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JEL Classification: D43, D61, D62, L13

Keywords: Procompetitive vs. Anticompetitive entry, Excessive vs. Insufficient entry, monopolistic competition, homothetic demand systems with gross substitutes

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When Does Procompetitive Entry Imply Excessive Entry?*

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1. Introduction

The monopolistic competition model with symmetric CES demand system with gross substitutes, developed in Dixit-Stiglitz (1977, Section I), is widely used as a building block across many applied general equilibrium fields, most notably in macroeconomics and international trade. Among its remarkable features are the invariance of the markup rate and the optimality of the free-entry equilibrium.[1] Of course, neither of these two features is robust. Once we depart from the knife-edge case of CES, the markup rate charged by each firm would change, as more firms enter in response to a market size increase. The markup rate may either go down (the case of *procompetitive* entry) or go up (the case of *anticompetitive* entry). Departure from CES would also lead to the inefficiency of the free-entry equilibrium. There may be either too many firms operating and hence too much product variety being offered (the case of *excessive* entry), or too few firms operating and too little product variety being offered (the case of *insufficient* entry).

But how is the condition for procompetitive vs. anticompetitive entry related to that for excessive vs. insufficient entry? One might think that all four combinations (procompetitive-excessive, procompetitive-insufficient, anticompetitive-excessive, and anticompetitive-insufficient) are feasible. After all, CES leads to the markup rate invariance and the optimality of the free-entry equilibrium for different reasons. The markup rate is invariant under CES, because each firm faces demand curve whose price elasticity is exogenously constant. It depends entirely on the *local* property of the demand curve, that is, it continues to hold as long as the price elasticity is constant around the point chosen by each firm. In contrast, the optimality of the free-entry equilibrium depends on the *global* property of the demand system. To understand this, recall that there are two sources of externalities in monopolistic competition with entry, as discussed in Tirole (1988, Chapter 7), Matsuyama (1995; Section 3E) and Dhingra and Morrow (2019) among many others. They are the inability of a firm to fully appropriate its social surplus its entry generates, which creates positive externalities, and its failure to account for business stealing from other firms, which creates negative externalities[2]. It turns out that

[1]To be precise, there exists an outside competitive sector in Dixit and Stiglitz (1977). Due to intersectoral distortion, there may be too little entry to the monopolistically competitive sector. Yet, they showed that the resource allocation *within* the monopolistically competitive sector is optimal at the free-entry equilibrium under CES.
[2]Mankiw and Whinston (1986) pointed out that such a business stealing effect causes excessive entry in a homogenous goods industry in partial equilibrium setting.

these two sources of externalities, one positive and one negative, exactly cancel out each other under CES.

We argue, however, that there is a tight connection between the condition for procompetitive vs. anticompetitive entry and that for excessive vs. insufficient entry. Starting from the knife-edge case of CES, where the equilibrium entry is optimal, making entry procompetitive exacerbates negative externalities to other firms by reducing their profit margin, which leads to excessive entry. Likewise, starting from the knife-edge case of CES, making entry anticompetitive mitigates negative externalities to other firms by increasing their profit-margin, which leads to insufficient entry. This suggests that procompetitive entry tends to be excessive, while anticompetitive entry tends to be insufficient.

So, the question to ask is: "when does procompetitive entry imply excessive entry? and when does anticompetitive entry imply insufficient entry?" To investigate this question, we extend the Dixit-Stiglitz monopolistic competition model with symmetric CES with gross substitutes to three classes of symmetric homothetic demand systems with gross substitutes, each named for its defining properties, *Homotheticity with a Single Aggregator* (H.S.A.), *Homotheticity with Direct Implicit Additivity* (H.D.I.A.), and *Homotheticity with Indirect Implicit Additivity* (H.I.I.A.). We have chosen these three classes for several reasons.

First of all, they are all *homothetic*. Although there have been many attempts to develop monopolistic competition models without CES, they have typically done so by making the demand system nonhomothetic.[3] However, in order to *isolate* the efficiency implications of the markup rate being responsive to entry caused by a market size change, we need to avoid introducing the scale effect of a market size change operating through nonhomotheticity. In addition, we need to maintain homotheticity to keep our departure from CES useful for most applications in macroeconomics, where monopolistic competition is used to model an intermediate inputs industry, which sells differentiated inputs to the *competitive* final goods industry, whose *constant returns to scale* (CRS) technology generates *homothetic* demand for those inputs.

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[3]Indeed, Dixit and Stiglitz (1977, Section II) already considered such an extension. Although they called this extension, "Variable Elasticity Case," the well-known Bergson's Law states that, within the class of demand systems they considered, they are homothetic if and only if they are CES. In other words, any departure from CES within this class introduces nonhomotheticity.

Second, these three classes are mutually exclusive except that they all contain CES as a knife-edge case; see Figure 1, adopted from Matsuyama and Ushchev (2017)[4]. Thus, they offer three alternative ways of departing from CES, while respecting the homotheticity requirement.

Third, they are all tractable and yet flexible. With some additional restrictions, there exists a unique symmetric free-entry equilibrium for any level of market size, which is analytically solvable, thereby facilitating the comparative statics and welfare analysis. Most importantly for our purpose, both the condition for procompetitive vs, anticompetitive entry and the condition for excessive vs. insufficient entry can be obtained explicitly and compared with each other.

Here are our main findings. In all three classes, entry is excessive when it is *globally* procompetitive. By "globally" procompetitive, we mean that the markup rate goes down *whenever* more firms enter, which occurs if and only if Marshall's second law of demand holds (i.e., the price elasticity increases with the price) *everywhere* along the demand curve. Likewise, in all three classes, entry is insufficient when it is globally anticompetitive, that is, when the markup rate goes up *whenever* more firms enter.[5] Between these two cases lies the borderline case of CES, where entry is always efficient because the markup rate is globally independent of market size. One important implication of these findings, as visualized in Figure 2, is that, for those who believe that procompetitive entry is the empirically relevant case, entry is excessive, which suggests that (small) regulation of entry is welfare-improving, at least in the absence of any other distortions.

We also show that entry is procompetitive and excessive for a sufficiently large market size in the presence of the choke price.[6] This is because the price elasticity goes to infinity at the choke price. This means that, as market size increases and more firms enter, each firm is forced to operate close to the choke price, that is, in the range where the price elasticity is increasing and the markup rate is decreasing in market size.

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[4]Matsuyama and Ushchev (2017, Proposition 4) proved that these classes are pairwise-disjoint with the sole exception of CES, even without restricting to be symmetric with gross substitutes. However, in this paper, we impose these restrictions to make them applicable to the Dixit-Stiglitz environment.

[5]The qualification that the markup rate responds to entry monotonically is important. In all three classes, we show by means of counter examples that, if the markup rate responds *nonmonotonically*, entry can be procompetitive and yet insufficient or anticompetitive and yet excessive in some range of parameter values.

[6]A choke price exists if demand for a product goes to zero at a finite price. There exists no choke price under CES. There exist choke prices under translog.

As already indicated, there have been many attempts to extend the Dixit-Stiglitz monopolistic competition models under CES to non-CES demand systems, starting from Dixit and Stiglitz (1977, Section II): see the survey by Thisse and Ushchev (2018) for an extensive reference. However, virtually all of them have done so by making the demand system nonhomothetic. Feenstra (2003) is an exception. He used symmetric homothetic translog as an alternative to CES and showed how it exhibits the procompetitive effect with a choke-price. However, he did not investigate how the equilibrium and optimal allocations differ from each other. Since symmetric homothetic translog is a special case of H.S.A., our analysis suggests excessive entry and hence a welfare-improving entry regulation under translog. Kimball (1995) considered the class of demand systems identical to H.D.I.A., except that he assumed an exogenous set of firms producing an exogenous set of products. By ruling out entry by assumption, he did not need to worry about ensuring the existence and uniqueness of the free-entry equilibrium, as we do, and he did not address any of the issues we are interested in. To the best of our knowledge, this is the first paper to offer a full characterization of the free-entry equilibrium of monopolistic competition models under H.S.A., H.D.I.A., and H.I.I.A.

Indeed, very few have ever investigated the question of excessive vs. insufficient entry in monopolistic competition under non-CES demand systems, whether homothetic or not. Two exceptions are Dixit and Stiglitz (1977, Section II) under nonhomothetic, non-CES demand systems, and Dhingra and Morrow (2019), which further extended their analysis to the case of heterogeneous firms *a la* Melitz. However, the class of demand systems they used makes it difficult to see to what extent their results are due to the endogeneity of the markup rate or due to nonhomotheticity of the demand systems.

The rest of the paper is organized as follows. In Section 2, we present what we call the Dixit-Stiglitz environment, the common setting across all three classes. Then, Sections 3, 4, and 5 deal with H.S.A., H.D.I.A., and H.I.I.A., respectively. These three sections are written in such a way that they can be read independently and in any order. And they are structured in the same way. In each section, we first define the class of homothetic demand systems, and explain its key properties. Then, we address the firm's behavior, identify the conditions that ensure the existence and uniqueness of symmetric free-entry equilibrium, and solve for it explicitly (Propositions 1, 4, and 7, respectively). Then, we conduct the comparative statics to identify the condition for procompetitive vs. anticompetitive entry (Propositions 2, 5, and 8, respectively), and perform the welfare analysis to identify the condition for excessive vs. insufficient entry (Propositions 3, 6, and 9, respectively). Then, we investigate the connection between the two conditions (Theorems 1,2, and 3), followed by three examples (one with global monotonicity, one with a choke price, and one without global monotonicity) to illustrate the theorems. Section 6 concludes. Technical proofs for some lemmas are gathered in the two appendices.

