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The Mathematical Appendix to
RESIDENTIAL INVESTMENT AND THE CURRENT ACCOUNT*

by

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* Abstract :
This is the mathematical appendix to the article entitled "Residential Investment and the Current Account," Journal of International Economics, February 1990.

Appendix

This appendix solves the local dynamics of residential investment and the current account, using the Laplace transform techniques, popularized by Judd (1985, 1987a, 1987b). The Laplace transform method is powerful in solving a linear system of differential equations, especially when the variables are also subject to some integral constraints.

The Laplace transform of a measurable function y_t defined for positive t is another function Y(z) defined for sufficiently large positive z, where

$$\text{(A.1)} \qquad Y(z) = \mathcal{L}\{y_t\} = \int_0^{+\infty} y_t e^{-zt} dt \quad .$$

It is defined for any positive z, if y_t is bounded. It is a linear operator on measurable functions and satisfies,

$$\text{(A.2)} \qquad \mathcal{L}\{\dot{y}_t\} = zY(z) - y_0$$

$$\text{(A.3)} \qquad \lim_{z\to+\infty} zY(z) = y_0 \quad .$$

See Judd (1985) for some technical issues in detail.

Now, let us first rewrite the system (3a-d) to:

$$\text{(A.4)} \qquad \dot{h}_t = I(q_t;\gamma_t) - \delta h_t \qquad ; \qquad h_0\text{: given} \quad ,$$

$$\text{(A.5)} \qquad \dot{q}_t = (r+\delta)q_t - \Lambda(h_t;\mu) - \alpha_t \qquad ; \qquad \lim_{t\to+\infty} q_t e^{-rt} = 0 \quad ,$$

$$\text{(A.6)} \qquad \int_0^{+\infty} f(h_t;\mu)e^{-rt}dt = b_0 + \int_0^{+\infty} X(q_t;\gamma_t)e^{-rt}dt - \int_0^{+\infty} g_t e^{-rt}dt \quad ,$$

where $c = f(h;\mu)$ is implicitly defined by $u_1(c,h) = \mu$ and $\Lambda(h;\mu) \equiv u_2(f(h;\mu),h)/\mu$. They satisfy,

$$f_h = - u_{12}/u_{11} > - u_2/u_1 \quad , \quad f_\mu = 1/u_{11} < 0$$

$$\Lambda_h = \frac{u_{11}u_{22}-u_{12}u_{21}}{u_1 u_{11}} < 0 \quad , \quad \Lambda_\mu = \frac{u_{12}u_1-u_2u_{11}}{(u_1)^2 u_{11}} < 0 \quad ,$$

where use has been made of the normality assumption: $u_2/u_1 > u_{12}/u_{11}$

Assume that the system is originally in the steady state ($q_t = q_\infty$, $h_t = h_\infty$) and consider a linear perturbation, due to an infinitesimal change, $\partial\alpha_t$, $\partial\gamma_t$, and ∂g_t. Then, (A.4) and (A.5) becomes:

$$\text{(A.7)} \qquad \begin{bmatrix} \partial\dot{h}_t \\ \partial\dot{q}_t \end{bmatrix} = \begin{bmatrix} -\delta & I' \\ -\Lambda_h & r+\delta \end{bmatrix} \begin{bmatrix} \partial h_t \\ \partial q_t \end{bmatrix} + \begin{bmatrix} I_\gamma \partial\gamma_t \\ -\Lambda_\mu \partial\mu - \partial\alpha_t \end{bmatrix} \quad ,$$

where all derivatives are evaluated with steady state values. Let H(z), Q(z), Γ(z) and A(z) denote the Leplace transforms of ∂h_t, ∂q_t, $\partial\gamma_t$ and $\partial\alpha_t$, respectively. Using (A.2), the linearity property, and the initial condition on h, $\partial h_0 = 0$, (A.7) can be rewritten to,

