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Efficiency And Equilibrium With Dynamic Increasing Aggregate Returns Due To Demand Complementarities

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Abstract. When do dynamic nonconvexities at the disaggregate level translate into dynamic nonconvexities at the aggregate level? We address this question in a framework where the production of differentiated intermediate inputs is subject to dynamic nonconvexities and show that the answer depends on the degree of Hicks-Allen complementarity (substitutability) between differentiated inputs. In our simplest model, a generalization of Judd (1985) and Grossman and Helpman (1991) among many others, there are dynamic nonconvexities at the aggregate level if and only if differentiated inputs are Hicks-Allen complements. We also compare dynamic equilibrium and optimal allocations in the presence of aggregate dynamic nonconvexities due to Hicks-Allen complementarities between differentiated inputs.

Keywords. Dynamic Nonconvexities at the Disaggregate Level, Dynamic Nonconvexities at the Aggregate Level, Hicks-Allen Complements (Substitutes), Optimal Intertemporal Allocations, Dynamic Equilibrium Allocations, Dynamic Inefficiencies.

Journal of Economic Literature Classifications. C61, C62, D2.

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1 Introduction

After many years of intensive research our understanding of optimal and equilibrium intertemporal allocations in convex economic models has become fairly complete. The recent literature has turned its focus to characterizing intertemporal allocations in models with dynamic nonconvexities. At least two approaches can be distinguished. The first approach—referred to as the aggregative framework by Majumdar and Mitra (1982)—starts from dynamic nonconvexities at the aggregate level by assuming that aggregate production is a convex-concave function of the aggregate capital stock, see Skiba (1978), Majumdar and Mitra (1982), Dechert and Nishimura (1983), and Brock and Malliaris (1989) among others. There are two main drawbacks to this approach: First, nothing can be said about decentralized market equilibrium allocations; second, the assumption of a convex-concave production-function, quite natural at the disaggregate level because of fixed costs or indivisibilities for example, is difficult to defend at the aggregate level. The second approach starts from dynamic nonconvexities at the disaggregate level, see Judd (1985), Romer (1987, 1990), and Grossman and Helpman (1991) among many others; despite the dynamic nonconvexities at the disaggregate level, there are no dynamic nonconvexities at the aggregate level in this approach.

This paper reexamines the aggregate implications of dynamic nonconvexities at the disaggregate level for optimal as well as decentralized market equilibrium allocations. We do so in a framework where the production of differentiated intermediate inputs is subject to dynamic nonconvexities due to start-up costs. The main question we ask is: When will dynamic nonconvexities at the disaggregate level translate into dynamic nonconvexities at the aggregate level? We show that the answer depends on the degree of Hicks-Allen complementarity (substitutability) between differentiated inputs. In our simplest model, an extension of Judd (1985), Grossman and Helpman (1991) and many others, there are dynamic nonconvexities at the aggregate level if and only if differentiated inputs are Hicks-Allen complements.

Dynamic nonconvexities in the production of differentiated intermediate inputs and Hicks-Allen complementarities between differentiated inputs can therefore provide microeconomic foundations for the convex-concave aggregate production-function in Skiba (1978), Majumdar and Mitra (1982), and Dechert and Nishimura (1983). These microeconomic foundations of aggregate nonconvexities allow us to define and characterize decentralized market equilibria following Judd (1985) and Grossman and Helpman (1991). We find that Hicks-Allen complementarities imply that the private return to investment increases with the aggregate level of investment and that decentralized market equilibrium allocations are qualitatively similar to allocations in the aggregative framework. We also find that economies may get inefficiently stuck at very low levels of income when there are Hicks-Allen complementarities between differentiated inputs produced with dynamic increasing returns.

The rest of the paper is organized in the following way. Section 2 describes the framework we use. Section 3 characterizes the production possibility set and links dynamic nonconvexities at the aggregate level to the degree of Hicks-Allen complementarity between differentiated inputs produced with dynamic increasing returns. Section 4 determines the optimal intertemporal allocation. Section 5 defines and describes decentralized market equilibria and compares optimal and equilibrium allocations. Section 6 summarizes.

2 The Framework

The total quantity of labor in the economy is normalized to unity. The economy produces three types of goods: Investment goods, consumption goods, and an endogenous variety of differentiated intermediate inputs. Differentiated inputs are indexed by $i \geq 0$. Although the space of differentiated inputs is unbounded, only a finite range $i \leq n$ is produced at any moment in time (all endogenous variables, like n for example, depend on time but time subscripts will generally be suppressed). Over time, this range can be increased by allocating I units of the investment good to start-up operations; the technology for start-up operations is $\dot{n} = I$ (dots denote time derivatives). All differentiated intermediate inputs $i \leq n$ are produced with constant returns to scale at the margin; x_i units of labor used to produce input $i \leq n$ produce $m_i = x_i$ units of the input.

The consumption-goods technology is given most generally by $C = F(m_i^C : i \geq 0, L^C)$ where L^C denotes the quantity of labor and m_i^C the quantity of intermediate input i employed in the production of consumption goods. We assume that the consumption-goods technology can be rewritten as

$$C = F(M^C, L^C) \tag{1}$$

where $F(\bullet)$ is a linear homogenous, concave, and twice continuously differentiable function, and

$$M^C = \left(\int_0^\infty \left(m_i^C\right)^{(\sigma-1)/\sigma} di \right)^{\sigma/(\sigma-1)} = \left(\int_0^n \left(m_i^C\right)^{(\sigma-1)/\sigma} di \right)^{\sigma/(\sigma-1)} \quad \text{with } \sigma > 1. \tag{2}$$

where the second equality makes use of the fact that only differentiated inputs $i \leq n$ are available at any moment in time. We refer to M^C as intermediate-input composites. This specification implies weak separability between differentiated inputs and labor. The assumption that $\sigma > 1$ ensures that no single differentiated input is essential for producing intermediate-input composites. All differentiated inputs enter symmetrically into the production of the intermediate-input composite, and the elasticity of substitution between any pair of inputs in the production of intermediate-input composites is constant and equal to $\sigma > 1$.

The production of investment goods also employs differentiated intermediate inputs and labor, with m_i^I the quantity of input i, M^I the quantity of intermediate-input composites (produced according to (2) using m_i^I, $i \leq n$, as inputs) and L^I the quantity of labor used in the production of investment goods. The production of investment goods may, however, employ a different technology than the production of consumption goods,

$$I = G(M^I, L^I), \tag{3}$$

where $G(\bullet)$ is a linear homogenous, concave, and twice continuously differentiable function.

The elasticity of substitution between differentiated inputs in the production of investment goods and consumption goods will depend on the investment-goods technology $G(\bullet)$ and the consumption-goods technology $F(\bullet)$. The model is sufficiently general to allow differentiated inputs to be either Hicks-Allen complements or Hicks-Allen substitutes. We can therefore use the model to discuss the relationship between dynamic nonconvexities at the aggregate level and the degree of Hicks-Allen complementarity (substitutability) between differentiated inputs produced with dynamic increasing returns.

The form of product differentiation specified in the intermediate-input composite in (2) has an important property for the analysis of intertemporal allocations: Total factor productivity increases with the variety of differentiated inputs available. To see this, let X be the total amount of labor used in the production of differentiated inputs. Because of symmetry and convexity, it is optimal to produce the same quantity m_i of each existing variety $m_i = m = x_i = X/n$ for all $i \le n$. The quantity of intermediate-input composites that can be produced with X units of labor is therefore $M = n^{\sigma/(\sigma-1)}m = n^{1/(\sigma-1)}X$. Since $\sigma > 1$, the average productivity of labor in producing intermediate-input composites increases with the variety of differentiated inputs n. Ethier (1982) describes this property as increasing returns due to specialization (and $1/(\sigma-1)$ is sometimes referred to as the degree of increasing returns due to specialization in the production of intermediate-input composites), and Romer (1987) observes that this captures Young's (1928) notion of increasing returns due to the progressive specialization of industries.

The basic specification of intermediate-input composites in (2) always has a built-in link between the degree of increasing returns due to specialization $1/(\sigma-1)$ and the elasticity of substitution between differentiated inputs in the production of intermediate-input composites σ. We will—however—extend the discussion of the relationship between aggregate dynamic nonconvexities and Hicks-Allen complementarities between differentiated inputs produced with dynamic increasing returns to a specification of the intermediate-input composite that unlinks the degree of increasing returns due to specialization and the elasticity of substitution between differentiated inputs in the production of intermediate-input composites.

3 The Shape of the Aggregate Production Possibility Frontier

The analysis of the shape of the aggregate production possibility frontier will proceed in two steps. First, we determine the rate of transformation between consumption and investment at the same point in time as a function of the variety of differentiated inputs (the shape of the "static production possibility frontier"). Second, we determine the rate of transformation between consumption at different points in time as a function of the variety of differentiated inputs (the shape of the "dynamic production possibility frontier").

3.1 The Shape of the Static Production Possibility Frontier

We start with the rate of transformation between consumption and investment goods at the same point in time as a function of the variety of differentiated inputs n.

PROPOSITION 3.1. The production possibility frontier is

$$1 = C/\hat{F}(n) + I/\hat{G}(n) \tag{4}$$

where

$$\hat{F}(n) = \underset{x,\ 0\le x\le 1}{\mathrm{Max}} \quad F(n^{1/(\sigma-1)}x, 1-x), \tag{5}$$

and

$$\hat{G}(n) = \underset{x,\ 0\le x\le 1}{\text{Max}}\ G(n^{1/(\sigma-1)}x, 1-x)\,. \qquad (6)$$

PROOF: In the appendix.

Making use of (1), (2), (3), and the intermediate-input technology, it is straightforward to show that $\hat{F}(n)$ and $\hat{G}(n)$ correspond to the maximum amount of consumption and investment goods that can be produced with one unit of labor when a variety n of differentiated inputs is available; (4) implies that the rate of transformation between consumption and investment goods at the same point in time is $\hat{F}(n)/\hat{G}(n)$.