2. The Dixit-Stiglitz Environment

Consider the economy endowed with L units of the single factor of production, which we shall call labor and take as the numeraire. Labor is used to produce a continuum of varieties of differentiated intermediate inputs, which are in turn assembled to produce the single final good.

2.1. Competitive Final Goods Producers and Their Demand for Intermediate Inputs

The final good is produced competitively by using CRS technology, given by $X = X(\mathbf{x})$, where $\mathbf{x} = \{x(\omega);\ \omega \in \Omega\}$ is a quantity vector of intermediate inputs, with ω being the index of a particular input variety, and Ω being the set of input varieties available. It is assumed that $X(\mathbf{x})$ satisfies linear homogeneity, strict monotonicity, quasi-concavity, and symmetry, for each Ω. The unit cost function corresponding to $X = X(\mathbf{x})$ can be obtained by:

$$P = P(\mathbf{p}) \equiv \min_{\mathbf{x}} \left\{ \mathbf{px} = \int_{\Omega} p(\omega)x(\omega)d\omega \,\middle|\, X(\mathbf{x}) \geq 1 \right\}, \tag{1}$$

where $\mathbf{p} = \{p(\omega);\ \omega \in \Omega\}$ is a price vector of intermediate inputs, and $P(\mathbf{p})$ also satisfies linear homogeneity, strict monotonicity, quasi-concavity, and symmetry, for each Ω. Conversely, starting from any linear homogeneous, strictly monotonic, quasi-concave and symmetric $P(\mathbf{p})$, one could recover the underlying linear homogenous, strictly monotonic, quasi-concave and symmetric production function as follows:

$$X = X(\mathbf{x}) \equiv \min_{\mathbf{p}} \left\{ \mathbf{px} = \int_{\Omega} p(\omega)x(\omega)d\omega \,\middle|\, P(\mathbf{p}) \geq 1 \right\}. \tag{2}$$

Thus, either $X = X(\mathbf{x})$ or $P = P(\mathbf{p})$ can be used as a primitive of this CRS technology.

As is well-known from the duality theory, the cost minimization by competitive producers generates the demand curve and the inverse demand curve for each input,

$$x(\omega) = X(\mathbf{x})\frac{\partial P(\mathbf{p})}{\partial p(\omega)};\ p(\omega) = P(\mathbf{p})\frac{\partial X(\mathbf{x})}{\partial x(\omega)},$$

from either of which one could show, using Euler's theorem on linear homogeneous functions,

$$\mathbf{px} = \int_{\Omega} p(\omega)x(\omega)d\omega = P(\mathbf{p})X(\mathbf{x}).$$

Furthermore, the market share of each input can be expressed as

$$\frac{p(\omega)x(\omega)}{\mathbf{px}} = \frac{p(\omega)x(\omega)}{P(\mathbf{p})X(\mathbf{x})} = \frac{\partial \log P(\mathbf{p})}{\partial \log p(\omega)} = \frac{\partial \log X(\mathbf{x})}{\partial \log x(\omega)},$$

and the condition for a pair of inputs to be *gross substitutes* (i.e., the Hicks-Allen elasticity of substitution between the two is greater than one) can be written as:

$$-\frac{\partial \log\left[\frac{x(\omega_1)}{x(\omega_2)}\right]}{\partial \log\left[\frac{p(\omega_1)}{p(\omega_2)}\right]} = -\frac{\partial \log\left[\frac{\partial P(\mathbf{p})/\partial p(\omega_1)}{\partial P(\mathbf{p})/\partial p(\omega_2)}\right]}{\partial \log\left[\frac{p(\omega_1)}{p(\omega_2)}\right]} = -\frac{\partial \log\left[\frac{x(\omega_1)}{x(\omega_2)}\right]}{\partial \log\left[\frac{\partial X(\mathbf{x})/\partial x(\omega_1)}{\partial X(\mathbf{x})/\partial x(\omega_2)}\right]} > 1.$$

2.2. Monopolistically Competitive Differentiated Intermediate Inputs Producers

There is a continuum of intermediate input producing firms, also indexed by $\omega \in \Omega$, each producing a single variety of its own. They share the same IRS technology: producing $x > 0$ units of input requires $\psi x + F$ units of labor, where $F > 0$ is the fixed cost of entry, and $\psi > 0$ the marginal cost of production. (Recall that labor is taken as the *numeraire*.) Being monopolistically competitive, each firm sets its price and/or its quantity to maximize profit, subject to the downward-sloping demand curve it faces with the aggregate variables taken as given. There is free entry/exit, so that the maximized profit is equal to the fixed cost of entry, F, and hence the net profit is equal to zero in equilibrium. Thus, for each active firm $\omega \in \Omega$, $p(\omega)x(\omega) = \psi x(\omega) + F$ holds, and hence

$$P(\mathbf{p})X(\mathbf{x}) = \mathbf{px} = \int_{\Omega} p(\omega)x(\omega)d\omega = \int_{\Omega} (\psi x(\omega) + F)d\omega = \psi \int_{\Omega} x(\omega)d\omega + VF = L,$$

where $V \equiv |\Omega|$ is the Lebesgue measure of Ω. Thus, the aggregate market size is given by the total labor supply, $P(\mathbf{p})X(\mathbf{x}) = L$.

2.3. CES Benchmark

The above setup is ubiquitous as a building block in many applied general equilibrium fields, particularly in international trade and macroeconomics (both in business cycles and economic growth). In addition, the vast majority of studies in these literatures assumes the assembly technology of the final good to be symmetric CES with gross substitutes:

$$X = X(\mathbf{x}) = Z\left[\int_{\Omega} [x(\omega)]^{1-\frac{1}{\sigma}}d\omega\right]^{\frac{\sigma}{\sigma-1}} \Leftrightarrow P = P(\mathbf{p}) = \frac{1}{Z}\left[\int_{\Omega} [p(\omega)]^{1-\sigma}d\omega\right]^{\frac{1}{1-\sigma}},$$

which implies

$$\frac{p(\omega)x(\omega)}{P(\mathbf{p})X(\mathbf{x})} = \left(\frac{x(\omega)}{X(\mathbf{x})/Z}\right)^{1-\frac{1}{\sigma}} = \left(\frac{p(\omega)}{ZP(\mathbf{p})}\right)^{1-\sigma},$$

where $\sigma > 1$ is the (exogenous and constant) elasticity of substitution between each pair of inputs, and $Z > 0$ a productivity parameter.

It is well-known (and will be demonstrated later in this paper) that the CES assumption has some strong implications in this setup. First, it guarantees the equilibrium is unique and symmetric, $p(\omega) = p$ and $x(\omega) = x$ for all $\omega \in \Omega$. Second, at this unique equilibrium,

- each firm sells its own variety at the (common) exogenous markup rate; in particular, it is independent of market size, L;
- the equilibrium allocation is optimal; in particular, the equilibrium mass of firms that enter (and that of input varieties offered) is optimal.

Of course, neither of these two results, the market size neutrality on the markup rate, and the optimality of equilibrium entry, is robust. Depending on how we depart from the knife-edge CES assumption, we could have either the case of procompetitive entry or the case of anticompetitive entry, in which the markup rate goes either down or up in response to entry caused by a market size increase, as well as the case of excessive entry or the case of insufficient entry. But how are the cases of procompetitive or anticompetitive entry related to the cases of excessive or insufficient entry? We explore this question, using three alternative classes of CRS technologies, which are pairwise disjoint except that each contains CES as a knife-edge case.

3. Dixit-Stiglitz under H.S.A.

3.1.H.S.A. Demand System

We call CRS technology, $X = X(\mathbf{x})$ or $P = P(\mathbf{p})$, *homothetic with a single aggregator* (H.S.A.) if the market share of any input ω, as a function of $\mathbf{p}$, can be written as:

$$\frac{p(\omega)x(\omega)}{\mathbf{px}} = \frac{p(\omega)x(\omega)}{P(\mathbf{p})X(\mathbf{x})} = \frac{\partial \log P(\mathbf{p})}{\partial \log p(\omega)} = s\left(\frac{p(\omega)}{A(\mathbf{p})}\right). \tag{3}$$

Here, $s: \mathbb{R}_{++} \to \mathbb{R}_{+}$ is the *market share function,* and it is assumed to be twice continuously differentiable and *strictly decreasing* as long as $s(z) > 0$, with $\lim_{z\to 0} s(z) = \infty$ and $\lim_{z\to \bar{z}} s(z) = 0$, where $\bar{z} \equiv \inf\{z > 0 | s(z) = 0\}$, and $A(\mathbf{p})$ is linear homogenous in $\mathbf{p}$, defined implicitly and uniquely by

$$\int_{\Omega} s\left(\frac{p(\omega)}{A(\mathbf{p})}\right) d\omega \ = 1, \tag{4}$$

which ensures, by construction, that the market shares of all inputs are added up to one.

Eqs.(3)-(4) state that the market share of any input ω is decreasing in its *relative price*, which is defined as its own price, $p(\omega)$, divided by the *common price aggregator,* $A(\mathbf{p})$. Notice that $A(\mathbf{p})$ is independent of ω; it is "the average price" against which the relative prices of *all* inputs are measured. In other words, one could keep track of all the cross-price effects in the demand system by looking at a single aggregator, $A(\mathbf{p})$, which is the key feature of H.S.A.[7] The monotonicity of $s(\cdot)$, combined with the assumptions, $\lim_{z\to 0} s(z) = \infty$ and $\lim_{z\to \bar{z}} s(z) = 0$, ensures that $A(\mathbf{p})$ is defined uniquely by eq.(4), no matter what V (the measure of Ω) is.

