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Endogenous Inequality

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Abstract

Does the market economy exacerbate inequality across households? In a capitalistic society, does the rich maintain a high level of wealth at the expense of the poor? Or would an accumulation of the wealth by the rich eventually trickle down to the poor and pull the latter out of poverty? This paper presents a theoretical framework, in which one can address these questions in a systematic way. The model focuses on the role of credit market, which determines the joint evolution of the distribution of wealth and the interest rate. A complete characterization of the steady states is provided. Under some configurations of the parameter values, the model predicts an endogenous and permanent separation of the population into the rich and the poor, where the rich maintains a high level of wealth partially due to the presence of the poor. Under others, the model predicts the Kuznets curve, i.e., the wealth eventually trickles down from the rich to the poor, eliminating inequality in the long run.

JEL Classification Numbers: D31(Personal Income and Wealth Distribution); O11(Macroeconomic Analysis of Economic Development)

Keywords: Imperfect Credit Markets, Distribution of Wealth, Endogenous Inequality, Trickle-Down, The Kuznets Curve

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1. Introduction

Does the market economy exacerbate inequality across households? In a capitalistic society, does the rich accumulate and maintain a high level of wealth at the expense of the poor? Or would an accumulation of the wealth by the rich eventually trickle down to the poor in one way or another and pull the latter out of the misery of poverty, as the Kuznets U-curve hypothesis suggests? The empirical findings for the Kuznets curve appear mixed. Then, what are the important factors for the determination of wealth distribution in the long run?

This paper presents a theoretical framework, in which one can address these questions in a systematic way. The model economy consists of a stationary population of infinitely-lived households, which are inherently identical. At any point in time, the level of the wealth is the only possible source of heterogeneity across households. The distribution of wealth in one period affects the supply and demand for credit, which in turn affects the distribution of wealth in the following period. The dynamics of the economy is described by the joint evolution of the equilibrium interest rate and of the distribution of wealth.

The two critical assumptions are; i) the project that generates a higher return requires a minimum level of investment; ii) because of the possibility of default (and imperfect sanctions against it), anyone who wants to invest in the project faces the borrowing constraint. Because of these two restrictions, the minimum level of investment and the borrowing constraint, the households that are *relatively* rich become borrowers and invest in the project, while the *relatively* poor households become lenders. The threshold level of wealth, which divides the poor lenders from the rich borrowers, is endogenously determined, as it depends on the equilibrium interest rate. This allocation of credits, allowing only the relatively rich to invest in the profitable project, creates a tendency of the rich getting richer and of the poor lagging behind, thereby magnifying inequality.

It turns out that the gap between the rich and the poor will never disappear in certain cases. Under some configurations of the parameter values, the analysis shows that the household's wealth is concentrated in two points in all the steady states. In other words, the population is polarized into the rich and the poor. In these steady states, the poor households are unable to borrow and have no choice but to lend their wealth to the rich, at an interest rate strictly lower than the project return. The rich households, who can borrow and invest, maintain a high level of wealth, not only because they are engaged in the profitable project, but also because they have access to the cheap credit supplied by the poor households. Furthermore, the fraction of the households that remain poor has a positive lower bound, which is independent of the initial distribution of wealth. In other words, even if there was perfect equality at the beginning, there is inequality of wealth across the households in the steady state. The model thus explains how the market interactions lead to endogenous inequality and offers some theoretical justification for the left-wing view that the rich households owe their high level of wealth partially to the presence of the poor households.

Endogenous, permanent inequality is not, however, the only prediction of the model. In certain cases, the model predicts a Kuznets U-curve, i.e., the gap between the rich and the poor will eventually disappear. This can occur when the growing demand for credit by the rich pushes up the interest rate so much that the poor lenders, benefiting from a higher interest rate, eventually become able to catch up with the rich. Under some configurations of the parameter values, the analysis shows that there is the unique steady state, in which all the household's wealth converges to the same level, which is high enough that any household can borrow and invest. The model predicts in this case that the wealth accumulated by the rich eventually trickles down to the poor; a rapid accumulation of the wealth by the rich pulls the poor out of the misery of poverty. This case thus provides some theoretical justification for the right-wing view that an accumulation of the wealth by the rich is beneficial for the society as a whole, including the poor.

Demonstrating the possibility of these two alternative scenarios is important enough, providing justifications for the two opposing views of the world. More importantly, which of these two alternative scenarios will materialize depends on a few key parameters in an interesting way. The present theoretical framework thus helps to provide a simple intuition-building device on these seemingly intractable issues.

Starting with Bernanke and Gertler (1989), the recent work in macroeconomics emphasizes the role of imperfect capital markets, which make investment be constraint by the wealth. The threshold level of wealth, which give one access to credits, is exogenous in most studies. Here, the threshold level of wealth adjusts endogenously to satisfy the balance between the supply and demand for credits. This means that those who obtain credit do not have to be rich by any absolute standard; they simply have to be richer than others. In other words, it is the distribution of wealth, not the absolute level of wealth, which determines the allocation of credits.

The literature on income and wealth distribution is vast, but they are mostly concerned with the effects of distribution on macroeconomic performances.[2] Among those studies that

[2] For a survey, see Bénabou (1996), who also provides a unified approach to this problem.

offer a theory of long run distribution of wealth, Aghion & Bolton (1997) is most closely related. They, too, focus on the role of credit market, where the interplay between the current distribution of wealth and the interest rate determines the allocation of credits, which in turn affects the future distribution of wealth. There are some significant differences. First, their analysis aims to identify the condition, under which a trickle down takes place. They left open the question of what might happen when the condition is not met. Second, they focused on a different mechanism through which the wealth trickles down from the rich to the poor. In the Aghion and Bolton model, investment demand satiates quickly for each household. This implies that, once the rich accumulated enough wealth to finance their own investment, any additional wealth accumulation by the rich leads to a *lower* interest rate, which makes it possible for the poor households to borrow and to escape from the poverty. In the present study, the rich households continue to borrow, as they accumulate wealth. An accumulation of the wealth by the rich helps the poor, because a higher demand for credit by the rich leads to a *higher* interest rate, which in turn helps the poor lenders to accumulate their wealth faster and to escape from the poverty.[3]

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[3] In addition, the two studies follow different modeling strategies. Aghion & Bolton modeled the credit market imperfection due to as a moral-hazard problem, which arises from the imperfect observability of the effort level that affects the probability of project success. Because each project is subject to an idiosyncratic shock, the steady state distribution of wealth never become degenerate. In their model, equality means the ergodicity of a steady state distribution. (The same can be said about Piketty (1997), who considered a model similar to Aghion & Bolton. Unlike Aghion & Bolton, Piketty is concerned with the possibility of multiple ergodic steady state distribution.) While this feature of their model had advantage of enabling them to discuss redistributive policies even when the trickle down take places, it makes the model so complicated that it is difficult to analyze the situation where the condition for the trickle down is violated. In contrast, this paper models the credit market imperfection due to a potential threat of default, which never takes place in equilibrium. The absence of any idiosyncratic shock keeps the model so simple that a complete characterization of the steady states can be done for a wide range of parameter values.

The rest of the paper is organized as follows. Section 2 presents the basic model, and derives the conditions for the credit market equilibrium and for the joint evolution of the interest rate and the distribution of wealth. Section 3 offers a complete characterization of the steady states. Section 4 extends the basic model. Section 4A introduces a storage technology, which has no minimum investment requirement, and hence offers the poor an alternative to lending in the credit market. This modification does not affect any of the steady states identified in section 3, if the return on a storage technology is not too high. However, the storage, unless its return is not too low, creates another, different type of the steady state, in which all the households are equally poor, and uses only the storage technology, being too poor to invest in the profitable project. Section 4B discusses the effect of diminishing returns to investing in the project. Unless the project is not too productive, this change in specification does not affect the essential features of the basic model. On the other hand, when the project is sufficiently productive, the modified model predicts a trickle-down mechanism similar to the Aghion & Bolton model. Section 5 concludes.

2. The Model

Time is discrete and extends to infinity. In any period the economy is populated by a unit mass of identical agents. Each agent is active for one period as a head of an infinitely-lived household (or dynasty). The only possible source of heterogeneity across households is their wealth. At the beginning of each period, the agents receive their initial wealth, $w \geq 0$, in the form of a bequest from the immediate predecessors (or parents). Let $G_t(w)$ denote the wealth distribution across households, inherited by the agents active in period t.

At the beginning of period t, the agents allocate their inherited wealth to maximize its return. They have two options. First, they may lend it in the competitive credit market, which earns the gross return equal to r_t per unit. Second, they may become an entrepreneur and start a project. The project technology is described by the following production function,

$$\text{(A1)}\quad F(k_t) = \begin{cases} 0 & \text{if} \quad 0 \leq k_t < 1, \\ Rk_t & \text{if} \quad k_t \geq 1, \end{cases}$$

where k_t is a scale of investment, and the minimum investment level is normalized to be one. To invest at the scale greater than the inherited wealth (i.e., $k_t > w_t$), the agent needs to borrow in the competitive credit market at the rate equal to r_t. After allocating the inherited wealth to maximize its return, each agent earns $y > 0$ units of the output during the period, which is independent of the inherited wealth and of their investment decision.

