consider an equilibrium that satisfies symmetry across states and uniform market clearing. There exists a period  $T$  and a scalar  $\bar{\varepsilon} > 0$  such that, in all periods  $t \geq T$ , one of the following holds: (i) the long-run marginal rates of substitution of the informed agents are sufficiently different in the two states:*

$$|\kappa_t(s_1) - \kappa_t(s_2)| \geq \bar{\varepsilon},$$

*or (ii) sufficiently many agents have a marginal rate of substitution different from  $\kappa_t(s)$ :*

$$P\left(\left|\frac{\pi(\delta_t)u'(x_t^1)}{(1 - \pi(\delta_t))u'(x_t^2)} - \kappa_t(s)\right| \geq 2\bar{\varepsilon} \mid s\right) \geq \bar{\varepsilon}$$

*for  $s \in \{s_1, s_2\}$ .*

The proof of this proposition is in the [Appendix](#). The argument is as follows. If both (i) and (ii) are violated, we can always find a period  $t$  in which all the agents' marginal rates of substitution are concentrated around some value  $\kappa$  which is independent of the state  $s$ . The point of the proof is to show that this leads to a violation of market clearing. We first show that the distribution of beliefs of the uninformed is always biased in the direction of the true signal. That is, in  $s_1$ , there are more uninformed agents with  $\delta_t \geq 1/2$  than uninformed agents with  $\delta_t < 1/2$ . Using symmetry across states, we can then show that, in  $s_1$ , market clearing holds if we sum the asset holdings of the informed agents and of the subset of uninformed agents with beliefs  $\delta_t \geq 1/2$ . Next, we show that if the marginal rates of substitution of all agents are the same and independent of  $s$ , this implies that, in state  $s_1$ , all agents with  $\delta_t \geq 1/2$  would hold weakly more of asset 1 than of asset 2, and a positive mass of informed agents with  $\delta_t > 1/2$  would hold strictly more of asset 1. Since the endowments of the two assets are the same, this leads to a contradiction.

### 3.3.1. Experimentation

We now show how uninformed agents can experiment and acquire information on the state  $s$  by making small offers. In the proof of Proposition 3, we construct a sequence of offers with the following property: given any  $\varepsilon > 0$ , if the uninformed agent makes the offers  $\{\hat{z}_{t+j}\}_{j=0}^{J-1}$  at times  $t, t+1, \dots, t+J-1$ , and receives the “right” string of responses (e.g.,  $\{\hat{r}_{t+j}\}_{j=0}^{J-1} = \{0, 1, 1, 0, \dots, 1\}$ ), then the probability he assigns to  $s_1$  at time  $t+J$  will be larger than  $1 - \varepsilon$ . That is, this sequence of offers allows the experimenter to acquire arbitrarily precise information on state  $s_1$  (a similar construction can be done for  $s_2$ ). Here we make the crucial step in the construction of this sequence of offers. Namely, we find a single offer  $z$  such that if the right response is received, the proposer's belief increases by a sufficient amount.

Consider an uninformed agent who assigns probability  $\delta \in (0, 1)$  to signal  $s_1$  at the beginning of period  $t$  and makes offer  $z$ . Recall that the probability of acceptance of  $z$ , conditional on  $s$ , is  $\chi_t(z|s)$ . Bayes's rule implies that if the offer is accepted, the agent's updated belief  $\delta'$  satisfies

$$\frac{\delta'}{1 - \delta'} = \frac{\delta}{1 - \delta} \frac{\chi_t(z|s_1)}{\chi_t(z|s_2)},$$

while if the offer is rejected, his updated belief satisfies

$$\frac{\delta'}{1 - \delta'} = \frac{\delta}{1 - \delta} \frac{1 - \chi_t(z|s_1)}{1 - \chi_t(z|s_2)}.$$

If  $\chi_t(z|s_1) > \chi_t(z|s_2)$ , the acceptance of offer  $z$  provides a signal in favor of  $s_1$ ; if  $\chi_t(z|s_1) < \chi_t(z|s_2)$ , the rejection of offer  $z$  provides a signal in favor of  $s_1$ . Our objective is to find a constant  $\rho > 1$ , such that we can always find an offer  $z$  such that either

$$\frac{\chi_t(z|s_1)}{\chi_t(z|s_2)} > \rho$$

or

$$\frac{1 - \chi_t(z|s_1)}{1 - \chi_t(z|s_2)} > \rho.$$

In this way, if the agent makes offer  $z$  and receives the right response (a “yes” in the first case, a “no” in the second), his beliefs satisfy

$$\frac{\delta'}{1 - \delta'} > \rho \frac{\delta}{1 - \delta}.$$

Since  $\rho > 1$ , this ensures that we can choose a long enough sequence of offers such that, if the right responses are received, the agent's belief will converge to 1.

The following lemma shows how to construct the offer  $z$ .

**LEMMA 3:** *Consider an equilibrium that satisfies symmetry across states and uniform market clearing. There are two scalars  $\beta > 0$  and  $\rho > 1$  with the following property: for all  $\theta > 0$ , there is a time  $T$  such that, for all  $t \geq T$ , there exists a trade  $z$  with  $\|z\| < \theta$  that satisfies either*

$$(4) \quad \chi_t(z|s_1) > \beta, \quad \chi_t(z|s_1) > \rho \chi_t(z|s_2),$$

or

$$(5) \quad 1 - \chi_t(z|s_1) > \beta, \quad 1 - \chi_t(z|s_1) > \rho(1 - \chi_t(z|s_2)).$$



PROOF: We provide a sketch of the argument here and leave the details to the Supplemental Material. We distinguish two cases. By Proposition 2, one of the following must be true in any period  $t$  following some period  $T$ : (i) either the long-run marginal rates of substitutions of informed agents  $\kappa_t(s_1)$  and  $\kappa_t(s_2)$  are sufficiently different from each other, or (ii) there is a sufficiently large mass of agents with marginal rates of substitution sufficiently different from  $\kappa_t(s)$ . The proof proceeds differently in the two cases.

*Case 1.* Suppose that there is a large enough difference between  $\kappa_t(s_1)$  and  $\kappa_t(s_2)$ . Assume without loss of generality that  $\kappa_t(s_1) > \kappa_t(s_2)$ . Suppose the uninformed agent offers to sell a small quantity  $\zeta$  of asset 1 at the price  $p = (\kappa_t(s_1) + \kappa_t(s_2))/2$ , which lies between the two marginal rates of substitutions  $\kappa_t(s_1)$  and  $\kappa_t(s_2)$ . That is, he offers the trade  $z = (\zeta, -p\zeta)$ . We now make two observations on offer  $z$ :

*Observation 1.* In state  $s_1$ , there is a positive mass of informed agents willing to accept offer  $z$ , provided  $\zeta$  is small enough and  $t$  is sufficiently large. Combining Lemma 2 and Proposition 1, we can show that in state  $s_1$ , for  $t$  large enough, there is a positive mass of informed agents with marginal rates of substitution sufficiently close to  $\kappa_t(s_1)$ , who are close enough to their long-run utility. These agents are better off accepting  $z$ , as they are buying asset 1 at a price smaller than their marginal valuation.

*Observation 2.* Conditional on signal  $s_2$ , the offer  $z$  cannot be accepted by any agent, informed or uninformed, except possibly by a vanishing mass of agents. Suppose, to the contrary, that a positive fraction of agents accepted  $z$  in state  $s_2$ . By an argument symmetric to the one above, informed agents in state  $s_2$  are strictly better off *making* the offer  $z$ , if this offer is accepted with positive probability, given that they would be selling asset 1 at a price higher than their marginal valuation (which converges to  $\kappa_t(s_2)$  by Proposition 1). But then an optimal deviation on their part is to make such an offer and strictly increase their expected utility above its equilibrium level, leading to a contradiction.

The first observation can be used to show that the probability of acceptance  $\chi_t(z|s_1)$  can be bounded from below by a positive number. The second observation can be used to show that the probability of acceptance  $\chi_t(z|s_2)$  can be bounded from above by an arbitrarily small number. These two facts imply that we can make  $\chi_t(z|s_1) > \beta$  for some  $\beta > 0$  and  $\chi_t(z|s_1)/\chi_t(z|s_2) > \rho$  for some  $\rho > 1$ . So, in this case, we can always find a trade such that (4) is satisfied, that is, such that the acceptance of  $z$  is good news for  $s_1$ . However, when we turn to the next case, this will not always be true, and we will need to allow for the alternative condition (5), that is, rejection of  $z$  is good news for  $s_1$ .

