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Playing Multiple Complementarity Games Simultaneously

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Abstract

This paper analyzes the situation, in which a continuum of identical players is engaged in more than one activity and each activity is characterized by a complementarity game. The player's intensity levels across different activities are linked in such a way that the marginal cost of increasing her intensity in one activity increases with her own intensity levels in other activities. Compared to the case where these games are played independently, a smaller degree of complementarity in each game is required to generate multiple stable Nash equilibria, which are all asymmetric in that the players operate at different levels in different activities. The implications of these and other results, which have a close connection with the Frobenius theory of positive matrices, are discussed in the context of two macroeconomic applications: endogenous inequality of nations and endogenous business cycles.

Keywords: Strategic Complementarities, Multiple Activities, Bifurcation Analysis, The Structure of the Equilibrium Set, Globalization and Inequality of Nations, Intertemporal Substitution and Business Cycles
JEL Classification Numbers: C72 (Noncooperative Games)

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[1] Department of Economics, Northwestern University, 2003 Sheridan Road, Evanston, IL 60208, USA. This is a significantly revised version of the paper, previously titled, "Multidimensional Complementarity Games." The author thanks Eddie Dekel for discussion.

1. Introduction.

This paper proposes and analyzes games, which describe the situation where the players participate in two or more complementarity games simultaneously. In the standard game of strategic complementarities, there is a continuum of identical players and each player chooses a single variable, which may be called "action," "effort," or "intensity". The game is said to exhibit strategic complementarities, when a player's marginal benefit of increasing her level of intensity is an increasing function of the other players' levels of intensity over a certain range. It is well known that the presence of sufficiently strong complementarities often lead to multiple (stable) Nash equilibria. In the class of games proposed in this paper, the players are engaged in more than one activity, and each activity is characterized by a game of strategic complementarities. Each player chooses her level of intensity in each activity, so that the dimensionality of her strategy space is equal to the number of activities. Symmetry across all the activities is also assumed.

If each player's cost of choosing an intensity level in one activity does not depend on her choices of intensity in other activities, this class of games can be reduced to and be analyzed as independently-played, identical, one-dimensional complementarity games. Hence, from the standard result of the complementarity game, one can conclude the following. For a small degree of complementarities, there is a unique Nash equilibrium, in which the players choose the same level of intensity in all the activities. For a large degree of complementarities, there are multiple Nash equilibria in each of the one-dimensional games. This means that the players choose different levels of intensity in different activities in *some* (stable) Nash equilibria. These equilibria co-exist with other (stable) Nash equilibria, where the players choose the same intensity in all the activities.

In many situations, however, it is more natural to assume that a player's marginal cost of increasing her intensity level in one activity increases with her intensity levels in other activities. In other words, different activities compete for a player's effort. In the presence of such interdependence in the cost function, strategic interactions among the players in all the activities must be solved simultaneously, even though strategic interdependence itself is limited within each activity. The goal of the analysis is to understand how the interdependence of the player's intensity choices across different activities affects the structure of (stable) Nash equilibria. Among the main results are:

(1) The greater the interdependence, the smaller degree of strategic complementarity within each activity is required to generate multiple (stable) Nash equilibria.

(2) For an intermediate range of strategic complementarity, the players choose different levels of intensity in different activities in *all* (stable) Nash equilibria.

The games proposed in this paper are abstract enough to encompass many different applications, and the significance of the results stated above may be better appreciated in a more concrete setting.

For example, in macroeconomics, many authors have argued that strategic complementarities are crucial for understanding business cycles. To present this idea, they typically build a game of strategic complementarities, where a player controls a single variable, say "investment" or "job search effort", and demonstrate the existence of multiple stable Nash equilibria (e.g., a High equilibrium and a Low equilibrium). Then, it is argued that the economy is in boom when the players play H, and it is in recession when they play L. What is implicit in the argument is the assumption that there is no intertemporal linkage, which enables us to analyze the allocations of the economy in different periods independently. Aside from being unrealistic, this assumption generates two additional drawbacks. First, without any intertemporal linkage, the degree of strategic complementary necessary to generate multiple equilibria in each period must be sufficiently large, which makes it empirically less plausible. Second, business cycles, where the players play H in one period and L in another, are one of many possible equilibrium configurations. It is also possible that the players play H in every period, or that they play L in every period, hence there are no business fluctuations. Let us now introduce the possibility that the players may be able to substitute their intensity levels across periods. In the presence of such intertemporal linkages, the game becomes inherently multidimensional in that one must solve for the equilibrium allocation of a multi-period economy. This extension not only makes the model more realistic, but also helps to remove the two drawbacks of the model. That is, with a high degree of substitution across periods, even a small degree of strategic complementarity within periods is sufficient to generate cycles and that the economy fluctuates over time in all (stable) Nash equilibria; i.e., the economy *must* play L in some periods and H in others.

The notion of strategic complementarity also plays a prominent role in the growth and development literature. The typical approach is to model a closed economy and to show multiple equilibria (H and L) in the presence of strong complementarities. Then, to explain the diversity of economic performance across economies, it is argued that different equilibria prevail in different economies. Some economies play H, while others play L. Again, implicit in this argument is that there is no interaction across the economies. With the closed economy assumption, the degree of complementarity necessary for the multiplicity must be larger, and the model does not offer any compelling reason why different economies must play different equilibria. Alternatively, one can model a global economy consisting of inherently identical regional economies, where international investors allocate their resources across regional economies and the return from investing in one economy depends on the aggregate investment in that economy. Then, with a high degree of resource movement across borders, even a small degree of complementarity within each economy would suffice to generate multiple stable (Nash) equilibria, all of which exhibit the diversity of economic performance across economies, i.e., some economies *must* play L, while others play H.

One may view the present study as an attempt to analyze complementarity games in a multidimensional setting. Although a large number of studies have developed a multidimensional extension of complementarity games, none seems to have considered the extension discussed in this paper. The existing literature is exclusively concerned with the case where there are some complementarities across different activities. For example, Blume (1993), Ellison (1993), and Goyal and Janssen (1997) considered local interaction games, where the players are engaged in multiple coordination games with different set of opponents. In these models, each player is constrained to play the same strategy in all the games she plays. Gale (1995), Matsui and Matsuyama (1995), and Matsuyama (1991) considered dynamic coordination games, where the players participate in a series of coordination games over time. In these games, the choice of actions taken in one period is constraint to be similar across time. In all these studies, intensity levels across different games (tend to) move together. In the present study, the players who choose to play intensively in one game tend to play less intensively in other games.

To avoid possible misunderstandings, it should be pointed out that the present study is *not* an attempt to analyze a complementarity game with a multidimensional strategy space.[2] It is an attempt to analyze a multidimensional game, which captures the notion of playing multiple complementarity games simultaneously. There exist strategic complementarities within each component game. No complementarities are assumed to exist across these component games.

Some recent studies considered an extension of complementarity games, which *reduces* the possibility of multiplicity for a given level of complementarity. For example, it is shown that certain types of stochastic shocks (Frankel and Pauzner 1997) or a sufficiently large heterogeneity of players (Herrendorf, Valentinyi, Waldman 1998) would eliminate multiplicity in Matsuyama's (1991) model. In contrast, this paper may be viewed as an extension of complementarity games, which *increases* the possibility of multiplicity for a given level of complementarity.

The work closest to this paper in spirit is Holmstrom and Milgrom (1991), who explored the idea that an increase in the level of intensity in one activity raises the marginal cost of increasing the intensity level in other activities in the context of principal-agency theory. They asked why real-life incentive schedules are less high-powered than as suggested in the standard agency model where the agent's effort is a one-dimensional variable. To address this question, they developed a model, where the agent is engaged in multiple tasks. Their answer is that the fact that many tasks compete for the agent's effort makes the agent's supply of effort to each task more elastic, which induces the principal to reduce the power of incentives he provides to the agent. The results presented below can be understood in a similar way. The fact that many games compete for the player's effort makes the player's supply of effort in each complementarity game more elastic, which generates multiple equilibria even for a small degree of complementarity.

The paper is organized as follows. Section 2 is a preliminary step. It presents a basic complementarity game, reviews its key features, and discusses some drawbacks of analyzing such a game in isolation. Section 3 discusses the case, where two complementarity games are played simultaneously. In this case, the strategy space of the game is two-dimensional, which permits a detailed analysis of the structure of the set of equilibria. Section 4 discusses the results obtained in section 3 in the context of some economic applications. Section 5 considers a higher-dimensional case, where more than two complementarity games are played simultaneously. Although our understanding of the structure of equilibria in a higher-dimensional case is far from complete, it can be shown that the main results carry over to this case, as well. This section also reveals a close connection between the main results of this paper and the Frobenius theory of positive matrices. Section 6, the concluding section, discusses possible extensions.

2. The Background.

2.A. *The Basic Complementarity Game.*

Consider the game played by a continuum of identical players. Each player is engaged in a single activity and chooses its intensity level, $x \in I \equiv [0, \infty)$. The payoff of the player is given by

$$V(x;\bar{x}) = F(\bar{x})x - x^2/2 .$$

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[2] The author apologizes if his unfortunate choice of the title in an earlier version has created such misunderstandings.