The assumption that $s(\cdot)$ is strictly decreasing means that inputs are *gross substitutes.* To see this, one could show from eq.(3) that the elasticity of substitution between a pair of inputs, ω_1 and ω_2, evaluated at the same price, is

$$-\frac{\partial \ln(x(\omega_1)/x(\omega_2))}{\partial \ln(p(\omega_1)/p(\omega_2))}\bigg|_{p(\omega_1)=p(\omega_2)=p} = \zeta\left(\frac{p}{A(\mathbf{p})}\right) > 1$$

where $\zeta: (0, \bar{z}) \to (1, \infty)$ is defined by:

$$\zeta(z) \equiv 1 - \frac{zs'(z)}{s(z)} > 1.$$

Note that $\zeta(\cdot)$ is continuously differentiable for $z \in (0, \bar{z})$, and $\lim_{z\to\bar{z}} \zeta(z) = \infty$ if $\bar{z} < \infty$. Conversely, from any continuously differentiable $\zeta: (0, \bar{z}) \to (1, \infty)$, satisfying $\lim_{z\to\bar{z}} \zeta(z) = \infty$ if $\bar{z} < \infty$, one could recover the market share function as follows:

$$s(z) = \exp\left[\int_{z_0}^{z} \frac{1-\zeta(\xi)}{\xi} d\xi\right],$$

where $z_0 \in (0, \bar{z})$ is a constant. [8] Hence, we could also use $\zeta(\cdot)$ as a primitive of symmetric H.S.A. with gross substitutes, instead of the market share function, $s(\cdot)$.

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[7]On the other hand, the assumption that $s(\cdot)$, is independent of ω is not a defining feature of H.S.A.; this is due to the symmetry of the production technology. For asymmetric H.S.A., $s(\cdot)$ could depend on ω.

[8]This constant implies that $s(\cdot)$ is determined up to a positive scalar multiplier. However, $\gamma s(z)$ with $\gamma > 0$ generate the same H.S.A. technology. All we need is to renormalize the indexation of varieties, as $\int_{\Omega} \gamma s(p(\omega)/A) d\omega \ = \int_{\Omega} \gamma s(p(\omega')/A) d\omega' = 1$, with $\omega' = \gamma\omega$.

Note also that we allow for the possibility of $\bar{z} < \infty$, that is, the existence of the choke (relative) price; if $\bar{z} = \infty$, the choke price does not exist and demand for each input always remains positive for any positive price vector.

Symmetric CES with gross substitutes is a special case of H.S.A., generated by $s(z) = \gamma z^{1-\sigma}$ $(\sigma > 1)$. In this case, $P(\mathbf{p}) = cA(\mathbf{p})$, where $c > 0$ is a constant. Symmetric translog is another special case, generated by $s(z) = \max\{-\gamma \log(z), 0\}$. In this case, $P(\mathbf{p}) \neq cA(\mathbf{p})$ for any constant c. More generally, one can show, by integrating eq.(3), that the common price aggregator, $A(\mathbf{p})$, is related to the unit cost function, $P(\mathbf{p})$, as follows:

$$\log\left(\frac{P(\mathbf{p})}{A(\mathbf{p})}\right) = const. - \int_{\Omega} \left[\int_{p(\omega)/A(\mathbf{p})}^{\bar{z}} \frac{s(\xi)}{\xi} d\xi \right] d\omega. \tag{5}$$

In the case of CES, the RHS of eq.(5) is independent of $\mathbf{p}$, hence $P(\mathbf{p}) = cA(\mathbf{p})$, where $c > 0$ is a constant. However, CES is unique in this regard. It turns out that, with the sole exception of CES, the RHS of eq.(5) depends on $\mathbf{p}$, and hence $P(\mathbf{p}) \neq cA(\mathbf{p})$ for any constant c, as will be shown in the Corollary 2 of Lemma 2.[9] This should not come as a total surprise. After all, $A(\mathbf{p})$ captures the *cross-price effects* in the demand system, while $P(\mathbf{p})$ captures the *productivity (or welfare) effects* of price changes; there is no reason to think that they should move together in general.

Remark 1: Eqs.(3)-(4) define H.S.A. by restricting the properties of the implied demand system. The natural question is then: do CRS production functions that generate such a demand system exist? The answer is yes. For each demand system satisfying these properties, there exist strictly increasing, CRS production functions, which is strictly quasi-concave in the interior, uniquely determined up to a positive scalar.[10]

Remark 2: For any market share function, $s: \mathbb{R}_{++} \to \mathbb{R}_{+}$, satisfying the above conditions, a class of the market share functions, $s_\lambda(z) \equiv s(\lambda z)$ for $\lambda > 0$, generate the same demand system, with $A_\lambda(\mathbf{p}) = \lambda A(\mathbf{p})$, because $s_\lambda\left(\frac{p(\omega)}{A_\lambda(\mathbf{p})}\right) = s\left(\frac{\lambda p(\omega)}{A_\lambda(\mathbf{p})}\right) = s\left(\frac{p(\omega)}{A(\mathbf{p})}\right)$. In this sense, $s_\lambda(z) \equiv s(\lambda z)$ for $\lambda > 0$ are all equivalent. This equivalence gives us freedom to select λ to simplify the notation when discussing parameteric examples.

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[9]This holds also for asymmetric H.S.A., as well as H.S.A. with gross complements. See Matsuyama and Ushchev (2017; Proposition 1-iii))

[10]See Matsuyama and Ushchev (2017; Proposition 1-i)), which proved the existence of the underlying CRS production functions for more general cases, including the cases of asymmetry and gross complementarity.

3.2.Profit Maximization By Input Producing Firms under H.S.A.

The profit of firm $\omega \in \Omega$ is given by $\pi(\omega) = (p(\omega) - \psi)x(\omega) - F$, which can be written, using eq.(3), as:

$$\pi(\omega) = \left(1 - \frac{\psi/A(\mathbf{p})}{z(\omega)}\right) s\big(z(\omega)\big) P(\mathbf{p}) X(\mathbf{x}) - F,$$

where $z(\omega) \equiv p(\omega)/A(\mathbf{p})$ is its relative price. Firm ω chooses its relative price $z(\omega)$ to maximize $\pi(\omega)$, taking the aggregate variables, $A(\mathbf{p}), P(\mathbf{p})$, and $X(\mathbf{x})$ as given. The FOC is

$$z(\omega)\left(1 - \frac{1}{\zeta\big(z(\omega)\big)}\right) = \frac{p(\omega)}{A(\mathbf{p})}\left(1 - \frac{1}{\zeta\big(z(\omega)\big)}\right) = \frac{\psi}{A(\mathbf{p})}, \tag{6}$$

and the SOC is

$$\zeta\big(z(\omega)\big) - 1 + z(\omega)\frac{\zeta'\big(z(\omega)\big)}{\zeta\big(z(\omega)\big)} > 0.$$

In what follows, we keep it simple by imposing the following assumption to ensure that the FOC is sufficient for the global optimum.

Assumption S1: *For all* $z \in (0, \overline{z})$,

$$\frac{d\log(s(z)/\zeta(z))}{d\log z} = 1 - \zeta(z) - \frac{z\zeta'(z)}{\zeta(z)} < 0.$$

Under **S1**, the LHS of eq.(6) is strictly decreasing in $z(\omega)$. Hence, eq.(6) gives the unique profit-maximizing price for each firm. Thus, all firms set the same price, $p(\omega) = p$, or $z(\omega) = z$, and produce the same amount, $x(\omega) = x$. Hence, under **S1**, asymmetric equilibria do not exist. Note also **S1** means that $s(z)/\zeta(z)$ is strictly decreasing; this ensures the uniqueness of the symmetric equilibrium, as will be seen below.

3.3.Symmetric Free-Entry Equilibrium under H.S.A.

A symmetric free-entry equilibrium under H.S.A. satisfies the following conditions:

H.S.A. integral condition, given by eq. (4) under symmetry:

$$s(z)V = 1. \tag{7}$$

Firm's pricing formula, given by FOC, eq.(6) under symmetry:

$$1-\frac{\psi}{p}=\frac{1}{\zeta(z)} \tag{8}$$

Zero-profit (free-entry) condition:

$$(p-\psi)x=F \tag{9}$$

Resource constraint:

$$(\psi x+F)V=L. \tag{10}$$

Note that, from eq.(9) and eq.(10),

$$pxV=PX=L. \tag{11}$$

By combining eqs.(7), (8), (9) and (11),

$$\frac{F}{L}=\frac{(p-\psi)x}{L}=\left(1-\frac{\psi}{p}\right)\frac{px}{L}=\frac{1}{\zeta(z)}\frac{1}{V}=\frac{s(z)}{\zeta(z)}.$$

Under **S1**, RHS of this equation is strictly decreasing in $z\in(0,\bar{z})$. Furthermore, $\lim_{z\to 0}s(z)/\zeta(z)=\infty$ and $\lim_{z\to\bar{z}}s(z)/\zeta(z)=0$. Hence, *for each* $L/F>0$, the equilibrium value of z, z^E, is uniquely pinned down by

$$\frac{s(z^E)}{\zeta(z^E)}=\frac{F}{L}, \tag{12}$$

and z^E is increasing in L/F, with the range $(0,\bar{z})$. By inserting this value into eqs.(7), (8), and (9),

$$\begin{aligned} V^E&=\frac{1}{s(z^E)}=\frac{1}{\zeta(z^E)}\frac{L}{F},\\ p^E&=\frac{\zeta(z^E)\psi}{\zeta(z^E)-1}>0,\\ x^E&=\frac{[\zeta(z^E)-1]}{\psi}F>0,\end{aligned} \tag{13}$$

from which one could also show

$$\frac{1}{A^E}=\frac{z^E}{p^E}=\frac{z^E}{\psi}\left(1-\frac{1}{\zeta(z^E)}\right)=\frac{z^E}{\psi}\left(1-\frac{F/L}{s(z^E)}\right),$$

and, using eq.(5),

$$\log\frac{X^E}{L}=\log\frac{1}{P^E}=\log\frac{z^E}{\psi}\left(1-\frac{F/L}{s(z^E)}\right)+\frac{1}{s(z^E)}\int_{z^E}^{\bar{z}}\frac{s(\xi)}{\xi}d\xi+const.$$

Thus, we have shown:

Proposition 1. *Under **S1**, no asymmetric equilibria exist. Furthermore, there exists a unique symmetric free-entry equilibrium under H.S.A. for each* $L/F > 0$*, given by eq.*(12) *and eq.*(13).