*Case 2.* Consider now the second case where the long-run marginal rates of substitution of the informed agents  $\kappa_t(s_1)$  and  $\kappa_t(s_2)$  are close enough but there is a large enough mass of uninformed agents whose marginal rates of substitution are far from  $\kappa_t(s_1)$ , conditional on  $s_1$ .

This means that we can find a price  $p$  such that the marginal rates of substitution of a group of uninformed agents are on one side of  $p$  and the long-run

marginal rates of substitution of informed agents  $\kappa_t(s_1)$  and  $\kappa_t(s_2)$  are on the other side. Consider the case where the MRS of a group of uninformed agents is greater than  $p$ , and  $\kappa_t(s_1)$  and  $\kappa_t(s_2)$  are smaller than  $p$  (the other case is symmetric). Then the uninformed agents in this group can make a small offer to buy asset 1 at a price  $p$  and the informed will accept this offer conditional on both signals  $s_1$  and  $s_2$ . If the probabilities of acceptance conditional on  $s_1$  and  $s_2$  were sufficiently close to each other, this would be a profitable deviation for the uninformed, since then their ex post beliefs would be close to their ex ante beliefs. In other words, the uninformed would not learn from the trade, but making the offer would increase their expected utility relative to their equilibrium strategy, leading to a contradiction. It follows that the probabilities of acceptance of this trade must be sufficiently different in the two states  $s_1$  and  $s_2$ . This leads to either (4) or (5), completing the proof. *Q.E.D.*

### 3.3.2. Convergence of Marginal Rates of Substitution

We now characterize the properties of the long-run marginal rates of substitution of uninformed agents. The next proposition shows that the convergence result established for informed agents (Proposition 1) extends to uninformed agents.

In what follows, instead of looking at the ex ante marginal rate of substitution, given by  $\pi(\delta_t)u'(x_t^1)/(1 - \pi(\delta_t))u'(x_t^2)$ , we establish convergence for the ex post marginal rate of substitution  $\phi(s)u'(x_t^1)/(1 - \phi(s))u'(x_t^2)$ . This is the marginal rate of substitution at which an agent would be willing to trade asset 2 for asset 1 if he could observe the signal  $s$ . As we will see, this is the appropriate convergence result given our objective, which is to establish the ex post efficiency of the equilibrium allocation.

**PROPOSITION 3—Convergence of MRS for Uninformed Agents:** *Consider an equilibrium that satisfies symmetry across states and uniform market clearing. Conditional on each  $s$ , the marginal rate of substitution of any agent, evaluated at the full information probabilities  $\phi(s)$  and  $1 - \phi(s)$ , converges in probability to  $\kappa_t(s)$ :*

$$(6) \quad \lim_{t \rightarrow \infty} P\left(\left|\frac{\phi(s)u'(x_t^1)}{(1 - \phi(s))u'(x_t^2)} - \kappa_t(s)\right| > \varepsilon \mid s\right) = 0 \quad \text{for all } \varepsilon > 0.$$

**PROOF:** We provide a sketch of the proof, leaving the details to the [Appendix](#). Suppose condition (6) fails to hold. Without loss of generality, we focus on the case where (6) fails for  $s = s_1$ . This means that there is a period  $T$  in which, with a positive probability, an uninformed agent has ex post marginal rate of substitution sufficiently far from  $\kappa_T(s_1)$  and is sufficiently close to his long-run utility. Without loss of generality, suppose his marginal rate of substitution is larger than  $\kappa_T(s_1)$ . To reach a contradiction, we construct a profitable deviation for this agent.

Before discussing the deviation, it is useful to clarify that, at time  $T$ , the uninformed agent has all the necessary information to check whether he should deviate or not. He can observe his own allocation  $x_T$ , compute  $\phi(s_1)u'(x_T^1)/(1 - \phi(s_1))u'(x_T^2)$ , and verify whether this quantity is sufficiently larger than  $\kappa_T(s_1)$  (which is known, since it is an equilibrium object).

The deviation then consists of two stages:

*Stage 1.* This is the experimentation stage, which lasts from period  $T$  to period  $T + J - 1$ . As stated in Lemma 3, the agent can construct a sequence of small offers  $\{\hat{z}_j\}_{j=0}^{J-1}$  such that, if these offers are followed by the appropriate responses, the agent's ex post belief on signal  $s_1$  will converge to 1. To be precise, for this to be true it must be the case that the agent does not start his deviation with a belief  $\delta_T$  too close to 0. Otherwise, a sequence of  $J$  signals favorable to  $s_1$  is not enough to bring  $\delta_{T+J}$  sufficiently close to 1. Therefore, when an agent starts deviating, we also require  $\delta_T$  to be larger than some positive lower bound  $\underline{\delta}$ , appropriately defined.

*Stage 2.* At date  $T + J$ , if the agent has been able to make the whole sequence of offers  $\{\hat{z}_j\}_{j=0}^{J-1}$  and has received the appropriate responses (i.e., the responses that bring the probability of  $s_1$  close to 1), he then makes one final offer  $z^*$ . This is an offer to buy a small quantity  $\zeta^*$  of asset 1 at a price  $p^*$ , which is in between the agent's own marginal rate of substitution and  $\kappa_T(s_1)$ . By choosing  $T$  large enough, we can ensure that there is a positive mass of informed agents close enough to their long-run marginal rate of substitution, who are willing to sell asset 1 at that price.<sup>9</sup> Therefore, the offer is accepted with a positive probability. The utility gain for the uninformed agent, conditional on reaching Stage 2 and conditional on  $z^*$  being accepted, can be approximated by

$$U\left(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 1\right) - U(x_T, 1),$$

given that, after the experimentation stage, the agent's ex post belief approaches 1. Moreover, by making the final offer  $z^*$  and the experimenting offers  $\hat{z}_j$  sufficiently small, this utility gain can be approximated by

$$\begin{aligned} & U(x_T + z^*, 1) - U(x_T, 1) \\ & \approx \phi(s)u'(x_T^1)\zeta^* - (1 - \phi(s))u'(x_T^2)p^*\zeta^* > 0. \end{aligned}$$

The last expression is positive because  $p^*$  was chosen smaller than the marginal rate of substitution  $\phi(s)u'(x_T^1)/(1 - \phi(s))u'(x_T^2)$ . In the [Appendix](#), we show

<sup>9</sup>Notice that the uninformed agent is using  $\kappa_T(s_1)$  as a reference point for the informed agents' marginal rate of substitution, while offer  $z^*$  is made in period  $T + J$ . Lemma 9 in the [Appendix](#) ensures that  $\kappa_T(s_1)$  and  $\kappa_{T+J}(s_1)$  are sufficiently close, so that at  $T + J$  enough informed agents have marginal rate of substitution near  $\kappa_T(s_1)$ .

that this utility gain is large enough that the deviation described is ex ante profitable, that is, it is profitable from the point of view of period  $T$ . To do so, we must ensure that the utility losses that may happen along the deviating path (e.g., when some of the experimenting offers do not generate a response favorable to  $s_1$  or when the agent is not selected as the proposer) are small enough. To establish this, we use again the fact that the experimenting offers are small. The argument in the [Appendix](#) makes use of the convergence of utility levels in Lemma 2, to show that a utility gain relative to the current utility  $u_t$  leads to a profitable deviation relative to the expected utility  $\hat{v}_t$ . Since we found a profitable deviation for the uninformed agents, a contradiction is obtained which completes the argument. *Q.E.D.*

### 3.4. Main Result

Having characterized the portfolios of informed and uninformed agents in the long run, we can finally derive our efficiency result.