$$\begin{bmatrix} zH(z) \\ zQ(z) - \partial q_0 \end{bmatrix} = \begin{bmatrix} -\delta & I' \\ -\Lambda_h & r+\delta \end{bmatrix} \begin{bmatrix} H(z) \\ Q(z) \end{bmatrix} + \begin{bmatrix} I_\gamma \Gamma(z) \\ -(\Lambda_\mu/z)\partial\mu - A(z) \end{bmatrix} \quad ,$$

or,

$$\text{(A.8)} \qquad \begin{bmatrix} H(z) \\ Q(z) \end{bmatrix} = \frac{1}{(z-\lambda^+)(z-\lambda^-)} \begin{bmatrix} z-r-\delta & I' \\ -\Lambda_h & z+\delta \end{bmatrix} \begin{bmatrix} I_\gamma \Gamma(z) \\ -(\Lambda_\mu/z)\partial\mu - A(z) + \partial q_0 \end{bmatrix} \quad ,$$

where λ^+ and λ^- are the two roots of $\lambda^2 - r\lambda - \delta(r+\delta) + I'\Lambda_h = 0$ and satisfy $\lambda^+ > r + \delta > 0 > - \delta > \lambda^-$, $r = \lambda^+ + \lambda^-$, $\lambda^+\lambda^- = I'\Lambda_h - \delta(r+\delta)$. We need to determine the value of ∂q_0, the initial change in q. This is tied down by invoking the stability condition. For a bounded change, $\{\partial\alpha_t, \partial\gamma_t\}$, the transversality condition implies that ∂h_t and ∂q_t are also bounded, or H(z) and

$Q(z)$ are defined for any positive z. In particular, $H(\lambda^+)$ and $Q(\lambda^+)$ need to be finite. Since the denominator in (A.8) is zero at $z=\lambda^+$, this requires that the numerator in (A.8) be also zero at $z=\lambda^+$. Thus,

$$\text{(A.9)} \qquad \partial q_0 = A(\lambda^+) + (\Lambda_\mu/\lambda^+)\partial\mu + [(\lambda^- + \delta)/I']I_\gamma \Gamma(\lambda^+) \quad .$$

The budget constraint (A.6) becomes:

$$\text{(A.10)} \qquad f_h H(r) + f_\mu \frac{\partial\mu}{r} = - q_\infty I' Q(r) + X_\gamma \Gamma(r) - G(r) \ ,$$

where use has been made of $q_\infty I' + X' = 0$ and the initial condition on b, $\partial b_0 = 0$. From (A.8), (A.9) and (A.10), we can solve for $H(z)$, $Q(z)$ and $\partial\mu$.

In order to solve for ∂b_t, or its Laplace transform $B(z)$, note that the Bellman's optimality principle implies:

$$b_t = \int_t^{+\infty} [f(h_s;\mu) - X(q_s;\gamma_s) + g_s] e^{-r(s-t)} ds$$

or

$$\text{(A.11)} \qquad B(z) = \frac{D(z) - D(r)}{r-z} = \frac{D(z)}{r-z} \quad ,$$

where $D(z) \equiv f_h H(z) + f_\mu(\partial\mu/z) + I' q_\infty Q(z) - X_\gamma \Gamma(z) + G(z)$. Note that $D(r) = 0$ from (A.10). The impact effect on the current account is therefore determined by:

$$\text{(A.12)} \qquad \partial\dot{b}_0 = \lim_{z\to+\infty} z\left[\mathcal{L}\{\partial\dot{b}_t\}\right] = \lim_{z\to+\infty} z\left[zB(z) - \partial b_0\right]$$

$$= \lim_{z\to+\infty} z^2 B(z) = - \lim_{z\to+\infty} zD(z) \quad ,$$

from (A.2), (A.3), (A.11) and $\partial b_0 = 0$.

In what follows, the short run impacts on q and $\dot{b}$ are examined in three different cases.