3.2 The Shape of the Dynamic Production Possibility Frontier

We now turn to the analysis of the intertemporal rate of transformation of consumption. Our main results link the intertemporal rate of transformation of consumption to the degree of Hicks-Allen complementarity (substitutability) between differentiated inputs.

3.2.1 Aggregate Dynamic Nonconvexities: Definitions

To determine the intertemporal rate of transformation of consumption in the framework in Section 2 it is useful to rewrite the production possibility frontier in (4) as

$$\hat{F}(n) = C + \frac{\hat{F}(n)}{\hat{G}(n)} I. \qquad (7)$$

Equation (7) can be interpreted in the following way: $\hat{F}(n)$ is the maximum output of consumption goods (keeping in mind that the total quantity of labor is normalized to unity) and $\hat{F}(n)/\hat{G}(n)$ is the cost of one unit of the investment good in terms of the consumption good. This implies that $\hat{G}(n)/\hat{F}(n)$ is the increase in the variety of differentiated intermediate inputs that can be achieved by consuming one unit less "today" (keeping in mind that the start-up technology is $\dot{n} = I$: we use quotation marks because time is continuous in the model) and that $\hat{F}'(n)\hat{G}(n)/\hat{F}(n)$ is the increase in maximum output of consumption goods "tomorrow" that can be achieved by consuming one unit less "today." The intertemporal rate of transformation of consumption $\hat{F}'(n)\hat{G}(n)/\hat{F}(n)$ is a function of the existing variety of differentiated inputs n only.

DEFINITION 3.1 (Intertemporal rate of transformation of consumption; dynamic aggregate returns schedule). The intertemporal rate of transformation of consumption, denoted by $\hat{r}(n)$, is

$$\hat{r}(n) = \hat{F}'(n)\hat{G}(n)/\hat{F}(n)\,. \qquad (8)$$

We also refer to $\hat{r}(n)$ as the dynamic aggregate returns schedule.

This definition allows us to distinguish between dynamic decreasing and dynamic increasing aggregate returns.

> **DEFINITION 3.2 (Dynamic decreasing aggregate returns; dynamic increasing aggregate returns schedule)** Dynamic decreasing (increasing) aggregate returns refers to instances where the intertemporal rate of transformation of consumption decreases (strictly increases) with the variety of differentiated inputs, $\hat{r}'(n) \leq 0$ ($\hat{r}'(n) > 0$).

We also refer to dynamic increasing aggregate returns as aggregate dynamic nonconvexities.

3.2.2 Aggregate Dynamic Nonconvexities: An Example

Before turning to the general analysis of aggregate dynamic nonconvexities, it may be useful to illustrate dynamic increasing aggregate returns in an example that allows us to determine the production possibility set in (4) explicitly. To do so, consider the case where the production of consumption goods and investment goods uses identical, perfectly symmetric, constant-elasticity-of-substitution technologies, $C = I = (M^{(\varepsilon-1)/\varepsilon} + L^{(\varepsilon-1)/\varepsilon})^{\varepsilon/(\varepsilon-1)}$; suppose also that the elasticity of substitution between intermediate-input composites and labor is equal to 5, $\varepsilon = 5$, while the elasticity of substitution between differentiated inputs in (2) is equal to 3, $\sigma = 3$. In this case, the production possibility set in (4) becomes

$$C + I \leq \hat{F}(n) = (1 + n^2)^{1/4},$$

and $\hat{F}(n)$, which can be interpreted as the "aggregate production-function," is convex-concave:[1] There will be dynamic increasing aggregate returns ($\hat{r}(n) = \hat{F}'(n)$ strictly increasing) if $n < \sqrt{2}$ and dynamic decreasing aggregate returns ($\hat{r}(n) = \hat{F}'(n)$ decreasing) if $n \geq \sqrt{2}$. (See Section 3.2.4.A for the more general case.)

3.2.3 Aggregate Dynamic Nonconvexities and Hicks-Allen Complementarities

When do dynamic nonconvexities in the production of differentiated inputs translate into aggregate dynamic nonconvexities, i.e. dynamic increasing aggregate returns? This section uses the well-known concept of complementarities due to Hicks and Allen (1934) to relate the degree of complementarity (substitutability) between differentiated inputs to the existence of dynamic increasing (decreasing) aggregate returns. The next subsection develops some useful preliminary results.

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[1] We thank one of the referees for suggesting this example.

3.2.3.A Hicks-Allen Complementarities: Definitions and Preliminary Results

To apply the concept of complementarities due to Hicks and Allen (1934) to the framework in Section 2, it is necessary to first define the cost-minimizing intermediate-input demand.

PROPOSITION 3.2. Denote the opportunity-cost of one unit of labor in terms of consumption goods with a and the opportunity-cost of one unit of intermediate input i in terms of consumption goods with a_i. The quantity of input i that minimizes the cost of producing one unit of the consumption good, $\hat{m}_i$, can be written as

$$\hat{m}_i^C(a_i, a_M, a) = \left(a_i / a_M\right)^{-\sigma} \hat{A}(a_M / a) / a_M \text{ for } i \leq n. \tag{9}$$

where a_M denotes the opportunity-cost of one unit of the intermediate-input composite in (2)

$$a_M = \left(\int_0^n a_i^{1-\sigma} di\right)^{1/(1-\sigma)}. \tag{10}$$

and $\hat{A}(a_M / a)$ denotes the cost-share of intermediate inputs

$$\hat{A}(a_M / a) = \left\{F_M(M^C, L^C) M^C / F(M^C, L^C) : F_M(M^C, L^C) / F_L(M^C, L^C) = a_M / a\right\}. \tag{11}$$

PROOF: In the appendix.

The definition of complementarities that we use is due to Hicks and Allen (1934).

DEFINITION 3.3 (Hicks-Allen substitutes; Hicks-Allen complements). Intermediate inputs i and j, $i \neq j$, are Hicks-Allen substitutes (complements) if the Hicks-Allen partial elasticity of substitution $\partial \log \hat{m}_i^C(a_i, a_M, a) / \partial \log a_j$ is positive (strictly negative).

Finally, it will be useful to introduce the Hicks-Allen partial elasticity of substitution between differentiated intermediate inputs and the intermediate-input composite, $\partial \log \hat{m}_i^C(a_i, a_M, a) / \partial \log a_M$, and to relate this elasticity to some of the parameters of the consumption-goods technology.

PROPOSITION 3.3. The Hicks-Allen partial elasticity of substitution between differentiated intermediate inputs and the intermediate-input composite satisfies

$$\partial \log \hat{m}_i^C(a_i, a_M, a) / \partial \log a_M = \xi(a_M / a) = (\sigma - 1) - \left(\varepsilon(a_M / a) - 1\right)(1 - \hat{A}(a_M / a)), \tag{12}$$

where $\varepsilon(a_M / a)$ denotes the elasticity of substitution between intermediate-input composites and labor in the production of consumption goods as a function of their relative opportunity-cost a_M / a.

PROOF: In the appendix.

The definition of Hicks-Allen substitutes (complements) combined with (9) and (10) implies that differentiated inputs are Hicks-Allen substitutes (complements) if $\xi(a_{xd}/a) \geq 0$ ($\xi(a_{xd}/a) < 0$).

3.2.3.B Aggregate Dynamic Nonconvexities and Hicks-Allen Complementarities: Results

The next proposition allows us to prove the main results linking the degree of Hicks-Allen complementarity (substitutability) to dynamic increasing (decreasing) aggregate returns.

PROPOSITION 3.4. There will be dynamic increasing aggregate returns if and only if

$$\xi(n^{1/(1-\sigma)}) < \hat{x}^I(n), \tag{13}$$

where $\hat{x}^I(n) = \underset{x,\, 0\leq x\leq 1}{\text{Argmax}}\ G(n^{1/(\sigma-1)}x, 1-x)$.

PROOF: In the appendix.

The intuition is that there are two forces that can potentially result in dynamic increasing aggregate returns. The first is the Hicks-Allen complementarity between differentiated inputs; this is because new differentiated inputs will increase the productivity of existing inputs if differentiated inputs are Hicks-Allen complements. The second force arises when the production of investment goods also uses differentiated inputs; the introduction of new differentiated inputs will, in this case, decrease the opportunity-cost of investment because of increasing returns due to increasing specialization.

Proposition 3.4 yields the next two results.

PROPOSITION 3.5. There will be dynamic increasing aggregate returns if differentiated inputs are Hicks-Allen complements.

This follows because Hicks-Allen complementarity between differentiated inputs, (9), (10), and the intermediate-input technology imply $\xi(a_{xd}/a) < 0$ and $a_{xd}/a = n^{1/(1-\sigma)}$.

The reverse of Proposition 3.5 is not always true. (Later we show that the reverse holds in a class of models used widely in modern growth theory). But there will be dynamic decreasing aggregate returns if differentiated inputs are good enough (strong) substitutes.

PROPOSITION 3.6. There will be dynamic decreasing aggregate returns if differentiated inputs are strong Hicks-Allen substitutes in the sense that the Hicks-Allen partial elasticity of substitution between differentiated intermediate inputs and the intermediate-input composite is larger than unity, $\xi(n^{1/(1-\sigma)}) \geq 1$.

This follows from Proposition 3.4 and the fact that $\hat{x}^I(n) \leq 1$.

Proposition 3.6 implies that a sufficient condition for globally decreasing dynamic aggregate returns is that differentiated inputs are strong substitutes for any variety of available inputs n; using Proposition 3.3, it is straightforward to show that this will be the case if the elasticity of substitution between intermediate-input composites and labor in the production of consumption goods is smaller than $\sigma - 2$, $\varepsilon(a_{xl}/a) \leq \sigma - 2$.