**THEOREM 1:** *All symmetric equilibrium allocations that satisfy symmetry across states and uniform market clearing converge to ex post efficient allocations in the long run, that is,*

$$(7) \quad \lim_{t \rightarrow \infty} P(|x_t^1 - x_t^2| > \varepsilon) = 0 \quad \text{for all } \varepsilon > 0.$$

*The long-run marginal rates of substitution  $\kappa_t(s)$  converge to the ratios of the conditional probabilities of states  $S_1$  and  $S_2$ :*

$$(8) \quad \lim_{t \rightarrow \infty} \kappa_t(s) = \phi(s)/(1 - \phi(s)) \quad \text{for all } s \in \{s_1, s_2\}.$$

**PROOF:** We provide a sketch of the proof and leave the formal details to the [Appendix](#). First, suppose that  $\kappa_t(s) > (1 + \varepsilon)\phi(s)/(1 - \phi(s))$  for some  $\varepsilon > 0$ , for infinitely many periods. Then Proposition 3 can be used to show that the agents' holdings of asset 1 will be larger than their holdings of asset 2. This, however, violates market clearing. In a similar way, we rule out the case in which  $\kappa_t(s) < (1 - \varepsilon)\phi(s)/(1 - \phi(s))$  for some  $\varepsilon > 0$ , for infinitely many periods. This proves (8). Then, using this result and Proposition 3, we can show that  $u'(x_t^1)/u'(x_t^2)$  converges in probability to 1, which implies (7). *Q.E.D.*

This theorem establishes that equilibrium allocations converge to ex post Pareto efficient allocations. It also shows, in (8), that the prices supporting the long-run allocation are the same as the rational expectations competitive equilibrium prices of the same economy, which are proportional to the probabilities of the two states.

It is useful to clarify that our results do not necessarily imply that uninformed agents will learn the value of the signal  $s$  in the long run. The proof of Proposition 3 shows that, in the long run, uninformed agents have the possibility to

learn the value of  $s$  with arbitrary precision at an arbitrarily low cost. However, as all agents' asset holdings converge to a perfectly diversified portfolio, the incentive to acquire this information also goes to zero. The reason is that uninformed agents know that when all asset holdings are converging to a perfectly diversified portfolio, further profitable trades are no longer possible. Therefore, our results imply that one of two possible outcomes are possible: either uninformed agents perfectly learn the signal or the value of learning the signal goes to zero. In the Supplemental Material, we present two examples showing that both these outcomes are indeed possible in equilibrium, depending on the model parameters.

To conclude this section, it is useful to compare our result to [Wolinsky \(1990\)](#). In our model, in the long run, all agents are only willing to trade at a single price, which corresponds to the rational expectations competitive equilibrium price  $\phi(s)/(1 - \phi(s))$ . [Wolinsky \(1990\)](#) also analyzed a dynamic trading game with asymmetric information and showed that, in steady state, different trades can occur at different prices, so a fraction of trades can occur at a price different from the rational expectations competitive price. That paper studied a game where a fraction of traders enter and exit at each point of time, focused on steady state equilibria, and took limits as discounting goes to zero. We consider a game with a fixed set of participants and a fixed probability of ending the game  $\gamma$  and study long-run outcomes. The key difference is that the model in [Wolinsky \(1990\)](#) features an indivisible good which can only be traded once. Our environment features perfectly divisible goods (assets) which are traded repeatedly. This makes the process of experimentation by market participants very different in the two environments. In [Wolinsky \(1990\)](#), agents only learn if their offers are rejected. Once the offer is accepted, they trade and exit the market. In our environment, agents keep learning and trading along the equilibrium play. In particular, they can learn by making small trades (as shown in Lemma 3) and then use the information acquired to make Pareto improving trades with informed agents (as shown in Proposition 3).

#### 4. CONCLUDING REMARKS

This paper analyzes long-run efficiency and the value of information in a dynamic trading game with private information. The main difficulty with our environment is that, due to private information, agents hold diverse beliefs about asset values in equilibrium and need to update these beliefs both on and off the equilibrium path. This means that standard arguments used in decentralized bargaining environments with full information cannot be applied. Nonetheless, proceeding by contradiction, we built arguments on learning and experimentation that are sufficiently powerful to characterize the long-run properties of the equilibrium without imposing additional restrictions on belief updating. To achieve this goal, we had to rely on some simplifying assumptions. We conclude with some remarks on the role of these assumptions.

We derived Theorem 1 in an environment with two states and two signals, so it is useful to discuss how the logic of our argument could be extended to more states and signals. The experimentation and deviation arguments in Proposition 3 can be easily extended to the case of finitely many states and signals, as long as markets are complete and there is an Arrow security for each state  $S$ . Take a signal  $s$  that carries information about two Arrow securities which pay in states  $S$  and  $S'$ . Partition the signal space in two subsets: a singleton that includes only  $s$  and the subset of all the other signals. Then the arguments in our binary environment can be adapted to prove that the marginal rates of substitution between assets  $S$  and  $S'$  must converge to the same value for all agents, informed and uninformed, conditional on  $s$ . However, other steps used to arrive to Theorem 1 are harder to generalize. In particular, Proposition 2 shows that the agents' marginal rates of substitution cannot converge to the same value in states  $s_1$  and  $s_2$ . The argument is by contradiction and shows that otherwise market clearing would be violated. For that argument, we use our two-signal environment and our assumption of symmetry across states to deal with the fact that, along the equilibrium path, uninformed agents hold, in general, a range of beliefs about the signal.<sup>10</sup> How to extend that argument to more general environments is an interesting open issue.

Another important simplifying assumption is that the informed agents have nothing to learn from trading, as they all obtain the only relevant piece of information at the beginning of the game. The benefit of this assumption is to have *some* agents in the economy holding fixed beliefs. Our argument is then built starting from the convergence of the marginal rates of substitution of these agents and then using these marginal rates of substitution as reference points for our experimentation steps. A challenging open question is what would happen in an environment in which different informed agents receive different pieces of information.

Finally, notice that throughout the paper we kept a fixed level of frictions in trading, by choosing a fixed value of  $\gamma$ . This parameter determines the random number of trading rounds before the game ends. All our long-run results implicitly depend on  $\gamma$ . That is, for given  $\gamma$ , there is a large enough  $T(\gamma)$  such that, for all periods  $t \geq T(\gamma)$ , efficiency holds with probability near 1. An important open question is what happens in our model as  $\gamma$  goes to 1 and the economy approaches frictionless trading. The Supplemental Material analyzes examples for which we can fully characterize the equilibrium for  $\gamma$  going to 1.

## APPENDIX

### A.1. Preliminary Results

The following three lemmas are used throughout the Appendix. The first is an elementary probability result. The second shows that in the long run,

<sup>10</sup>This is discussed in the paragraph following Proposition 2.

portfolios are in a compact set  $X$  with probability arbitrarily close to 1. This set will be used to ensure that some optimization problems used in the proofs are well defined. The third lemma shows that, given any two agents with portfolios in a compact set  $X$  and marginal rates of substitution that differ by at least  $\varepsilon$ , there is a trade that achieves a gain in current utility of at least  $\Delta$ , for some positive  $\Delta$ . The proofs of Lemmas 5 and 6 are in the Supplemental Material.

The function  $\mathcal{M}$  is defined as the ex ante marginal rate of substitution between the two assets:

$$\mathcal{M}(x, \delta) \equiv \frac{\pi(\delta)u'(x^1)}{(1 - \pi(\delta))u'(x^2)}.$$

LEMMA 4: *Take two sets  $A, B \subset \Omega$  such that  $P(A|s) \geq 1 - \varepsilon$  and  $P(B|s) > 1 - \eta$  for some positive scalars  $\varepsilon$  and  $\eta$ . Then,  $P(A \cap B|s) > 1 - \varepsilon - \eta$ .*

LEMMA 5: *For any  $\varepsilon > 0$  and any state  $s$ , there are a compact set  $X \subset R_{++}^2$  and a time  $T$  such that  $P(x_t \in X | s) \geq 1 - \varepsilon$  for all  $t \geq T$ .*

LEMMA 6: *Take a compact set  $X \in R_{++}^2$ . For any  $\varepsilon > 0$  and  $\theta > 0$ , there are  $\Delta > 0$  and  $\zeta > 0$  with the following property. Take any two agents with portfolios  $x_A, x_B \in X$  and beliefs  $\delta_A, \delta_B \in [0, 1]$  with marginal rates of substitution that differ by more than  $\varepsilon$ ,  $\mathcal{M}(x_B, \delta_B) - \mathcal{M}(x_A, \delta_A) > \varepsilon$ . Choose any price sufficiently close to the middle of the interval between the two marginal rates of substitution:*

$$p \in [\mathcal{M}(x_A, \delta_A) + \varepsilon/2, \mathcal{M}(x_B, \delta_B) - \varepsilon/2].$$