3.4. Comparative Statics under H.S.A.: Procompetitive versus Anticompetitive

We now turn to the comparative statics.

Proposition 2. *Assume **S1**. At the unique symmetric equilibrium in monopolistic competition under H.S.A., given by eq.*(12) *and eq.*(13)*,*

Procompetitive:
$$\zeta'(z^E) > 0 \Rightarrow \frac{\partial p^E}{\partial L} < 0;\ \frac{\partial \log V^E}{\partial \log L} < 1;\ \frac{\partial x^E}{\partial L} > 0;$$

Neutral (CES):
$$\zeta'(z^E) = 0 \Rightarrow \frac{\partial p^E}{\partial L} = 0;\ \frac{\partial \log V^E}{\partial \log L} = 1;\ \frac{\partial x^E}{\partial L} = 0;$$

Anticompetitive:
$$\zeta'(z^E) < 0 \Rightarrow \frac{\partial p^E}{\partial L} > 0;\ \frac{\partial \log V^E}{\partial \log L} > 1;\ \frac{\partial x^E}{\partial L} < 0.$$

Proof: Since eq.(12) implies $\partial z^E/\partial L > 0$ under **S1**, this follows from eq.(13). ■

It is well-known that, in the knife-edge case of CES, the market size effect is neutral on the markup rate ($\partial p^E/\partial L = 0$) and the mass of firms increases proportionally ($\partial \log V^E/\partial \log L = 1$) without any effect on the firm size ($\partial x^E/\partial L = 0$). Thus, the expansion takes place only at the extensive margin. In the case of $\zeta'(z^E) > 0$, the market size effect is procompetitive, ($\partial p^E/\partial L < 0$) i.e., an increase in L reduces the markup rate. This forces each firm to operate at a larger scale in order to break even ($\partial x^E/\partial L = 0$), and hence some expansion also takes place at the intensive margin($\partial \log V^E/\partial \log L < 1$). In the opposite case of $\zeta'(z^E) < 0$, the market size effect is anticompetitive, i.e., the markup rate increases in response to an increase in L, which causes a more-than-proportionate increase in the mass of firms and forces each firm to operate at a smaller scale.

It should be also pointed out that, when the condition for the procompetitive effect holds globally, $\zeta'(\cdot) > 0$, it automatically implies **S1.** However, the opposite does not hold. Hence, **S1** does not rule out the anticompetitive case, $\zeta'(\cdot) < 0$.

3.5. Welfare Analysis under H.S.A.: Excessive versus Insufficient

We now turn to the welfare analysis under H.S.A.. Because $X = X(\mathbf{x})$ is strictly quasi-concave in the interior and symmetric, the optimal allocation that maximizes $X = X(\mathbf{x})$ must be symmetric, $x(\omega) = x$ for all $\omega \in \Omega$. Or equivalently, the optimal allocation that minimizes $P = P(\mathbf{p})$ must be symmetric, $p(\omega) = p$ for all $\omega \in \Omega$. Hence, from eq.(5), the optimal allocation can be obtained by choosing $z = p/A$ to maximize

$$\log\frac{X}{L} = \log\frac{1}{P} = \log\frac{1}{A} + V\int_{z}^{\bar{z}} \frac{s(\xi)}{\xi} d\xi + const.$$

subject to the constraints, eq.(7), eq.(10), and eq.(11) which can be combined to yield:

$$\frac{1}{A} = \frac{z}{\psi}\left(1 - \frac{F/L}{s(z)}\right) \tag{14}$$

Hence, the optimal allocation can be obtained by choosing z to maximize:

$$\log\frac{X}{L} = \log\frac{1}{P} = W(z) \equiv \log\frac{z}{\psi}\left(1 - \frac{F/L}{s(z)}\right) + \Phi(z) + const. \tag{15}$$

where $W(z)$ is the objective function and

$$\Phi(z) \equiv \frac{1}{s(z)}\int_{z}^{\bar{z}} \frac{s(\xi)}{\xi} d\xi. \tag{16}$$

The following lemma shows that z^E, the equilibrium value of z, given by eq.(12) generally fails to maximize the RHS of eq.(15). Indeed, it maximizes the RHS of eq.(14) instead. In other words, *the unique symmetric equilibrium minimizes* $A = A(\mathbf{p})$, not $P = P(\mathbf{p})$.

Lemma 1. *Under **S1**, eq.(14) is unimodal, and reaches the maximum at* $z = z^E$.

Proof. Differentiating the RHS of eq.(14) yields

$$1 - \frac{F/L}{s(z)} + z\frac{F/L}{\left(s(z)\right)^2}s'(z) = 1 - \frac{F/L}{s(z)}\left(1 - \frac{zs'(z)}{s(z)}\right) = 1 - \frac{F}{L}\frac{\zeta(z)}{s(z)}.$$

From **S1**, this is strictly decreasing in z, and hence the RHS of eq.(14) is strictly concave and reaches its maximum with respect to z when

$$1 - \frac{F}{L}\frac{\zeta(z)}{s(z)} = 0$$

which is equivalent to eq.(12), satisfied if and only if $z = z^E$. This completes the proof. ■

Lemma 1 states that the equilibrium allocation maximizes only the first term of $W(z)$ in eq. (15). The second term, $\Phi(z)$, given in eq.(16), represents externalities that are ignored in a decentralized equilibrium. To understand the property of this externality term, notice that it can be rewritten as:

$$1+\frac{1}{\Phi(z)} \equiv 1+\frac{s(z)}{\int_z^{\bar{z}} s(\xi)/\xi \, d\xi} = 1+\frac{-\int_z^{\bar{z}} s'(\xi)\, d\xi}{\int_z^{\bar{z}} s(\xi)/\xi \, d\xi} = \frac{\int_z^{\bar{z}} \zeta(\xi)\frac{s(\xi)}{\xi} d\xi}{\int_z^{\bar{z}} s(\xi)/\xi \, d\xi} = \int_z^{\bar{z}} \zeta(\xi) w(\xi)\, d\xi,$$

where $w(\xi) \equiv \frac{s(\xi)/\xi}{\int_z^{\bar{z}} s(\xi')/\xi' d\xi'}$, satisfying $\int_z^{\bar{z}} w(\xi)\, d\xi = 1$. Hence, log-differentiating eq.(16) yields

$$\frac{z\Phi'(z)}{\Phi(z)} = -\frac{zs'(z)}{s(z)} - \frac{1}{\Phi(z)} = \zeta(z) - 1 - \frac{1}{\Phi(z)} = \zeta(z) - \int_z^{\bar{z}} \zeta(\xi) w(\xi)\, d\xi,$$

from which the next lemma and its two corollaries follow:

Lemma 2.

$$\Phi'(z) \lesseqgtr 0 \Leftrightarrow \zeta(z) \lesseqgtr \int_z^{\bar{z}} \zeta(\xi) w(\xi)\, d\xi.$$

Corollary 1: *Assume that* $\zeta'(\cdot)$ *does not change sign over* $(z_0, \bar{z})$*, where* $0 < z_0 < \bar{z}$*. Then, for all* $z \in (z_0, \bar{z})$

$$\zeta'(\cdot) \gtreqless 0 \;\Rightarrow \Phi'(\cdot) \lesseqgtr 0.$$

Corollary 2. $P(\mathbf{p}) = cA(\mathbf{p})$ *with* $c > 0$ *is a constant, only in the case of CES.*

Proof. [11] Using eqs.(14)-(15), we obtain $\log A(\mathbf{p})/P(\mathbf{p}) = \Phi(z) + const$. Hence, $P(\mathbf{p}) = cA(\mathbf{p}) \Leftrightarrow \Phi'(z) = 0$ for all $z \in (0, \bar{z}) \Leftrightarrow \zeta'(z) = 0$ for all $z \in (0, \bar{z}) \Leftrightarrow$ CES. ■

Lemma 3. $W(z)$ *is unimodal, and reaches its peak at* z^O*, given by the unique solution to*

$$\frac{L}{F} = \frac{1}{s(z)} + \frac{1}{\int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi} = \frac{1}{s(z)}\left(1+\frac{1}{\Phi(z)}\right) = \frac{1}{s(z)} \int_z^{\bar{z}} \zeta(\xi) w(\xi)\, d\xi. \tag{17}$$

and z^O *is increasing in* L/F.

Proof: Differentiating $W(z)$, defined in eq.(15), yields

[11] As already discussed, Corollary 2 is a special case of Matsuyama and Ushchev (2017; Proposition 1-iii)). Nevertheless, we offer this proof, because it is much simpler due to the symmetry and gross substitutability.

$$W'(z) = \frac{1}{z} + \frac{\frac{s'(z)}{s(z)^2}}{\frac{L}{F} - \frac{1}{s(z)}} + \Phi'(z).$$

Differentiating $\Phi(z)$, eq.(16), yields

$$\Phi'(z) = -\frac{s'(z)}{s(z)^2}\int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi - \frac{1}{z}.$$

By combining these two expressions,

$$W'(z) = \frac{s'(z)}{s(z)^2}\left[\frac{1}{\frac{L}{F} - \frac{1}{s(z)}} - \int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi\right].$$

Because the term in the square bracket is strictly increasing, $W(z)$ is unimodal with

$$W'(z) \gtreqless 0 \Leftrightarrow z \lesseqgtr z^O,$$

where z^O is given by

$$\frac{L}{F} = \frac{1}{s(z)} + \frac{1}{\int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi},$$

whose RHS is increasing in z, because $s(z)$ and $\int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi$ are both decreasing in z. Hence, z^O is uniquely defined and is increasing in L/F. ■

We are now ready to state the welfare property of the equilibrium entry under H.S.A..