3.2.4 Aggregate Dynamic Nonconvexities in One and Two Sector Models

The previous section has yielded some insight into the determinants of dynamic increasing aggregate returns. Now we want to better characterize the dynamic aggregate returns schedule in two widely used classes of models. First, the one sector model of standard neoclassical growth theory, see for example Solow (1956) and Cass (1965). Second, the two sector model that has become the workhorse in more recent work in growth theory, see Judd (1985) and Grossman and Helpman (1991) among many others.

3.2.4.A The One Sector Model

In the one sector model, consumption and investment goods are produced with identical technologies, or $F(M,L) = G(M,L)$ in terms of the framework in Section 2. In this case, the production possibility set is $C + I \leq \hat{F}(n)$ where $\hat{F}(n)$ plays the role of the aggregate production-function in the standard neoclassical growth model, see Cass (1965) for example. Propositions 3.3 and 3.4 can now be used to obtain sufficient conditions for dynamic decreasing (increasing) aggregate returns: it can be shown that they imply that there will be dynamic increasing aggregate returns if and only if $\sigma - 2 < (\varepsilon(n^{1/(1-\sigma)}) - 2)(1 - \hat{A}(n^{1/(1-\sigma)}))$.[2] Hence, there will be globally decreasing dynamic aggregate returns if $\sigma \geq \varepsilon(\circ)$ and $\sigma \geq 2$: in this case, the effect due to the substitutability between differentiated inputs outweighs the effect due to the decreasing opportunity-cost of investment goods. There will be globally increasing dynamic aggregate returns if $\sigma < \varepsilon(\circ)$ and $\sigma < 2$.

To determine simple sufficient conditions for a convex-concave "aggregate production-function" $\hat{F}(n)$, we specify the consumption- and investment-goods technology in (1) and (3) as

$$F(M,L) = G(M,L) = \left(M^{(\varepsilon-1)/\varepsilon} + \beta^{1/\varepsilon} L^{(\varepsilon-1)/\varepsilon}\right)^{\varepsilon/(\varepsilon-1)}, \qquad (14)$$

where ε is the constant elasticity of substitution between intermediate-input composites and labor. In this case, the "aggregate production-function" becomes

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[2] This is because in the one sector model $\hat{x}^I(n) = \hat{A}(n^{1/(1-\sigma)})$; this follows from $F(M,L) = G(M,L)$ and (A9) in the appendix.

$$\hat{F}(n) = \left(\beta + n^{(\varepsilon-1)/(\sigma-1)} \right)^{1/(\varepsilon-1)}.[3] \tag{15}$$

Using either the "aggregate production-function" in (15) or Propositions 3.3 and 3.4, yields that there will be dynamic increasing aggregate returns if and only if

$$\beta(\varepsilon - \sigma) n^{-(\varepsilon-1)/(\sigma-1)} > \sigma - 2. \tag{16}$$

Our disaggregate framework therefore provides microeconomic foundations for the convex-concave aggregate production-function in Skiba (1978), Majumdar and Mitra (1982), Dechert and Nishimura (1983), and Brock and Malliaris (1989) if $\varepsilon > \sigma > 2$. The convex-concave "aggregate production-function" arises because differentiated inputs produced with dynamic increasing returns are Hicks-Allen complements when few are available and become strong Hicks-Allen substitutes as the available variety increases.

The specification in (14) allows us to fully characterize the global shape of the "aggregate production-function" in (15) in terms of the elasticities of substitution ε, σ in (14) and (2). The results in the beginning of this subsection imply that the "aggregate production-function" will be globally concave if $\varepsilon \leq \sigma$ and $\sigma \geq 2$ and globally convex if $\varepsilon > \sigma$ and $\sigma < 2$. The "aggregate production-function" will be concave-convex if $2 > \sigma > \varepsilon$ and convex-concave if $\varepsilon > \sigma > 2$. Figure 1 summarizes the relationship between the elasticities of substitution ε, σ in (14) and (2) and the global shape of the "aggregate production-function" in (15).

3.2.4.B The Two Sector Model

Judd (1985) presents a growth model that has been widely used in recent contributions to growth theory, see Grossman and Helpman (1991) for examples and references. The version of Judd (1985) in Grossman and Helpman (1991) has two sectors as consumption and investment goods are produced with different technologies. In particular, investment goods are produced with labor only, or $G(M, L) = L$ in terms of the framework in Section 2. In this two sector model, $\hat{x}^I(n) = 0$ and Proposition 3.4 therefore implies that Hicks-Allen complementarities between differentiated inputs are necessary and sufficient for dynamic increasing aggregate returns. Propositions 3.3 and 3.4 yield simple sufficient conditions for the global shape of the dynamic aggregate returns schedule in terms of parameters of the consumption-goods technology: There will be globally decreasing dynamic aggregate returns if $\varepsilon(a_{xx} / a) \leq \sigma$ (which includes the Grossman and Helpman (1991) case where

[3] Equation (15) is calculated as $\hat{F}(n) = \underset{X+L\leq 1}{\mathrm{Max}} \left(n^{(\varepsilon-1)/\varepsilon(\sigma-1)} X^{(\varepsilon-1)/\varepsilon} + \beta^{1/\varepsilon} L^{(\varepsilon-1)/\varepsilon} \right)^{\varepsilon/(\varepsilon-1)}$, where use has been made of (2), (14), and the intermediate-input technology.

consumption goods are produced with a Cobb-Douglas technology); differentiated inputs can never be Hicks-Allen complements in this case. If, on the other hand, $b \geq \varepsilon(a_M / a) > \sigma$, where b is some constant, then there will be dynamic increasing aggregate returns for a low variety of differentiated inputs, but dynamic decreasing aggregate returns for a large variety of differentiated inputs;[4] this is because differentiated inputs will be Hicks-Allen complements when few are available and become Hicks-Allen substitutes as the available variety increases. The two sector model with an elasticity of substitution $\varepsilon(a_M / a)$ that may be larger than σ therefore extends Judd (1985) and Grossman and Helpman (1991) to include the case of the inverted U-shaped dynamic aggregate returns schedule typical of the aggregative framework.

3.2.5 Aggregate Dynamic Nonconvexities and Returns to Specialization

The specification of the intermediate-input composite in (2) has a built-in link between the elasticity of substitution between differentiated inputs σ and the degree of increasing returns due to specialization in the production of intermediate-input composites. This is why we now turn to a specification of intermediate-input composites—due to Ethier (1982)—that separates the degree of increasing returns due to specialization from the elasticity of substitution between differentiated intermediate inputs in the production of intermediate-input composites, see also Benassy (1996). In particular, we replace (2) by

$$M = n^{\gamma}\left(\int_0^n m_i^{(\sigma-1)/\sigma} di\right)^{\sigma/(\sigma-1)} \tag{17}$$

where

$$\gamma = (\sigma(\alpha-1)-\alpha)/(\sigma-1), \tag{18}$$

and $\alpha > 1$ and $\sigma > 1$. The specification in (17) introduces a "direct specialization effect" of the variety of differentiated inputs n that is captured by γ; the introduction of new inputs has a direct effect on the productivity of other inputs that is independent of the quantity of new inputs actually used in production. This direct specialization effect separates the degree of increasing returns due to specialization from the elasticity of substitution between differentiated intermediate inputs in the production of intermediate-input composites. The specification in (18) implies that the degree of increasing returns due to specialization is $\alpha - 1$ while the elasticity of substitution between differentiated inputs in the production of intermediate-input composites is σ.

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[4] To see this, notice that $\varepsilon(a_M / a) > \sigma > 1$ implies that $A(a_M / a) \to 1$ as $a_M / a \to 0$ and that (10) implies that $a_M / a \to 0$ as $n \to \infty$. Combined with (12) and (13), this yields the result.

It can be shown that the basic relationship between the degree of Hicks-Allen complementarity (substitutability) and dynamic increasing (decreasing) aggregate returns developed in Propositions 3.5 and 3.6 generalizes—although it must of course be augmented to take into account the additional considerations formalized in (17) and (18). To see this in the simplest way possible, notice first that Proposition 3.5 will hold with (17) replacing (2) if $\gamma \geq 0$. This is because the intuition behind Proposition 3.5 is that if there are Hicks-Allen complementarities between differentiated inputs, then the introduction of new inputs makes other inputs more productive, and thus, results in dynamic increasing aggregate returns. The specification in (17) with a positive direct specialization effect, $\gamma \geq 0$ (new inputs directly increase the productivity of other inputs through the positive direct specialization effect), reinforces this effect of Hicks-Allen complementarities between differentiated inputs. If, on the other hand, $\gamma < 0$, then the complementarity between differentiated inputs needs to be sufficiently strong to outweigh the negative direct specialization effect. Similarly, Proposition 3.6 will hold with (17) replacing (2) if $\gamma \leq 0$. The intuition is that if differentiated inputs are Hicks-Allen substitutes, then the introduction of new inputs makes other inputs less productive, and thus, results in dynamic decreasing aggregate returns. The specification in (17) with a negative direct specialization effect, $\gamma \leq 0$ (new inputs directly decrease the productivity of other inputs through the negative direct specialization effect), reinforces this effect of Hicks-Allen substitutability between differentiated inputs. If, on the other hand, $\gamma > 0$, then the substitutability between differentiated inputs needs to be sufficiently strong to outweigh the positive direct specialization effect. These results are proven in the appendix.

4 Dynamic Optimality with a Representative Consumer

We now turn to the issue of dynamic optimality when there is a representative consumer. The intertemporal preferences of the consumer are $\int_0^\infty e^{-\rho\tau}U(C_\tau)d\tau$, with $U(C_\tau)$ twice continuously differentiable, strictly concave, $U'(C) \to \infty$ as $C \to 0$, and $\rho > 0$. We analyze dynamic optimality in two steps. We first derive necessary conditions for dynamic optimality and then develop a criterion to determine the path that yields higher intertemporal utility among any two paths satisfying those necessary conditions.