The credit market is competitive in the sense that both lenders and borrowers take the equilibrium rate, r_t, given. It is not competitive, however, in the sense that one cannot borrow any amount at the equilibrium rate. The borrowing limit exists because of the enforcement problem: the payment is made only when it is the borrower's interest to do so. More specifically, the entrepreneur, after borrowing b_t, would refuse to honor its payment obligation, r_tb_t, if it is greater than the cost of default, which is taken to be a fraction of the project output λRk_t.[4] Knowing this, the lender would allow the entrepreneur to borrow only up to $\lambda Rk_t/r_t$. The parameter, $0 \leq \lambda < 1$, can be naturally taken to be the degree of the efficiency of the credit market. Note that there is no default in equilibrium. It is the possibility of default that makes the credit market imperfect. It should also be noted that the same enforcement problem rules out the possibility that different agents may pool their wealth to overcome the borrowing constraint.[5]

The investment demand by the agent, who has inherited w_t, is given by k_t that maximizes the end-of-the-period wealth, given by

$$(1) \qquad F(k_t) - r_t k_t + r_t w_t + y = \begin{cases} r_t (w_t - k_t) + y, & \text{if } 0 \leq k_t < 1, \\ (R - r_t)k_t + r_t w_t + y & \text{if } k_t \geq 1, \end{cases}$$

subject to the borrowing constraint,

$$(2) \qquad r_t(k_t - w_t) \leq \lambda R k_t.$$

The equilibrium interest rate in period t is determined by the credit market equilibrium condition that the aggregate net borrowing is equal to zero, or equivalently, that the aggregate investment in period t is equal to the aggregate wealth at the beginning of period t. (See Equation (3) below.) The following lemma shows how the equilibrium condition imposes the restriction over the range in which the interest rate can change.

Lemma 1. Let $\lambda < 1$. Then, $\lambda R < r_t \leq R$ in equilibrium.

Proof. Suppose that $r_t > R$. Then, the end-of-the-period wealth, (1), is strictly decreasing in k_t. That is, all the agents would rather lend without investing, instead of borrowing and becoming an entrepreneur. Hence there would be an excess supply of credit. Suppose now that $r_t \leq \lambda R < R$. Then, (1) is strictly increasing in k_t, while the borrowing constraint, (2), is not binding. Hence,

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[4] One possible interpretation of the cost is that the creditor seizes a fraction λ of the project output in the event of default. Alternatively, one may interpret that this fraction of the output will be dissipated in the borrower's effort to default.

every agent would want to borrow by an infinite amount, and there would be excess demand for credit. Therefore, the equilibrium interest rate must satisfy $\lambda R < r_t \leq R$.

Having established $\lambda R < r_t \leq R$, let us now look more closely at the optimal behavior of the agents. First, consider the case, $\lambda R < r_t < R$. Equation (1) is then strictly increasing in k_t, so that every agent wants to invest as much as possible. Equation (2) suggests that any would-be entrepreneur can borrow only up to $\lambda R k_t / r_t$, or is required to contribute $100(1-\lambda R/r_t)\%$ of investment as a down payment. Since the minimum investment level is equal to one, this implies that the agent, whose wealth satisfies $w_t \geq 1-\lambda R/r_t$, becomes an entrepreneur and invests by $k_t = w_t/(1-\lambda R/r_t) \geq 1$ and borrows by $b_t = k_t - w_t = [(\lambda R/r_t)/(1-\lambda R/r_t)]w_t > 0$. On the other hand, if $w_t < 1-\lambda R/r_t$, the agent is unable to become an entrepreneur and becomes a lender.

Consider now the case, $r_t = R$. Then, (1) becomes simply $Rw_t + y$ for $k_t = 0$ and for all $k_t \geq 1$, while $0 < k_t < 1$ is strictly dominated. All the agents are hence indifferent between lending without investing and borrowing to invest at $k_t \geq 1$. If $w_t \geq 1-\lambda$, the agent can borrow and invest up to $w_t/(1-\lambda)$. Therefore, the demand for credit is given by $k_t \in \{0, [1, w_t/(1-\lambda)]\}$. On the other hand, if $w_t < 1-\lambda$, the agent cannot borrow and does not invest.

The above discussion implies that the equilibrium condition in the credit market can be expressed as

$$\int \quad = \left(1 - \frac{\lambda R}{r_t}\right)^{-1} \int_{1-\lambda R/r_t}^{\infty} w\,dG_t(w) \qquad \text{if } \lambda R < r_t < R,$$

[5]See Kiyotaki and Moore (1997) for another study, which addresses the problem of imperfect enforcement in lending contracts in the context of macroeconomics.

$$(3) \qquad \int_0^\infty w dG_t(w) \quad \left\{ \quad \leq (1-\lambda)^{-1} \int_{1-\lambda}^\infty w dG_t(w) \qquad \text{if } r_t = R, \right.$$

which is depicted in Figure 1. The left-hand side is simply the aggregate wealth of the economy at the beginning of period t. It is independent of the interest rate and given by the vertical line in Figure 1. The right hand side is the total demand for investment. Note that, at $r_t = R$, all the agents are indifferent between borrowing to invest and lending, and hence the demand curve is flat. When $r_t = R$ in equilibrium, some qualified agents do not obtain credit up to the limit, but this does not mean that they are credit rationed, because they do not strictly prefer obtaining credit. If $r_t < R$, all the agents prefer to invest as much as possible. It is the borrowing constraint that determines investment demand. The demand for investment declines with a high interest rate, which implies a tighter borrowing constraint. The decline comes not only by a smaller demand by those entrepreneurs who can continue to borrow, but also by the fact that a fewer agents would be qualified to borrow. The demand curve may be flat at $r_t < R$, if there is a positive measure of the agents, who inherited wealth is equal to $1-\lambda R/r_t$. As long as the vertical line intersects with the downward sloping part of the demand curve, as depicted in Figure 1, all the agents borrow up to the credit limit.[6]

[6]What happens if the vertical line intersects with a flat part of the investment demand curve at $r_t < R$? Then, some agents are credit-rationed. That is, they cannot borrow up to the limit, even though they want to do so and they are equally qualified as others. This introduces an element of chances in the dynamics of the household wealth. To deal with this situation, one needs to introduce some kind of ad-hoc rationing rule. The analysis and discussion in the text ignores such a possibility of credit rationing, because this situation never arises in the steady state. (Also, the steady states are independent of any credit-rationing rule.)