Proposition 3. *Assume **S1**. Then, at the unique symmetric equilibrium in monopolistic competition under H.S.A., given by eq.*(12) *and eq.*(13), V^E*, the equilibrium mass of firms that enter = the equilibrium mass of varieties produced, and* V^O*, the mass of the optimal mass of firms that enter = the optimal mass of varieties produced, satisfy*

$$V^E \gtreqless V^O \Leftrightarrow \zeta(z^E) \lesseqgtr \int_{z^E}^{\bar{z}} \zeta(\xi) w(\xi)\, d\xi.$$

In particular,

Excessive Entry: $\zeta'(z) > 0\ for\ all\ z \in (z^E, \bar{z}) \Rightarrow V^E > V^O$

Optimal Entry (CES): $\zeta'(z) = 0\ for\ all\ z \in (z^E, \bar{z}) \Rightarrow V^E = V^O$

Insufficient Entry: $\zeta'(z) < 0\ for\ all\ z \in (z^E, \bar{z}) \Rightarrow V^E < V^O$

Proof. By combining eq.(12) and eq.(17),

$$\frac{\zeta(z^E)}{s(z^E)} = \frac{L}{F} = \frac{1}{s(z^O)} + \frac{1}{\int_{z^O}^{\bar{z}} \frac{s(\xi)}{\xi} d\xi}.$$

Since $s(z)$ and $\int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi$ are both decreasing in z,

$$z^E \gtreqless z^O \Leftrightarrow \frac{\zeta(z^E)}{s(z^E)} = \frac{1}{s(z^O)} + \frac{1}{\int_{z^O}^{\bar{z}} \frac{s(\xi)}{\xi} d\xi} \lesseqgtr \frac{1}{s(z^E)} + \frac{1}{\int_{z^E}^{\bar{z}} \frac{s(\xi)}{\xi} d\xi} = \frac{1}{s(z^E)} \int_{z^E}^{\bar{z}} \zeta(\xi) w(\xi)\, d\xi.$$

Hence,

$$V^E = \frac{1}{s(z^O)} \gtreqless V^O = \frac{1}{s(z^O)} \Leftrightarrow z^E \gtreqless z^O \Leftrightarrow \zeta(z^E) \lesseqgtr \int_{z^E}^{\bar{z}} \zeta(\xi) w(\xi)\, d\xi,$$

which completes the proof. ■

3.6. Main H.S.A. Theorem and Some Examples

We are now ready to state the main properties of H.S.A. in the following theorem, by consolidating Propositions 1, 2, and 3. In doing so, we take into account that z^E is monotonically increasing in L/F and takes any value in $(0, \bar{z})$, as L/F varies from zero to infinity, and that the existence of the choke price, $\bar{z} < \infty$, implies $\lim_{z \to \bar{z}} \zeta(z) = \infty$, and hence $\zeta'(z) > 0$ for z sufficiently close to $\bar{z}$, which means that entry is procompetitive and excessive for a sufficiently large $L/F > 0$.

Theorem 1: *Consider monopolistic competition under symmetric H.S.A. with gross substitutes. Assume* ***S1*** *to ensure that there exists a unique equilibrium, which is symmetric and given by eq.*(12) *and eq.*(13)*. At this unique symmetric equilibrium, entry is,*

- *procompetitive and excessive for any* $L/F > 0$*, if* $\zeta'(z) > 0$ *for all* $z \in (0, \bar{z})$*;*
- *neutral and optimal for any* $L/F > 0$*, if* $\zeta'(z) = 0$ *for all* $z \in (0, \infty)$*, that is, under CES;*
- *anticompetitive and insufficient for any* $L/F > 0$*, if* $\zeta'(z) < 0$ *for all* $z \in (0, \infty)$.

Furthermore, in the presence of the choke price, $\bar{z} < \infty$*, entry is procompetitive and excessive for a sufficiently large* $L/F > 0$.

One important implication of this theorem is that, for those who believe in the empirical *Marshall's second law of demand*, i.e., the price elasticity of demand goes up as its price goes up, holding the aggregates fixed, entry is not only procompetitive but also excessive under H.S.A.

We now turn to some examples to illustrate Theorem 1.

Example 1: Perturbed CES, H.S.A. with global monotonicity

$$s(z) = z^{1-\sigma} \exp\left[-\delta(\sigma-1)\int_c^z \frac{g(\xi)}{\xi} d\xi\right] \Leftrightarrow \zeta(z) = \sigma + \delta(\sigma-1)g(z),$$

where $g(z)$ satisfies $g'(z) > 0$ for all $z > 0$ with $g(0) = 0$ and $g(\infty) = 1$ and $\kappa \equiv \sup\{zg'(z)|z > 0\} < \infty$. For example,

$$g(z) = \frac{z}{\eta + z}, \qquad \eta > 0 \Rightarrow \kappa = \frac{1}{4};$$

$$g(z) = 1 - e^{-\mu z}, \qquad \mu > 0 \Rightarrow \kappa = e^{-1}.$$

We also impose the restrictions that $\sigma > 1$ and $\delta > -\sigma/(\kappa + 2\sigma - 1) > -1$ to ensure the gross substitutability, i.e., $\zeta(z) > 1$ for all $z \in (0, \infty)$ as well as **S1.**[12] Clearly, $\delta = 0$ corresponds to the knife-edge case of CES, where entry is neutral and optimal. If $\delta > 0$, entry is procompetitive and excessive. And, if $-\sigma/(\kappa + 2\sigma - 1) < \delta < 0$, entry is anticompetitive and insufficient.

Example 2: Homothetic Translog, H.S.A. with a choke price

Homothetic symmetric translog is a special case of symmetric H.S.A. To see this, look at eq. (19') of Feenstra (2003), which gives the expression for the market share for each product under homothetic translog as

$$\frac{p(\omega)}{P(\mathbf{p})}\frac{\partial P(\mathbf{p})}{\partial p(\omega)} = \frac{1}{V} - \gamma\left[\log p(\omega) - \frac{1}{V}\int_\Omega \log p(\omega')\, d\omega'\right], (\gamma > 0)$$

in our notation. This can be rewritten as

$$\frac{p(\omega)}{P(\mathbf{p})}\frac{\partial P(\mathbf{p})}{\partial p(\omega)} = -\gamma \log\left(\frac{p(\omega)}{A(\mathbf{p})}\right), \qquad \text{where } \log A(\mathbf{p}) \equiv \frac{1}{\gamma N} + \frac{1}{N}\int_\Omega \log p(\omega')\, d\omega',$$

[12] For **S1**, note that it can be rewritten as $[\zeta(z) - 1]\zeta(z) + z\zeta'(z) = [\zeta(z) - 1]\zeta(z) + \delta z g'(z) > 0$. Clearly, this holds for $\delta \geq 0$. For $\delta < 0$, $\zeta(z) > \sigma + \delta$ and $\delta z g'(z) \geq \delta\kappa$ for all z, and hence $[\zeta(z) - 1]\zeta(z) + z\zeta'(z) > (\sigma + \delta - 1)(\sigma + \delta) + \delta\kappa = \delta^2 + (2\sigma - 1 + \kappa)\delta + (\sigma - 1)\sigma > (2\sigma - 1 + \kappa)\delta + (\sigma - 1)\sigma > 0$.

which shows that it is a H.S.A. with $s(z) \equiv -\gamma \log(z)$ for $z \in (0,1)$, with the choke price, $\overline{z} = 1$. Hence,

$$\zeta(z) = 1 + \frac{\gamma}{s(z)} = 1 - \frac{1}{\log(z)} > 1, \text{for } z \in (0,1).$$

which is strictly increasing in $z \in (0,1)$, which also implies **S1**. Hence, in the case of translog, entry is always procompetitive and excessive.

The assumption of the global monotonicity of $\zeta(\cdot)$ in Theorem 1 is important. Otherwise, entry could be procompetitive and yet insufficient, or anticompetitive and yet excessive, as the next example illustrates.

Example 3: **Perturbed CES, H.S.A. without global monotonicity**

Consider the following family of H.S.A technologies:

$$\zeta(z) \equiv 1 - \frac{zs'(z)}{s(z)} = 1 + (\sigma - 1)\frac{\delta z g'(z) + 1}{1 + \delta(\sigma - 1)g(z)},$$

where $\sigma > 1$, δ can be either positive or negative (but sufficiently small in absolute value to ensure that $\zeta(z)$ satisfies S1), while $g(z)$ is twice continuously differentiable, single-peaked, and satisfies $g(0) = g(\infty) = 0$, and $\sup|g'(z)| < \infty$. Let $\tilde{z} > 0$ be the maximizer of $g(z)$, with $g'(\tilde{z}) = 0 > g''(\tilde{z})$. For example,

$$g(z) = z/(\lambda + z^2), \lambda > 0 \implies \tilde{z} = \sqrt{\lambda};$$

$$g(z) = ze^{-\mu z}, \mu > 0 \implies \tilde{z} = 1/\mu.$$

It is readily verified that the externality term in the welfare function is given by

$$\Phi(z) \equiv \frac{1}{s(z)}\int_z^\infty \frac{s(\xi)}{\xi} d\xi = \frac{1}{\sigma - 1} + \delta g(z).$$

From Lemma 1, $W'(z^E) = \Phi'(z^E) = \delta g'(z^E)$, and from Lemma 3, $W'(z^E) \gtreqless 0 \Leftrightarrow z^E \lesseqgtr z^O \Leftrightarrow V^E \lesseqgtr V^O$. Hence, $\delta g'(z^E) \gtreqless 0 \Leftrightarrow V^E \lesseqgtr V^O$. Thus, entry is insufficient for $z^E < \tilde{z}$ and excessive for $z^E > \tilde{z}$ for $\delta > 0$, while entry is excessive for $z^E < \tilde{z}$ and insufficient for $z^E > \tilde{z}$ for $\delta < 0$. On the other hand, evaluating $\zeta'(z)$ at $z = \tilde{z}$ yields:

$$\zeta'(\tilde{z}) = \delta\frac{\tilde{z}g''(\tilde{z})}{\Phi(\tilde{z})} \gtreqless 0 \Leftrightarrow \delta \lesseqgtr 0.$$

Thus, entry is anticompetitive in the vicinity of $\tilde{z}$ for $\delta > 0$, while entry is procompetitive in the vicinity of $\tilde{z}$ for $\delta < 0$.