*Then the trade  $z = (\zeta, -p\zeta)$  satisfies  $\|z\| < \theta$ , and the utility gains of the two agents satisfy*

$$(9) \quad U(x_A - z, \delta_A) - U(x_A, \delta_A) \geq \Delta,$$

$$(10) \quad U(x_B + z, \delta_B) - U(x_B, \delta_B) \geq \Delta.$$

*Moreover, there is a constant  $\lambda > 0$ , which depends on the set  $X$  and on the difference between the marginal rates of substitution  $\varepsilon$ , but not on the size of the trade  $\theta$ , such that the potential loss in current utility associated with the trade  $z$  is bounded below by  $-\lambda\Delta$  for all beliefs  $\delta$ :*

$$(11) \quad U(x_A - z, \delta) - U(x_A, \delta) \geq -\lambda\Delta \quad \text{for all } \delta \in [0, 1].$$

## A.2. Proof of Proposition 1

Proceeding by contradiction, suppose (3) does not hold. Without loss of generality, let us focus on state  $s_1$ . If (3) is violated in  $s_1$ , then there exist an  $\varepsilon > 0$  and an  $\eta \in (0, 1)$  such that the following holds for infinitely many periods  $t$ :

$$(12) \quad P(|\mathcal{M}(x_t, \delta_t) - \kappa| > \varepsilon, \delta_t = 1 | s_1) > \eta P(\delta_t = 1 | s_1) \quad \text{for all } \kappa.$$



We want to show that (12) implies a profitable deviation for informed agents. The informed agent starts deviating at some date  $T$  to be defined if three conditions are satisfied: (a) his marginal rate of substitution is below some level  $\kappa^*$  to be defined:  $\mathcal{M}(x_T, \delta_T) < \kappa^*$ ; (b) his utility is close enough to its long-run level:  $u_T \geq \hat{v}_T - \alpha\eta\Delta/4$ , for some  $\Delta > 0$  to be defined; (c) his portfolio  $x_T$  is in some compact set  $X$  to be defined. When (a)–(c) hold, the agent makes an offer  $z^*$  to be defined, which is accepted with probability  $\chi_T(z^*|s_1) \geq \alpha\eta/4$  and gives him a utility gain of at least  $\Delta$ . The expected payoff of this strategy at time  $T$  is then

$$u_T + \chi_T(z^*|s_1)(U(x_T - z^*, \delta_T) - u_T) > u_T + \alpha\eta\Delta/4 \geq \hat{v}_T.$$

Since  $\hat{v}_T$  is, by definition, the expected payoff of a proposer who follows an optimal strategy, this leads to a contradiction.

To complete the proof, we need to define the scalars  $\kappa^*$  and  $\Delta$ , the set  $X$ , the deviating period  $T$ , and the offer  $z^*$  and check that they satisfy the desired properties. Applying Lemma 5, choose a compact set  $X$  such that, for some  $T'$ , we have  $P(x_t \in X | s_1) \geq 1 - \alpha\eta/4$  for all  $t \geq T'$ . Applying Lemma 6, choose  $\Delta > 0$  to be the minimal gain from trade for two agents with marginal rates of substitution that differ by at least  $\varepsilon$  with portfolios in  $X$ . Applying Lemmas 2 and 4, choose a  $T'' \geq T'$  such that

$$P(u_t \geq \hat{v}_t - \alpha\eta\Delta/4, x_t \in X | s_1) > 1 - \alpha\eta/2 \quad \text{for all } t \geq T''.$$

Using (12) and the fact that there are at least  $\alpha$  informed agents, choose  $T \geq T''$  such that

$$\begin{aligned} P(|\mathcal{M}(x_T, \delta_T) - \kappa| > \varepsilon, \delta_T = 1 | s_1) \\ > \eta P(\delta_t = 1 | s_1) \geq \alpha\eta \quad \text{for all } \kappa. \end{aligned}$$

Using Lemma 4, it follows that

$$\begin{aligned} (13) \quad P(|\mathcal{M}(x_T, \delta_T) - \kappa| \leq \varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 | s_1) \\ < P(u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 | s_1) - \alpha\eta/2 \quad \text{for all } \kappa. \end{aligned}$$

Define

$$\begin{aligned} \kappa^* = \sup \{ \kappa : P(\mathcal{M}(x_T, \delta_T) > \kappa + (3/2)\varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, \\ x_T \in X, \delta_T = 1 | s_1) \geq \alpha\eta/4 \}. \end{aligned}$$

This definition implies that there are fewer than  $\alpha\eta/4$  informed agents with marginal rate of substitution above  $\kappa^* + 2\varepsilon$  who satisfy (b)–(c),

$$\begin{aligned} (14) \quad P(\mathcal{M}(x_T, \delta_T) > \kappa^* + 2\varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 | s_1) \\ < \alpha\eta/4, \end{aligned}$$



given that  $\kappa^* + \varepsilon/2 > \kappa^*$ . Consider the following chain of equalities and inequalities:

$$\begin{aligned}
 & P(\mathcal{M}(x_T, \delta_T) \geq \kappa^*, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1) \\
 &= P(\kappa^* \leq \mathcal{M}(x_T, \delta_T) \leq \kappa^* + 2\varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, \\
 &\quad x_T \in X, \delta_T = 1 \mid s_1) \\
 &\quad + P(\mathcal{M}(x_T, \delta_T) > \kappa^* + 2\varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, \\
 &\quad x_T \in X, \delta_T = 1 \mid s_1) \\
 &< P(u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1) - \alpha\eta/4,
 \end{aligned}$$

where the equalities are immediate and the inequality follows from (13) (with  $\kappa = \kappa^* + \varepsilon$ ) and (14). This implies

$$(15) \quad P(\mathcal{M}(x_T, \delta_T) < \kappa^*, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1) > 0,$$

which shows that conditions (a)–(c) are met with positive probability.

To choose the deviating offer  $z^*$ , notice that, by the definition of  $\Delta$ , there exists an offer  $z^* = (\zeta^*, -p^*\zeta^*)$ , with price  $p^* = \kappa^* + \varepsilon/2$ , such that

$$(16) \quad U(x - z^*, \delta) \geq U(x, \delta) + \Delta \quad \text{if } \mathcal{M}(x, \delta) < \kappa^* \text{ and } x \in X,$$

$$(17) \quad U(x + z^*, \delta) \geq U(x, \delta) + \Delta \quad \text{if } \mathcal{M}(x, \delta) > \kappa^* + \varepsilon \text{ and } x \in X.$$

Condition (16) shows that an informed proposer who satisfies (a)–(c) gains at least  $\Delta$  if offer  $z^*$  is accepted.

Finally, the definition of  $\kappa^*$  implies that there must be at least  $\alpha\eta/4$  agents with marginal rate of substitution above  $\kappa^* + \varepsilon$ ,

$$\begin{aligned}
 (18) \quad & P(\mathcal{M}(x_T, \delta_T) > \kappa^* + \varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/2, x_T \in X, \delta_T = 1 \mid s_1) \\
 & \geq \alpha\eta/4,
 \end{aligned}$$

given that  $\kappa^* - \varepsilon/2 < \kappa^*$ . Recall that  $\hat{v}_t$  represents, by definition, the maximal expected utility the responder can get from rejecting all offers and behaving optimally in the future. A responder who receives  $z^*$  has the option to accept it and stop trading from then on, which yields expected utility  $U(x_T + z^*, \delta_T)$ . For all informed agents who satisfy  $\mathcal{M}(x_T, \delta_T) \geq \kappa^* + \varepsilon$ ,  $u_T \geq \hat{v}_T - \alpha\eta\Delta/4$ , and  $x_T \in X$ , we have the chain of inequalities

$$U(x_T + z^*, \delta_T) \geq u_T + \Delta > u_T + \alpha\eta\Delta/4 \geq \hat{v}_T,$$

where the first inequality follows from (17). This shows that rejecting  $z^*$  at time  $T$  is a strictly dominated strategy for these informed agents. Since there are at least  $\alpha\eta/4$  of them, by (18), the probability that  $z^*$  is accepted must then satisfy  $\chi_T(z \mid s_1) \geq \alpha\eta/4$ .