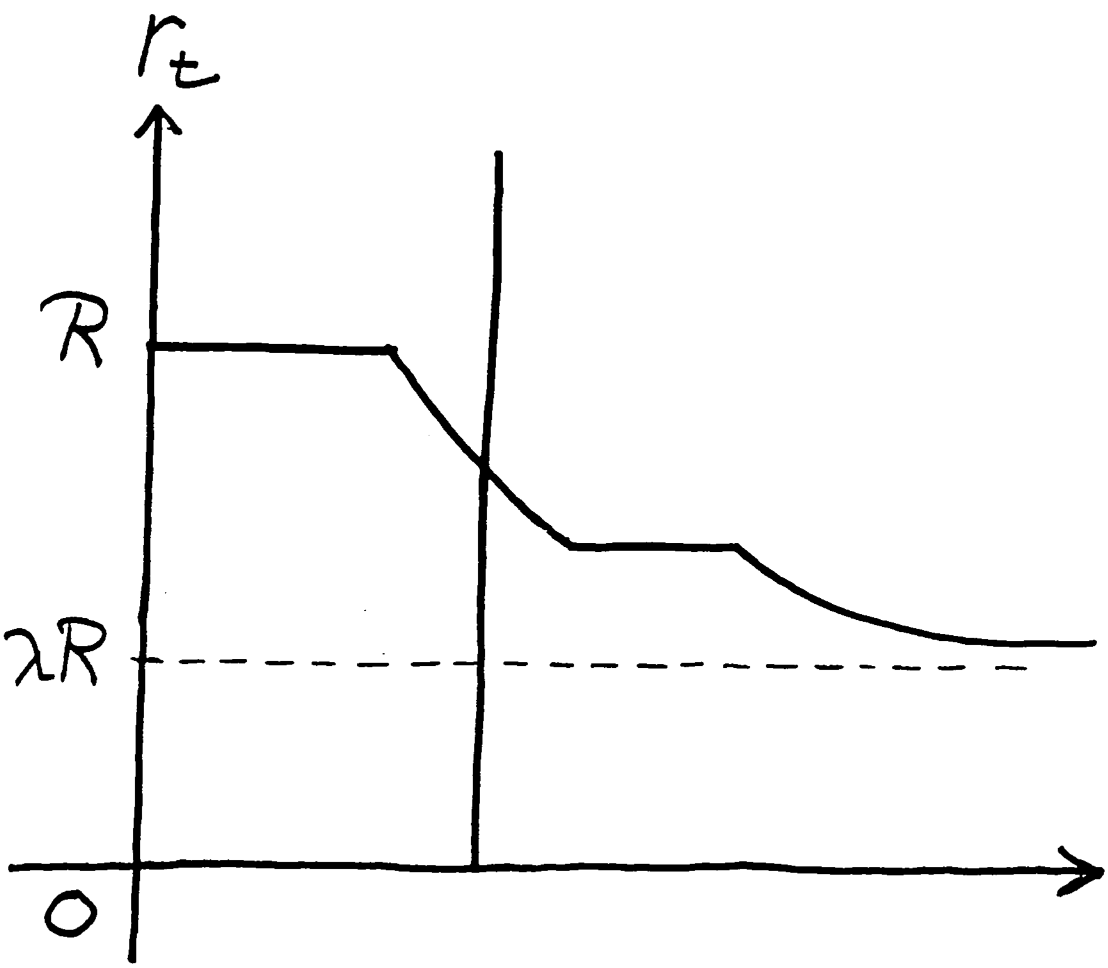

fig.1

To close the model, the bequest rule of the agents must be specified. To keep the matter simple, it is assumed that the agents consume $(1-\beta)$ fraction of the end-of-the wealth and leaves β fraction to the next generation (sibling).[7] Then, the wealth of each household changes according to the following dynamics:

$$(4)\quad w_{t+1} = \begin{cases} \beta\{(R-r_t)k_t + r_t w_t + y\} = \beta\{[(1-\lambda)R/(1-\lambda R/r_t)]w_t + y\}, & \text{if } w_t \geq 1-\lambda R/r_t\,, \\ \\ \beta(r_t w_t + y), & \text{if } w_t < 1-\lambda R/r_t\,. \end{cases}$$

Figure 2 depicts Equation (4). There is a threshold level of the wealth, $1-\lambda R/r_t$, below which the agent cannot borrow nor invest. Note that the wealth of the poor household, the lender, generates a return strictly lower than R. The wealth of the rich household, the borrower/entrepreneur, generates a return equal to $(1-\lambda)R/(1-\lambda R/r_t)$, which is strictly higher than R. They enjoy the higher return, because they can borrow at a rate strictly lower than the return on the project. It is in this sense that the rich accumulates wealth at the expense of the poor, who has no choice but to lend to the rich at a rate lower than the project return.

The arrows indicate the effects of a rise in the interest rate. A higher interest rate means a more favorable terms-of-trade for the poor lender and a less favorable terms-of-trade for the rich borrower. Hence, with a high interest rate, the poor accumulates wealth faster, while the rich's rate of accumulation becomes smaller (though it is still larger than the poor's.) A higher interest rate also makes the threshold higher, because the present value of the penalty is smaller, and as a result, entrepreneurs need to contribute more in the form of a down payment. This suggests that

[7] Aghion & Bolton (1997) and Piketty (1997), who make the same assumption, offer some justification for this assumption.

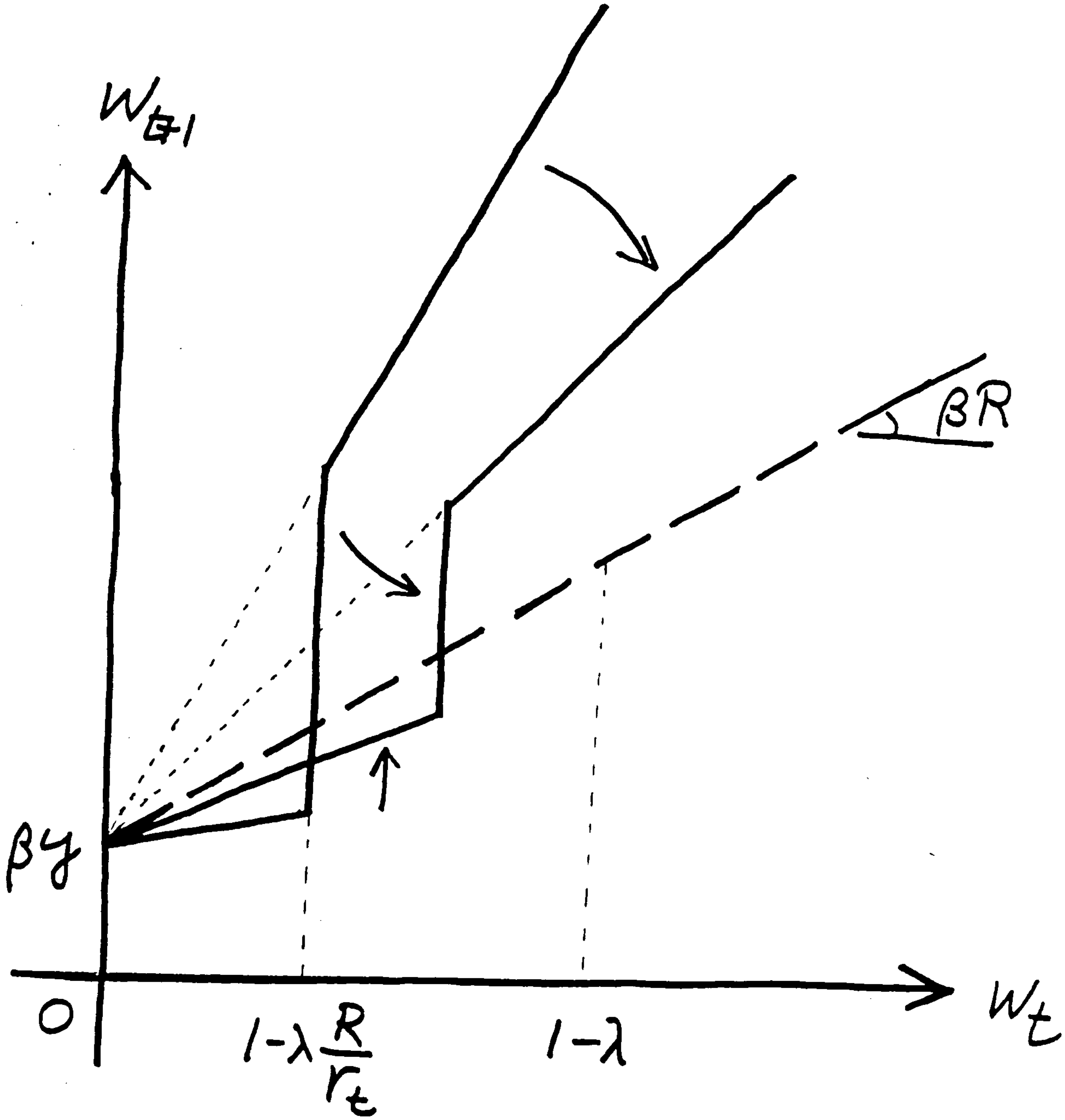

fig. 2

a high interest rate is good for the very poor household, which cannot borrow in any case, while it is bad for the middle class household, which could borrow at a lower rate, but not at a higher rate.

Finally, the dashed line, $w_{t+1} = \beta(Rw_t + y)$, depicts the dynamics when $r_t = R$. In this case, all the households earn the same return equal to the project return, R.

To summarize,

<u>Proposition 1.</u>

a) If $\lambda R < r_t < R$, the rich agent, whose inherited wealth is greater than or equal to the threshold, $1-\lambda R/r_t$, borrows and becomes an entrepreneur. The poor agent, whose inherited wealth is less than the threshold, cannot invest and becomes a lender. The poor lenders earn the rate of return lower than R, the project return, while the rich entrepreneurs/borrowers earn a return higher than R, the project return.

b) If $r_t = R$, the rich agent, whose inherited wealth is greater than or equal to the threshold, $1-\lambda$, may borrow and become an entrepreneur. The poor agent, whose inherited wealth is less than the threshold, becomes a lender. All the households earn the same rate of return on their wealth.

Equations (3) and (4) contain all the information necessary for solving the dynamics of the economy. Given the distribution of wealth at the beginning of period t, G_t, (3) determines the interest rate, r_t. Then, (4) can be used to calculate G_{t+1}. Applying (3) and (4) iteratively would determine the dynamic evolution of the economy.

Before proceeding, the following assumption,

(A2) $\beta y < 1$,

is imposed solely to rule out the trivial case.[8]

3. The Steady State

Let us now look at the behavior of the economy in the long run. The steady state is associated with the limit distribution, $G_\infty(w)$, and the limit interest rate, r_∞. It is the state, which replicates itself over time, once the economy is settled in, and where all the households hold a constant level of wealth. The following condition is necessary and sufficient to ensure the existence of the steady states.

(A3) $\beta R < 1$.

Note that βR can be interpreted as the rate of aggregate wealth accumulation, and also as the lower bound for the rate of wealth accumulation by the rich household, which earns the return on their wealth higher than or equal to R.