By combining these two observations, we conclude that entry is procompetitive and yet insufficient for $\delta < 0$ and z^E slightly higher than $\tilde{z}$, or equivalently, L/F slightly higher than $\zeta(\tilde{z})/s(\tilde{z})$, while it is anticompetitive and yet excessive for $\delta > 0$ and z^Eslightly lower than $\tilde{z}$, or equivalently, L/F slightly lower than $\zeta(\tilde{z})/s(\tilde{z})$.

3.7.H.S.A. Demand System: An Alternative Formulation

Before proceeding, it should be pointed out that there exists an alternative (but equivalent) definition of H.S.A.. That is, $X = X(\mathbf{x})$ or $P = P(\mathbf{p})$ is called *homothetic with a single aggregator* (H.S.A.) if the market share of input ω, as a function of $\mathbf{x}$, can be written as:

$$\frac{p(\omega)x(\omega)}{\mathbf{px}} = \frac{p(\omega)x(\omega)}{P(\mathbf{p})X(\mathbf{x})} = \frac{\partial \log X(\mathbf{x})}{\partial \log x(\omega)} = s^*\left(\frac{x(\omega)}{A^*(\mathbf{x})}\right).$$

Here, $s^*: \mathbb{R}_{++} \to \mathbb{R}_+$ is the *market share function,* and it is assumed to be twice continuously differentiable with $0 < ys^{*\prime}(y)/s^*(y) < 1$, $s^*(0) = 0$ and $s^*(\infty) = \infty$, and $A^*(\mathbf{x})$ is linear homogenous in $\mathbf{x}$, defined implicitly and uniquely by

$$\int_\Omega s^*\left(\frac{x(\omega)}{A^*(\mathbf{x})}\right) d\omega \ = 1,$$

which ensures that the market shares of all inputs are added up to one. Thus, the market share of input ω is a function of its *relative quantity*, defined as its own quantity $x(\omega)$ divided by the *common quantity aggregator* $A^*(\mathbf{x})$, which is strictly increasing with the elasticity less than one. This common quantity aggregator, $A^*(\mathbf{x})$, is related to the production function, $X(\mathbf{x})$, as follows:

$$\log\left(\frac{X(\mathbf{x})}{A^*(\mathbf{x})}\right) = const. + \int_\Omega \left[\int_0^{x(\omega)/A^*(\mathbf{x})} \frac{s^*(\xi)}{\xi} d\xi\right] d\omega,$$

and $X(\mathbf{x}) = cA^*(\mathbf{x})$, with a positive constant $c > 0$, if and only if $s^*(y) = \gamma y^{1-1/\sigma}$, which is the case of CES.

These two alternative definitions of H.S.A. are isomorphic to each other via the one-to-one mapping between $s(z) \leftrightarrow s^*(y)$, defined by:

$$s^*(y) = s\left(\frac{s^*(y)}{y}\right); \ s(z) = s^*\left(\frac{s(z)}{z}\right).$$

With this mapping, the relative quantity, $y(\omega) \equiv x(\omega)/A^*(\mathbf{x})$, and the relative price, $z(\omega) \equiv p(\omega)/A(\mathbf{p})$, are negatively related as $z(\omega) = s^*(y(\omega))/y(\omega)$ and $y(\omega) = s(z(\omega))/z(\omega)$, with

$\lim_{y\to 0} s^*(y)/y = s^{*\prime}(0) = \bar{z}$. Furthermore, by differentiating either of the two equalities above, one could obtain the identity,

$$\left(1 - \frac{ys^{*\prime}(y)}{s^*(y)}\right)\left(1 - \frac{zs'(z)}{s(z)}\right) = 1,$$

which shows that the condition, $0 < ys^{*\prime}(y)/s^*(y) < 1$, is equivalent to $s'(z) < 0$, the condition for gross substitutability.[13]

Under this alternative (but equivalent) formulation of H.S.A., the price elasticity function can be expressed as a function of the relative quantity, $y(\omega) \equiv x(\omega)/A^*(\mathbf{x})$, given by

$$\zeta^*(y) \equiv \left[1 - \frac{ys^{*\prime}(y)}{s^*(y)}\right]^{-1} = \zeta(z) \equiv 1 - \frac{zs'(z)}{s(z)} > 1,$$

and, under the following assumption, which is equivalent to **S1**,

$$\frac{d\log(s^*(y)/\zeta^*(y))}{d\log y} = 1 - \frac{1}{\zeta^*(y)} - \frac{y\zeta^{*\prime}(y)}{\zeta^*(y)} > 0,$$

there exists a unique symmetric equilibrium, in which all firms choose $y(\omega) \equiv y^E$, given by the condition,

$$\frac{s^*(y^E)}{\zeta^*(y^E)} = \frac{F}{L},$$

where y^E is decreasing in L/F. Furthermore, entry is procompetitive and excessive for any $L/F > 0$, if $\zeta^{*\prime}(y) < 0$ for all $y \in (0,\infty)$; neutral and optimal for any $L/F > 0$, if $\zeta^{*\prime}(y) = 0$ for all $y \in (0,\infty)$, (i.e., under CES) and anticompetitive and insufficient for any $L/F > 0$, if $\zeta^{*\prime}(y) > 0$ for all $y \in (0,\infty)$. Furthermore, in the presence of the choke price, $s^{*\prime}(0) = \bar{z} < \infty$, $\zeta^{*\prime}(0) = \infty$, and hence $\zeta^{*\prime}(y) < 0$ for a sufficiently small y, which means that entry is procompetitive and excessive for a sufficiently large $L/F > 0$.

4. Dixit-Stiglitz under H.D.I.A. (Homothetic Direct Implicit Additivity)

4.1.H.D.I.A. Demand System

We call a symmetric CRS technology, $X = X(\mathbf{x})$ or $P = P(\mathbf{p})$, *homothetic with direct implicit additivity* (H.D.I.A.)[14] if $X = X(\mathbf{x})$ can be defined implicitly by:

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[13]This isomorphism has been shown for the broader class of H.S.A., which allows for asymmetry as well as gross complements; see Matsuyama and Ushchev (2017, Section 3, Remark 3).

[14] More generally, $X = X(\mathbf{x})$ satisfies *direct implicit additivity* (D.I.A.) if it is defined implicitly by $\int_\Omega \tilde{\phi}(x(\omega), X)d\omega = 1$. See Hanoch (1975; Section 2). Clearly, H.D.I.A. is a subclass of D.I.A., where

$$\int_{\Omega} \phi\left(\frac{x(\omega)}{X}\right) d\omega = 1, \tag{18}$$

where $\phi(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, strictly concave, and at least thrice continuously differentiable with $\phi(0) = 0$ and $\phi(\infty) = \infty$.

In the following analysis, both the elasticity of $\phi(\cdot)$,

$$0 < \mathcal{E}_{\phi}(y) \equiv \frac{y\phi'(y)}{\phi(y)} < 1, \tag{19}$$

and the elasticity of $\phi'(\cdot)$ in its absolute value,

$$0 < r_{\phi}(y) \equiv -\frac{y\phi''(y)}{\phi'(y)} < 1, \tag{20}$$

play important roles. The monotonicity of $\phi(y)$ and $\phi(0) = 0$ jointly ensure $\mathcal{E}_{\phi}(y) > 0$ and the concavity of $\phi(y)$ ensures $\mathcal{E}_{\phi}(y) < 1$. The monotonicity and concavity of $\phi(y)$ jointly ensure $r_{\phi}(y) > 0$. In addition, it is necessary to assume that $r_{\phi}(y) < 1$ to ensure that inputs are gross substitutes, as will be seen below. Note that one could recover $\phi(\cdot)$ either from any $\mathcal{E}_{\phi}(\cdot)$ or any $r_{\phi}(\cdot)$ satisfying the bounds in eq. (19) and eq.(20) as follows:

$$\phi(y) = exp\left[\int_{y_0}^{y} \mathcal{E}_{\phi}(\xi)\frac{d\xi}{\xi}\right];$$

$$\phi(y) = \int_{0}^{y} exp\left[-\int_{y_0'}^{\xi} r_{\phi}(\xi')\frac{d\xi'}{\xi'}\right] d\xi,$$

where $y_0 > 0$ and $y_0' > 0$ are both constants.[15] One could also verify from eq. (19) and eq.(20) that $\mathcal{E}_{\phi}(y)$ and $r_{\phi}(y)$ are related as follows:

$$\frac{y\mathcal{E}_{\phi}{}'(y)}{\mathcal{E}_{\phi}(y)} = 1 - r_{\phi}(y) - \mathcal{E}_{\phi}(y).$$

Clearly, CES with gross substitutes is a special case with $\phi(y) = Ay^{1-\frac{1}{\sigma}}$, and $0 < r_{\phi}(y) = 1 - \mathcal{E}_{\phi}(y) = 1/\sigma < 1$.