### A.3. Proof of Proposition 2

The proof makes use of the following lemma, which is proved in the Supplemental Material.

LEMMA 7: *Given the equilibrium measure  $P$ , for all  $\varepsilon > 0$ , there are a scalar  $M$  and a sequence of measures  $G_t$  on the space of portfolios and beliefs  $R_+^2 \times [0, 1]$  that satisfy the following properties: (i) the measure is zero for all beliefs smaller than or equal to  $1/2$ :*

$$G_t(x, \delta) = 0 \quad \text{if} \quad \delta \leq 1/2;$$

(ii)  $G_t$  corresponds to the distribution generated by the measure  $P$  conditional on  $s_1$  for informed agents:

$$G_t(x, 1) = P(\omega : x_t(\omega) = x, \delta_t(\omega) = 1 \mid s_1) \quad \text{for all } x \text{ and } t;$$

(iii) *the average holdings of asset 1 exceed the average holdings of asset 2, truncated at any  $m \geq M$ , by less than  $\varepsilon$ :*

$$(19) \quad \int_{x^2 \leq m} (x^1 - x^2) dG_t(x, \delta) \leq \varepsilon \quad \text{for all } m \geq M \text{ and all } t.$$

To prove Proposition 2, we proceed by contradiction and suppose that, for all  $\varepsilon > 0$ , there are infinitely many periods  $t$  in which

$$(20) \quad |\kappa_t(s_1) - \kappa_t(s_2)| < \varepsilon \quad \text{and} \\ P(|\mathcal{M}(x_t, \delta_t) - \kappa_t(s)| < 2\varepsilon \mid s) > 1 - \varepsilon \quad \text{in some } s.$$

By symmetry, the long-run marginal rates of substitutions of informed agents in states  $s_1$  and  $s_2$  satisfy  $\kappa_t(s_1) = 1/\kappa_t(s_2)$ . Some algebra shows that  $|\kappa_t(s_1) - \kappa_t(s_2)| < \varepsilon$  implies  $|\kappa_t(s_1) - 1| < \varepsilon$ . Moreover, by the triangle inequality,  $|\mathcal{M}(x_t, \delta_t) - \kappa_t(s_1)| < 2\varepsilon$  and  $|\kappa_t(s_1) - 1| < \varepsilon$  imply  $|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon$ . Therefore, (20) implies

$$(21) \quad P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon \mid s) > 1 - \varepsilon.$$

Without loss of generality, we focus on the case in which this condition holds for infinitely many periods in state  $s_1$ .

We want to show that, for some appropriately chosen positive scalars  $m$  and  $\zeta$ , the following inequality holds for some  $t^*$ :

$$(22) \quad \int_{x^2 \leq m} (x^1 - x^2) dG_{t^*} > \zeta,$$

and then show that this contradicts (19). The argument proceeds in two steps.

*Step 1.* Since there is at least a mass  $\alpha$  of informed agents, using Lemmas 4 and 5, we can find a compact set  $X \subset R_{++}^2$  and a time  $T$  such that, for all  $\varepsilon > 0$ , we have

$$(23) \quad P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t \in X \mid s_1) > (5/6)\alpha - \varepsilon$$

for all periods  $t \geq T$  in which  $P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon \mid s_1) > 1 - \varepsilon$ . Consider the minimization problem:

$$d_I(\varepsilon) = \min_{x \in X} (x^1 - x^2) \quad \text{s.t.} \quad |\mathcal{M}(x, 1) - 1| \leq 3\varepsilon.$$

Notice that  $d_I(\varepsilon)$  is continuous, from the theorem of the maximum. Consider this problem at  $\varepsilon = 0$ . Let us prove that  $d_I(0) > 0$ . If  $x^1 \leq x^2$ , then  $u'(x^1) \geq u'(x^2)$  and, therefore, the marginal rate of substitution

$$\mathcal{M}(x, 1) = \frac{\pi(1)u'(x^1)}{(1 - \pi(1))u'(x^2)} \geq \frac{\pi(1)}{1 - \pi(1)} > 1.$$

Therefore, all  $x$  that satisfy  $|\mathcal{M}(x, 1) - 1| \leq 0$  must also satisfy  $x^1 > x^2$ . In other words, given that informed agents have a signal favorable to state 1, if their marginal rate of substitution is exactly 1 they must hold strictly more of asset 1. We can now define the constant  $\zeta$  (to be used in expression (22)) as

$$\zeta = \frac{\alpha}{6} d_I(0).$$

Next, we define the quantity  $m$ . Applying uniform market clearing and Lemma 7, we can find an  $m \geq d_I(0)$  such that the following inequalities hold for all  $t$ :

$$(24) \quad \int_{x_t^2 > m} x_t^2 dP(\omega \mid s_1) \leq \zeta$$

and

$$(25) \quad \int_{x^2 \leq m} (x^1 - x^2) dG_t \leq \zeta.$$

From (24), we have

$$mP(x_t^2(\omega) > m) \leq \int_{x_t^2(\omega) > m} x_t^2(\omega) dP(\omega \mid s_1) \leq \zeta \quad \text{for all } t,$$

which, given the definition of  $\zeta$  and the fact that  $m \geq d_I(0)$ , implies

$$P(x_t^2(\omega) > m) \leq \frac{\alpha}{6} \frac{d_I(0)}{m} \leq \frac{\alpha}{6} \quad \text{for all } t.$$

We then obtain the following chain of equalities and inequalities:

$$\begin{aligned}
 & P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t \in X \mid s_1) \\
 &= P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t^2 \leq m, x_t \in X \mid s_1) \\
 &\quad + P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t^2 > m, x_t \in X \mid s_1) \\
 &\leq P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t^2 \leq m, x_t \in X \mid s_1) + \alpha/6,
 \end{aligned}$$

and combine it with (23) to conclude that

$$(26) \quad P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t^2 \leq m, x_t \in X \mid s_1) > (2/3)\alpha - \varepsilon$$

for all  $t \geq T$  in which  $P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon \mid s_1) > 1 - \varepsilon$ .

*Step 2.* Consider the problem

$$d_U(\varepsilon) = \min_{\substack{x^2 \leq m \\ \delta \geq 1/2}} (x^1 - x^2) \quad \text{s.t.} \quad |\mathcal{M}(x, \delta) - 1| \leq 3\varepsilon.$$

The theorem of the maximum implies that  $d_U(\varepsilon)$  is continuous. Moreover,  $d_U(\varepsilon)$  is negative for all  $\varepsilon > 0$  and  $d_U(0) = 0$ . Recall from Step 1 that  $d_I(\varepsilon)$  is continuous and  $d_I(0) > 0$ . It is then possible to find a positive  $\varepsilon^*$ , smaller than both  $\alpha/6$  and  $\zeta/m$ , such that

$$(27) \quad \frac{\alpha}{2}d_I(\varepsilon^*) + d_U(\varepsilon^*) > \frac{\alpha}{3}d_I(0) = 2\zeta$$

(the second equality comes from the definition of  $\zeta$ ).

Since, by construction,  $\varepsilon^* < \alpha/6$ , it follows from (26) that the mass of informed agents with marginal rates of substitution near 1 (within  $3\varepsilon^*$ ) and a portfolio that satisfies  $x_t^2 \leq m$  and  $x_t \in X$  is sufficiently high:

$$(28) \quad P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon^*, \delta_t = 1, x_t^2 \leq m, x_t \in X \mid s_1) > \alpha/2$$

for all  $t \geq T$  in which  $P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon^* \mid s_1) > 1 - \varepsilon^*$ .