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[8] If $\beta y \geq 1$, every household, regardless of its initial wealth, becomes rich enough to be able to invest just after one period, without participating the credit market.

3A. *The Steady State with Wealth Equality*

First, suppose that the wealth distribution is degenerate in a steady state. This is possible only when all the households earn the same rate of return of their wealth, and hence, $r_\infty = R$. From (4), this implies that the household's steady state wealth is given by the fixed point of the map, $w_{t+1} = \beta(Rw_t + y)$, or

(5) $w_\infty = \beta y/(1-\beta R) \geq 1-\lambda$.

As long as the inequality in Equation (5) is satisfied, there exists a steady state, in which all the households maintain the same level of wealth and are rich enough to be able to borrow and invest. Furthermore, the rate of return of lending is equal to the rate of return on the project investment, and hence the households are indifferent between being lenders and being borrowers, so that the credit market equilibrium condition, (3), is satisfied. Thus, we have

<u>Proposition 2.</u> If $\beta y/(1-\beta R) \geq 1-\lambda$, there exists a steady state, in which $r_\infty = R$ and all the households maintain the wealth equal to $w_\infty = \beta y/(1-\beta R)$.

A sufficiently high R and a sufficiently large λ make the existence of this steady state more likely. If the parameters violate the condition in Proposition 2, the steady state with the equal distribution of wealth cannot exist; some inequality of wealth across households must be present in a steady state.[9]

[9] The reader may wonder what would happen if the economy starts with a perfectly equal distribution of the wealth, when the condition in Proposition 2 is violated. Suppose that the household's initial wealth is less than $1-\lambda$. Then, the equilibrium interest rate in the first period is determined in such a way that credit rationing will take place immediately. (See the discussion

3B. The Steady State with Wealth Inequality

Consider now a steady state with an unequal distribution of wealth. That is, some households are rich enough to be able to borrow and invest, while others remain poor and have no choice but to lend. The existence of persistent inequality requires $r_\infty < R$. In such a steady state, the wealth of all the households that invest must converge to the fixed point of the map, $w_{t+1} = \beta\{[(1-\lambda)R/(1-\lambda R/r_\infty)]w_t+y\}$ or,

(6) $\quad w^B_\infty = B(r_\infty) \equiv \beta y/\{1-\beta[(1-\lambda)R/(1-\lambda R/r_\infty)]\} \geq 1-\lambda R/r_\infty$,

where the inequality in (6) is the condition that these households are indeed rich enough to borrow and invest. (B stands for the borrower.) As seen in Equation (6), the wealth of the rich borrower converges only when the steady state interest rate satisfies $r_\infty > \lambda R/[1-\beta(1-\lambda)R] > \lambda R$. If $r_\infty \leq \lambda R/[1-\beta(1-\lambda)R]$, the wealth of the rich, and hence their demand for credit, would grow unbounded. In other words, an accumulation of wealth by the rich, and growing credit demand will drive the interest rate above $\lambda R/[1-\beta(1-\lambda)R]$. Next, the wealth of all the households that cannot invest must converge to the fixed point of the map, $w_{t+1} = \beta(r_\infty w_t+y)$ or,

(7) $\quad w^L_\infty = L(r_\infty) \equiv \beta y/(1-\beta r_\infty) < 1-\lambda R/r_\infty$,

where the inequality in (7) is the condition that the poor is indeed too poor to be able to invest, and hence has no choice but to lend. (L stands for the lender.) Note that the above argument also

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in footnote 3.) The lucky households obtain credit and accumulate wealth faster than the others that are denied credit. This breaks the perfect equality. If the household's initial wealth is greater than $1-\lambda$, the wealth declines monotonically, and after a finite period, it goes below $1-\lambda$, at which point credit rationing will take place to break the equality.

establishes that the wealth of the households is concentrated on two points in a steady state with inequality. To summarize,

Lemma 2. In a steady state with inequality, $r_\infty \in I \equiv (\lambda R/[1-\beta(1-\lambda)R], R)$, and a household's wealth is equal to either $w^B_\infty = B(r_\infty) \equiv \beta y/\{1-\beta[(1-\lambda)R/(1-\lambda R/r_\infty)]\}$ or $w^L_\infty = L(r_\infty) \equiv \beta y/(1-\beta r_\infty)$.

Note that the interval $I \equiv (\lambda R/[1-\beta(1-\lambda)R], R)$ is empty if $\lambda = 1$. Thus, the lemma also implies that that $\lambda < 1$ is a necessary condition for the existence of a steady state with inequality.

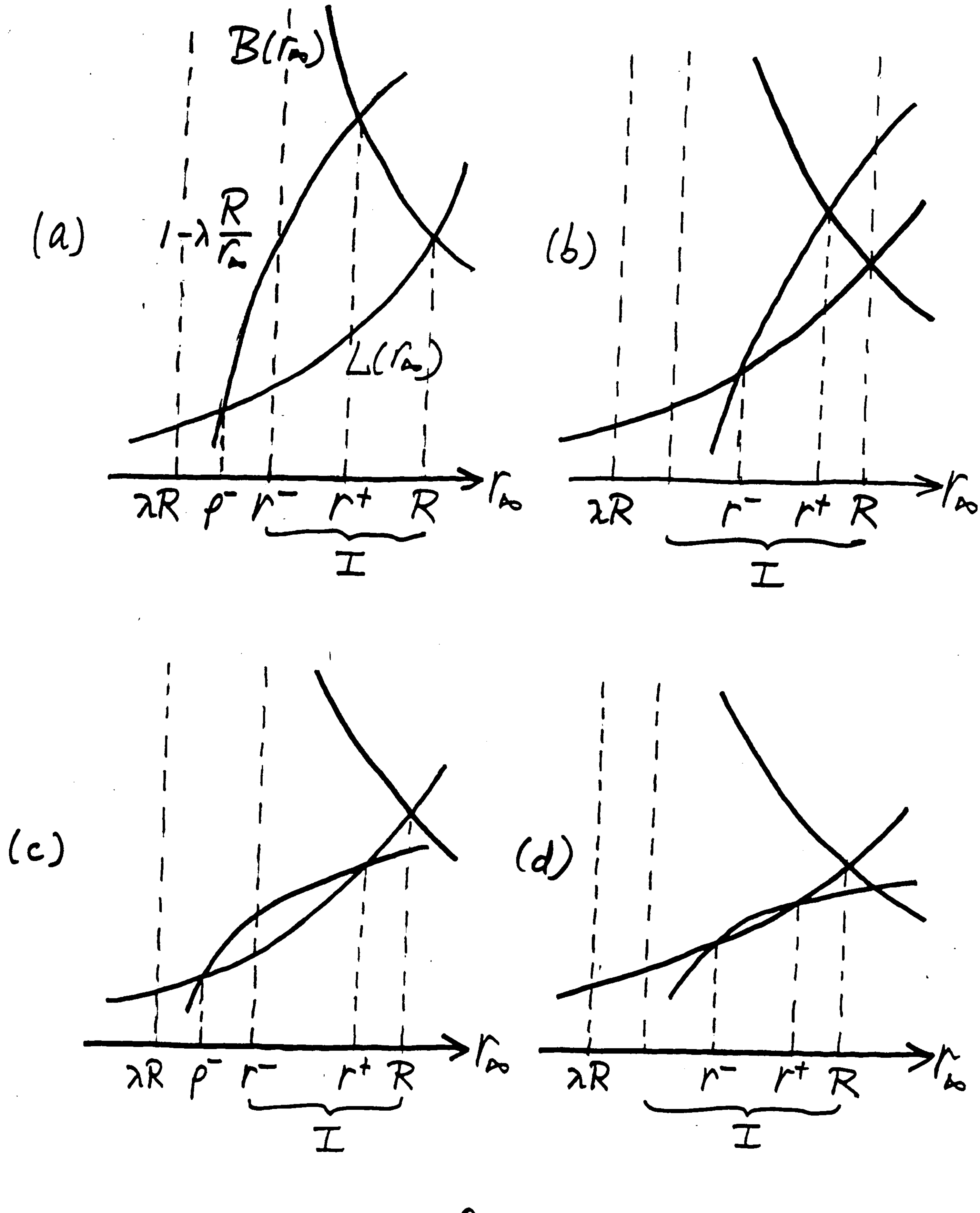
The credit market equilibrium condition, (3), becomes $X_\infty w^L_\infty + (1-X_\infty)w^B_\infty = [(1-X_\infty)/(1-\lambda R/r_\infty)]w^B_\infty$, where $0 < X_\infty < 1$ is the steady state fraction of the poor households, whose wealth is given by w^L_∞. From (6) and (7), this condition can be rewritten to

(8) $X_\infty = X(r_\infty) \equiv (\lambda R)(1-\beta r_\infty)/(1-\beta R)r_\infty$,

which is strictly decreasing in $r_\infty \in I$, with the range equal to $(\lambda, 1)$. This suggests that, for any steady state interest, $r_\infty \in I$, one can always find a unique value of $X_\infty \in (\lambda, 1)$ that satisfies Equation (8). Therefore, to demonstrate the existence of a two-point steady state distribution, it suffices to check the inequalities in (6) and (7), which is reproduced as follows:

(9) $L(r_\infty) < 1-\lambda R/r_\infty \le B(r_\infty)$.