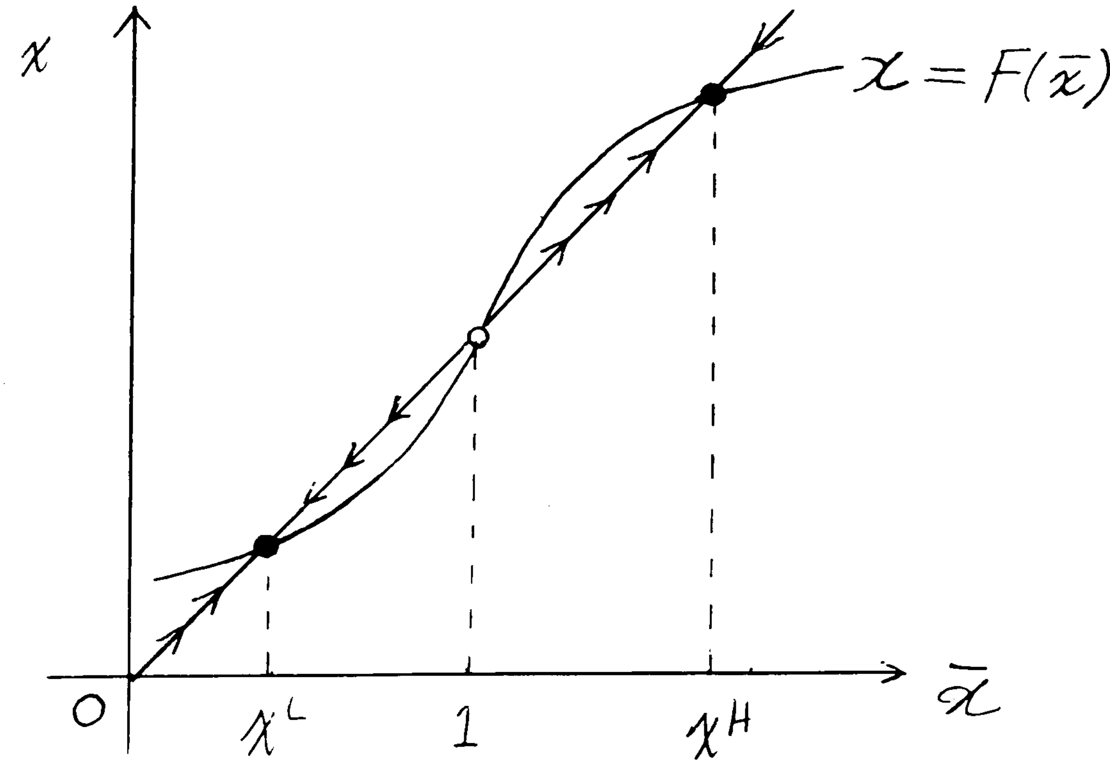

Fig. 1A

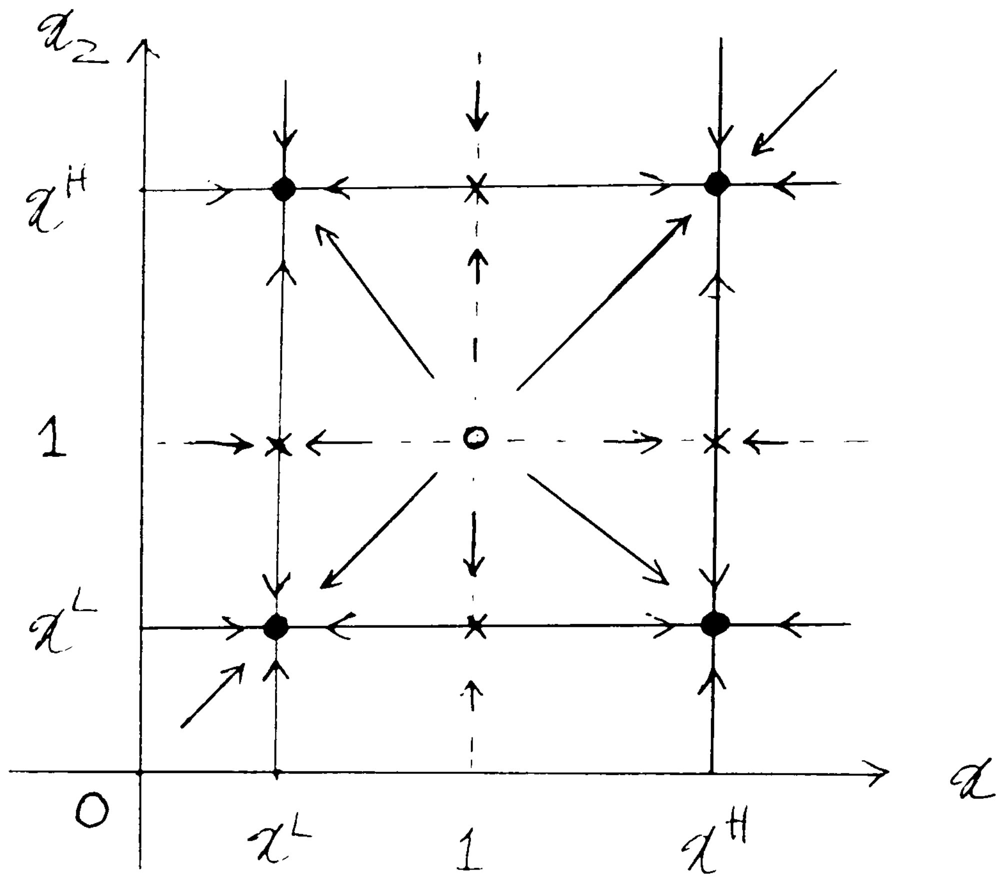

Fig 1B

where $F : I \to R$ is a continuous function, representing the marginal return of an additional intensity, which depends on the average level of intensity chosen by others, $\bar{x} \in I$. The second quadratic term represents the cost. The best response of the player, $Arg_x V(x;\bar{x})$, is simply given by $x = F(\bar{x})$. The set of Nash Equilibria, or simply "Equilibria," of this game, is hence given by the fixed points of F:

$$E = \left\{x \in I \middle| x = F(x)\right\}.$$

In the case depicted in Figure 1A, it is given by $E = \{x^L, 1, x^H\}$. In order to ensure the existence of an equilibrium in the interior, it is assumed that

(A1) $\quad \lim_{x \to 0}\{F(x) - x\} > 0 \qquad$ and $\qquad \lim_{x \to \infty}\{F(x) - x\} < 0.$

The game is said to exhibit strategic complementarity when the marginal return to a player's intensity increases with the average intensity of other players. In our notation, there is strategic complementarity, whenever F is increasing in $\bar{x}$. Strategic complementarity is a necessary condition for this game to have multiple equilibria.

When there are multiple equilibria, they may differ in the stability property. For example, consider the following dynamical system, defined by the gradient system,

(D1) $\quad \dot{x}_t = V_x(x_t;x_t) = F(x_t) - x_t,$

where subscript x here implies that the derivative of V with respect to its first argument. The logic is that, taking the intensity level chosen by the other players, each player moves in the direction that increases its payoff at the speed proportional to its slope of the payoff function. This particular form of tâtonnement processes is chosen for the expositional purpose. Many alternative tâtonnement processes could also be used, and would not affect the following discussion.[3]

Any equilibrium, $x^* \in E$, is a stationary point of the dynamical system (D1). It is locally stable if $F'(x^*) < 1$ and locally unstable if $F'(x^*) > 1$. It is for this reason that most discussion in the literature focus on the set of Stable Equilibria:

$$SE = \{x \in I \mid x = F(x), F'(x) < 1\}.$$

In Figure 1A, it is given by $SE = \{x^L, x^H\}$. This is not to say that an unstable equilibrium is of no interest. On the contrary, an unstable equilibrium plays an important role in the analysis, as it suggests the existence of multiple stable equilibria. To state this argument formally, let the choice of normalization be such that

(A2) $F(1) = 1$,

which implies that $x = 1$ is an equilibrium. Then,

Proposition 0. *Suppose (A1) and (A2). Then, if $F'(1) \equiv \theta > 1$, the equilibrium, $x = 1$, is unstable and there are at least two stable equilibria: $x = x^L < 1$ and $x = x^H > 1$.*
Proof: It should be obvious from Figure 1A. ∎

In other words, the presence of sufficiently strong complementarity, $F'(1) \equiv \theta > 1$, together with the boundary condition, implies multiple stable equilibria.

2.B. *Some Limitations of Analyzing the Complementarity Game in Isolation.*

The existence of multiple stable equilibria with differing intensity levels, the case depicted in Figure 1A, is arguably the most important message of the literature on macroeconomic complementarities. For it helps to generate endogenously the diversity of economic performance across

[3] For example, $\dot{x}_t = \varphi(F(x_t) - x_t)$, where φ: $\boldsymbol{R} \to \boldsymbol{R}$ is any smooth function satisfying $\varphi(0) = 0$, and $\varphi' > 0$, leads

space and across time. In the growth and development literature, for example, it is argued that rich countries are in a high-level equilibrium, while poor countries find themselves in a low-level equilibrium. Or business cycles are explained by arguing that the economy is a high-level equilibrium in one period and a low-level equilibrium in another period.

Although the basic complementarity game presented above is a simple and powerful way of delivering this important message, it also has some drawbacks.

First, the mere presence of strategic complementarity is not enough to generate multiple stable equilibria. For example, if $0 < F' < 1$ everywhere, the equilibrium is unique and stable. Multiple stable equilibria requires that $F' \geq 1$ at least for some range in I. Furthermore, a generic existence of multiple stable equilibria requires the existence of an unstable equilibrium, $x^* = F(x^*)$, which in turn requires the presence of strong complementarity, $F'(x^*) > 1$. This requirement makes the case of multiple equilibria empirically less plausible.

Second, although the multiplicity is consistent with the observed diversity of economic performance, the basic complementarity game offers no compelling reason why we should expect to see the diversity. Nothing in this game tells us that different economies have to play different equilibria or that the economy has to play different equilibria in different periods. For example, consider the economy that lasts for two periods. When one argues that each period can be depicted by Figure 1A, and that the economy plays $x = x^H > 1$ in one period and $x = x^L < 1$ in the other, it is implicitly assumed that there is no interaction across the two periods. In other words, the model is actually a two-dimensional one, where the player controls the intensity in the two periods independently, to maximize the payoff,

$$V(x;\bar{x}) = \sum_{i=1}^{2} \left\{ F(\bar{x}_i) x_i - x_i^2/2 \right\}.$$

However, the no-intertemporal linkage assumption makes it equally likely (and, some may argue, more likely) that the same equilibrium prevails in both periods. Hence, this approach for explaining the business cycles is subject to the most common criticism against a model of multiple equilibria; the model can explain anything, both the presence of business cycles and the absence of business cycles, which make the model empirically irrefutable.