$\tilde{\phi}(x(\omega), X) = \phi(x(\omega)/X)$. In contrast, $X = X(\mathbf{x})$ satisfies *direct explicit additivity* (D.E.A.) if it can be written as $X = \mathcal{M}\left(\int_{\Omega} \phi(x(\omega))d\omega\right)$, where $\mathcal{M}(\cdot)$ is a monotone transformation. This is the class of demand systems used by Dixit and Stiglitz (1977, Section II) and Dhingra and Morrow (2019). Although D.E.A. is another subclass of D.I.A., it cannot be homothetic unless it is CES.

[15] These constants imply that $\phi(y)$ is determined up to a positive scalar multiplier. However, $\gamma\phi(y)$ with $\gamma > 0$ generate the same CRS technology. All we need is to renormalize the indices of varieties, as $\int_{\Omega} \gamma\phi(x(\omega)/X)d\omega = \int_{\Omega} \gamma\phi(x(\omega')/X)d\omega' = 1$, with $\omega' = \gamma\omega$.

The cost minimization problem, eq.(1) subject to eq.(18) implies that the inverse demand curve for each $\omega \in \Omega$ can be written as:

$$p(\omega) = B(\mathbf{p})\phi'\left(\frac{x(\omega)}{X(\mathbf{x})}\right), \tag{21}$$

where $B(\mathbf{p})$ is the Lagrange multiplier associated with eq. (18), and it is the linear homogenous function in $\mathbf{p}$, given by

$$\int_{\Omega} \phi\left((\phi')^{-1}\left(\frac{p(\omega)}{B(\mathbf{p})}\right)\right) d\omega \equiv 1.$$

From eq.(21), the market share of each input can be expressed either as a function of $\mathbf{p}$, or as a function of $\mathbf{x}$, as follows:

$$\frac{p(\omega)}{P(\mathbf{p})}\frac{x(\omega)}{X(\mathbf{x})} = \frac{p(\omega)}{P(\mathbf{p})}(\phi')^{-1}\left(\frac{p(\omega)}{B(\mathbf{p})}\right) = \frac{x(\omega)}{C^*(\mathbf{x})}\phi'\left(\frac{x(\omega)}{X(\mathbf{x})}\right),$$

where the unit cost function is given by:

$$P(\mathbf{p}) = \int_{\Omega} p(\omega)(\phi')^{-1}\left(\frac{p(\omega)}{B(\mathbf{p})}\right) d\omega,$$

and $C^*(\mathbf{x})$ is a linear homogenous function of $\mathbf{x}$, given by

$$C^*(\mathbf{x}) \equiv \int_{\Omega} x(\omega)\phi'\left(\frac{x(\omega)}{X(\mathbf{x})}\right) d\omega,$$

and it satisfies

$$\frac{P(\mathbf{p})}{B(\mathbf{p})} = \int_{\Omega} \frac{p(\omega)}{B(\mathbf{p})}(\phi')^{-1}\left(\frac{p(\omega)}{B(\mathbf{p})}\right) d\omega = \int_{\Omega} \phi'\left(\frac{x(\omega)}{X(\mathbf{x})}\right)\frac{x(\omega)}{X(\mathbf{x})} d\omega = \frac{C^*(\mathbf{x})}{X(\mathbf{x})}.$$

These two expressions for the market share under H.D.I.A. show that it is a function of the two relative prices, $p(\omega)/P(\mathbf{p})$ and $p(\omega)/B(\mathbf{p})$, or a function of the two relative quantities, $x(\omega)/X(\mathbf{x})$ and $x(\omega)/C^*(\mathbf{x})$, unless $P(\mathbf{p})/B(\mathbf{p}) = C^*(\mathbf{x})/X(\mathbf{x}) = c > 0$ for a constant c, which occurs if and only if it is CES. Thus, H.D.I.A. and H.S.A. do not overlap with the sole exception of CES. [16]

From the inverse demand curve, eq.(21), the elasticity of substitution between a pair of inputs, ω_1 and ω_2, evaluated at the same quantity (hence at the same price) can be expressed as:

[16] See Proposition 2-(ii) in Matsuyama and Ushchev (2017).

$$-\frac{\partial \ln(x(\omega_1)/x(\omega_2))}{\partial \ln(p(\omega_1)/p(\omega_2))}\bigg|_{x(\omega_1)=x(\omega_2)=x} = \frac{1}{r_\phi(x/X)} > 1,$$

hence $r_\phi(\mathcal{y}) < 1$ ensures that inputs are gross substitutes. It should be also clear from eq.(21) that the choke price exists if and only if

$$\phi'(0) = \lim_{\mathcal{y}\to 0} exp\left[\int_{\mathcal{y}}^{\mathcal{y}_0'} \frac{r_\phi(\xi)}{\xi} d\xi\right] < \infty,$$

which implies $\lim_{\mathcal{y}\to 0} r_\phi(\mathcal{y}) = 0$, as well as $\lim_{\mathcal{y}\to 0} \mathcal{E}_\phi(\mathcal{y}) = \lim_{\mathcal{y}\to 0} \frac{\phi'(\mathcal{y})}{\phi(\mathcal{y})/\mathcal{y}} = \frac{\phi'(0)}{\phi'(0)} = 1$.

4.2.Profit Maximization by Input Producing Firms under H.D.I.A.

From the inverse demand curve, eq.(21), the profit of firm $\omega \in \Omega$ is given by:

$$\pi(\omega) = \left(B(\mathbf{p})\phi'\left(\frac{x(\omega)}{X}\right) - \psi\right)x(\omega) - F.$$

Firm ω chooses its output, $x(\omega)$, to maximize its profit $\pi(\omega)$, taking the aggregate variables, $B(\mathbf{p})$ and Y as given. Or equivalently, it chooses $\mathcal{y}(\omega) \equiv x(\omega)/Y$ to maximize

$$(B(\mathbf{p})\phi'(\mathcal{y}(\omega)) - \psi)\mathcal{y}(\omega).$$

The FOC is:

$$B(\mathbf{p})[\phi'(\mathcal{y}(\omega)) + \mathcal{y}(\omega)\phi''(\mathcal{y}(\omega))] = B(\mathbf{p})\phi'(\mathcal{y}(\omega))[1 - r_\phi(\mathcal{y}(\omega))] \tag{22}$$
$$= p(\omega)[1 - r_\phi(\mathcal{y}(\omega))] = \psi.$$

In what follows, we keep it simple by imposing the following assumption to ensure that the FOC is sufficient for the global optimum.

Assumption D1: *For all* $\mathcal{y} > 0$,

$$\frac{\mathcal{y}\phi'''(\mathcal{y})}{\phi''(\mathcal{y})} + 2 > 0 \Leftrightarrow \frac{\mathcal{y}r_\phi'(\mathcal{y})}{r_\phi(\mathcal{y})} + 1 - r_\phi(\mathcal{y}) > 0.$$

Under **D1**, the LHS of eq.(22) is strictly decreasing in $\mathcal{y}(\omega)$. Hence, eq.(22) gives the unique profit-maximizing output for each firm. Thus, all firms set the same price, $p(\omega) = p$, and produce the same amount, $x(\omega) = x$. Hence, under **D1**, asymmetric equilibria do not exist. Unlike in the case of H.S.A., the condition that rules out asymmetric equilibria does not ensure the uniqueness of a symmetric equilibrium under H.D.I.A., which needs to be introduced separately; see **D2** below.

4.3.Symmetric Free-Entry Equilibrium under H.D.I.A.

A *symmetric free-entry equilibrium* under H.D.I.A. satisfies the following conditions:

H.D.I.A. integral condition, eq.(18) under symmetry:

$$V\phi\left(\frac{x}{X}\right) = 1; \tag{23}$$

Firm's pricing formula, given by FOC, eq.(22) under symmetry:

$$1 - \frac{\psi}{p} = r_\phi\left(\frac{x}{X}\right), \tag{24}$$

in addition to the zero-profit (free-entry) condition, eq.(9) and the resource constraint, eq.(10).

For the uniqueness of a symmetric equilibrium, we introduce the following condition:

Assumption D2: *For all* $\mathcal{y} > 0$,

$$\frac{\mathcal{y}\phi'''(\mathcal{y})}{\phi''(\mathcal{y})} + 1 + r_\phi(\mathcal{y}) + \mathcal{E}_\phi(\mathcal{y}) > 0 \iff \frac{\mathcal{y}r_\phi{}'(\mathcal{y})}{r_\phi(\mathcal{y})} + \mathcal{E}_\phi(\mathcal{y}) > 0.$$

Clearly, **D1** implies **D2** if

$$\frac{\mathcal{y}\mathcal{E}_\phi{}'(\mathcal{y})}{\mathcal{E}_\phi(\mathcal{y})} = 1 - r_\phi(\mathcal{y}) - \mathcal{E}_\phi(\mathcal{y}) < 0,$$

and **D2** implies **D1**, if

$$\frac{\mathcal{y}\mathcal{E}_\phi{}'(\mathcal{y})}{\mathcal{E}_\phi(\mathcal{y})} = 1 - r_\phi(\mathcal{y}) - \mathcal{E}_\phi(\mathcal{y}) > 0.$$

And, **D1** and **D2** are equivalent if and only if

$$\frac{\mathcal{y}\mathcal{E}_\phi{}'(\mathcal{y})}{\mathcal{E}_\phi(\mathcal{y})} = 1 - r_\phi(\mathcal{y}) - \mathcal{E}_\phi(\mathcal{y}) = 0,$$

that is, under and only under CES.