I. <u>Government Purchase Increases</u>: $G(r) > 0$, $A(z) = \Gamma(z) = 0$.

From (A.8), (A.9) and (A.10), it is straightforward to derive equation (7a) in the text:

$$\text{(7a)} \qquad \partial q_0 = -\Omega_1[rG(r)] \quad ,$$

where

$$\Omega_1 \equiv \frac{\Lambda_\mu}{\lambda^+ f_\mu + K\Lambda_\mu} > 0 \qquad \text{where } K \equiv -\mu\Lambda_\mu I'/\lambda^+ = [f_h + q_\infty(r+\delta)]I'/\lambda^+ > 0$$

Therefore, (7a) states that an increase in (the discounted value of) government purchases on the tradeable good causes a drop in the housing price. Note that, if housing services have zero income elasticity ($u_2 u_{11} = u_1 u_{12}$,) $\Lambda_\mu = K = 0$, and therefore, $\Omega_1 = 0 = \partial q_0$.

It is also easy to show, from (A.12), that the effect on the current account is given by (7b) or (7c):

$$\partial \dot{b}_0 = [1 - \Omega_1 L]rG(r) - \partial g_0$$

$$= rG(r) - \partial g_0 - \Omega_1 LrG(r) \quad ,$$

where

$$L \equiv [f_h + q_\infty(\lambda^- + \delta)]I'/\lambda^+ = K - q_\infty I'$$

and satisfies,

$$1 - \Omega_1 L = 1 - \Omega_1(K - q_\infty I') = \frac{\lambda^+ f_\mu + q_\infty I'\Lambda_\mu}{\lambda^+ f_\mu + K\Lambda_\mu} > 0 \quad .$$

Therefore, (i) any anticipated increase in government purchases ($rG(r) > 0$, $\partial g_0 = 0$) leads to a surplus ($\partial \dot{b}_0 > 0$), while (ii) the effect of an unanticipated permanent increase ($rG(r) = \partial g_0$) depends on the sign of L. From the definition, if f_h is nonpositive or if housing services and the tradeable good are Edgeworth-Pareto substitutes ($u_{12} \leq 0$), then $L < 0$ and $\partial \dot{b}_0 > 0$. If f_h is positive, then L can be positive, because $\lambda^- + \delta$ can be made arbitrarily close to zero by choosing I' sufficiently small. And this can be done without changing the steady state. In this case, $\partial \dot{b}_0 < 0$.

II. <u>Housing Subsidy Increases</u>: $A(z) > 0$, $G(z) = \Gamma(z) = 0$.

The short impacts on subsidy increases are given by:

$$\text{(A.13)} \qquad \partial q_0 = A(\lambda^+) + rK\Omega_1 \frac{A(\lambda^+) - A(r)}{\lambda^+ - r} ,$$

$$\text{(A.14)} \qquad \partial \dot{b}_0 = I' q_\infty A(\lambda^+) + rK(1 - \Omega_1 L) \frac{A(\lambda^+) - A(r)}{\lambda^+ - r} .$$

From these formulae, the effects of an unanticipated permanent increase ($A(z) = \partial\alpha/z$) are,

$$\text{(9a)} \qquad \partial q_0 = \Omega_2 \partial\alpha > 0$$

$$\text{(9b)} \qquad \partial \dot{b}_0 = \Omega_2 L \partial\alpha \ (< 0 \ \text{ if } L < 0 \ ) \quad ,$$

where

$$\Omega_2 \equiv \frac{f_\mu}{\lambda^+ f_\mu + K\Lambda_\mu} > 0$$

The signs of ∂q_0 and $\partial \dot{b}_0$ would be the same even if the normality assumption is dropped ($K = \Lambda_\mu = 0$).

In general, however, subsidy policies could work in the opposite direction, as long as housing services are normal. From (A.13), the condition for $\partial q_0 < 0$ with $A(z) > 0$ is ,

$$\text{(A.15)} \qquad \frac{rA(r)}{\lambda^+ A(\lambda^+)} > \left[1 - \frac{\lambda^-}{r}\frac{\Omega_2}{\Omega_1 K}\right] > 1 \quad ,$$

and, from (A.14), the condition for $\partial\dot{b}_0 > 0$ with $A(z) > 0$ is,

$$\text{(A.16)} \qquad \frac{rA(r)}{\lambda^+ A(\lambda^+)} > \left[1 + \frac{\lambda^-\Omega_2 L}{K(1-\Omega_1 L)}\right] \qquad (> 1 \text{ if } L < 0) \quad ,$$