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to the same result.

The limitations of the basic complementarity game can clearly be seen as Figure 1B. If each period of the two-period economy can be analyzed independently by means of Figure 1A, the two-period economy has indeed nine equilibria, four of which are stable equilibria: (x^L, x^H), (x^H, x^L), (x^L, x^L), and (x^H, x^H), indicated by black dots. Of these four stable equilibria, the only first two, (x^L, x^H), (x^H, x^L), the asymmetric ones, exhibit the business cycles, while the last two, (x^L, x^L), and (x^H, x^H), the symmetric ones, do not. For the purpose of explaining the business cycles, the only asymmetric equilibria are useful. The symmetric ones are indeed nothing but a nuisance, which merely weakens the prediction of the theory. However, as long as one treats the two periods as being played independently, there is no way of generating asymmetric equilibria without generating symmetric ones.

The same criticism can be made against the use of the basic complementarity game as a way of explaining the diversity of economic performance across countries. When one models each economy as a closed economy, the model with multiple equilibria is consistent with the diversity of economic performance, but fails to explain it.

The following sections develop and analyze multidimensional games, where the players participate in two or more basic complementarity games simultaneously, and different games compete for the player's effort. This extension helps to improve and sharpen the prediction of the theory for two reasons. First, it is capable of generating asymmetric equilibria without generating symmetric ones. Second, it is capable of doing so, with a degree of complementarity smaller than in the case where each complementarity game is played independently.

3. Playing Two Complementarity Games Simultaneously.

As before, the game is played by a continuum of identical players. The strategy space is now two-dimensional. Players are engaged in two different activities, and control their levels of intensity, $x = (x_1, x_2) \in I^2 = [0, \infty)^2$. The payoff function is given by

$$V(x;\bar{x}) = F(\bar{x}_1)x_1 + F(\bar{x}_2)x_2 - C(x_1, x_2)$$

$$= F(\bar{x}_1)x_1 + F(\bar{x}_2)x_2 - \left\{\left(1-\frac{\beta}{2}\right)\frac{x_1^2}{2} + \left(1-\frac{\beta}{2}\right)\frac{x_2^2}{2} + \frac{\beta}{2}x_1x_2\right\}$$

where $\beta \in [0, 1)$ measures the extent to which the player's intensity levels across the two activities are related in the cost function. The restriction on β ensures the strict convexity of the cost function, as well as the uniqueness of the best response. Note that, if $\beta = 0$, the payoff function becomes simply

$$V(x;\bar{x}) = \sum_{i=1}^{2} \left\{ F(\bar{x}_i) x_i - x_i^2 / 2 \right\}.$$

In this case, the player chooses her intensity levels in the two activities independently, and hence one can analyze this two-dimensional game as two separate one-dimensional games. On the other hand, if $\beta = 1$ (although this violates the strict convexity), then

$$V(x;\bar{x}) = F(\bar{x}_1) x_1 + F(\bar{x}_2) x_2 - \left(\frac{x_1 + x_2}{2} \right)^2$$

In this case, each player's efforts in the two activities are perfect substitutes. By controlling $\beta \in [0, 1)$, this specification allows us to look at the whole range of intermediate cases between these two extremes.[4] Note that the two complementarity games are linked only through the cost function of each player; there exists no direct strategic dependence across the games. (This assumption is necessary to capture the notion that players are participating in "different" complementarity games simultaneously.) Nevertheless, the linkage through the player's cost functions makes it necessary to solve for the two complementarity games simultaneously.

It should also be noted that the game is symmetric across the two activities. Hence, the existence of a symmetric equilibrium, where $x_1{}^* = x_2{}^*$ holds, should be expected. It is not obvious, however, whether asymmetric equilibria ($x_1{}^* \neq x_2{}^*$) exist. (Note that, because the game is symmetric across the

[4] Brief mention should be made for the case, $\beta < 0$. In this case, the cost function exhibits the economies of scope and the player's choice of intensity levels across two activities become (nonstrategic) complements to each other. Not only such a situation has been analyzed extensively in the literature (see the studies referred in the introduction), but also $\beta < 0$ makes this game one two-dimensional complementarity game, whose property is very similar to one-dimensional complementarity game, discussed in the previous section. The analysis below is restricted to the case, $\beta > 0$, because the purpose of this paper is to analyze a two-dimensional game, which captures the situation, where the players allocate their effort across two one-dimensional complementarity games.

two activities, the set of equilibria is symmetric. Hence, if asymmetric equilibria exist, they always appear in pairs.)

The best response of the player is characterized by the two first-order conditions for the payoff maximization,

$$\left(1-\frac{\beta}{2}\right)x_1+\frac{\beta}{2}x_2=F(\overline{x_1}) \qquad \text{and} \qquad \frac{\beta}{2}x_1+\left(1-\frac{\beta}{2}\right)x_2=F(\overline{x_2}),$$

or by the two-dimensional best response map, Ψ: $I^2 \to I^2$,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x = \Psi(\bar{x}) = \begin{bmatrix} \Psi_1(\bar{x}) \\ \Psi_2(\bar{x}) \end{bmatrix} \equiv \frac{1}{1-\beta}\begin{bmatrix} 1-\beta/2 & -\beta/2 \\ -\beta/2 & 1-\beta/2 \end{bmatrix}\begin{bmatrix} F(\overline{x_1}) \\ F(\overline{x_2}) \end{bmatrix}$$

The equilibria of this game are the fixed points of Ψ: $E=\left\{x \in I^2 \middle| x=\Psi(x)\right\}$.

As for the stability, we consider the dynamical system, defined by the gradient method:

$$\text{(D2)} \qquad \dot{x}_t = gradV(x_t;x_t) = \begin{bmatrix} F(x_{1t}) \\ F(x_{2t}) \end{bmatrix} - \begin{bmatrix} 1-\beta/2 & \beta/2 \\ \beta/2 & 1-\beta/2 \end{bmatrix} x_t$$

As in one-dimensional case, the choice of this particular tâtonnement process is not essential.[5]

Again, $x = x^*$ is an equilibrium of this game if and only if it is a stationary point of the dynamical system (D2). The local stability of an equilibrium can be examined by linearizing (D2) around the equilibrium and calculating the eigenvalues of its Jacobian matrix,

$$J(x^*) \equiv \frac{\partial}{\partial x} gradV(x^*;x^*) = \begin{bmatrix} F'(x_1^*)-(1-\beta/2) & -\beta/2 \\ -\beta/2 & F'(x_2^*)-(1-\beta/2) \end{bmatrix}$$

[5] For example, all the propositions in this paper hold true, when alternative processes, $\dot{x}_{it} = \varphi_i(\Psi_i(x_t)-x_{it})$, where φ_i: $\boldsymbol{R} \to \boldsymbol{R}$ is any smooth function satisfying $\varphi_i(0)=0$, and $\varphi_i' > 0$, are assumed.

The equilibrium is stable if both eigenvalues have negative real parts. If one of the eigenvalues has a positive real part, then the equilibrium is unstable.

Under (A2), $(x_1^*, x_2^*) = (1, 1)$ is always an equilibrium of the game and hence a stationary point of (D2). As in the case of the one-dimensional game, the stability of this equilibrium is of particular interest, because the presence of an unstable equilibrium suggests multiple stable equilibria. Unlike in the case of the one-dimensional game, however, one needs to pay particular attention to the direction in which the symmetric equilibrium $(x_1^*, x_2^*) = (1, 1)$ loses its stability. If it becomes unstable in such a way that the local dynamics around it magnify $|(x_1 - 1) - (x_2 - 1)| = |x_1 - x_2|$, then it suggests the appearance of a pair of asymmetric stable equilibria, which helps to explain the diversity. On the other hand, the local dynamics around the unstable symmetric equilibrium may magnify $|(x_1 - 1) + (x_2 - 1)|$, which suggests the appearance of a pair of symmetric stable equilibria. Recall that the multiplicity of symmetric equilibria does not help to explain the diversity. It merely weakens the predictive content of the theory.

Using $F'(1) \equiv \theta$, the Jacobian matrix at the symmetric equilibrium $(x_1^*, x_2^*) = (1, 1)$ is

$$J(x^*) = \begin{bmatrix} \theta - (1 - \beta/2) & -\beta/2 \\ -\beta/2 & \theta - (1 - \beta/2) \end{bmatrix},$$

and its two eigenvalues are given by $\lambda_1 = \theta - 1 < \lambda_2 = \theta - 1 + \beta$, with the associated eigenvectors, $(1, 1)$, and $(1, -1)$, respectively. We thus have the following proposition.

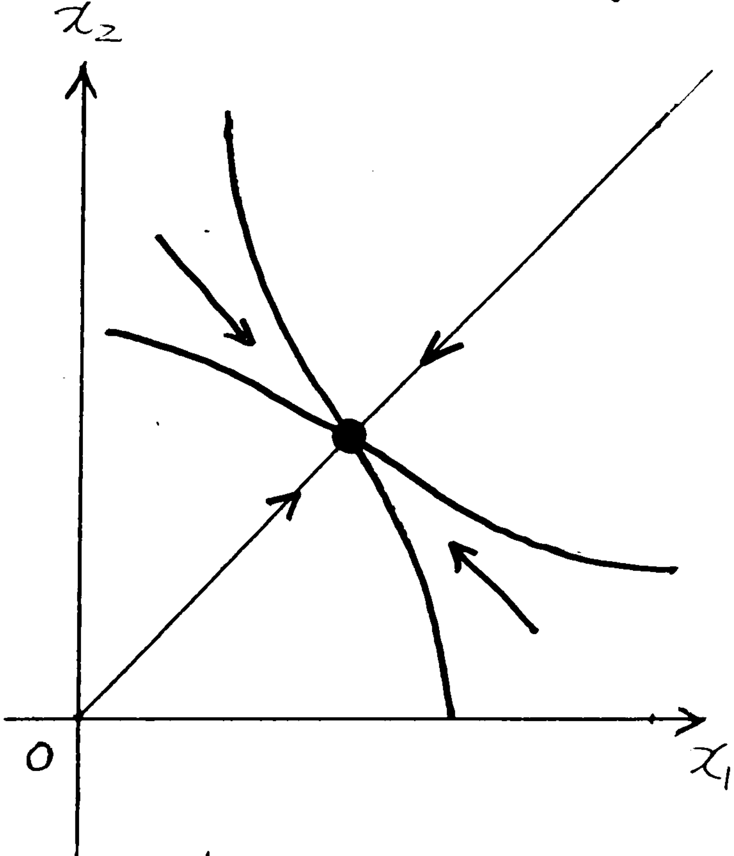
<u>Proposition 1</u>. *Suppose (A2). Then, $(x_1^*, x_2^*) = (1, 1)$ is a symmetric equilibrium of the game. Furthermore,*

i) if $\theta < 1 - \beta$, the Jacobian matrix has two negative eigenvalues; $\lambda_1 < \lambda_2 < 0$. Hence, $(x_1^*, x_2^*) = (1, 1)$ is a sink of (D2).