Figures 3a-d illustrate the condition, (9). The upward, convex curve depicts the steady state wealth of the poor lender, $L(r_\infty)$, while the downward-sloping curve depicts that of the rich borrower, $B(r_\infty)$, both as functions of r_∞. As expected, the wealth of the poor lender increases



(a)
$B(r_\infty)$
$1-\lambda\frac{R}{r_\infty}$
$L(r_\infty)$
λR ρ^- r^- r^+ R
r_∞
I
(b)
λR r^- r^+ R
r_∞
I
(c)
λR ρ^- r^- r^+ R
r_∞
I
(d)
λR r^- r^+ R
r_∞
I

fig.3

with the interest rate, while the wealth of the rich borrower declines, and the wealth gap between the two would disappear, as r_∞ becomes close to R, at which $L(R) = B(R) = \beta y/(1-\beta R)$. The third curve, the concave, upward-sloping one, depicts the threshold level of wealth, $1-\lambda R/r_\infty$, which divides the rich borrower from the poor lender. Figures 3a-d show four generic ways, in which this curve intersects with $L(r_\infty)$ and $B(r_\infty)$. For example, in Figure 3b, the curve, $1-\lambda R/r_\infty$, intersects with $L(r_\infty)$ at $r^- > \lambda R/[1-\beta(1-\lambda)R]$ and with $B(r_\infty)$ at $r^+ < R$. Therefore, there exists a continuum of steady states, with $r_\infty \in (r^-, r^+]$, and $X_\infty = X(r_\infty) \in [X(r^+), X(r^-))$, where $X(r^+) > \lambda$ and $X(r^-) < 1$. All these steady states are characterized by a two-point distribution of wealth, and the degree of inequality differ across the steady states. A low interest rate is associated with greater inequality. A lower interest rate implies not only that the wealth gap between the rich and the poor, $B(r_\infty) - L(r_\infty)$, is bigger. It also implies that a large fraction of the households is poor. The intuition behind is easy to grasp. The presence of a large number of poor households, which cannot borrow, keeps the interest rate low. A lower interest rate favors the rich at the expense of the poor, which increases the wealth gap. A larger demand for credit by each rich household, backed by its large wealth, can be met by a smaller supply of credit by each poor household, only when only a fraction of the households are rich. In all the four cases depicted in Figures 3a-d, the curve, $1-\lambda R/r_\infty$, stays above the curve, $L(r_\infty)$, over a subinterval of $I = (\lambda R/[1-\beta(1-\lambda)R], R)$, which implies that (9) is satisfied, and hence there exists a continuum of steady states with a two-point wealth distribution.

On the other hand, if $1-\lambda R/r_\infty$ stays below $L(r_\infty)$ everywhere over $I = (\lambda R/[1-\beta(1-\lambda)R], R)$, there exists no steady state with a two-point wealth distribution. Therefore, the only steady state is the one characterized in Proposition 2.

The conditions for the four cases, portrayed in Figures 3a-d, are summarized in the following proposition.

Proposition 3.[10]

There exists a continuum of steady states with a two-point distribution of wealth, $(w^L_\infty, w^B_\infty) = (L(r_\infty), B(r_\infty)) \equiv (\beta y/(1-\beta r_\infty), \beta y/\{1-\beta[(1-\lambda)R/(1-\lambda R/r_\infty)]\})$, with $r_\infty \in (r^-, r^+] \subset I \equiv (\lambda R/[1-\beta(1-\lambda)R], R)$ and $X_\infty = X(r_\infty) \in [X(r^+), X(r^-))$,

a) if $y/R(\beta y+1-\beta R), \beta y/(1-\beta R) < 1-\lambda$, where $r^- = \lambda R/[1-\beta(1-\lambda)R]$ and r^+ is a unique solution to $1-\lambda R/r = B(r)$ in I;

b) if $y/R(\beta y+1-\beta R) > 1-\lambda > \beta y/(1-\beta R)$, where r^- is a unique solution to $1-\lambda R/r = L(r)$, and r^+ is a unique solution to $1-\lambda R/r = B(r)$, both in I;

c) if $y/R(\beta y+1-\beta R) < 1-\lambda < \beta y/(1-\beta R)$, where $r^- = \lambda R/[1-\beta(1-\lambda)R]$, and r^+ is a unique solution to $1-\lambda R/r = L(r)$ in I;

d) if $y/R(\beta y+1-\beta R), \beta y/(1-\beta R) > 1-\lambda > 1-(1-(\beta y)^{1/2})^2/(\beta R)$, where r^- and r^+ are two solutions to $1-\lambda R/r = L(r)$ in I.

Propositions 2 and 3 summarize all the steady states in this model.

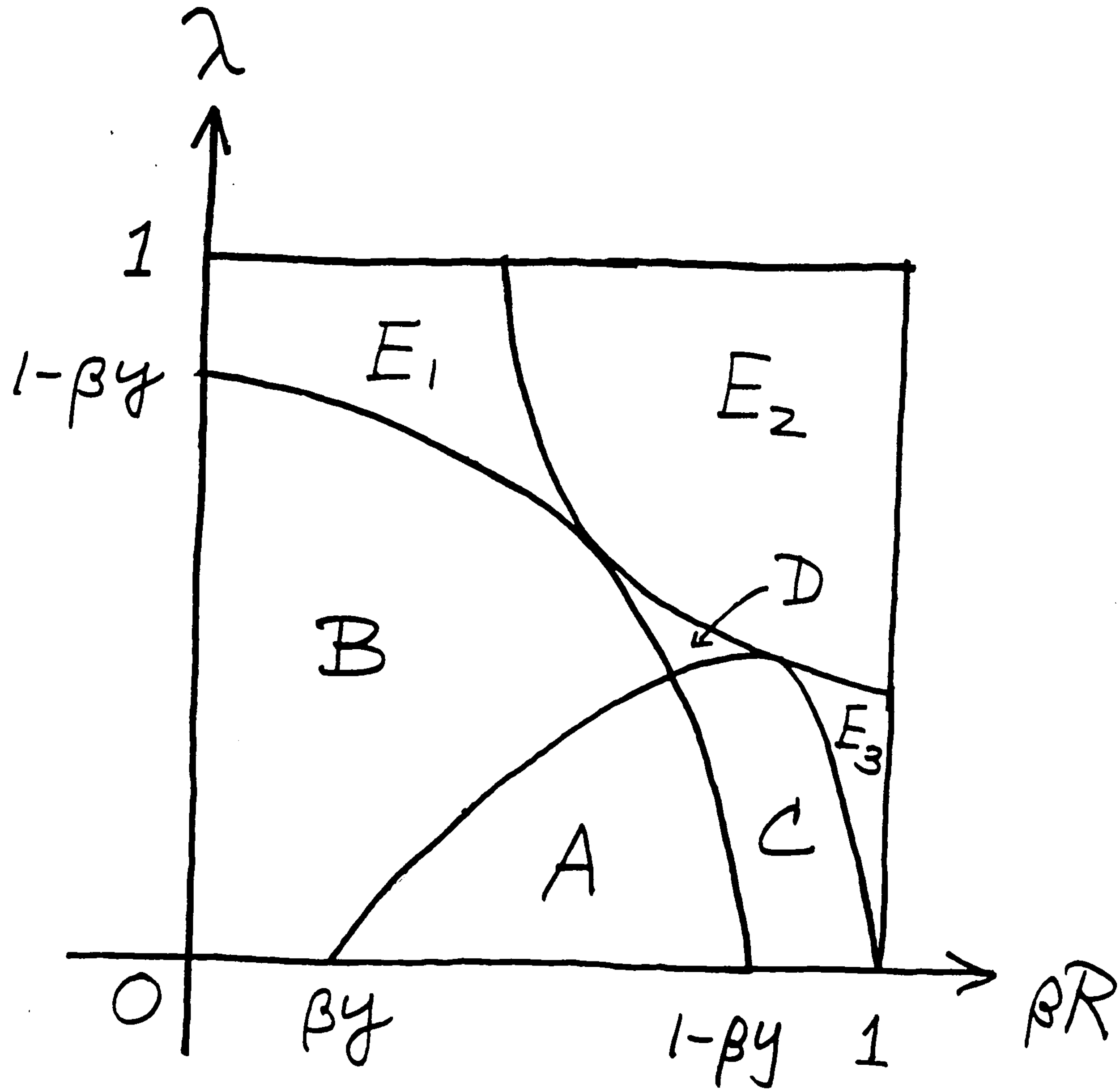


fig. 4

Figure 4 depicts the parameter configurations, given in Propositions 2 and 3, in terms of $\beta R \in (0,1)$ and $\lambda \in (0,1)$, for a given $\beta y \in (0,1)$. The regions marked by A, B, C, and D correspond to the conditions stated in Proposition 3a), 3b), 3c), and 3d), respectively. Regions A and B violate the condition in Proposition 2, which are satisfied in Regions C, D, and E = $E_1+E_2+E_3$.[11]

In Regions A and B, with a small βR and a small λ, there is a separation of the rich and the poor in all the steady states. In these cases, the model generates endogenous inequality. The fraction of the poor in the population is strictly greater than λ. In Region A, the fraction of the rich must be positive, but can be arbitrarily small. In Region B, it has a positive lower bound.