We focus throughout on convex-concave dynamic models and assume that dynamic aggregate returns fall below the rate of time preference ρ as the variety of differentiated inputs becomes sufficiently large. More precisely, we assume that there is a variety of differentiated inputs $n_m > 0$ that satisfies:

ASSUMPTION 1. $\hat{r}(n_m) = \rho$ and $\hat{r}(n) < \rho$ for $n > n_m$. (19)

This assumption also ensures the existence of a dynamically optimal path.

For a given initial variety of differentiated inputs $n_0 > 0$, the dynamically optimal allocation solves

$$\underset{\{C_\tau\}}{\text{Maximize}} \quad \int_0^\infty e^{-\rho\tau} U(C_\tau) d\tau \tag{20}$$

subject to

$$0 \leq \dot{n} = I = \hat{G}(n) - \frac{\hat{G}(n)}{\hat{F}(n)} C. \tag{21}$$

The next proposition gives necessary conditions for dynamic optimality.

PROPOSITION 4.1. The following conditions are necessary for dynamic optimality.

$$\frac{\dot{C}}{C} = \begin{cases} \gamma(C)\left(\dfrac{\hat{F}'(n)\hat{G}(n)}{\hat{F}(n)} - \rho\right) & \text{if} \quad C < \hat{F}(n) \\ 0 & \text{if} \quad C = \hat{F}(n) \quad \text{and} \quad \rho \geq \dfrac{\hat{F}'(n)\hat{G}(n)}{\hat{F}(n)} \end{cases} . \tag{22}$$

where $\gamma(C) = -U'(C)/U''(C)C$ denotes the intertemporal elasticity of substitution. The transversality condition $\lim_{\tau\to\infty} e^{-\rho\tau}(\hat{F}(n_\tau)U'(C_\tau)/\hat{G}(n_\tau))n_\tau = 0$ is also necessary.

PROOF: In the appendix.

It can be shown that the necessary conditions in Proposition 4.1 and (21) have a solution for all $n_0 > 0$ if (19) holds. The condition in (22) confirms that the dynamic aggregate returns schedule $\hat{r}(n)$ defined in (8) plays a crucial role for the optimal intertemporal allocation.

The main complication introduced by dynamic increasing aggregate returns is that the necessary conditions for dynamic optimality in Proposition 4.1 and (21) may have more than one solution. The next proposition selects the consumption profile with higher intertemporal utility between any two consumption profiles that satisfy the necessary conditions in Proposition 4.1 and (21).

PROPOSITION 4.2. Consider any two paths, indexed by *i* and *ii*, that for a given $n_0 > 0$ satisfy the necessary conditions for dynamic optimality in Proposition 4.1 and (21). Then, path *ii* yields strictly higher intertemporal utility than path *i* if and only if it has a strictly lower initial level of consumption.

PROOF: In the appendix.

4.3 Dynamic Optimality in the One Sector Model

When consumption goods and investment goods are produced with the same technology, then the dynamic system in (21) and (22) simplifies to

$$\begin{aligned} \dot{C}/C &= \gamma(C)(\hat{F}'(n)-\rho) \\ \dot{n} &= \hat{F}(n)-C \end{aligned} \tag{23}$$

in the interior. This system is identical to the dynamic system of the standard neoclassical growth model, see Cass (1965) for example, with $\hat{F}(n)$ taking the place of the aggregate production-function and $\hat{r}(n)=\hat{F}'(n)$ taking the place of the marginal product of capital.

Specifying the consumption- and investment-goods technology as in (14), allows us to better characterize optimal intertemporal allocations. If $\varepsilon \leq \sigma$ and $\sigma \geq 2$, then the "aggregate production-function" $\hat{F}(n)$ will be globally concave. As a result, there will be globally decreasing dynamic aggregate returns and optimal intertemporal allocations will be qualitatively similar to optimal intertemporal allocations in the standard neoclassical, convex growth model in the following sense: Optimal intertemporal allocations in economies that start with a variety of differentiated inputs n strictly below the values for n that satisfy $\hat{F}'(n)=\rho$ will be characterized by an increasing variety of differentiated inputs and an increasing level of consumption, and converge to the same steady-state level of income.

If, on the other hand, $\varepsilon > \sigma > 2$, then our disaggregate framework provides microeconomic foundations for the convex-concave aggregate production-function and hence optimal intertemporal allocations in the aggregative framework in Skiba (1978), Majumdar and Mitra (1982), Dechert and Nishimura (1983), and Brock and Malliaris (1989). The dynamic aggregate returns schedule will be inverted U-shaped, and, generally, there will be either no value or two values for n that satisfy $\hat{F}'(n)=\rho$. Assuming that there are two values, optimal intertemporal allocations are: Optimal intertemporal allocations in economies that start with a variety of differentiated inputs n strictly between the two values that satisfy $\hat{F}'(n)=\rho$ are characterized by a rising variety of differentiated inputs and increasing levels of consumption, and converge to a variety of differentiated inputs equal to the larger of the two solutions of $\hat{F}'(n)=\rho$; economies that start just below the lower of the two values that satisfy $\hat{F}'(n)=\rho$ will experience temporarily falling levels of consumption and rising levels of investment, and also converge to a variety of differentiated inputs equal to the larger of the two solutions of $\hat{F}'(n)=\rho$; optimal investment will be zero and consumption will be constant in economies that start with a very low variety of differentiated inputs. Optimal allocations in the one sector model with consumption and investment produced according to (14) and $\varepsilon > \sigma > 2$ are illustrated in Figure 2 which makes use of Proposition 4.2 to select optimal intertemporal allocations whenever there are multiple paths that satisfy the necessary conditions in Proposition 4.1 and (21).

The results for the two sector model with the consumption-goods technology as specified in (14) are similar. If $\varepsilon \leq \sigma$, then there are globally decreasing dynamic aggregate returns and optimal intertemporal allocations are qualitatively similar (in the sense explained in the previous subsection) to optimal intertemporal allocations in the standard neoclassical, convex growth model. If $\varepsilon > \sigma$, then the dynamic aggregate returns schedule has an inverted U-shape and optimal intertemporal allocations will therefore be qualitatively similar to optimal intertemporal allocations in the aggregative framework with a convex-concave aggregate production-function.

5 Dynamic Equilibrium Allocations

One advantage of our approach to aggregate dynamic nonconvexities—compared to the approach in Skiba (1978), Majumdar and Mitra (1982), Dechert and Nishimura (1983), and Brock and Malliaris (1989)—is that it allows us to define and characterize decentralized dynamic market equilibria and assess their optimality. We define and characterize dynamic market equilibria in the framework in Section 2 using the following market structure: Markets for consumption goods, investment goods, and labor are perfectly competitive but the market for differentiated intermediate inputs produced with dynamic increasing returns is monopolistically competitive; this market structure is common to the many recent contributions to growth theory that, following Judd (1985), Romer (1987,1990), and Grossman and Helpman (1991), associate economic growth with the introduction of new inputs. Our objective is to analyze how the degree of Hicks-Allen complementarity (substitutability) between differentiated inputs affects the characteristics and optimality of dynamic market equilibria.

There are two related main results in this section. First, Hicks-Allen complementarities play a similar role for dynamic market equilibrium allocations than for optimal dynamic allocations: they imply that the private return to investment increases with the aggregate level of investment and therefore result in dynamic market equilibrium allocations that are qualitatively similar to optimal intertemporal allocations in the aggregative framework. Second, Hicks-Allen complementarities give rise to dynamic equilibrium allocations that are "globally inefficient" in the following sense: Economies that start with low levels of income may be stuck in equilibrium, although the optimal intertemporal allocation involves strictly positive investment and growth, and convergence to the level of income of economies that started with high levels of income. These dynamic inefficiencies cannot be undone by marginal, Pigouvian tax policies but must be addressed with nonlinear policy instruments.

5.1 Characterization of Dynamic Market Equilibrium Allocations

Denote the wage rate, the price of investment goods, and the price of differentiated input i, all in terms of consumption goods, with w, q, and p_i. The fact that investment- and consumption-goods producers take prices as given, then implies that profit-maximizing intermediate-input and labor demands satisfy: $F_l(M^C, L^C) = w$, $F_M(M^C, L^C) = p_M$, and $m_i^C / M^C = (p_i / p_M)^{-\sigma}$ in the production of consumption goods and $qG_L(M^I, L^I) = w$, $qG_M(M^I, L^I) = p_M$, and $m_i^I / M^I = (p_i / p_M)^{-\sigma}$ in the production of investment goods; all variables are defined analogously to the optimal case and

$$p_M = \left(\int_0^{\infty} p_i^{1-\sigma} di \right)^{1/(1-\sigma)}$$

denotes the minimum cost of purchasing sufficient differentiated inputs to produce one unit of the intermediate-input composite defined in (2).

Differentiated inputs are produced by monopolistically competitive firms: Each firm produces one input and maximizes profits by setting its price taking all other prices as given. This implies that profit-maximizing prices for available differentiated inputs are given by a constant markup over marginal cost, $p = \mu w$ for $i \le n$ where $\mu = \sigma / (\sigma - 1) > 1$, as each firm faces an intermediate-input demand with constant price elasticity. The price of intermediate-input composites relative to labor is therefore equal to $p_M / w = \mu n^{1/(1-\sigma)}$ in equilibrium. Combined with constant returns to scale of the consumption-goods technology, this implies that the factor share of differentiated inputs in the production of consumption goods, $p_M M^C / C$, can be written as

$$A(n) = \left\{ F_M(M^C, L^C) M^C / F(M^C, L^C) : F_M(M^C, L^C) / F_l(M^C, L^C) = \mu n^{1/(1-\sigma)} \right\}, \tag{24}$$

where we made use of the fact that $F_M(M^C, L^C) = p_M$ and $F(M^C, L^C) = C$ in equilibrium. Defining the income share of differentiated inputs in the production of investment goods, $B(n)$, analogously to (24), yields the next proposition.