Moreover, by Lemma 4, in all periods  $t \geq T$  in which almost all agents have marginal rate of substitution close to 1, that is,  $P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon^* \mid s_1) > 1 - \varepsilon^*$ , almost all agents with beliefs higher than  $1/2$  and portfolios satisfying  $x_t^2 \leq m$  also have a marginal rate of substitution close to 1:

$$\begin{aligned}
 (29) \quad & P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon^*, \delta_t > 1/2, x_t^2 \leq m \mid s_1) \\
 & > P(\delta_t > 1/2, x_t^2 \leq m \mid s_1) - \varepsilon^*.
 \end{aligned}$$

By hypothesis, that is, by (21), we can choose a  $t^* \geq T$  such that

$$P(|\mathcal{M}(x_{t^*}, \delta_{t^*}) - 1| < 3\varepsilon^* \mid s_1) > 1 - \varepsilon^*$$

so that both (28) and (29) are satisfied.

Define the following three groups of agents:

$$\begin{aligned} A_1 &= \{(x, \delta) : |\mathcal{M}(x, \delta) - 1| < 3\varepsilon^*, \delta = 1, x^2 \leq m, x \in X\}, \\ A_2 &= \{(x, \delta) \notin A_1 : |\mathcal{M}(x, \delta) - 1| < 3\varepsilon^*, \delta > 1/2, x^2 \leq m\}, \\ A_3 &= \{(x, \delta) \notin A_1 \cup A_2 : \delta > 1/2, x^2 \leq m\}. \end{aligned}$$

*Step 3.* Now we split the integral (22) in three parts, corresponding to the three sets  $A_j$  defined above, and determine a lower bound for each of them. First, we have

$$(30) \quad \int_{A_1} (x^1 - x^2) dG_{t^*} = \int_{(x_{t^*}, \delta_{t^*}) \in A_1} (x_{t^*}^1(\omega) - x_{t^*}^2(\omega)) dP(\omega | s_1) \geq \frac{\alpha}{2} d_I(\varepsilon^*),$$

where the equality follows from property (ii) of the distribution  $G_t$  (in Lemma 7) and the inequality follows from the definition of  $d_I(\varepsilon^*)$  and condition (28). The definition of  $d_U(\varepsilon^*)$  implies that

$$(31) \quad \int_{A_2} (x^1 - x^2) dG_{t^*} \geq d_U(\varepsilon^*) P(A_2) \geq d_U(\varepsilon^*),$$

since  $d_U(\varepsilon^*) < 0$  and  $P(A_2) \leq 1$ . Finally, the definition of the measure  $G_t$  and condition (29) imply that

$$\begin{aligned} G_{t^*}(A_3) &\leq P((x_{t^*}, \delta_{t^*}) \in A_3 | s_1) \\ &\leq P(\delta_{t^*} > 1/2, x_{t^*}^2 \leq m | s_1) - P((x_{t^*}, \delta_{t^*}) \in A_1 \cup A_2 | s_1) \\ &\leq \varepsilon^* < \zeta/m, \end{aligned}$$

where the last inequality follows from the definition of  $\varepsilon^*$ . We then have the following lower bound:

$$(32) \quad \int_{A_3} (x^1 - x^2) dG_{t^*} \geq -m G_{t^*}(A_3) \geq -\zeta.$$

We can now combine (30), (31), and (32) and use inequality (27) to obtain a lower bound for the whole integral (22):

$$\int_{x^2 \leq m} (x^1 - x^2) dG_{t^*} \geq \frac{\alpha}{2} d_I(\varepsilon^*) + d_U(\varepsilon^*) - \zeta > \zeta.$$

Comparing this inequality and (25) leads to the desired contradiction.

#### A.4. Proof of Proposition 3

The following two lemmas are used in the proof. They are proved in the Supplemental Material.

LEMMA 8: *For all  $\varepsilon > 0$ , the probability that the belief  $\delta_t$  is above the threshold  $\varepsilon/(1 + \varepsilon)$  conditional on signal  $s_1$  is bounded below for all  $t$ :*

$$P(\delta_t \geq \varepsilon/(1 + \varepsilon) \mid s_1) > 1 - \varepsilon.$$

LEMMA 9: *For any integer  $J$ , the sequence  $\kappa_t(s_1)$  satisfies the property*

$$\lim_{t \rightarrow \infty} |\kappa_{t+J}(s_1) - \kappa_t(s_1)| = 0.$$

*For all  $\varepsilon > 0$  and all integers  $J$ , it is possible to find a  $T$  such that*

$$P(|\mathcal{M}(x_{t+J}, \delta_{t+J}) - \kappa_t(s_1)| < \varepsilon, \delta_{t+J} = 1 \mid s) > \alpha - \varepsilon \quad \text{for all } t \geq T.$$

To prove Proposition 3, suppose, by contradiction, that there exists an  $\varepsilon > 0$  such that, for some state  $s$ , the following condition holds for infinitely many  $t$ :

$$P(|\mathcal{M}(x_t, \delta^I(s)) - \kappa_t(s)| > \varepsilon \mid s) > \varepsilon,$$

where  $\mathcal{M}(x_t, \delta^I(s))$  is the marginal rate of substitution of an agent (informed or uninformed) evaluated at the belief of the informed agents  $\delta^I(s)$ . Without loss of generality, let us focus on state  $s_1$  and suppose

$$(33) \quad P(\mathcal{M}(x_t, 1) - \kappa_t(s_1) > \varepsilon \mid s_1) > \varepsilon$$

for infinitely many  $t$ . The other case is treated in a symmetric way.

We want to show that if (33) holds, we can construct a profitable deviation in which:

- (i) The player follows the equilibrium strategy  $\sigma$  up to some period  $T$ .
- (ii) At  $T$ , if his portfolio satisfies  $\mathcal{M}(x_T, 1) > \kappa_t(s) + \varepsilon$ , his beliefs  $\delta_T$  are above some positive lower bound  $\underline{\delta}$  and some other technical conditions are satisfied, he moves to the experimentation stage (iii); otherwise, he keeps playing  $\sigma$ .
- (iii) The experimentation stage lasts between  $T$  and  $T + J - 1$  for some  $J$ . An agent makes a sequence of offers  $\{\hat{z}_j\}_{j=0}^{J-1}$  as long as he is selected as the proposer. The “favorable” responses to the offers  $\{\hat{z}_j\}_{j=0}^{J-1}$  are given by the binary sequence  $\{\hat{r}_j\}_{j=0}^{J-1}$ . If at any point during the experimentation stage the agent is not selected as the proposer or fails to receive response  $\hat{r}_j$  after offer  $\hat{z}_j$ , he stops trading. Otherwise, he goes to (iv).
- (iv) At time  $T + J$ , after making all the offers  $\{\hat{z}_j\}_{j=0}^{J-1}$  and receiving responses equal to  $\{\hat{r}_j\}_{j=0}^{J-1}$ , if the player is selected as the proposer he makes an offer  $z^*$  and stops trading at  $T + J + 1$ . Otherwise, he stops trading right away.

The expected payoff of this strategy, from the point of view of a deviating agent at time  $T$ , is

$$\begin{aligned}
 (34) \quad w = & u_T - \hat{L} + \delta_T \gamma^J 2^{-J-1} \xi_1 \chi_{T+J}(z^* | s_1) \\
 & \times \left[ U \left( x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 1 \right) - U(x_T, 1) \right] \\
 & + (1 - \delta_T) \gamma^J 2^{-J-1} \xi_2 \chi_{T+J}(z^* | s_2) \\
 & \times \left[ U \left( x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 0 \right) - U(x_T, 0) \right],
 \end{aligned}$$

where the term  $\hat{L}$  captures the expected utility losses if the player makes some or all of the offers in  $\{\hat{z}_j\}_{j=0}^{J-1}$  but not the last offer  $z^*$  and the following two terms capture the expected utility gains in states  $s_1$  and  $s_2$ , if all the deviating offers, including  $z^*$ , are accepted. The factors  $\xi_1$  and  $\xi_2$  denote the probabilities, in states  $s_1$  and  $s_2$ , that the player receives the sequence of responses  $\{\hat{r}_j\}_{j=0}^{J-1}$ . Notice that  $\gamma^J$  is the probability that the game does not end between periods  $T$  and  $T+J$  and  $2^{-J-1}$  is the probability of being selected as the proposer in all these periods.

In order to show that the strategy above is a profitable deviation, we need to show that the utility gain in the first square brackets is large enough, by choosing  $z^*$  to be a profitable trade with informed agents in  $s_1$ , and that the remaining terms are sufficiently small. In the rest of the proof, we choose the time  $T$ , the lower bound  $\underline{\delta}$ , and the offers  $\{\hat{z}_j\}_{j=0}^{J-1}$  and  $z^*$  to achieve this goal.