where the right hand side of (A.16) is greater than one if L is negative. If $K = \Lambda_\mu = \Omega_1 = +0$, then, the right hand sides of (A.15) and (A.16) are infinite, so that we have always $\partial q_0 > 0$ and $\partial\dot{b}_0 < 0$. Otherwise, the opposite results could arise. In particular, if we consider an anticipated permanent increase ($\partial\alpha_t = 0$ for $t \le T$ and $\partial\alpha_t = \partial\alpha$ for $t > T$; $A(z) = (\partial\alpha/z)e^{-zT}$), then (A.15) and (A.16) become:

$$\text{(10a)} \qquad T > -\frac{1}{\lambda^-}\log\left[1 - \frac{\lambda^-}{r}\frac{\Omega_2}{\Omega_1 K}\right] > 0 \quad ,$$

and

$$\text{(10b)} \qquad T > -\frac{1}{\lambda^-}\log\left[1 + \frac{\lambda^-\Omega_2 L}{K(1-\Omega_1 L)}\right] \quad (> 0 \text{ if } L < 0) \ ,$$

respectively. Therefore, an increase in housing subsidies, if anticipated sufficiently in advance, leads to a decline in the housing price and residential investment and a current account surplus.

III. <u>Pruductivity Increase in the Housing Sector</u>: $\Gamma(z) > 0$, $A(z) = G(z) = 0$, $I_\gamma > 0$, $X_\gamma < 0$, $I_\gamma < 0$, $q_\infty I_\gamma + X_\gamma > 0$.

The general formulae are:

$$\text{(A.17)} \qquad \partial q_0 = \left[(\lambda^-+\delta)/I'\right] I_\gamma \Gamma(\lambda^+) + r\Omega_1\left[\omega\Gamma(r)+(\lambda^+-r)\Omega_3\Gamma(\lambda^+)+\delta\Omega_3\{\Gamma(r)-\Gamma(\lambda^+)\}\right]$$

and

$$(A.18)\qquad \partial\dot{b}_0 = X_\gamma \partial\gamma_0 - r(1-\Omega_1 L)\Big[\omega\Gamma(r)+(\lambda^+ - r)\Omega_3\Gamma(\lambda^+)+\delta\Omega_3\{\Gamma(r)-\Gamma(\lambda^+)\}\Big]$$

$$-q_\infty I_\gamma(\lambda^- +\delta)\Gamma(\lambda^+) \quad ,$$

where

$$\omega \equiv q_\infty I_\gamma + X_\gamma > 0 \quad \text{and} \quad \Omega_3 \equiv \frac{I_\gamma K}{\lambda^- I'} < 0 \quad .$$

Equations (11a-c) are immediate from (A.17) amd (A.18) by setting $K = \Omega_1 = \Omega_3 = 0$. With the normality assumption, almost anything goes. For example, for an unanticipated permanent change ($\Gamma(z) = \partial\gamma/z$), we have

$$\frac{\partial q_0}{\partial\gamma} = I_\gamma\left[\frac{\lambda^- +\delta}{\lambda^+ I'} + \Omega_1\left\{q_\infty\left\{1 - \frac{(r+\delta)^2}{(\lambda^+)^2}\right\} + \frac{X_\gamma}{I_\gamma} - \frac{r+\delta}{(\lambda^+)^2} f_h\right\}\right]$$

$$\frac{\partial\dot{b}_0}{\partial\gamma} = I_\gamma\left[\frac{X_\gamma}{I_\gamma} - (1 - \Omega_1 L)\left\{q_\infty\left\{1-\frac{(r+\delta)^2}{(\lambda^+)^2}\right\} + \frac{X_\gamma}{I_\gamma} - \frac{r+\delta}{(\lambda^+)^2} f_h\right\} - q_\infty\frac{\lambda^- +\delta}{\lambda^+}\right]$$

The terms in the brakets can be shown to have indeterminate signs. This is because one can set X_γ/I_γ to be of any negative value greater than $-q_\infty$ independent of other parameters.