PROPOSITION 5.1. Labor market clearing implies that

$$1 = C / F(n) + I / G(n), \tag{25}$$

where

$$F(n) = F((\mu^{1-\sigma} n)^{1/(\sigma-1)} A(n), (1 - A(n)) / (1 - A(n) / \sigma)) \tag{26}$$

and

$$G(n) = G((\mu^{1-\sigma} n)^{1/(\sigma-1)} B(n), (1 - B(n)) / (1 - B(n) / \sigma)) . \tag{27}$$

PROOF: In the appendix.

The variety of available differentiated inputs n increases over time through entry of new firms. These new firms must start up production by purchasing one unit of the investment good; this start-up investment is irreversible. There is free entry into the production of new differentiated inputs: the market value of differentiated input firms v—not indexed by i as all firms have the same market value in equilibrium—will therefore never exceed the start-up cost q in equilibrium, $q \geq v$. Free entry also implies that there will be no entry if the market value of differentiated input firms is strictly below the start-up cost, $v < q$; when there is entry in equilibrium, then the market value of differentiated input firms and the start-up cost will be equalized. Summarizing, we obtain that in equilibrium, $\dot{n} \geq 0$, $q \geq v$, and

$$\dot{n}(q - v) = 0 . \tag{28}$$

Households supply labor and own all firms. They choose their consumption profile to maximize intertemporal utility subject to their intertemporal budget constraint,

$$\int_0^{\infty} e^{-R_\tau} C_\tau d\tau \leq n_t v_t + \int_0^{\infty} e^{-R_\tau} w_\tau d\tau , \tag{29}$$

where $R_\tau = \int_0^\tau r_t dt$ is the real interest rate between periods 0 and τ. Their optimal consumption profile satisfies the Euler condition,

$$\dot{C} / C = \gamma(C)(r - \rho) , \tag{30}$$

where $\gamma(C) = -U'(C) / U''(C)C$ denotes the intertemporal elasticity of substitution. The next result summarizes the equilibrium conditions in terms of the level of consumption and differentiated inputs available, (C, n).

PROPOSITION 5.2. For any initial variety of differentiated inputs $n_o > 0$, dynamic market equilibria are characterized by levels of consumption C_τ and differentiated input varieties n_τ, $\tau \geq 0$, that satisfy

$$
\frac{\dot{C}}{C} = \begin{cases} \gamma(C)\left(\frac{\dot{q}(n)}{q(n)} + \frac{A(n)C + B(n)q(n)\dot{n}}{\sigma q(n)n} - \rho\right) & \text{if } C < F(n) \\ 0 & \text{if } C = F(n) \text{ and } h(n) \le \rho \end{cases} \quad (a) \tag{31}
$$

$$
\dot{n} = \operatorname{Max}\{G(n)(1 - C/F(n)), 0\} \quad (b)
$$

and

$$
\lim_{\tau\to\infty} e^{-\rho\tau} U'(C_\tau) n_\tau F(n_\tau)/G(n_\tau) = 0 \tag{32}
$$

where

$$
h(n) = A(n)F(n)/\sigma q(n)n \tag{33}
$$

and

$$
q(n) = (\sigma - A(n))F(n)/(\sigma - B(n))G(n). \tag{34}
$$

PROOF: In the appendix.

5.2 Dynamic Equilibrium and Optimality in the One Sector Model

We now prove existence and further characterize dynamic market equilibria, and also compare dynamic equilibrium and optimal allocations, in the one sector model in Section 3; the results for the two sector model are similar and therefore omitted.

In the one sector model, the equilibrium dynamic system in (31) simplifies to

$$
\begin{aligned} \dot{C}/C &= \gamma(C)(r(n) - \rho) \\ \dot{n} &= F(n) - C \end{aligned} \tag{35}
$$

in the interior, where $r(n) = A(n)F(n)/\sigma n$ is the rate of return to differentiated input firms' investment; $r(n)$ will be called the aggregate dynamic private returns schedule. Focusing on convex-concave models by assuming that there is a $n_e > 0$, such that $r(n_e) = \rho$ and $r(n) < \rho$ for all $n > n_e$, makes it straightforward to prove existence of a dynamic equilibrium. If n_0 satisfies $r(n_0) \le \rho$, then there is a stationary allocation with $C_\tau = F(n_0)$, $\tau \ge 0$, that satisfies all equilibrium conditions. If, on the other hand, $r(n_0) > \rho$, then there is a dynamic equilibrium with increasing levels of consumption and an increasing differentiated input variety that converges to a variety of differentiated inputs equal to the solution of $r(n) = \rho$ closest above n_0.

Hicks-Allen complementarities between differentiated inputs affect the aggregate dynamic private returns schedule $r(n)$ in the same way as the aggregate dynamic returns schedule $\hat{r}(n)$:

PROPOSITION 5.3. There will be dynamic increasing aggregate private returns $r'(n) > 0$ if differentiated inputs are complements in the sense of Hicks-Allen.

PROOF: In the appendix.

To better characterize dynamic market equilibria and assess their optimality in the presence of Hicks-Allen complementarities between differentiated inputs, we return to the case where consumption goods and investment goods are produced as in (14). In this case, we can determine $F(n) = G(n)$ defined in (26) and (27) explicitly as

$$F(n) = (\beta' / \mu + n^{(\varepsilon-1)/(\sigma-1)})^{1/(\varepsilon-1)} / \theta(n) \tag{35}$$

where $\beta' = \beta\mu^{\varepsilon}$ and

$$1 \leq \theta(n) = (\beta' + n^{(\varepsilon-1)/(\sigma-1)}) / (\beta' / \mu + n^{(\varepsilon-1)/(\sigma-1)}) \leq \mu . \tag{37}$$

This yields that there are dynamic increasing aggregate private returns ($r'(n) > 0$) if and only if

$$(\varepsilon - \sigma)\beta' n^{-(\varepsilon-1)/(\sigma-1)} > (\sigma - (1 + \theta(n))) . \tag{38}$$

This, combined with the fact that (37) implies that $\theta(n) \to 1$ as $n \to \infty$, implies that the aggregate dynamic private returns schedule $r(n)$ will have the inverted U-shape typical of the convex-concave aggregate production-function in the aggregative framework if $\varepsilon > \sigma > 2$; notice that this condition is identical to the condition than ensured the inverted U-shape of the aggregate returns schedule $\hat{r}(n)$. In this case, differentiated inputs are Hicks-Allen complements when few inputs are available but become strong substitutes as the available variety increases. Generally, there will be either no or two solutions to $r(n) = \rho$ when the necessary condition for Hicks-Allen complementarities, $\varepsilon > \sigma > 2$, is satisfied and $r(n)$ is inverted U-shaped. We focus on the case with two solutions where dynamic equilibrium allocations take the following form: In economies with n_0 larger or equal than the larger of the two values that satisfy $r(n) = \rho$, the unique equilibrium allocation is $C_\tau = F(n_0)$, $\tau \geq 0$, and there is no investment. Dynamic equilibria in economies with n_0 strictly between the two values that satisfy $r(n) = \rho$ are characterized by an increasing variety of differentiated inputs and rising levels of consumption and production: all these economies converge to the same steady-state variety of differentiated inputs—equal to the larger of the two values that satisfy $r(n) = \rho$—and the same level of income. There are two equilibria in economies that start with n_0 equal or just below the smaller of the two values that satisfy $r(n) = \rho$; the first equilibrium has a constant level of consumption $C_\tau = F(n_0)$, $\tau \geq 0$, and no investment; the second dynamic equilibrium is characterized by falling levels of consumption and rising levels of investment in the beginning and converges to a steady-state variety of differentiated inputs equal to the larger of the two values that satisfy $r(n) = \rho$. The dynamic equilibrium allocations in the one sector model when the necessary condition for Hicks-Allen complementarities, $\varepsilon > \sigma > 2$, is satisfied are illustrated in Figure 3. Comparing Figure 3 to Figure 2 illustrates that dynamic equilibrium allocations are qualitatively similar to optimal allocations in the following sense: They either involve no investment and a constant level of consumption, or a level of consumption (investment) that is falling (rising) when the variety of intermediate inputs is low and a level of consumption that is rising when the variety of intermediate inputs is large. Dynamic equilibrium allocations in the model with Hicks-Allen complementarities are therefore qualitatively similar to optimal allocations in the aggregative framework with a convex-concave aggregate production-function.

To compare dynamic equilibrium and optimal allocations in the one sector model when the necessary condition for Hicks-Allen complementarities, $\varepsilon > \sigma > 2$, is satisfied, we compare the optimal dynamic system in (23) with the equilibrium dynamic system in (35). Notice that the $n-$isocline in the equilibrium dynamic system in (23) lies below the $n-$isocline in the optimal dynamic system in (35) as $F(n) < \hat{F}(n)$ ($F(n) \leq \hat{F}(n)$ follows from the definition of $\hat{F}(n)$ as the largest quantity of consumption goods that can be produced with one unit of labor combined with the fact that $F(n)$ is the average labor productivity in the production of consumption goods in the market equilibrium; $F(n) < \hat{F}(n)$ follows because the markup charged by intermediate input producers drives a wedge between the marginal cost and the price of intermediate inputs in the market equilibrium). Furthermore, it can be shown that the aggregate private returns schedule $r(n)$ lies always strictly below the aggregate returns schedule $\hat{r}(n)$.[5] We focus on two implications of these observations about dynamic market equilibrium and optimal allocations. First, the "high" equilibrium steady-state with a differentiated input variety equal to the larger of the two values that satisfy $r(n) = \rho$ is characterized by an inefficiently low variety of differentiated inputs. The reasons for this are well understood since the contribution of Judd (1985): The relative price distortion due to the markup, $\mu > 1$, charged by differentiated input producers in the market equilibrium combined with an elasticity of substitution larger than unity, $\varepsilon > 1$; this "local inefficiency" can be undone by subsidizing differentiated input purchases so as to equalize their purchase price with their marginal cost of production. Second, and more interestingly from our point of view, there are "global

[5] To see this, notice that $F(n) < \hat{F}(n)$ and that $A(n) < \hat{A}(n^{1/(1-\sigma)})$ because the elasticity of substitution between intermediate-input composites and labor is larger than unity, $\varepsilon > 1$, and the price of the intermediate-input composite relative to labor is higher in the market equilibrium than in the optimal allocation. Combined with the fact that (A6), (A7), and (A9) in the appendix imply that $\hat{r}(n) = \hat{F}'(n) = \hat{A}(n^{1/(1-\sigma)})\hat{F}(n)/(\sigma-1)n$, this yields $\hat{r}(n) > r(n) = A(n)F(n)/\sigma n$.

inefficiencies" in the following sense: Economies where the optimal intertemporal allocation would involve strictly positive investment and growth, and convergence to a "high" steady-state (the steady-state that corresponds to a variety of differentiated inputs equal to the larger of the two values that satisfy $\hat{F}'(n)=\rho$), may instead be stuck at a "low" steady-state in equilibrium (steady-states with a variety of differentiated inputs smaller or equal than the smaller of the two values that satisfy $r(n)=\rho$). These global inefficiencies arise because Hicks-Allen complementarities between differentiated inputs imply that the rate of return to investment in new differentiated inputs increases with aggregate investment. Global inefficiencies cannot be undone with subsidies that equalize the purchase price of intermediate inputs to their marginal cost of production but must be addressed with non-linear policy instruments.