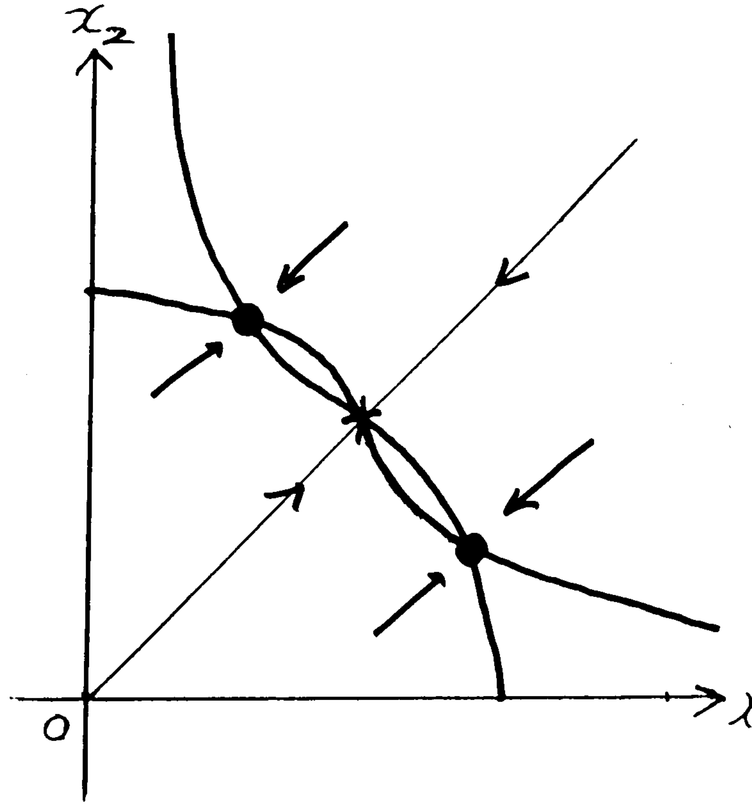
ii) if $1 - \beta < \theta < 1$, the Jacobian matrix has one positive and one negative eigenvalues; $\lambda_1 < 0 < \lambda_2$. Hence, $(x_1^*, x_2^*) = (1, 1)$ is a saddle of (D2), whose one-dimensional unstable manifold is tangential to the hyperplane, $x_1 + x_2 = 1$, and orthogonal to the 45-degree line, $x_1 = x_2$.

iii) if $\theta > 1$, the Jacobian matrix has two positive eigenvalues; $0 < \lambda_1 < \lambda_2$. Hence, $(x_1^*, x_2^*) = (1, 1)$ is a source of (D2).

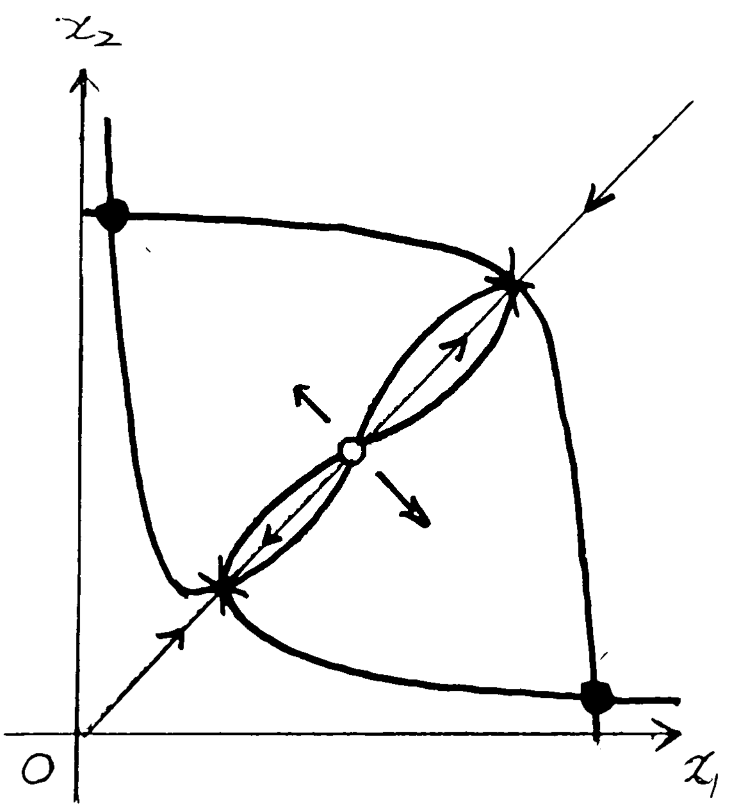
Fig. 2A - 2D.



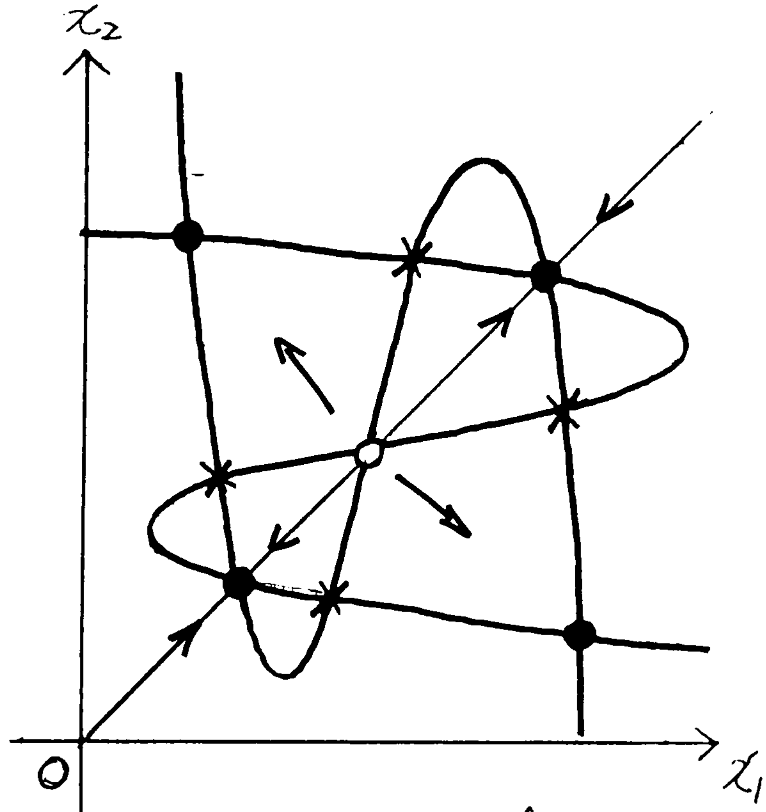
A: $0 < \theta < 1-\beta$



B: $1-\beta < \theta < 1$



C: $1 < \theta < 1+\frac{\beta}{2}$



D: $\theta > 1+\frac{\beta}{2}$

Proposition 1 states that $(x_1^*, x_2^*) = (1, 1)$ loses its stability when $\theta > 1-\beta$. The degree of complementarity necessary is smaller than in the case where the two activities are independent. Furthermore, when $\theta < 1$, $(x_1^*, x_2^*) = (1, 1)$ remains stable along the 45-degree line, and the instability around $(x_1^*, x_2^*) = (1, 1)$ causes x_1 and x_2 to diverge. In other words, for an intermediate range of complementarity, $1-\beta < \theta < 1$, complementarity is strong enough to generate multiple asymmetric equilibria, but weak enough that it does not generate multiple symmetric equilibria. Note that such an intermediate range of complementarity would not exist, if there were no interaction, i.e., $\beta = 0$.

To illustrate the above point further, let us consider the following example, which is simple enough to permit the characterization of the global structure of the equilibrium set.

$$\text{(A3)} \qquad F(x) = 1 + \theta(x-1) - \mu(x-1)^3, \qquad (\theta, \mu > 0; \mu > \theta - 1)$$

which satisfies (A1) and (A2), and $F'(1) \equiv \theta$. Figures 2A-2D depict four generic cases and illustrate how the dynamical system (D2) and its set of equilibria change as the complementarity parameter θ gets larger. In these figures, the equilibria are depicted by black dots (when they are sinks), crosses (when they are saddles) or circles (when they are sources).

In Figure 2A ($\theta < 1-\beta$), $(x_1^*, x_2^*) = (1, 1)$ is a unique equilibrium and it is stable. As the complementarity becomes stronger ($1-\beta < \theta < 1$), $(x_1^*, x_2^*) = (1, 1)$ loses its stability and becomes a saddle, as shown in Figure 2B. The local dynamics around $(x_1^*, x_2^*) = (1, 1)$ is unstable in the direction of $(1, -1)$, so that it tends to magnify $|x_1 - x_2|$. This bifurcation generates a pair of stable asymmetric equilibria:

$$(x_1^*, x_2^*) = (1 \pm \sqrt{(\theta - 1 + \beta)/\mu},\ 1 \mp \sqrt{(\theta - 1 + \beta)/\mu}),$$

whose two eigenvalues, $\lambda_1 = -2(\theta - 1 + \beta) - \beta < \lambda_2 = -2(\theta - 1 + \beta) < 0$, are both negative. It should be noted, however, that the local dynamics around $(x_1^*, x_2^*) = (1, 1)$ remains stable in the direction of $(1, 1)$, or along the 45 degree line, and there is no symmetric equilibrium other than $(x_1^*, x_2^*) = (1, 1)$. Hence, in this case, the pair of asymmetric equilibria is the only stable equilibria.