The reason for inequality should be easy to grasp. Because of the imperfect enforcement problem, the interest rate must become sufficiently low to make it possible for some household to be able to borrow and invest and become rich. In order to keep the interest rate low, however, some households must stay poor, so that they are unable to borrow and forced to lend. In these steady states, the rich households are capable to maintain their wealth only because there are some poor households in the economy. And the rich never accumulate enough wealth to pull the poor out of the poverty.

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[10]To avoid a taxonomic presentation, this Proposition does not include non-generic, borderline cases.

[11] Figure 4 is drawn under the assumption $0 < \beta y < 1/2$. If $1/2 < \beta y < 1$, Region A is empty. In Region E = $E_1+E_2+E_3$, $1-\lambda R/r_-$ stays below $L(r_-)$ everywhere over $I = (\lambda R/[1-\beta(1-\lambda)R], R)$. In Region E_1, $1-\lambda R/r_-$ intersects $L(r_-)$ only in $r_- \geq R$. In Region E_2, $1-\lambda R/r_-$ never intersect with $L(r_-)$. In Region E_3, $1-\lambda R/r_-$ intersects $L(r_-)$ only in $r_- \leq \lambda R/[1-\beta(1-\lambda)R]$. These distinctions become important, when the model is extended to allow for a storage technology in the next section.

In Region E, with a combination of a high βR and a high λ, wealth inequality disappears in the steady state. In this case, therefore, the model predicts a trickle-down phenomenon, in which an accumulation of the wealth by the rich pulls the poor out of the poverty, and the poor households will eventually catch up the rich households. In other words, the model predicts Kuznets's famous U-curve pattern over economic development.

Although an explicit analysis of the dynamics is beyond the scope of this paper, the transition process is not difficult to imagine. Suppose that the economy starts at an underdeveloped state, where all the households are poor. There is little inequality, but some households are richer than others. Initially, the equilibrium interest rate is very low, so that the relatively rich households, while poor in absolute terms, are able to borrow and invest. Their wealth then starts growing faster than others, magnifying inequality. In the cases of Regions A and B, this is the end of a story, and the Kuznets U-curve fails to materialize. In the cases of Regions E, however, the rich's growing demand for credit will increase the interest rate so much that the poor households, the lenders which benefit from a high interest rate, will be able to catch up with the rich, reducing inequality.

Note that the mechanism through which the wealth trickles down from the rich to the poor in the case of Region E differs from the trick-down mechanism discussed by Aghion & Bolton (1997). In their model, investment demand satiates quickly for each household. This implies that, once the rich households accumulate enough wealth, they start lending additional funds to the poor households at a *lower* interest rate, which makes it possible for the latter to borrow and to escape from the poverty. In the present model, the rich households continue to borrow, as they accumulate wealth. An accumulation of the wealth by the rich helps the poor, because the rich's increasing demand for credit leads to a *higher* interest rate, which in turn helps the poor lenders to accumulate their wealth faster and to escape from the poverty.

In Regions C and D, characterized by a combination of a high βR and a small λ, both endogenous inequality and a convergence of the wealth are possible in steady states. In Region C, the fraction of the rich in the population is positive, but can be arbitrarily small, while it has a positive lower bound in Region D.

4. Some Alternative Specifications

The model presented above makes strong assumptions on the investment technology, which helps to simplify the analysis considerably. This section discusses to what extent the main results obtained in the previous section will carry over under alternative assumptions.

4A. A Storage Technology

In the model discussed above, the poor agents, who are unable to borrow and invest, have no choice but to lend. Suppose now that they have an access to a storage technology, which earns the return equal to $\rho < R$ per unit, and has no minimum requirement. If $\rho \leq \lambda R$, storage is always dominated by lending, hence the analysis does not need to be changed. If $\rho > \lambda R$, the credit market equilibrium condition becomes, instead of (3),

$$(10) \qquad \int_0^\infty w dG_t(w) \begin{cases} \geq \left(1-\dfrac{\lambda R}{\rho}\right)^{-1} \displaystyle\int_{1-\lambda R/\rho}^\infty w dG_t(w) & \text{if } r_t = \rho . \\ = \left(1-\dfrac{\lambda R}{r_t}\right)^{-1} \displaystyle\int_{1-\lambda R/r_t}^\infty w dG_t(w) & \text{if } \rho < r_t < R. \end{cases}$$

$$\leq (1-\lambda)^{-1}\int_{1-\lambda}^{\infty} wdG_t(w) \qquad \text{if } r_t = R,$$

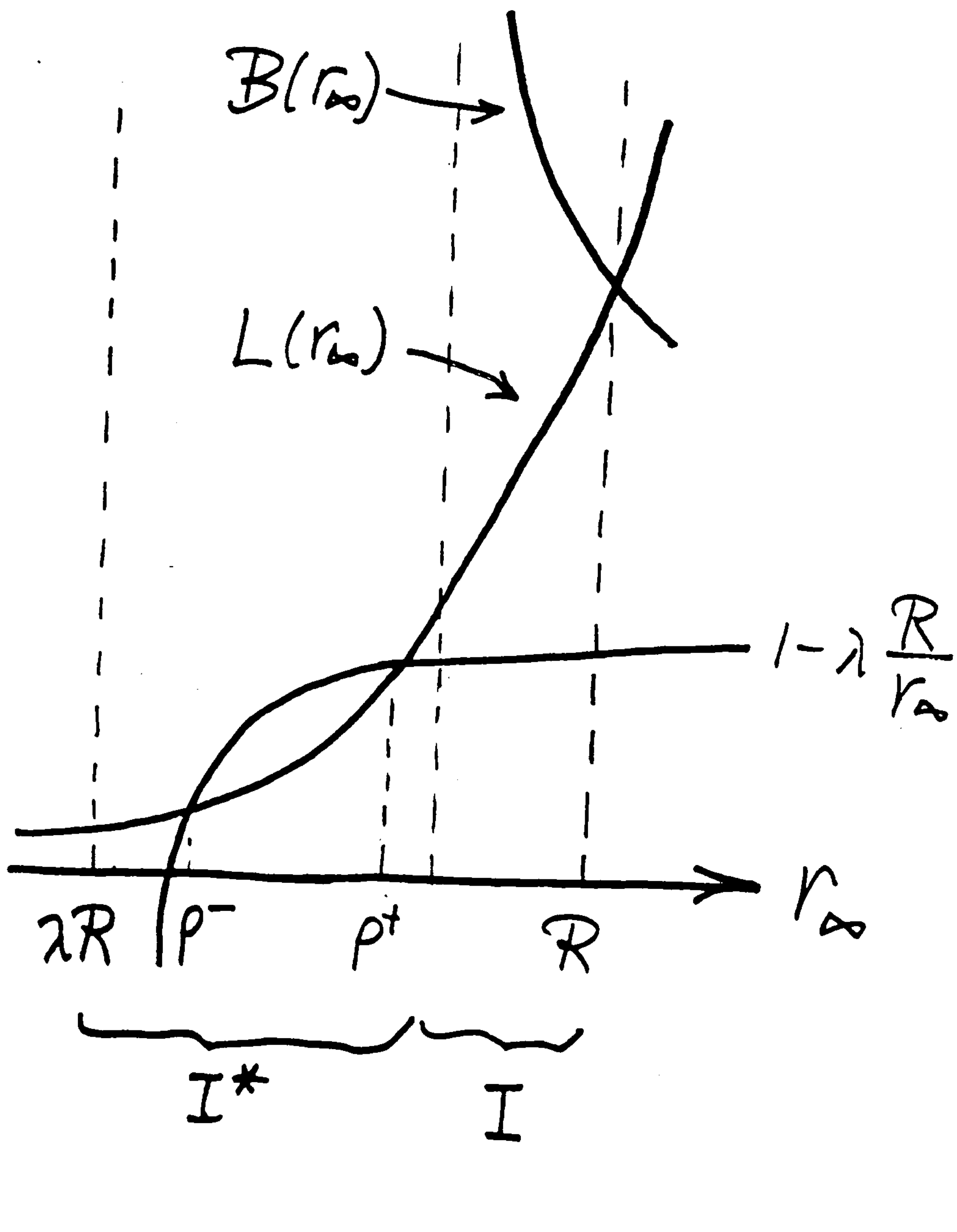
because the supply of credits is perfectly elastic at $r_t = \rho$. Equation (4) still describes the dynamic evolution of each household's wealth.