6 Summary

When do dynamic nonconvexities at the disaggregate level translate into dynamic nonconvexities at the aggregate level? We have addressed this question in a model where the production of differentiated intermediate inputs is subject to dynamic nonconvexities and shown that the answer depends on the degree of Hicks-Allen complementarity (substitutability) between differentiated inputs. In our simplest model, a generalization of Judd (1985) and Grossman and Helpman (1991) among many others, there are dynamic nonconvexities at the aggregate level if and only if differentiated inputs are Hicks-Allen complements.

We have also compared dynamic equilibrium and optimal allocations in the presence of dynamic increasing aggregate returns due to Hicks-Allen complementarities between differentiated inputs. Our main results are that Hicks-Allen complementarities imply that the private return to investment increases with the aggregate level of investment; that intertemporal equilibrium allocations are qualitatively similar to dynamic allocations in the aggregative framework; and that dynamic equilibria may be globally inefficient (economies may get inefficiently stuck at very low levels of income); these global inefficiencies cannot be eliminated by marginal, Pigouvian tax policies but must be addressed with nonlinear policy instruments.

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Appendix

Proof of Proposition 3.1. Define X^C as the amount of labor employed to produce differentiated inputs used in the production of consumption goods and N^C as the total amount of labor used in the production of consumption goods, i.e. $N^C = X^C + L^C$. Furthermore, define $\hat{f}(N^C, n)$ as the maximum amount of consumption goods that can be produced with N^C units of labor when a variety n of differentiated inputs is available. With these definitions we obtain that $\hat{f}(N^C, n) = \hat{F}(n)N^C$ because i) for a given variety of differentiated inputs n, the production of consumption goods is subject to constant returns to scale to the quantities of existing differentiated inputs and labor; ii) the production of existing differentiated inputs is subject to constant returns to scale to labor; iii) all intermediate inputs enter symmetrically into the production of intermediate-input composites in (2) and are produced in the same way. Proceeding in exactly the same way for investment goods, using analogous definitions, yields $\hat{g}(N^I, n) = \hat{G}(n)N^I$. This implies that the minimum amount of labor required to produce one unit of the consumption good is equal to $1/\hat{F}(n)$ and that the minimum amount of labor required to produce one unit of the investment good is equal to $1/\hat{G}(n)$. The minimum amount of labor required to produce the bundle (C, I) is therefore equal to $C/\hat{F}(n) + I/\hat{G}(n)$. Efficiency requires that all labor is used in the production of consumption goods or investment goods, which yields (4). ∎

Proof of Proposition 3.2. The minimization problem is to choose $\{L^C; m_i^C : i \le n\}$ to

$$\text{minimize } aL^C + \int_0^n a_i m_i^C \, di \text{ subject to } F(M^C, L^C) = 1 \text{ and } M^C = \left(\int_0^n \left(m_i^C\right)^{(\sigma-1)/\sigma} di \right)^{\sigma/(\sigma-1)}.$$

This minimization problem can be split in two stages. First, determine $\{m_i^C : i \le n\}$ to minimize the cost of producing M^C units of the intermediate-input composite. The solution $\{\hat{m}_i^C : i \le n\}$ to this problem is $\hat{m}_i^C = (a_i / a_M)^{-\sigma} M^C$. Second, determine $\{L^C, M^C\}$ to minimize the cost of producing 1 unit of the consumption good given the opportunity-cost of one unit of the intermediate-input composite a_M. The solution $\{\hat{L}^C, \hat{M}^C\}$ to this problem satisfies $F_M(\hat{M}^C, \hat{L}^C) = a_M$ and $F_L(\hat{M}^C, \hat{L}^C) = a$. Combined with constant returns to scale of $F(M^C, L^C)$, this implies that $\hat{M}^C F_M(\hat{M}^C, \hat{L}^C) / F(\hat{M}^C, \hat{L}^C)$ is a function of the relative opportunity-cost of intermediate-input composites a_M / a only. This establishes (11). Making use of (11) and $F_M(\hat{M}^C, \hat{L}^C) = a_M$ yields $\hat{M}^C a_M / F(\hat{M}^C, \hat{L}^C) = \hat{A}(a_M / a)$. Rearranging and making use of $F(\hat{M}^C, \hat{L}^C) = 1$ implies $\hat{M}^C = \hat{A}(a_M / a) / a_M$ which combined with $\hat{m}_i^C = (a_i / a_M)^{-\sigma} M^C$ yields (9). ∎

Proof of Proposition 3.3.[1] The proof of Proposition 3.2 implies that $\hat{A}(a_M / a) = \hat{M}^C a_M$ where $\hat{M}^C$ is the quantity of intermediate-input composites that minimizes the cost of producing 1 unit of the

[1] We thank Guy Laroque for suggesting this proof.

consumption good. Partially differentiating with respect to a_M yields

$$\partial \log \hat{A}(a_M / a) / \partial \log a_M = 1 + \partial \log \hat{M}^C / \partial \log a_M . \tag{A1}$$

By definition,

$$\varepsilon(a_M / a) = -\partial \log \hat{M}^C / \partial \log a_M + \partial \log \hat{L}^C / \partial \log a_M . \tag{A2}$$

Partially differentiating the Euler identity associated with constant returns to scale of the consumption-goods technology with respect to a_M yields

$$\hat{M}^C + a_M \partial \hat{M}^C / \partial a_M + a \partial \hat{L}^C / \partial a_M = 0 . \tag{A3}$$

Solving (A2) and (A3) implies $\partial \log \hat{M}^C / \partial \log a_M = -1 - (\varepsilon(a_M / a) - 1)(1 - \hat{A}(a_M / a))$ which substituted in (A1) yields

$$\partial \log \hat{A}(a_M / a) / \partial \log a_M = -(\varepsilon(a_M / a) - 1)(1 - \hat{A}(a_M / a)) . \tag{A4}$$

Partially differentiating (9) with respect to a_M and making use of (A4) implies (12). ∎

Proof of Proposition 3.4. Let

$$\hat{x}^C(n) = \underset{x,\ 0 \le x \le 1}{\text{Argmax}}\ F(n^{1/(\sigma-1)} x, 1 - x) . \tag{A5}$$

The envelope theorem applied to (5) yields that

$$\hat{F}' = \hat{F}_M n^{1/(\sigma-1)} \hat{x}^C / n(\sigma - 1) . \tag{A6}$$

The first-order condition of (5) and constant returns to scale of the consumption-goods technology imply that

$$\hat{F}_M n^{1/(\sigma-1)} = \hat{F} . \tag{A7}$$

Combining (A6) and (A7) with (8) yields

$$\hat{r}(n) = (\hat{x}^C(n) / n) \hat{G}(n) / (\sigma - 1) . \tag{A8}$$

According to (A8) there will be dynamic increasing aggregate returns, $\hat{r}'(n) > 0$, if and only if $d \log(\hat{x}^C(n) / n) / d \log n + d \log \hat{G}(n) / d \log n > 0$. The second term on the left-hand side of this inequality can be rewritten as $\hat{x}^I(n) / (\sigma - 1)$ using the analogues of (A5), (A6), and (A7) for investment goods. To rewrite $d \log(\hat{x}^C(n) / n) / d \log n$, we use that (A5) and (11) imply

$$\hat{x}^C(n) = \hat{A}(n^{1/(1-\sigma)}) . \tag{A9}$$

To see this, notice that (10) and the intermediate-input technology imply $a_M / a = n^{1/(1-\sigma)}$ which combined with (11) yields that $\hat{A}(n^{1/(1-\sigma)}) = F_M(\hat{M}^C, \hat{L}^C)\hat{M}^C / F(\hat{M}^C, \hat{L}^C) = a_M \hat{M}^C$ where we made use of $F_M(\hat{M}^C, \hat{L}^C) = a_M$ and $F(\hat{M}^C, \hat{L}^C) = 1$. Furthermore, constant returns to scale of the consumption-goods technology, the definition of $\hat{x}^C(n)$ in (A5), the definition of $\hat{F}(n)$ in (5), and (2) imply that $\hat{M}^C \hat{F}(n) = n^{1/(\sigma-1)} \hat{x}^C(n)$ as one unit of labor produces $\hat{F}(n)$ units of the consumption good. This, combined with $a_M / a = n^{1/(1-\sigma)}$, yields $\hat{A}(n^{1/(1-\sigma)}) = a_M \hat{M}^C = n^{1/(1-\sigma)} a(n^{1/(\sigma-1)} \hat{x}^C(n) / \hat{F}(n)) = a\hat{x}^C(n) / \hat{F}(n)$. Recall that a is the opportunity-cost of labor in terms of consumption goods and hence that (5) implies $\hat{F}(n) = a$ and $\hat{A}(n^{1/(1-\sigma)}) = \hat{x}^C(n)$. Differentiating (A9) with respect to n, making use of (A4) and $\xi(a_M / a)$ in (12), yields

$$d\log(\hat{x}^C(n)/n)/d\log n = -((\sigma-1) - (\varepsilon(n^{1/(1-\sigma)}) - 1)(1 - \hat{A}(n^{1/(1-\sigma)})) / (\sigma-1) = -\xi(n^{1/(1-\sigma)}) / (\sigma-1)$$

and hence that there will be dynamic increasing aggregate returns if and only if $\hat{x}^{\prime}(n)/(\sigma-1) - \xi(n^{1/(1-\sigma)})/(\sigma-1) > 0$, which yields (13). ∎

Discussion of the case where intermediate-input composites are produced according to (17): It is possible to generalize Propositions 3.5 and 3.6.