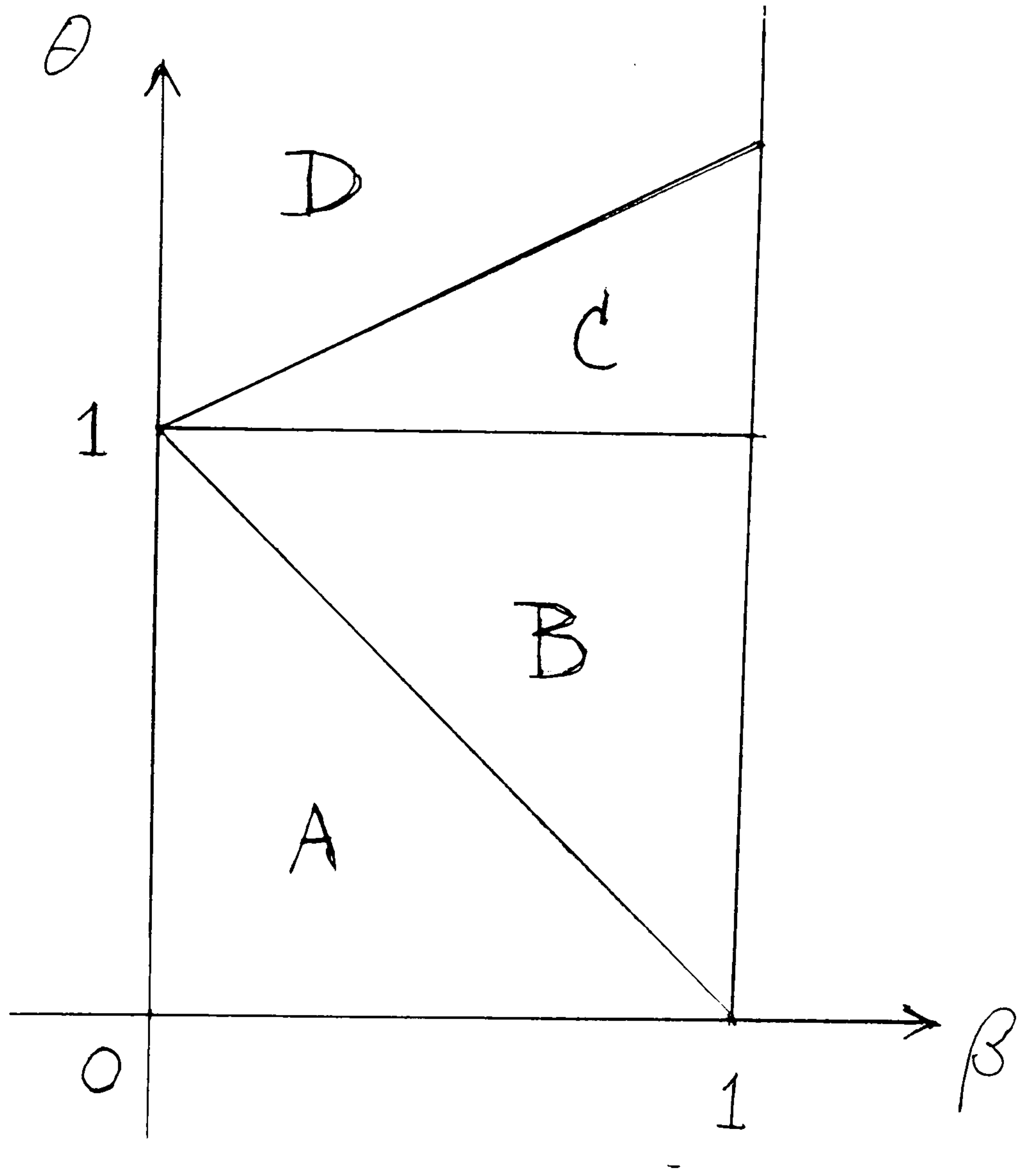


Fig 2E

As the complementarity gets even stronger ($1 < \theta < 1 + \beta/2$), $(x_1^*, x_2^*) = (1, 1)$ now becomes a source (see Figure 2C). This bifurcation generates a pair of symmetric (meaning $x_1 = x_2$) equilibria,

$$(x_1^*, x_2^*) = (1 \pm \sqrt{(\theta - 1)/\mu},\ 1 \pm \sqrt{(\theta - 1)/\mu}),$$

which are both saddles. Finally, as $\theta > 1 + \beta/2$, these two symmetric equilibria become stable, and this bifurcation generates four more asymmetric equilibria, all of which are saddles, as shown in Figure 2D. In the presence of such a strong complementarity, there are now four stable equilibria, only two of which are asymmetric. Figure 2E summarizes the results.[6]

Note that the prediction of the complementarity game, as a theory of endogenous diversity is most powerful in Case B. It is the weakest in the case depicted in Case D, in the sense that there are other stable equilibria, which do not predict the diversity. Furthermore, Case B does not necessarily require the presence of large complementarities. Case B can occur for an arbitrarily small θ, if the interdependence across the two activities is sufficiently high. Note that if one assumes no interaction, $\beta = 0$, only the cases qualitatively similar to Case A and Case D are possible. (Note, in particular, the similarity of Figure 1B and Figure 2D.) If $\theta < 1$, the complementarity is so weak that the equilibrium is unique. If $\theta > 1$, the complementarity is so strong that there are too many stable equilibria, both symmetric and asymmetric.

4. Applications

Before proceeding to higher-dimensional cases, this section discusses the implications of the results obtained in the previous section in two macroeconomic contexts.

4A. Globalization and Uneven Development

[6] If, instead of (A3), $F(x) = 1 + \theta(x-1) + \gamma(x-1)^2 - \mu(x-1)^3$, there will be another region between C and D, where there are three symmetric equilibria, one source, one saddle, and one sink.

Models with multiple equilibria offer a natural framework for explaining the diversity of economic performance across countries. In a model with a unique equilibrium, any attempt to explain the variations of per capita income across regions forces us to introduce variations in other variables, such as saving rates and education, as is commonly done in growth accounting exercises. Yet the variations in these variables themselves are left unexplained or need to be explained by introducing variations in another set of variables. This is not to deny the importance of growth accounting exercises as a useful way of summarizing the correlation of key variables across the regions. But they tell us little about why poor countries remain poor, because of the simultaneity of the key variables. On the other hand, the economics of macroeconomic complementarities, as a model with multiple equilibria, can explain the diversity across economies without assuming inherent differences, and hence they serve as a theory of endogenous inequality. Murphy, Shleifer and Vishny (1989) is probably the most influential work along this line.

One major drawback of models with multiple equilibria, as the critic might argue, is that they often seem to allow many possibilities for equilibrium behavior and hence to have little predictive content. Many previous studies on strategic complementarities in the area of economic development is subject to such a criticism. For example, Murphy, Shleifer and Vishny developed a closed economy model with multiple equilibria, and argued that the Rich countries somehow manage to achieve a Pareto-superior equilibrium, while the Poor countries fail to achieve a necessary coordination and are trapped in a Pareto-inferior equilibrium. While insightful, it is difficult to see within their framework why some countries have to be trapped in the bad state, while others find themselves in the good state, since the coordination in the good state is no more difficult than that in the bad state. This approach offers no compelling reason for which we have observed and continue to observe huge cross-country differences. As long as we treat each economy as a closed, isolated, independent entity, models cannot tell us anything about the degree of inequalities in the world economy. As shown in Figure 2E, the case where $\beta = 0$, i.e., two economies do not interact, is an extreme case of D, where the equilibrium condition imposes no restriction on the degree of inequality observed.

Furthermore, this approach gives one a false impression that the closedness of the national economy is responsible for the inequality among nations, and that one needs a significant degree of complementarity within each economy. Murphy, Shleifer and Vishny (1989, sec. II) indeed emphasized the closed economy assumption in their models for generating an underdevelopment trap, and discussed at length the empirical importance of the domestic demand spillover, which can be translated as a large θ in the present framework.

Case B shown in Figure 2E, however, suggests that neither the closed economy assumption ($\beta = 0$) nor the presence of a large degree of complementarities (a large θ) are essential for the uneven development. On the contrary, the greater interdependence between the two economies makes the inequality more likely. Even for an arbitrarily small degree of complementarity, one economy must be trapped into a lower state than the other economy, when β gets sufficiently close to one. Furthermore, this approach explains the inevitability, not just the possibility, of the uneven development. It also suggests a drastically different policy implication; a globalization, by increasing β, can cause inequality, rather than eliminating it.

Several authors have pursued this approach of analyzing the problem of uneven development from a global perspective. Among them, Grossman and Helpman (1992, Ch.8), Krugman (1981), Matsuyama (1992, 1996), Young (1991) demonstrated that international trade causes the inequality of nations in the presence of (arbitrarily small) country-specific externalities. In a model of imperfect capital markets, where higher initial wealth alleviates the moral-hazard costs, Gertler and Rogoff (1990) showed that international capital mobility leads to capital flow from poor to rich countries, thereby amplifying the inequality. In models of Krugman and Venables (1995) and Matsuyama (1998), international trade is costly, so that, as in Figure 2E, the openness is parameterized from zero to one, depending on the transport cost. Krugman and Venables showed that the inequality is created below the critical level of transport cost. In Matsuyama, geographical asymmetry across regions, however small, is magnified to create a bigger inequality, as the transport cost goes down.