*Step 1 (Bounds on gains and losses for the final trade).* Following steps similar to the ones in the proof of Proposition 1, we can use Lemmas 5 and 8 to find a compact set  $X \subset R_{++}^2$  and a period  $T'$  such that

$$(35) \quad P(\delta_t \geq \underline{\delta}, x_t \in X \mid s_1) > 1 - \varepsilon/2$$

for all  $t \geq T'$ , where  $\underline{\delta} = (\varepsilon/2)/(1 + \varepsilon/2) > 0$ . Pick a scalar  $\theta^* > 0$  such that  $x + z > 0$  when  $x \in X$  and  $\|z\| < \theta^*$ . Using Lemma 6, we can then find a  $\Delta^* > 0$  which is a lower bound for the gains from trade between two agents with marginal rates of substitution differing by at least  $\varepsilon/2$  and portfolios in  $X$ , making trades of norm smaller than  $\theta^*$ . This will be used as a lower bound for the gains from trading in state  $s_1$ . Define an upper bound for the potential losses of an uninformed agent who makes a trade of norm smaller than or equal to  $\theta^*$  in the other state,  $s_2$ :

$$L^* \equiv - \min_{x \in X, \|z\| \leq \theta^*} \{U(x + z, 0) - U(x, 0)\}.$$

Next, choose  $J$  to be an integer large enough that

$$\underline{\delta}(\alpha/2)\Delta^* - (1 - \underline{\delta})\rho^{-J}L^* > 0,$$

where  $\rho$  is the scalar defined in Lemma 3. This choice of  $J$  ensures that the experimentation phase is long enough that, when offering the last trade, the agent assigns sufficiently high probability to state  $s_1$ , so that the potential gain  $\Delta^*$  dominates the potential loss  $L^*$ .

*Step 2 (Bound on losses from experimentation).* To simplify notation, let

$$\tilde{\Delta} = \gamma^J 2^{-J-1} \beta^J (\underline{\delta}(\alpha/2)\Delta^* - (1 - \underline{\delta})\rho^{-J}L^*),$$

where  $\beta$  is the positive scalar defined in Lemma 3. Choose a scalar  $\hat{\theta} > 0$  such that, for all  $x \in X$ , all  $\|z_1\| < J\hat{\theta}$ , all  $\|z_2\| \leq \theta^*$ , and any  $\delta \in [0, 1]$ , the following inequality holds:

$$(36) \quad |U(x + z_1 + z_2, \delta) - U(x + z_2, \delta)| < \tilde{\Delta}/3.$$

Next, applying Lemma 3, we can find a time  $T'' \geq T'$  such that, in all  $t \geq T''$ , there is a trade of norm smaller than  $\hat{\theta}$  that satisfies either (4) or (5). Before using this property to define the offers  $\{\hat{z}_j\}_{j=0}^{J-1}$ , we need to define the time period  $T$  where the deviation occurs. To do so, using our starting hypothesis (33), condition (35), and applying Lemma 2, we can find a  $T''' \geq T''$  such that, for infinitely many periods  $t \geq T'''$ , there is a positive mass of uninformed agents who have: marginal rate of substitution sufficiently above  $\kappa_t(s_1)$ , utility near its long-run level, beliefs sufficiently favorable to  $s_1$ , and portfolio in  $X$ ; that is,

$$(37) \quad P(\mathcal{M}(x_t, 1) - \kappa_t(s_1) > \varepsilon, \delta_t \geq \underline{\delta}, u_t > \hat{v}_t - \tilde{\Delta}/3, x_t \in X \mid s_1) > 0.$$

Finally, applying Lemma 9, we pick a  $T \geq T'''$  so that (37) holds at  $t = T$  and, at time  $T + J$ , there is a sufficiently large mass of informed agents who have: marginal rate of substitution sufficiently near  $\kappa_T(s_1)$ , utility near its long-run level, and portfolio in  $X$ ; that is,

$$(38) \quad P(|\mathcal{M}(x_{T+J}, 1) - \kappa_T(s_1)| < \varepsilon/2, \delta_{T+J} = 1, \\ u_{T+J} > \hat{v}_{T+J} - \Delta/2, x_{T+J} \in X \mid s_1) > \alpha/2.$$

Having defined  $T$ , we can apply Lemma 3 to find the desired sequence of trades  $\{\hat{z}_j\}_{j=0}^{J-1}$  of norm smaller than  $\hat{\theta}$ , that satisfy either (4) or (5). For each trade  $\hat{z}_j$ , if (4) holds, we set  $\hat{r}_j = 1$  (accept). In this way, the probability of observing  $\hat{r}_j$  is  $\chi_{T+j}(\hat{z}_j | s_1) > \beta$  in state  $s_1$  and  $\chi_{T+j}(\hat{z}_j | s_2) < \rho^{-1} \chi_{T+j}(\hat{z}_j | s_1)$  in state  $s_2$ . Otherwise, if (5) holds, we set  $\hat{r}_j = 0$  and obtain analogous inequalities. This implies that the factors  $\xi_1$  and  $\xi_2$  in (34) satisfy

$$(39) \quad \xi_1 > \beta^J \quad \text{and} \quad \xi_2 < \xi_1 \rho^{-J}.$$



*Step 3 (Define  $z^*$  and check profitable deviation).* We can now define the final trade  $z^*$  to be a trade of norm smaller than  $\theta^*$ , such that

$$\begin{aligned} U(x - z^*, 1) &> U(x, 1) + \Delta^* \\ \text{if } \mathcal{M}(x, 1) &> \kappa_T(s_1) + \varepsilon \text{ and } x \in X, \\ U(x + z^*, 1) &> U(x, 1) + \Delta^* \\ \text{if } \mathcal{M}(x, 1) &< \kappa_T(s_1) + \varepsilon/2 \text{ and } x \in X, \end{aligned}$$

which is possible given the definition of  $\Delta^*$ . Finally, we check that we have constructed a profitable deviation. Let uninformed agents start deviating whenever the following conditions are satisfied at date  $T$ :

$$\mathcal{M}(x_T, 1) > \kappa_T(s_1) + \varepsilon, \quad \delta_T \geq \underline{\delta}, \quad u_T > \hat{v}_T - \tilde{\Delta}/3, \quad x_T \in X.$$

Equation (37) shows that this happens with positive probability. Let us evaluate the deviating strategy payoff (34), beginning with the last two terms. The triangle inequality implies  $\|\sum_{j=0}^{J-1} \hat{z}_j\| < J\hat{\theta}$ . Then the definition of  $z^*$  and (36) imply that the gain from trade of the uninformed agent, conditional on  $s_1$ , is bounded below:

$$\begin{aligned} &U\left(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 1\right) - U(x_T, 1) \\ &\geq U(x_T + z^*, 1) - U(x_T, 1) \\ &\quad - \left|U\left(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 1\right) - U(x_T + z^*, 1)\right| \\ &> \Delta^* - \tilde{\Delta}/3. \end{aligned}$$

The definition of  $L^*$  implies that the gain conditional on  $s_2$  is also bounded:

$$U\left(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 0\right) - U(x_T, 0) > -L^* - \tilde{\Delta}/3.$$

Moreover, condition (38) shows that the probability that informed agents accept  $z^*$  at  $T + J$  satisfies  $\chi_{T+J}(z^*|s_1) > \alpha/2$ . These results, together with the inequalities (39) and the fact that  $\chi_{T+J}(z^*|s_2) \leq 1$ , imply that the last two terms in (34) are bounded below by

$$\gamma^J 2^{-J-1} \beta^J [\underline{\delta}(\alpha/2)(\Delta^* - \tilde{\Delta}/3) - (1 - \underline{\delta})\rho^{-J}(L^* + \tilde{\Delta}/3)],$$

which, by the definition of  $\tilde{\Delta}$ , is greater than  $(2/3)\tilde{\Delta}$ . Finally, all the expected losses in  $\hat{L}$  in (34) are bounded above by  $\tilde{\Delta}/3$ , thanks to (36). Therefore,  $w > u_T + \tilde{\Delta}/3$ . Since  $u_T > \hat{v}_T - \tilde{\Delta}/3$ , we conclude that  $w > \hat{v}_T$  and we have found a profitable deviation.