Proposition 3.5 becomes:

Proposition A.1. There will be dynamic increasing aggregate returns if

$$\zeta(\tilde{a}_M / a) < \gamma(\sigma-1)/(\alpha-1)$$

where $\tilde{a}_M$ is the opportunity-cost of the intermediate-input composite in (17) in terms of consumption goods, $\tilde{a}_M = n^{-\gamma} a_M$, and $\zeta(\tilde{a}_M / a)$ is defined analogously to $\xi(a_M / a)$, $\zeta(\tilde{a}_M / a) = \partial \log \hat{m}_I^C(a_I, \tilde{a}_M, a) / \partial \log a_M$.

Proof of Proposition A.1. Using an argument that is analogous to the proof of Proposition 3.4, we can establish that there will be dynamic increasing aggregate returns if and only if $\zeta(\tilde{a}_M / a) < \hat{x}^{\prime}(n) + \gamma(\sigma-1)/(\alpha-1)$, where $0 \le \hat{x}^{\prime}(n) \le 1$. ∎

As expected, the inequality in Proposition A.1 shows that if there is a negative direct specialization effect, $\gamma < 0$, then it is no longer sufficient for dynamic increasing aggregate returns that differentiated inputs are Hicks-Allen complements. But there still is a link between the degree of Hicks-Allen complementarity and dynamic increasing aggregate returns: The complementarity between differentiated inputs now needs to be sufficiently strong, i.e. $\zeta(\tilde{a}_M / a) < \gamma(\sigma-1)/(\alpha-1)$, to outweigh the negative direct specialization effect.

Proposition 3.6 becomes:

Proposition A.2. There will be dynamic decreasing aggregate returns if

$$\zeta(\tilde{a}_M / a) \ge 1 + \gamma(\sigma-1)/(\alpha-1).$$

The interpretation and proof of this inequality is analogous to the interpretation and proof of Proposition A.1.

Proof of Proposition 4.1. From the Lagrangian of the problem

$$L(n,C,\lambda,\theta)=U(C)+(\lambda+\theta)\hat{G}(n)\left(1-C/\hat{F}(n)\right) \tag{A10}$$

we derive the necessary conditions for dynamic efficiency

$$\begin{aligned}
&\rho\lambda-\dot{\lambda} &&=(\lambda+\theta)\hat{G}(n)\left(\frac{\hat{G}'(n)}{\hat{G}(n)}\left(1-\frac{C}{\hat{F}(n)}\right)+\frac{\hat{F}'(n)}{\hat{F}(n)}\frac{C}{\hat{F}(n)}\right) && \text{(a)}\\
&U'(C) &&=(\lambda+\theta)\cdot\frac{\hat{G}(n)}{\hat{F}(n)} && \text{(b)}\\
&\theta\left(\hat{G}(n)-\frac{\hat{G}(n)C}{\hat{F}(n)}\right) &&=0 && \text{(c)}
\end{aligned} \tag{A11}$$

where λ and $\theta\geq 0$ denote the continuously differentiable adjoint variables and the Kuhn-Tucker multiplier associated with the non-negativity constraint on investment. To reduce the necessary conditions for optimality (A11a)-(A11c) to a dynamic system in the (n,C)-phase plane we distinguish two cases:

(i) Suppose that $C<\hat{F}(n)$ and hence $\dot{n}>0$ and $\theta=0$. Differentiating (A11b) with respect to time and substituting (21) and (A11a) yields

$$\frac{\dot{C}}{C}=-\gamma(C)\left(\frac{\dot{\lambda}}{\lambda}+\left(\frac{\hat{G}'(n)}{\hat{G}(n)}-\frac{\hat{F}'(n)}{\hat{F}(n)}\right)\dot{n}\right)=\gamma(C)\left(\frac{\hat{F}'(n)\hat{G}(n)}{\hat{F}(n)}-\rho\right). \tag{A12}$$

(ii) To characterize the dynamic system on the boundary note from (i) that the boundary is absorbing for all n such that $\rho<\hat{F}'(n)\hat{G}(n)/\hat{F}(n)$. This implies that, if $C_t=\hat{F}(n_t)$ and $\rho<\hat{F}'(n_t)\hat{G}(n_t)/\hat{F}(n_t)$ for some t, then $n_\tau=n_t$ and $C_\tau=\hat{F}(n_t)$ for all $\tau>t$. Integrating (A11a), and using (A11b) and $\theta\geq 0$ implies that $\lambda_\tau\leq U'(C_t)\hat{F}(n_t)/\hat{G}(n_t)$ for all $\tau>t$. We therefore obtain that θ_τ is equal to $(U'(C_t)\hat{F}(n_t)/\hat{G}(n_t)\rho)(\rho-\hat{F}'(n_t)\hat{G}(n_t)/\hat{F}(n_t))$ for all $\tau>t$, which is inconsistent with the non-negativity of θ. If, on the other hand, $C_t=\hat{F}(n_t)$ and $\rho\geq\hat{F}'(n_t)\hat{G}(n_t)/\hat{F}(n_t)$ for some t, then it is straightforward to check that $n_\tau=n_t$, $C_\tau=\hat{F}(n_t)$, $\lambda_\tau=U'(C_t)\hat{F}'(n_t)/\rho$, and $\theta_\tau=(U'(C_t)\hat{F}(n_t)/\hat{G}(n_t)\rho)(\rho-\hat{F}'(n_t)\hat{G}(n_t)/\hat{F}(n_t))$ for all $\tau>t$ satisfy (A11a)-(A11c).

Finally, to see that the transversality condition is necessary, first notice that the optimal path for n_τ is bounded above by $\max(n_0,n_m)$, where n_m is defined in (19), and bounded below by $n_0>0$. To see that the optimal path is bounded above, consider an optimal path that at time t satisfies that $C_t<\hat{F}(n_t)$ and $n_t>n_m$. The necessary condition in (22) combined with (19) implies that consumption would be falling forever in this case. This cannot be optimal, however, as the path

$C_\tau = \hat{F}(n_t)$, $\tau \geq t$, is feasible and always achieves higher levels of consumption. It follows that no optimal path can ever satisfy that $C_t < \hat{F}(n_t)$ and $n_t > n_m$ at some point in time, and that all optimal paths must be bounded above by $\max(n_0, n_m)$. The fact that the optimal path is bounded from below by $n_0 > 0$ follows from (21). This lower bound combined with $\hat{F}(n_0) > 0$ implies that the optimal path C_τ satisfies $U'(C_\tau) < d < \infty$, $\tau \geq t$, for some constant d and some t. The upper bound on $U'(C_\tau)$ for sufficiently large τ and the upper bound on the optimal path n_τ imply the transversality condition. (This argument is related to the proof of the necessity of the transversality in Dechert and Nishimura (1983). See also Majumdar (1975) and Majumdar and Mitra (1982).) ∎

Proof of Proposition 4.3. Skiba (1978) establishes that if $\{\hat{n}_\tau, \hat{C}_\tau, \hat{\lambda}_\tau, \hat{\theta}_\tau : 0 \leq \tau\}$ satisfies the necessary conditions for optimality in Proposition 4.1 and (21), and if (19) holds, then $\rho \int_0^\infty e^{-\rho\tau} U(\hat{C}_\tau) d\tau = L(\hat{n}_0, \hat{C}_0, \hat{\lambda}_0, \hat{\theta}_0)$, where $L(\bullet)$ is defined in (A10). Let $\hat{C}_0^i, \hat{C}_0^{ii}, \hat{\lambda}_0^i, \hat{\lambda}_0^{ii}, \hat{\theta}_0^i, \hat{\theta}_0^{ii}$ and $\hat{L}_0^i, \hat{L}_0^{ii}$ denote consumption, shadow prices, and the value of the Lagrangian at $\tau = 0$ along paths i and ii respectively. Then, Skiba's result and (A11b) imply that

$$\int_0^\infty e^{-\rho\tau} U(\hat{C}_\tau^{ii}) d\tau = \int_0^\infty e^{-\rho\tau} U(\hat{C}_\tau^i) d\tau$$

if $\hat{C}_0^{ii} = \hat{C}_0^i$. If $\hat{C}_0^{ii} \neq \hat{C}_0^i$, then Skiba's result, (A11b), and strict concavity of U yields

$$\begin{aligned}
&\rho\left(\int_0^\infty e^{-\rho\tau} U(\hat{C}_\tau^{ii}) d\tau - \int_0^\infty e^{-\rho\tau} U(\hat{C}_\tau^i) d\tau\right)\rho \\
&= U(\hat{C}_0^{ii}) - U(\hat{C}_0^i) + (\hat{\lambda}_0^{ii} + \hat{\theta}_0^{ii})\hat{G}_0(\hat{F}_0 - \hat{C}_0^{ii}) / \hat{F}_0 - (\hat{\lambda}_0^i + \hat{\theta}_0^i,)\hat{G}_0(\hat{F}_0 - \hat{C}_0^i) / \hat{F}_0 \\
&= U(\hat{C}_0^{ii}) - U(\hat{C}_0^i) + U'(\hat{C}_0^{ii})(\hat{F}_0 - \hat{C}_0^{ii}) - U'(\hat{C}_0^i)(\hat{F}_0 - \hat{C}_0^i) \\
&> U'(\hat{C}_0^{ii})(\hat{C}_0^{ii} - \hat{C}_0^i) + U'(\hat{C}_0^{ii})(\hat{F}_0 - \hat{C}_0^{ii}) - U'(\hat{C}_0^i)(\hat{F}_0 - \hat{C}_0^i) \\
&= (U'(\hat{C}_0^{ii}) - U'(\hat{C}_0^i))(\hat{F}_0 - \hat{C}_0^i).
\end{aligned}$$