4B. Intertemporal Substitution and Business Cycles

Models of multiple equilibria offer a natural framework for explaining business cycles. In a model with a unique equilibrium, economic fluctuation over time has to be driven by some exogenous shocks; in the real business cycle theory, they are aggregate productivity shocks. While useful for understanding the propagation mechanisms, this approach cannot possibly answer questions like, why are there booms and recessions? or are there any role of automatic stabilization policies? etc. One major advantage of the economics of macroeconomic complementarities is that it can generate business cycles without assuming any exogenous shocks. Cooper and John (1988) is probably the most influential work along this line. Any static model of strategic complementarities with multiple equilibria can easily extended to multi-period models of endogenous business cycles, by replicating the static model over time, and letting the players in the economy play different equilibria in different periods. A change in the behavior over time is coordinated by "sunspots," a random variable with no intrinsic effect on the economy. In this approach, sunspot fluctuations causes economic fluctuations, not because they affect some fundamentals of the economy, but because the players believe that they matter, or due to self-fulfilling prophecies.

One common criticism against models with multiple equilibria as an explanation of endogenous business cycles, is that they often seem to allow many possibilities for equilibrium behavior and hence to have little predictive content. The "sunspot" theory of business cycles is subject to such a criticism. Random realization of sunspots causes business cycles only in the subset of equilibria of the model; cyclical behaviors are not inevitable. There are other equilibria, including the one in which the sunspot variable is ignored by the players: the belief that sunspots are irrelevant is also self-fulfilling. In such an equilibrium, the players play the same equilibrium in each period, and there are no fluctuations. As long as we treat different periods independently ($\beta = 0$), the equilibrium condition imposes no restriction on the volatility of the economy.

One way of responding to such a criticism is to introduce intertemporal substitution in such a way that an increase in the level of activities in one period induces a decline in the level of activities in another. Not only can the presence of large intertemporal substitution (a large β) cause an instability of the no-fluctuation equilibrium and lead to cyclical behaviors. It also helps to generate fluctuations with a much smaller degree of intratemporal complementarities (a small θ), as suggested by Case B in Figure 2E.

This is exactly the approach taken by many recent studies of endogenous cycles. In Diamond and Fudenberg (1989), business cycles are generated not only by intratemporal complementarities due to search externalities, but also by the restriction that the players must alternate between production and search activities. The players choose the timing of innovation in Deneckere and Judd (1992) and Matsuyama (1998), or the timing of implementation in Gale (1996) and Shleifer (1986), and with some strategic complementarities, this feature leads to a synchronization of these activities. The presence of inventories, as in Cooper and Haltiwanger (1992), of intertemporal substitution of labor, as in Hall (1991), or of durable goods, as in Murphy, Shleifer and Vishny (1990), can lead to a production bunching with small complementarities within each period.

Case B not only offers a sharper prediction than Case D, as a theory of endogenous business cycles. Policy implications differ significantly. In Case D, the recession, i.e., the period in which the economy operates at a lower level, is a waste, and can be avoided. An attempt to eliminate the recession is thus desirable and feasible. In Case B, the recession is just one phase of inevitable business fluctuations, which may not be a waste in that sense that the recession might be a necessary stage for booms in the future. An attempt to eliminate the recession may not only be counterproductive. It may simply end up shifting the recession from one period to another.

It has been argued that one of the drawbacks of the real business cycle theory is that it has to rely on large intertemporal substitution. The above discussion suggests that, in this respect, the two theories of business cycles are very similar. Intertemporal substitution also helps the theory of business cycles based on strategic complementarities. Any model specification that helps the real business cycle theory along this dimension, such as nonseparable preferences over leisure, proposed by Kydland and Prescott (1982), should also helps the business cycle theory based on strategic complementarities.

5. Playing N Complementarity Games Simultaneously.

Let us now consider the case, where players participate in $N \geq 3$ complementarity games, and see to what extent that the result obtained in the two-dimensional case can carry over to a higher dimension. A continuum of identical players are engaged in N activities and each controls $x^T = (x_1, x_2, \ldots, x_N) \in I^N = [0, \infty)^N$. The payoff function has the following form:

$$V(x, \bar{x}) = \sum_{i=1}^{N} F(\overline{x_i}) x_i - \frac{1}{2} x^T A x$$

The second term in the payoff function represents the quadratic cost function, and the matrix $A = \left[a_{ij} \right]$ can be taken to be symmetric (i.e., $a_{ij} = a_{ji}$ for all i and j) without loss of generality. Furthermore, A is assumed to be:

(P1) Positive-Definite (to ensure the strict convexity of the cost function and the uniqueness of the best response),

(P2) Circulant (i.e., a_{ij} depends only on $|i-j|$, after relabeling the activities, if necessary. This imposes the symmetry across activities.)

(P3) Nonnegative (i.e., $a_{ij} \geq 0$ for all i and j, which captures the notion that different activities compete for the player's effort).

Note that (P2) implies $\sum_j a_{ij}$ is independent of i, which can be made equal to one by an appropriate choice of the intensity unit. In other words, (P1) through (P3) implies that A is a positive-definite, Markov matrix.

Furthermore, to make this multidimensional extension meaningful, it is assumed that all the activities are at least indirectly connected in the player's cost function.

(P4). For some integer $R > 0$. A^R is a positive matrix. (i.e., for any i and j, there is a sequence , $i = k_1$, $k_2, \ldots, k_R = j$, such that $a_{k_r k_{r+1}} > 0$ for all $r = 1, \ldots, R-1$.

Note that the positive-definiteness implies that all eigenvalues are positive real numbers. Furthermore, the largest eigenvalue is one, because A is a Markov. Hence, N eigenvalues of A can be denoted by $0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_N = 1$. Furthermore, from the first Frobenius theorem of nonnegative matrices, (P4) implies that $0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{N-1} < \alpha_N = 1$. Note that, if the player can chose intensity levels in the N-activities independently, then $A = \boldsymbol{I}_N$, where $\boldsymbol{I}_N$ is the NxN identity matrix, which satisfies (P1), (P2), and (P3). However, it does not satisfy (P4), and $\alpha_1 = \alpha_2 = \ldots = \alpha_{N-1} = \alpha_N = 1$.

The best response of the player is characterized by the first-order condition for the payoff maximization, $\left[F(\bar{x}_i)\right] = Ax$ or by the N-dimensional best response map, $\Psi: I^N \rightarrow I^N$:

$$x = \Psi(\bar{x}) \equiv A^{-1}\left[F(\bar{x}_i)\right].$$

The equilibria of this game are the fixed points of Ψ: $E = \left\{x \in I^N \middle| x = \Psi(x)\right\}$.

As before, consider the dynamical system defined by the gradient method in order to evaluate the stability of equilibria;

$$\text{(DN)} \qquad \dot{x}_t = gradV(x_t; x_t) = \left[F(x_{it})\right] - Ax_t.$$

As in the previous cases, this choice of tâtonnement process is not essential for the results.

Again, $x = x^*$ is an equilibrium of this game if and only if it is a stationary point of the dynamical system (DN). The local stability of an equilibrium depends on the signs of the eigenvalues of its Jacobian matrix,

$$J(x^*) \equiv \frac{\partial}{\partial x} gradV(x^*; x^*) = \left[F'(x_i^*)\delta_{ij}\right] - A$$

where δ_{ij} is Kronecker's delta (i.e., $\delta_{ij} = 1$ if $i = j$; $\delta_{ij} = 0$ if $i \neq j$). The equilibrium is stable if all the eigenvalues have negative real parts. If one of the eigenvalues has a positive real part, then the equilibrium is unstable.

Under (A2), $x^{*T} = (x_1^*, x_2^*, \ldots, x_N^*) = (1, 1, \ldots, 1)$ is always an equilibrium of this game. The question is then, when this symmetric equilibrium loses its stability in such a way to generate multiple asymmetric equilibria without generating multiple symmetric equilibria.

Using $F'(1) \equiv \theta$, the Jacobian matrix at the symmetric equilibrium, $(x_1^*, x_2^*, \ldots, x_N^*) = (1, 1, \ldots, 1)$, is equal to $J = \theta \boldsymbol{I}_N - A$, where $\boldsymbol{I}_N$ is the NxN identity matrix. The eigenvalues of the Jacobian matrix are given by $\lambda_i = \theta - \alpha_i$. Hence, the equilibrium is unstable if θ is strictly greater than the smallest eigenvalue of A, that is, if $\theta > \alpha_1$, and the equilibrium is a source if $\theta > 1$.

Proposition 2. *Suppose (A2). Then,* $(x_1^*, x_2^*, \ldots, x_N^*) = (1, 1, \ldots, 1)$ *is a symmetric equilibrium of this game. Furthermore,*

i) if $\theta < \alpha_1$, *the eigenvalues of the Jacobian matrix,* $J = \theta \boldsymbol{I}_N - A$, *are all negative. Hence,* $(x_1^*, x_2^*, \ldots, x_N^*) = (1, 1, \ldots, 1)$ *is a sink of (DN).*

ii) if $\alpha_1 < \theta < 1$, *the Jacobian matrix,* $J = \theta \boldsymbol{I}_N - A$, *has at least one negative and one positive eigenvalues. Hence,* $(x_1^*, x_2^*, \ldots, x_N^*) = (1, 1, \ldots, 1)$ *is a saddle of (DN), whose unstable manifold is orthogonal to the line,* $x_1 = x_2 = \ldots = x_N$. *If* $\alpha_{N-1} < \theta < 1$, *then the unstable manifold is (N−1)-dimensional and tangent to the hyperplane,* $x_1 + x_2 + \ldots + x_N = N$.

iii) if $\theta > 1$, *the eigenvalues of the Jacobian matrix,* $J = \theta \boldsymbol{I}_N - A$, *are all positive. Hence,* $(x_1^*, x_2^*, \ldots, x_N^*) = (1, 1, \ldots, 1)$ *is a source of (DN).*

The proof is straightforward, hence omitted. The first part of ii) states that, if there is any interdependence between the choices of the player's intensity across activities, the degree of complementarity necessary for the symmetric equilibrium to lose its stability is less than in the case of independence. The second part of ii) states that, if all the activities are at least indirectly connected in the player's cost function, then there exists a range of strategic complementarities, where the unstable local dynamics around the symmetric equilibrium causes $x_1, x_2, \ldots, x_N$ to diverge, without generating the multiple symmetric equilibria.