In the following discussion, let us focus on the case, where $\lambda R < \rho \leq \lambda R/[1-\beta(1-\lambda)R]$. Then, introducing the storage technology does not eliminate any of the steady states described in Propositions 2 and 3. This is because the interest rate in these steady states satisfy $r_\infty >$ $\lambda R/[1-\beta(1-\lambda)R]$. Once the economy reaches one of these steady states, the storage option becomes dominated, and never will be used. The introduction of the storage technology, however, may create another steady state, in which all the households use the storage technology to maintain the level of wealth, which is too small for them to be able to borrow. The existence of this steady state requires

(11) $\beta y/(1-\beta\rho) < 1-\lambda R/\rho$.

As can be verified easily from Figures 3a and 3c, one can find $\rho < \lambda R/[1-\beta(1-\lambda)R]$, which satisfies the condition (11), in Regions A and C. On the other hand, Figures 3b and 3d show that (11) cannot be satisfied with $\rho \leq \lambda R/[1-\beta(1-\lambda)R]$ in Regions B and D. The condition (11) can also be satisfied in Region E_3, where the two upward-sloping curves intersect below $\lambda R/[1-\beta(1-\lambda)R]$, as depicted in Figure 5.

It is also easy to see that in no steady state, both the investment technology and the storage technology are used. (Proof: The storage technology is used only when $r_\infty = \rho$. If some



$B(r_\infty)$
$L(r_\infty)$
$1-\lambda\frac{R}{r_\infty}$
r_∞
λR
ρ^-
ρ^+
R
I^*
I

fig. 5

households are rich enough to borrow and invest, their wealth grows unbounded and hence their demand for credit is infinity, which violates the credit market equilibrium.)

The above argument leads to the following proposition.

Proposition 4.

Suppose that $\rho \in I^* \equiv (\lambda R, \lambda R/[1-\beta(1-\lambda)R]]$. Then, all the steady states described in Propositions 2 and 3 exist. In addition,

i) if $y/R(\beta y+1-\beta R) < 1-\lambda$ and $\rho^- < \rho \leq \lambda R/[1-\beta(1-\lambda)R]$, where ρ^- is an unique solution to $1-\lambda R/r = \beta y/(1-\beta r)$ in I^*,

or

ii) if $y/R(\beta y+1-\beta R) > 1-\lambda > 1-(1-(\beta y)^{1/2})^2/(\beta R)$ and $\rho^- < \rho < \rho^+$, where ρ^- and ρ^+ are the two solutions to $1-\lambda R/r = \beta y/(1-\beta r)$ in I^*,

there exists another steady state, in which $r_\infty = \rho$ and the wealth of all the households is $w_\infty = \beta y/(1-\beta\rho) < 1-\lambda R/\rho$. In this steady state, all the households use the storage technology and remain too poor to be able to borrow and invest.

Thus, all the steady states discussed in Proposition 2 and 3 continue to exist, unless the storage technology is too productive. The storage technology may, however, create a different type of steady state, in which there is perfect equality, but unlike the steady state identified in Proposition 2, it is an equalization of misery, in which all the households remain poor. All the households hold a lower level of wealth in this steady state than any steady state in Propositions

2 and 3. In this sense, the storage technology creates a poverty trap, i.e.; a steady state of underdevelopment, which co-exists with steady states with development. Proposition 4 shows that this poverty trap can exist if the storage technology is not too unproductive and the capital markets are relatively inefficient. (For a sufficiently large λ, it can be ruled out. In particular, it does not exist in Regions B, D, E_1, and E_2.) In the case depicted in Figure 3a, all the steady states that dominate the poverty trap necessarily come with an unequal distribution of wealth. In the case depicted in Figure 3c, there is a steady state with perfect equality that dominates the poverty trap steady state. Finally, in the case depicted in Figure 5, there are only two steady states, both of which imply perfect equality. In one, all the households become rich enough to invest, while all the households are too poor to invest in the other. This result has some resemblance to Piketty (1997).

How difficult is it for the economy to escape from the poverty trap? The answer to this question turns out to be quite simple. If the starting wealth of all the households is less than $1-\lambda R/\rho$, the economy will be trapped forever. On the other hand, if a positive measure of the households has the starting wealth greater than $1-\lambda R/\rho$, they can invest and their wealth grow faster, because they can borrow at a low rate, $r_t = \rho$. Their growing demand for credit eventually causes the interest rate to rise above $\lambda R/[1-\beta(1-\lambda)R]] \geq \rho$. (Otherwise, the wealth of the rich households would grow unbounded.) This rise in the interest rate helps the other households to increase their wealth, as well. Whether this increase in the wealth of the poorer households is significant enough for them to be able to start investing depends on the parameter configurations.

The case, where $\lambda R/[1-\beta(1-\lambda)R] < \rho \leq R$, can be analyzed similarly. The main difference from the previous case is that some of the steady states characterized by Proposition 3 no longer exist, as an additional restriction $r_\infty \geq \rho$ must be satisfied. Indeed, if the storage technology is very productive, $r^+ < \rho < R$, then all the steady states are characterized by an equal distribution of wealth. In particular, if ρ is taken arbitrarily close to R, i) Regions A and B have only the steady state with $r_\infty = \rho$, where every household uses the storage technology; ii) Regions E_1 and E_2 has only the steady state with $r_\infty = R$, where every household can invest to the project; iii) In Regions C, D, and E_3, both types of the steady states co-exist, which, again, has a resemblance to Piketty's (1997) result.

4B. *Diminishing Returns to Investment*

In the basic model, the demand for credit by the rich never satiates, as long as $r_t = R$. Indeed, the richer they become, the more they borrow. This is because their investment never runs into diminishing returns. Let us now examine the effect of diminishing returns to investment. Such a situation may be more plausible, when investment is interpreted as education or a housing purchase.

To be more specific, suppose now that the project return is now given by the following production function,

$$(A1') \quad F(k_t) = \begin{cases} 0 & \text{if} \quad 0 \leq k_t < 1, \\ R+n(k_t-1) & \text{if} \quad k_t \geq 1. \end{cases}$$

where $n < R$. That is to say, once the minimum level of investment is made, additional investment to the project generates a smaller return. It is also assumed that, as in section 4A, the storage technology yielding the return, ρ, is available to all the agents, where $R > \rho > n$.[12] Then, the rich entrepreneurs never invest more than one unit to the project. If $w_t > 1$, they do not borrow and additional funds, $w_t - 1$, are either allocated to the storage technology or lent in the credit market. The equilibrium interest rate is now restricted to the range, $r_t \in [\rho, R]$. Note that λR is no longer the lower bound on the interest rate.

Since no agent invests more than one unit and only the agents whose inherited wealth satisfy $w_t \geq \max\{0, 1-\lambda R/r_t\}$ invest in the project, the aggregate demand for investment is equal to $1-G_t(\max\{0, 1-\lambda R/r_t\})$. The credit market equilibrium is thus given by

$$(12) \qquad \int_0^\infty w\,dG_t(w) \begin{cases} \geq 1 - G_t\left(\max\left\{0, 1 - \dfrac{\lambda R}{\rho}\right\}\right) & \text{if } r_t = \rho. \\[2ex] = 1 - G_t\left(\max\left\{0, 1 - \dfrac{\lambda R}{r_t}\right\}\right) & \text{if } \rho < r_t < R. \\[2ex] \leq 1 - G_t(1-\lambda) & \text{if } r_t = R. \end{cases}$$

The dynamics of each household's wealth now follows

$$(13) \qquad w_{t+1} = \begin{cases} \beta\{(R-r_t)+r_t w_t + y\} & \text{if } w_t \geq 1-\lambda R/r_t, \\ \beta(r_t w_t + y) & \text{if } w_t < 1-\lambda R/r_t. \end{cases}$$

Equation (13) is depicted in Figure 6, where the arrows indicate the effect of a higher interest rate.

It is also necessary to replace (A2) with

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[12] If $n > \rho$, the rich will continue to borrow and invest. The model is thus essentially the same with the model in section 4A, where the role of R is now replaced by n.