This strict inequality, combined with $\hat{F}_0 - \hat{C}_0^i \geq 0$ and strict concavity of U, implies that path ii yields strictly higher utility than path i if and only if $\hat{C}_0^{ii} < \hat{C}_0^i$. ∎

Proof of Proposition 5.1. In equilibrium, $A(n)C = npm^C = \mu w X^C$, where X^C is the amount of labor employed to produce differentiated inputs used in the production of consumption goods, and $(1 - A(n))C = wL^C$. These equations allow us to calculate the total labor used in the production of consumption goods as

$$X^C + L^C = (1 - A(n)/\sigma)(C/w). \tag{A13}$$

The same equations imply $C = F(n^{1/(\sigma-1)} X^C, L^C) = (C/w) F((\mu^{1-\sigma} n)^{1/(\sigma-1)} A(n), 1 - A(n))$ and hence, making use of (26), that wages are

$$w = (1 - A(n)/\sigma)F(n), \tag{A14}$$

and, making use of (A13), that $C = F(n)(X^C + L^C)$. Similarly,

$$w/q = (1 - B(n)/\sigma)G(n) \tag{A15}$$

and $I = G(n)(X^I + L^I)$. Combining these results with labor-market clearing, $X^I + L^I + X^C + L^C = 1$, implies (25). ∎

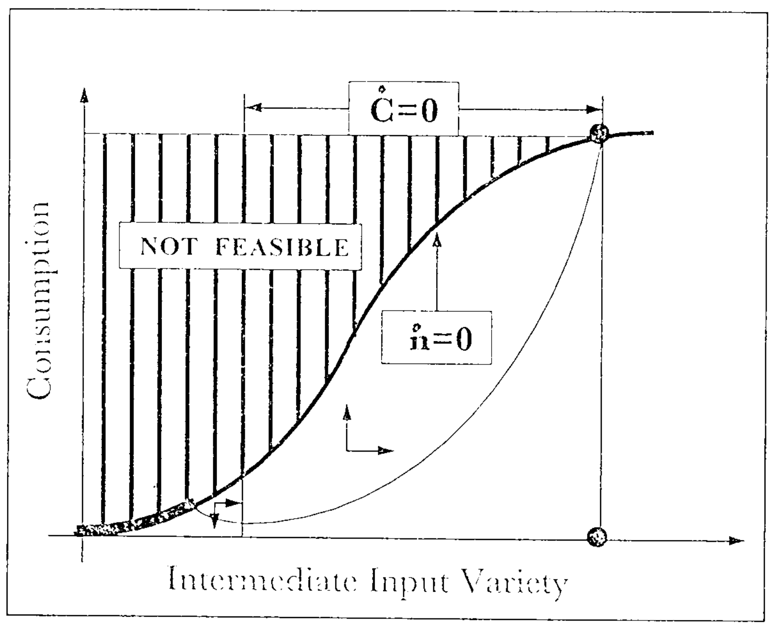
Proof of Proposition 5.2. It is straightforward to verify that if $C_t = F(n_t)$ and $b(n_t) \le \rho$ at time t, then $r_\tau = \rho$ and $C_\tau = F(n_t)$, $\tau \ge t$, satisfy all equilibrium conditions. If $C < F(n)$, then $\dot{n} = I > 0$ and (28) yield $q = v$. Combined with arbitrage between consumption loans and equity, this implies $r = \Pi/q + \dot{q}/q$, where Π denotes operating profits of differentiated input firms,

$$\Pi = (p - w)(m^C + m^I) = (A(n)C + qB(n)I)/\sigma n. \tag{A16}$$

Thus, $r = (AC + qB\dot{n})/\sigma qn + \dot{q}/q$. Substituting in (30) yields the first part of (31a). The labor-market clearing condition in (25) and the irreversibility of the start-up investment imply (31b). The price of the investment good q in (34) follows from (A14) and (A15). Finally, the national income account identity implies $\partial(n_\tau v_\tau)/\partial\tau = r_\tau n_\tau v_\tau + w_\tau - C_\tau$ and therefore $e^{-R_t}(n_t v_t) = n_0 v_0 + \int_0^t e^{-R_\tau}(w_\tau - C_\tau)d\tau$. The necessary condition for optimality of the consumption plan, $\eta e^{-R_\tau} = e^{-\rho\tau}U'(C_\tau)$ with η the marginal utility of wealth, and (30) with equality imply $e^{-\rho\tau}U'(C_\tau)n_\tau v_\tau \to 0$ as $\tau \to \infty$, which, combined with (28), (34), $0 \le A(n) \le 1$, and $0 \le B(n) \le 1$, implies (32). ∎

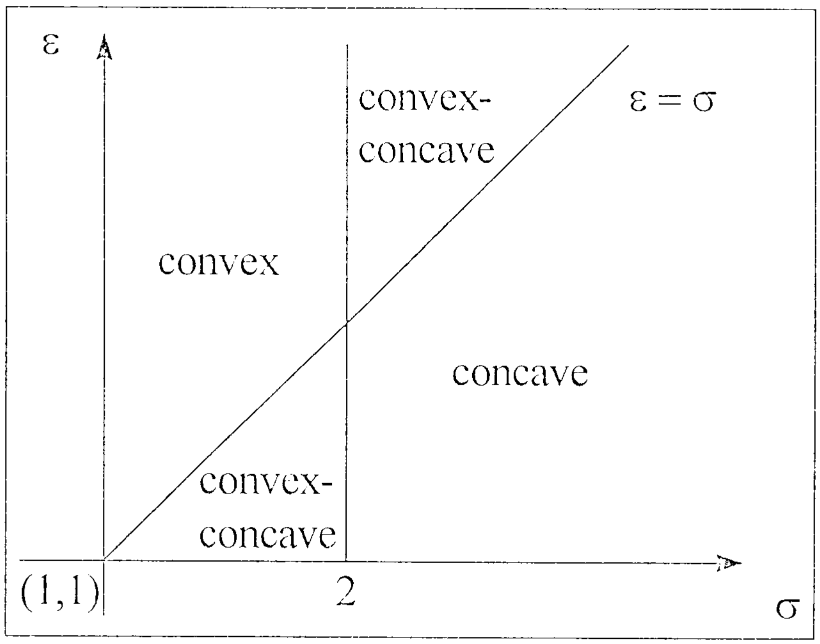
Proof of Proposition 5.3. Proposition 3.4 established that $d(\hat{x}^C(n)/n)/dn > 0$ if differentiated inputs are Hicks-Allen complements. Analogously, it can be shown that if differentiated inputs are Hicks-Allen complements then $d(A(n)/n)/dn > 0$, see Ciccone and Matsuyama (1996). This implies that if differentiated inputs are Hicks-Allen complements, then $A'(n) > 0$ which making use of (26), $F_L(M^C, L^C) = w$, and $F_M(M^C, L^C) = p_M = n^{1/(1-\sigma)}\mu w$ implies $F'(n) > 0$. Combined, these results imply $r(n) = A(n)F(n)/\sigma n$ is strictly increasing in n if differentiated inputs are Hicks-Allen complements. ∎

Figure 2: Dynamically Optimal Allocations with Input Demand Complementarities.



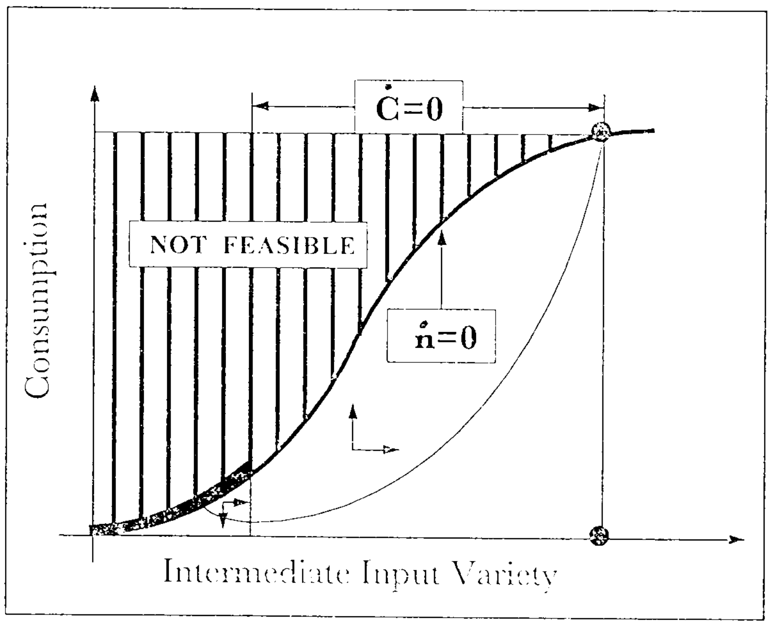
Notes: The thick line corresponds to the region where it is optimal to not invest in differentiated inputs because of low current and future returns.

Figure 1: The Shape of the "Aggregate Production Function."



Notes: For the one sector model with the consumption goods technology and investment goods technology as specified in (24).

<u>Figure 3:</u> Dynamic Market Equilibria with Input Demand Complementarities



<u>Notes:</u> The thick line corresponds to stationary equilibria. For differentiated input varieties where the thick line and the saddle path overlap, there are multiple equilibria.