To explore further, let us now consider some specific examples.

Example 1: For the three-dimensional case, note that all the 3x3 matrices satisfying (P1), (P2), and (P3) can be represented by

$$A = \begin{bmatrix} 1-2\beta/3 & \beta/3 & \beta/3 \\ \beta/3 & 1-2\beta/3 & \beta/3 \\ \beta/3 & \beta/3 & 1-2\beta/3 \end{bmatrix}$$

for $\beta \in [0, 1)$. If $\beta = 0$, $A = \boldsymbol{I}_3$, and hence the three activities are not connected. If $\beta \in (0, 1)$, (P4) is satisfied for $R = 1$. The eigenvalues of A are $0 < \alpha_1 = \alpha_2 = 1 - \beta < \alpha_3 = 1$.

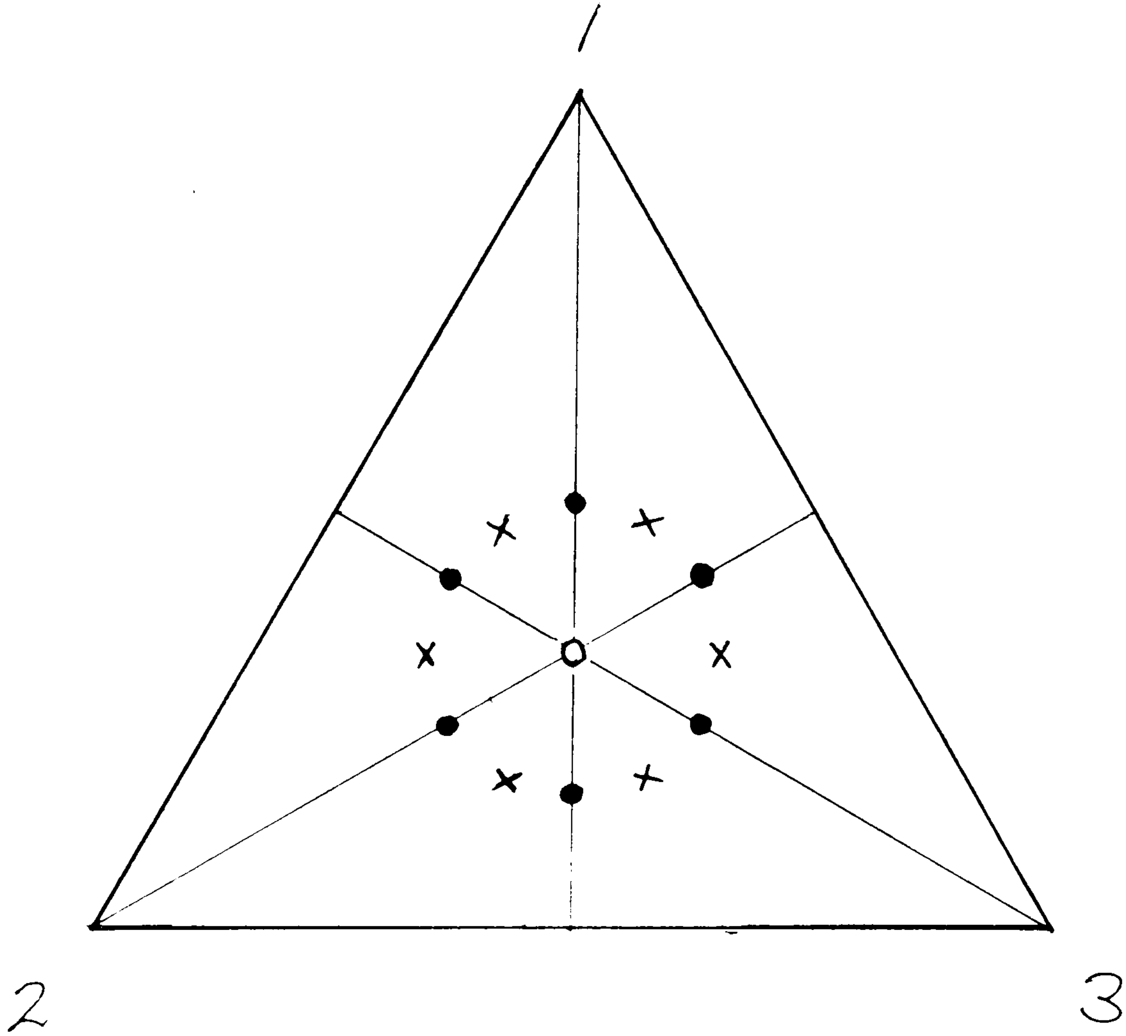


Figure 3

Figure 3 illustrates this example under (A3). When $\theta < 1-\beta$, the symmetric equilibrium, $(x_1*, x_2*, x_3*) = (1, 1, 1)$, is the unique, stable equilibrium. As $1-\beta < \theta < 1$, it loses its stability and becomes a saddle (with one negative and two positive eigenvalues). This bifurcation generates a 2-dimensional unstable manifold tangent to $x_1 + x_2 + x_3 = 3$. The local dynamics remains stable along the ray $x_1 = x_2 = x_3$. As the result of the bifurcation, there appear six unstable asymmetric equilibria (the saddles with two negative and one positive eigenvalues) and six stable asymmetric equilibria. (Indeed, from the logic of the bifurcation analysis, it is straightforward to show that (A3) is not restrictive, as long as F is sufficiently smooth around $x = 1$ and $\theta - (1-\beta)$ is sufficiently small.) Again, for an arbitrary small $\theta > 0$, the diversity appears if β is sufficiently close to one.

Example 2: The straightforward extension of the above example for an arbitrary N is given by

$$A = \left[\frac{\beta}{N} + (1-\beta)\delta_{ij}\right]$$

for $\beta \in [0, 1)$. If $\beta = 0$, $A = I_N$, and hence no activities is not connected to others. If $\beta \in (0, 1)$, (P4) is satisfied for $R = 1$. The eigenvalues of A are $0 < \alpha_1 = \alpha_2 = \ldots = \alpha_{N-1} = 1-\beta < \alpha_N = 1$.

In both Examples 1 and 2, $\alpha_1 = \alpha_2 = \ldots = \alpha_{N-1}$. Hence, when the symmetric equilibrium loses its stability, it immediately generates a $(N-1)$-dimensional unstable manifold tangential to $x_1 + x_2 + \ldots + x_N = N$ The following example shows the case where the unstable manifold can has the dimensionality less than $N-1$.

In the above example, there exists the same degree of interdependence across any pair of activities. For some economic applications, this may be too restrictive. The following example allows the possibility, where some activities are closer than others, hence there is a higher degree of interdependence.

Example 3: Consider a 4-dimensional case, where

$$A = \begin{bmatrix} 1-\beta/2 & \beta/4 & 0 & \beta/4 \\ \beta/4 & 1-\beta/2 & \beta/4 & 0 \\ 0 & \beta/4 & 1-\beta/2 & \beta/4 \\ \beta/4 & 0 & \beta/4 & 1-\beta/2 \end{bmatrix}$$

for $\beta \in [0, 1)$, which satisfies (P1), (P2), and (P3). This case may capture the situation, where four activities represents activities in four different regional economies, and the regions are located along the circle. Then, Region 1 is the neighbor of Regions 2 and 4, but not of Region 3.

If $\beta = 0$, $A = \boldsymbol{I_4}$. If $\beta \in (0, 1)$, (P4) is satisfied for $R = 2$. Thus, Activities 1 and 3 are indirectly connected via Activities 2 and 4. The eigenvalues of A are given by $0 < \alpha_1 = 1-\beta \leq \alpha_2 = \alpha_3 = 1-\beta/2 < \alpha_4 = 1$. In this case, if $1-\beta < \theta < 1-\beta/2$, then the local dynamics around the symmetric equilibrium causes x_1 to move away from x_2 and x_4, but it moves along with x_3.

For some applications, (P2) may be too stringent. For example, in temporal interpretation, different activities are located along the time line, and hence it is reasonable to assume not only that there are more interdependence across adjacent periods, but also that there are the first and the last periods, which violates the symmetry assumption implied by (P2). In spatial interpretation of the game, too, (P2) is violated if different regional economies are located along the coastline, rather than along the globe.

To accommodate these cases, let us remove (P2) and (P3) and instead assume directly that A is a Markov matrix. This does not affect Proposition 2. However, it should be pointed out that, because there is no underlying symmetry to justify that $\sum_j a_{ij}$ is independent of i, the assumption, $\sum_j a_{ij} = 1$, is no longer just a matter of normalization, and it is restrictive. This assumption should be now regarded as a matter of convenience, as a way of ensuring the existence of the symmetric equilibrium, $(x_1^*, x_2^*, \ldots, x_N^*) = (1, 1, \ldots, 1)$, a useful benchmark to discuss asymmetric equilibria.

The following example illustrates Proposition 2 under this alternative assumption.