#### A.5. Proof of Theorem 1

We begin from the second part of the theorem, proving (8), which characterizes the limit behavior of  $\kappa_t(s)$ .

Without loss of generality, let  $s = s_1$ . Suppose first that, for infinitely many periods, the long-run marginal rate of substitution  $\kappa_t(s_1)$  is larger than the ratio of the probabilities  $\phi(s_1)/(1 - \phi(s_1))$  by a factor larger than  $1 + \varepsilon$ :

$$\kappa_t(s_1) > (1 + \varepsilon)\phi(s_1)/(1 - \phi(s_1)) \quad \text{for some } \varepsilon > 0.$$

Proposition 3 then implies that, for all  $\eta > 0$  and  $T$ , there is a  $t$  such that almost all agents have portfolios that satisfy  $u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2$ :

$$(40) \quad P(u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2 \mid s_1) > 1 - \eta.$$

We want to show that this property violates uniform market clearing, since it implies that almost all agents hold more of asset 2 than of asset 1.

Uniform market clearing implies that, for any  $\zeta > 0$ , we can find an  $M$  such that

$$(41) \quad \int_{x_t^1(\omega) \leq m} x_t^1(\omega) dP(\omega \mid s_1) \geq 1 - \zeta \quad \text{for all } m \geq M \text{ and all } t.$$

Moreover, since  $\int x_t^2(\omega) dP(\omega \mid s_1) = 1$ , this implies that

$$(42) \quad \int_{x_t^1(\omega) \leq m} (x_t^2(\omega) - x_t^1(\omega)) dP(\omega \mid s_1) \leq \zeta \quad \text{for all } m \geq M \text{ and all } t.$$

The idea of the proof is to reach a contradiction by splitting the integral on the left-hand side of (42) in three pieces: a group of agents with a strictly positive difference  $x_t^2 - x_t^1$ , a group of agents with a nonnegative difference  $x_t^2 - x_t^1$ , and a small residual group. The argument here follows a similar logic as the proof of Proposition 2.

Using Lemma 5, find a compact set  $X$  and a period  $T$  such that, for all  $t \geq T$ , at least half of the agents have portfolios in  $X$ :

$$(43) \quad P(x_t \in X \mid s_1) \geq 1/2 \quad \text{for all } t \geq T.$$

Let us then find a lower bound for the difference between the holdings of asset 1 and 2 for agents with portfolios in  $X$  that satisfy  $u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2$ . We do so by solving the problem

$$d = \min_{x \in X} (x^2 - x^1) \quad \text{s.t.} \quad u'(x^1)/u'(x^2) \geq 1 + \varepsilon/2,$$

which gives a  $d > 0$ .

Let us pick  $\zeta = d/5$  and find an  $M$  such that (41) and (42) hold. Condition (42) (with  $\zeta = d/5$ ) is the market clearing condition that we will contradict below. Condition (41) is also useful, because it gives us a lower bound for  $P(x_t^1 \leq m)$ :

$$(44) \quad P(x_t^1 \leq m) \geq 1 - \zeta/m \quad \text{for all } m \geq M \text{ and all } t,$$

which follows from the chain of inequalities

$$mP(x_t^1 > m) \leq \int_{x_t^1(\omega) > m} x_t^1(\omega) dP(\omega|s_1) \leq \zeta.$$

Using our hypothesis (40), we know that, for any  $\eta > 0$ , we can find a period  $t \geq T$  in which more than  $1 - \eta$  agents satisfy  $u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2$ . Combining this with (43) and (44) (applying Lemma 4), we can always find a  $t \geq T$  in which almost all agents satisfy  $u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2$  and  $x_t \leq m$ :

$$(45) \quad P(u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2, x_t \leq m | s_1) > 1 - \eta - \zeta/m,$$

and almost half of them satisfy  $u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2$  and  $x_t \leq m$ , and have portfolios in  $X$ :

$$(46) \quad P(u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2, x_t \leq m, x_t \in X | s_1) > 1/2 - \eta - \zeta/m.$$

Define the three disjoint sets

$$A_1 = \{\omega : u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2, x_t \in X, x_t^1 \leq m\},$$

$$A_2 = \{\omega : u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2, x_t^1 \leq m\} / A_1,$$

$$A_3 = \{\omega : u'(x_t^1)/u'(x_t^2) < 1 + \varepsilon/2, x_t^1 \leq m\},$$

which satisfy  $A_1 \cup A_2 \cup A_3 = \{\omega : x_t^1 \leq m\}$ . We can then bound from below the following three integrals:

$$\int_{A_1} (x_t^2 - x_t^1) dP(\omega|s_1) \geq d \cdot (1/2 - \eta - \zeta/m),$$

$$\int_{A_2} (x_t^2 - x_t^1) dP(\omega|s_1) \geq 0,$$

$$\int_{A_3} (x_t^2 - x_t^1) dP(\omega|s_1) \geq -m \cdot (\eta + \zeta/m).$$

The first inequality follows from the definitions of  $d$  and  $A_1$  and the fact that  $P(A_1|s_1) > 1/2 - \eta - \zeta/m$  from (46). The second follows from the definition of  $A_2$  and the fact that  $u'(x_t^1)/u'(x_t^2) > 1$  implies  $x_t^2 > x_t^1$ . The third follows from the definition of  $A_3$  (which implies  $x_t^2 - x_t^1 \geq -m$ ) and the fact that  $P(A_3|s_1) < \eta + \zeta/m$  from (45). Summing term by term, we then obtain

$$\int_{x_t^1(\omega) \leq m} (x_t^2 - x_t^1) dP(\omega|s_1) \geq d \cdot (1/2 - \eta - \zeta/m) - m \cdot (\eta + \zeta/m).$$

Since we can choose an  $m$  arbitrarily large and an  $\eta$  arbitrarily close to 0 (in that order), we can make this expression as close as we want to  $d/2 - \zeta$ , which is strictly greater than  $\zeta$ , given that  $\zeta = d/5 < d/4$ . This contradicts the market clearing condition (42).

In a similar way, we can rule out the case in which  $\kappa_t(s_1) < (1 - \varepsilon)\phi(s_1)/(1 - \phi(s_1))$  for infinitely many periods. This completes the argument for  $\lim_{t \rightarrow \infty} \kappa_t(s_1) = \phi(s_1)/(1 - \phi(s_1))$ . An analogous argument can be applied to  $s_2$ .

To complete the proof, we need to prove the long-run efficiency of equilibrium portfolios, that is, property (7). Proposition 3 and  $\lim \kappa_t(s) = \phi(s)/(1 - \phi(s))$  imply, by the properties of convergence in probability, that

$$(47) \quad \lim_{t \rightarrow \infty} P(|u'(x_t^1)/u'(x_t^2) - 1| > \varepsilon) = 0.$$

We want to show that negating (7) leads to a contradiction of (47).

Suppose that, for some  $\varepsilon > 0$ , we have  $P(|x_t^1 - x_t^2| > \varepsilon) > \varepsilon$  for infinitely many periods. Then, as usual, we can use Lemmas 4 and 5 to find a compact set  $X$  such that the following condition holds for infinitely many periods:

$$P(|x_t^1 - x_t^2| > \varepsilon, x_t \in X) > \varepsilon/2.$$

But then the continuity of  $u'(\cdot)$  implies that there is a  $\delta > 0$  such that

$$|u'(x^1)/u'(x^2) - 1| > \delta \implies |x^1 - x^2| > \varepsilon \quad \text{for all } x \in X,$$

which implies

$$P(|u'(x_t^1)/u'(x_t^2) - 1| > \delta, x_t \in X) \geq P(|x_t^1 - x_t^2| > \varepsilon, x_t \in X) > \varepsilon/2.$$

Given that

$$P(|u'(x_t^1)/u'(x_t^2) - 1| > \delta) \geq P(|u'(x_t^1)/u'(x_t^2) - 1| > \delta, x_t \in X),$$

we conclude that there are  $\varepsilon, \delta > 0$  such that

$$P(|u'(x_t^1)/u'(x_t^2) - 1| > \delta) > \varepsilon/2,$$

contradicting (47) and completing the proof.

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