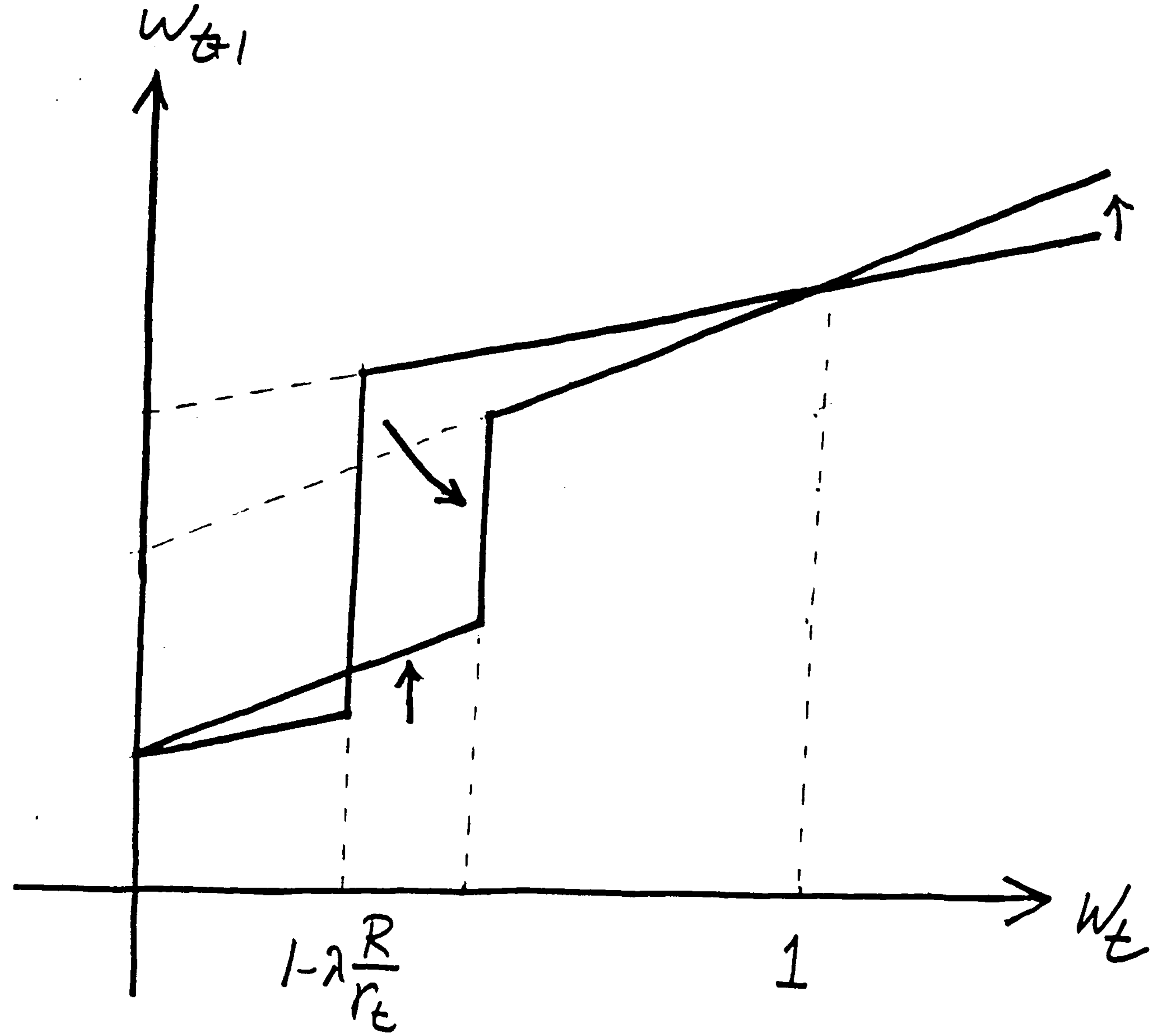

fig. 6

(A2') $\beta(\rho+y) < 1$,

to rule out the trivial case, where every household, regardless of its initial wealth, will eventually become wealthy enough to become able to self-finance investment, without ever participating the credit market.

Some critical differences between this model and the previous one deserve emphasis. First, the aggregate investment, the RHS of (12), is bounded from above. Thus, if the economy ever accumulates enough wealth, the interest rate will go down to ρ. This implies that, in order to ensure the existence of a steady state, it suffices to assume, instead of (A3),

(A3') $\beta\rho < 1$,

which is already implied by (A2'). Note that R can now take an arbitrarily large value without jeopardizing the existence of a steady state.

Second, unlike in Figure 4, the slope of the map in Figure 6 is the same, regardless of whether the household's wealth is below or above the threshold level. Both the rich and the poor earn the same return on their wealth, equal to $r_t \in [\rho, R]$. The rich still accumulates more wealth than the poor, when $r_t < R$, because the rich's non-interest income, $R-r_t+y$, is higher than the poor's non-interest income, y.

Third, there are now two types of the rich that need to be distinguished. One is the moderately rich, whose wealth satisfies $1-\lambda R/r_t \leq w_t < 1$. These households are rich enough to be able to borrow and invest, but not rich enough to be able to self-finance the investment. The other is the very rich, $w_t > 1$, who are rich enough to self-finance the investment and to lend. A rise in the interest rate thus hurts the moderately rich, the borrower, while it benefits the very rich, as well as the very poor, which are both lenders.

Solving for and characterizing all the steady states in this model is, although conceptually simple, algebraically cumbersome with a wide range of the cases to be distinguished, so that only the main results will be reported here.[13] It turns out that the prediction of the model is very similar to that of the model discussed in section 4A, as long as $\beta(R+y) < 1$. That is to say; i) a sufficiently small λ implies a two-point steady state distribution of wealth; ii) a sufficiently large λ implies the equal distribution of wealth, where all the households are rich enough to be able to borrow, and not rich enough that they need to borrow, and the steady state interest rate is equal to $r_\infty = R$; iii) when R is not too small, there is an intermediate range of λ, in which both types of steady states co-exist. Therefore, in spite that investment in this model is subject to diminishing returns, this model, when R is not too large, keeps the essential feature of the model where investment is not subject to diminishing returns. The reason is that, when $\beta(R+y) < 1$, even the rich households, which can invest, will never be able to maintain a level of wealth, which enables them to self-finance their investment. In other words, they continue to borrow to invest, in spite of the diminishing returns to investment.

On the other hand, when $\beta(R+y) > 1$--recall that R can be arbitrarily large in this model without jeopardizing the existence of a steady state--, the model has a feature similar to that of the Aghion & Bolton model. In this case, the rich households may accumulate enough wealth so that they stop borrowing and lending the remaining wealth to the poor, who in turn become rich.

[13] See Matsuyama (1998) for a detail.

In the long run, every household could become equally rich, and rich enough to self-finance, and the interest rate is equal to $r_\infty = \rho$. Furthermore, one can show that this is the only steady state, when λ is sufficiently large.

5. Concluding Remarks

This paper has presented a theoretical framework for understanding processes that determine the distribution of wealth across households in the long run. Under some parameter values, the model predicts endogenous, permanent inequality, in which the rich are able to maintain a high level of wealth partially due to the presence of the poor. Under others, the model predicts that wealth eventually trickles down from the rich to the poor; an accumulation of wealth by the rich thus pulls the poor out of misery of poverty, thereby generating the Kuznets U-curve pattern.

In the models presented above, the interest rate adjusts endogenously to keep the balance between the supply and demand for credit. Moreover, it is through a low equilibrium interest rate that the rich benefit from the presence of the poor, who have no choice but to lend. The interest rate is not, however, the only price that can play these roles. The wage rate may be at least equally as important as the interest rate. Imagine the following model, which is inspired by Banerjee and Newman (1993). The project requires use of labor, and hence its profitability depends on the equilibrium wage rate. The relatively wealthy agents can borrow and become entrepreneurs, hiring the workers, while the poor agents, being unable to borrow, have no choice but to become workers. The threshold level of wealth and the wage rate adjust in such a way to keep the balance between the supply and demand for labor. In some cases, endogenous inequality can arise. The rich entrepreneurs maintain a high level of wealth due to the presence of a large poor working class, which has no choice but to work at a low wage. The low wage rate in turn prevents the working class from accumulating wealth. In other cases, a strong demand for labor by the entrepreneurial class pushes up the wage rate to help the working class to escape from the poverty.[14] This suggests the robust of the results obtained in this paper.

Because the distribution of wealth is an infinite-dimensional object, an explicit analysis of the transitional dynamics in these models is mathematically demanding. Furthermore, to analyze the dynamics for an arbitrary initial condition, one cannot avoid the possibility of credit rationing, which introduces stochastic elements into the models. For these reasons, one may have to resort to numerical simulations. For a very limited set of initial conditions, however, the dynamics can be solved explicitly.[15]

One can think of many ways in which the models can be extended. Introducing long run growth is one. In the above models, the minimum level of investment required for the project plays a crucial role. If the economy as a whole is growing over time, perhaps due to some exogenous improvement in technology, and if the minimum level of investment is exogenously fixed, the mechanism identified in this paper for generating endogenous inequality loses its power in the long run. On the other hand, if the engine of growth is endogenous technological change due to investment, and if better technology requires a higher level of minimum investment, then long run growth may never eliminate inequality. Indeed, it is conceivable that some levels of inequality may be necessary for sustainable growth.

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[14] See Matsuyama (1998) for a sketch of these results.

[15] Matsuyama (1998) offers some preliminary analysis along this line.

Finally, the theoretical framework presented above, or some variations of it, should be useful for the policy analysis. For example, in the case of endogenous inequality, redistributing wealth from the rich to the poor may help the poor to escape out of the poverty. In other cases, it may push the economy into the poverty trap, by reducing investment by the rich, which could help the poor. It is hoped that the present paper would stimulate further research on these issues.

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