<u>Example 4</u>: Consider a 3-dimensional case, where

$$A = \begin{bmatrix} 1-\beta/3 & \beta/3 & 0 \\ \beta/3 & 1-2\beta/3 & \beta/3 \\ 0 & \beta/3 & 1-\beta/3 \end{bmatrix}$$

For $\beta \in [0, 1)$, A satisfies (P1). It is a Markov matrix, but does not satisfy (P2). If $\beta = 0$, $A = \boldsymbol{I}_3$, and hence no activities is not connected to others. If $\beta \in (0, 1)$, both Activity 1 and Activity 3 are indirectly connected through Activity 2; (P4) holds for $R = 2$. The eigenvalues of A are $0 < \alpha_1 = 1-\beta < \alpha_2 = 1-\beta/3 < \alpha_3 = 1$. When $\theta < 1-\beta$, the symmetric equilibrium, $(x_1^*, x_2^*, x_3^*) = (1, 1, 1)$, is the unique, stable equilibrium. As $1-\beta < \theta < 1-\beta/3$, it loses its stability and becomes a saddle (with one positive and two negative eigenvalues). This bifurcation generates a 1-dimensional unstable manifold to the symmetric equilibrium, which is tangent to the vector (1,–2,1) and orthogonal to (1,1,1). The dynamics magnifies $|x_1 + x_3 - 2x_2|$ along this manifold. As the result of the bifurcation, there appear two stable asymmetric equilibria, one of which satisfies $x_1^* = x_3^* > 1 > x_2^*$ and the other, $x_1^* = x_3^* < 1 < x_2^*$. As θ becomes larger than $1-\beta/3$, the symmetric equilibrium bifurcates again, but remain a saddle (with two positive and one negative eigenvalues). It now has a 2-dimensional unstable manifold, which is tangent to the line, $x_1 = x_2 = x_3$. This bifurcation generates another pair of asymmetric equilibria on this manifold, which are also saddles (with one positive and two negative eigenvalues). As long as $\theta < 1$, however, the line $x_1 = x_2 = x_3$, remains the stable manifold of the symmetric equilibrium, $(1, 1, 1)$, and hence there exists no other symmetric equilibrium. If θ becomes larger than 1, another bifurcation turns the symmetric equilibrium, $(1, 1, 1)$, from a saddle to a source, and generates a pair of stable symmetric equilibria, which co-exist with stable asymmetric equilibria.

The above results can be given the following temporal interpretation. This three-period economy has always a stationary equilibrium. When $\theta < 1-\beta$, it is the only equilibrium and stable. If $1-\beta < \theta < 1$, the stationary equilibrium loses its stability, and there appear two stable nonstationary equilibria. In one, the economy is in boom during periods 1 and 3 and in recession during period 2. In the other, the economy is in recession during periods 1 and 3 and in boom in period 2. In this case, the model predicts business cycles as the only stable equilibrium outcomes. If $\theta > 1$, there are two additional equilibrium outcomes. In one, the economy is always in boom. In the other, the economy is always in recession.

Note that the condition, $1-\beta < \theta$, suggests that, for an arbitrary small degree of complementarity within periods, there exist business cycle (nonstationary) equilibria, if the intertemporal substitution is sufficiently large. Also, if there were no intertemporal substitution, $\beta = 0$, the interval, $(1-\beta, 1)$ becomes empty. Thus, the degree of complementarity required to generate business cycle equilibria would also be strong enough to generate multiple stationary equilibria (i.e., a permanent boom, and a permanent recession).

6. Concluding Remarks and Suggestions for Further Extensions

When presenting models of macroeconomic strategic complementarities, one of the frequently asked questions is, "Do you really think strategic complementarities are that important in reality?" The author's short response is "yes." The reason for this is *not* because the presence of a large strategic complementarity is empirically plausible. Rather, the reason is that the presence of an *even small* complementarity makes a big difference, when the players can choose to adjust their participation levels across different complementarity games. In this respect, presenting a complementarity game and analyzing it in isolation can be very misleading, because it gives a false impression that a large degree of strategic complementarities would be necessary. In the class of the games presented in this paper, the players participate in multiple complementarity games, and they can adjust their participation levels across these component games. The larger the freedom to adjust their participation levels across the games, the larger is the effect of a given degree of strategic complementarities. It has also be shown that the presence of a large degree of strategic complementarities weakens the prediction of the model as a theory of endogenous diversity.

Naturally there are many ways in which the above analysis can be extended. Three of them will be suggested below, and some conjectures as well as some potential difficulties associated with such extensions will be discussed.

6.A. *Multidimensional Complementarity Games.*

In the class of the games discussed in this paper, the players play multiple one-dimensional complementarity games. More generally, one may consider the situation, where the players play N complementarity games, and each complementarity game has a K-dimensional strategy space. The strategy space of the entire game then become NK-dimensional. We conjecture that many of the implications of playing multiple complementarity games (i.e., $N \geq 2$) discussed in this paper would survive in one form or another, even K is greater than one. The exact form in which these implications are expressed should, however, depend on how the games become interdependent in the player's cost function. To illustrate this, consider the case, where $N = K = 2$. The player's strategy is now given by $X = (x_1, x_2) = (y_1, z_1; y_2, z_2) \in \Gamma^4 = [0, \infty)^4$, where y and z denote two choice variables in each 2-dimensional complementarity game. Let these complementarity games have the following form of the payoff function, when they are played in isolation: $F(x_i, \overline{x_i}) - [(y_i)^2 + (z_i)^2]/2$. One way of introducing the cost dependence across games is to assume that the cost function

$$C(x_1, x_2; \beta) = \left(1 - \frac{\beta}{2}\right)\frac{(y_1)^2 + (z_1)^2 + (y_2)^2 + (z_2)^2}{2} + \frac{\beta}{2}\sqrt{\left((y_1)^2 + (z_1)^2\right)\left((y_2)^2 + (z_2)^2\right)}$$

This specification may be appropriate when the player has a single type of resources, which she would first allocate between the games, and then allocate between y and z. Alternatively, one could assume that

$$C(x_1, x_2; \beta) = \left(1 - \frac{\beta}{2}\right)\frac{(y_1)^2 + (z_1)^2 + (y_2)^2 + (z_2)^2}{2} + \frac{\beta}{2}(y_1 y_2 + z_1 z_2).$$

This specification may be appropriate when the player has two distinct types of resources, y and z, which cannot be transformed from one to the other, and the player decides to allocate these two types of resources between the games. And, of course, one could also consider a whole range of combinations between these two specifications. One potential challenge of extending the game into the case where the component complementarity game has a multidimensional strategic space is to develop a simple classification of these different types of interdependence.

6.B. *Strategic Complementarity Across Games.*

In the class of the games discussed in this paper, the interdependence between the complementarity games exists only through the player's cost function. In particular, no strategic complementarities exist across games. This assumption was adopted so as to make it precise the notion that players are participating in "different" complementarity games simultaneously. For some economic applications, however, this assumption may be restrictive. For example, recall a spatial interpretation of the game, where the international investors allocate their resources across the regional economies, and there are strategic complementarities within each regional economy. In reality, of course, some strategic complementarities should exist at least across neighboring economies. A higher aggregate investment in Singapore should increase the marginal return on investment in Malaysia, if not in Thailand.

One way of extending the game to capture this possibility would be to change the payoff function as follows:

$$V(x,\bar{x}) = \sum_{i=1}^{N} F(w_i \bar{x})x_i - \frac{1}{2}x^T Ax$$

where $w_i = (w_{i1}, w_{i2}, \ldots, w_{iN})$ is a circulant vector, whose *j*-th element represents the extent to which the average intensity level in activity *j* affects the return of increasing in intensity in activity *i*. With an appropriate parameterization of w_i, say δ, which captures the degree of spillover across the activities, one should be able to go through bifurcation analyses in terms of θ, β, and δ. For example, consider the case where $N = 2$, and

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1-\delta/2 & \delta/2 \\ \delta/2 & 1-\delta/2 \end{bmatrix}$$

where $\delta \in [0,1)$ so that the diagonal elements are always greater than the off-diagonal ones. As long as $\beta > \delta$, the prediction of the model remains qualitatively the same. That is, if $\theta < (1-\beta)/(1-\delta)$, the symmetric equilibrium, $(x_1^*, x_2^*) = (1, 1)$, is stable. If $(1-\beta)/(1-\delta) < \theta < 1$, it becomes a saddle, and this bifurcation generates a pair of asymmetric stable equilibria, and, if $\theta > 1$, it bifurcates again to become a source, which generates a pair of symmetric stable equilibria. Therefore, there exists an intermediate range of complementarity, which predicts asymmetry as the unique stable outcomes, and this interval exists only if there are cost dependence across the activities. Furthermore, the condition, $\beta > \delta$, is automatically satisfied when the cost dependence is sufficiently large.[7] We conjecture that similar results can be obtained for $N \geq 3$.

6C. *Playing A Continuum of Complementarity Games Simultaneously.*

[7] If $\beta < \delta$, then asymmetric stable equilibria, when they exist, must co-exist with symmetric stable equilibria.

Perhaps the most restrictive assumption of the game developed in this paper, when applied to some economic issues, is the discreteness of the games. One way of eliminating the discreteness is to allow for a continuum of complementarity games, indexed by $i, j \in \boldsymbol{R/Z}$, the unit circle, and let the player's strategy space be the set of nonnegative functions defined on the unit circle, x: $\boldsymbol{R/Z} \to \boldsymbol{R}_+$. The payoff function is now given by

$$V(x,\bar{x}) = \int F\left[\int w(i,j)\overline{x(j)}dj\right]x(i)di - \int\left[\int A(i,j)x(j)dj\right]x(i)di$$

where w: $\boldsymbol{R}^2/\boldsymbol{Z}^2 \to \boldsymbol{R}_+$, and A: $\boldsymbol{R}^2/\boldsymbol{Z}^2 \to \boldsymbol{R}_+$, are symmetric, positive-definite, nonnegative, circulant functions on the 2-dimensional torus, satisfying

$$\int w(i,j)dj = \int A(i,j)dj = 1,$$

where all the integrals are over the unit circle. Then, under (A2), a constant function, $x^* = 1$, is a symmetric equilibrium of this game. Furthermore, if A and w depreciate at the rate equal to β and δ, respectively, the local dynamics around the symmetric equilibrium, $x^* = 1$, is expressed in terms of θ, β, and δ. Therefore, at least in principle, one should be able to proceed with the bifurcation analysis to find out the condition under which asymmetric stable equilibria exist and no symmetric equilibrium is stable.

The author hopes to explore these and other extensions in the future work.

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