To see why **D2** ensures the existence and the uniqueness of a symmetric free-entry equilibrium, note first that the pricing formula, eq.(24), and the free entry condition, eq.(9), can be combined to yield:

$$r_\phi(x/X)px = F. \tag{25}$$

From eq.(9) and eq.(10), $pVx = L$, which can be combined with eq.(25) to obtain:

$$\frac{L}{V} = px = \frac{F}{r_\phi(x/X)},$$

which becomes after using the H.D.I.A. condition, eq.(23):

$$r_\phi(x/X)\phi(x/X) = F/L.$$

The LHS of this equation is increasing in x/X, because **D2** implies

$$\frac{d\log[\phi(y)r_\phi(y)]}{d\log y} = \frac{y\phi'''(y)}{\phi''(y)} + 1 + r_\phi(y) + \mathcal{E}_\phi(y) > 0 \text{ for all } y > 0.$$

Furthermore, $\lim_{y\to 0} r_\phi(y)\phi(y) = 0$ and $\lim_{y\to\infty} r_\phi(y)\phi(y) = \infty$. Hence, *for each $L/F > 0$, the equilibrium value of y, y^E, is pinned down uniquely by,*

$$r_\phi(y^E)\phi(y^E) = F/L, \tag{26}$$

and $y^E$is decreasing in L/F, with the range, $(0,\infty)$. By inserting this value into eq.(23), eq.(24), and eq.(9),

$$V^E = \frac{1}{\phi(y^E)};$$

$$p^E = \frac{\psi}{1-r_\phi(y^E)}; \tag{27}$$

$$X^E = \frac{x^E}{y^E} = \frac{L}{y^E p^E V^E} = \frac{[1-r_\phi(y^E)]\phi(y^E)}{\psi y^E}L = \frac{\phi(y^E)L - F}{\psi y^E} > 0.$$

Thus, we have shown:

Proposition 4. *Under **D1**, no asymmetric equilibria exist. Furthermore, under **D1** and **D2**, there exists a unique symmetric free-entry equilibrium under H.D.I.A. for each $L/F > 0$, given by eq.*(26) *and eq.*(27).

4.4.Comparative Statics under H.D.I.A.: Procompetitive versus Anticompetitive

We now turn to the comparative statics.

Proposition 5. *Assume* **D1** *and* **D2**. *At the unique symmetric equilibrium in monopolistic competition under H.D.I.A., given by eq.*(26) *and eq.*(27),

Procompetitive: $$r'_\phi(y^E) > 0 \Rightarrow \frac{\partial p^E}{\partial L} < 0;\ \frac{\partial \log V^E}{\partial \log L} < 1;\ \frac{\partial x^E}{\partial L} > 0$$

Neutral (CES): $$r'_\phi(y^E) = 0 \Rightarrow \frac{\partial p^E}{\partial L} = 0;\ \frac{\partial \log V^E}{\partial \log L} = 1;\ \frac{\partial x^E}{\partial L} = 0$$

Anticompetitive: $$r'_\phi(y^E) < 0 \Rightarrow \frac{\partial p^E}{\partial L} > 0;\ \frac{\partial \log V^E}{\partial \log L} > 1;\ \frac{\partial x^E}{\partial L} < 0.$$

Proof: Since eq.(26) implies $\partial y^E/\partial L < 0$ under **D2**, this follows from eq.(27). ■

The conditions for the procompetitive vs. anticompetitive cases under H.D.I.A. are analogous to those under H.S.A. For example, recall that the condition for the procompetitive case under H.S.A. is $\zeta'(z^E) = \zeta'(p^E/A^E) > 0$, that is, the price elasticity of demand goes *up* as its *price* goes up, holding the aggregates fixed. This is nothing but *Marshall's 2nd law of demand.* Here, under H.D.I.A., the condition is $r'_\phi(y^E) = r'_\phi(x^E/X^E) > 0$; that is, the price elasticity of demand for an input goes *down* as its *quantity* goes up, holding the aggregate fixed. This is another way of stating *Marshall's 2nd law of demand.* Note also that, if the condition for the procompetitive case holds globally, $r'_\phi(\cdot) > 0$, **D1** and **D2** hold automatically. However, neither **D1** nor **D2** necessarily implies $r'_\phi(y) > 0$. This means that **D1** and **D2** do not rule out the anticompetitive case, $r'_\phi(y) < 0$.

4.5. Welfare Analysis under H.D.I.A.: Excessive versus Insufficient

We now turn to the welfare analysis under H.D.I.A. The social planner's problem is to maximize social welfare subject to the resource constraint. From the symmetry and strict quasi-concavity of $X = X(\mathbf{x})$, defined by eq.(18), the solution is clearly symmetric. The problem can be thus stated as:

$$\max_{(x,V)} X \qquad s.t. \;\; (\psi x + F)V = L; \;\; V\phi(x/X) = 1$$

Using $y = x/X$, this can be written as

$$\max_{(x,y)} \frac{x}{y} \qquad s.t. \qquad \psi x = \frac{L}{V} - F = \phi(y)L - F \geq 0$$

or, equivalently, as

$$\max_{y \geq \underline{y}} W(y) \equiv \frac{\phi(y) - F/L}{y}, where\ \underline{y} \equiv \phi^{-1}(F/L) > 0.$$

To make this social planner's problem well-defined, we need to introduce:

Assumption D3: $\lim_{y \to \infty} \mathcal{E}_\phi(y) < 1$.[17]

[17] Assumption **D3** rules out the pathological case, where the social planner can produce an unbounded output, X, by letting $V \longrightarrow 0$ and $x \longrightarrow \infty$. Note that **D3** does not rule out the choke price, which would imply $\lim_{y \to 0} \mathcal{E}_\phi(y) = 1$.

Lemma 4. *Under* **D3,** $W(y)$ *is unimodal,* with

$$W'(y) \gtreqless 0 \Leftrightarrow \frac{F}{L} \gtreqless \phi(y) - \phi'(y)y = \phi(y)[1 - \mathcal{E}_\phi(y)] \Leftrightarrow y \lesseqgtr y^O$$

where y^O is the socially optimal value of y, uniquely given by

$$\phi(y^O) - \phi'(y^O)y^O = \phi(y^O)[1 - \mathcal{E}_\phi(y^O)] = \frac{F}{L}$$

and y^O *is strictly decreasing in* L/F.

Proof: By differentiating $W(y)$, it is easily verified that

$$y^2 W'(y) = \frac{F}{L} - [\phi(y) - \phi'(y)y] = \frac{F}{L} - \phi(y)[1 - \mathcal{E}_\phi(y)],$$

which is strictly decreasing, because

$$\frac{d[\phi(y) - \phi'(y)y]}{dy} = -\phi''(y)y > 0.$$

Furthermore, $\underline{y}^2 W'(\underline{y}) = \underline{y}\phi'(\underline{y}) > 0$ and $y^2 W'(y) < 0$ for a sufficiently large y, because **D3** implies $\phi(y)[1 - \mathcal{E}_\phi(y)] \longrightarrow \infty$ as $y \longrightarrow \infty$. Hence, $W(y)$ reaches its global maximum at $y^O \in (\underline{y}, \infty)$, given by $W'(y^O) = 0 \Leftrightarrow \phi(y^O) - \phi'(y^O)y^O = F/L$, which is strictly decreasing in L/F, and $W'(y) \gtreqless 0 \Leftrightarrow y \lesseqgtr y^O$. ■

We are now ready to state the welfare property of the equilibrium allocation.

Proposition 6. *Assume* **D1, D2** *and* **D3**. *Then, at the unique symmetric equilibrium in monopolistic competition under H.D.I.A.,given by eq.*(26) *and eq.*(27)*,* V^E*, the equilibrium mass of firms that enter = the equilibrium mass of varieties produced and* V^O*, the mass of the optimal mass of firms that enter = the optimal mass of varieties produced,, satisfy*

Excessive Entry: $$\frac{y^E \mathcal{E}_\phi{}'(y^E)}{\mathcal{E}_\phi(y^E)} = 1 - r_\phi(y^E) - \mathcal{E}_\phi(y^E) < 0 \Leftrightarrow V^E > V^O$$

Optimal Entry (CES): $$\frac{y^E \mathcal{E}_\phi{}'(y^E)}{\mathcal{E}_\phi(y^E)} = 1 - r_\phi(y^E) - \mathcal{E}_\phi(y^E) = 0 \Leftrightarrow V^E = V^O$$

Insufficient Entry: $$\frac{y^E \mathcal{E}_\phi{}'(y^E)}{\mathcal{E}_\phi(y^E)} = 1 - r_\phi(y^E) - \mathcal{E}_\phi(y^E) > 0 \Leftrightarrow V^E < V^O$$

Proof: From eq.(26) and Lemma 4,

$$y^E \lesseqgtr y^O \Leftrightarrow r_\phi(y^E)\phi(y^E) = \frac{F}{L} \gtreqless \phi(y^E)[1 - \mathcal{E}_\phi(y^E)] \Leftrightarrow r_\phi(y^E) \gtreqless 1 - \mathcal{E}_\phi(y^E)$$

Since $V^E\phi(y^E) = 1 = V^O\phi(y^O), y^E \lesseqgtr y^O \Leftrightarrow V^E \gtreqless V^O$, this completes the proof. ■

Note that, in order for the equilibrium entry to be optimal for a range of the parameter values under H.D.I.A., $\frac{y\mathcal{E}_{\phi}'(y)}{\mathcal{E}_{\phi}(y)} = 1 - r_{\phi}(y) - \mathcal{E}_{\phi}(y) = 0$ must hold for the relevant range of y, that is, under and only under CES. Thus, CES offers the borderline case between the cases of excessive entry and insufficient entry within H.D.I.A..

4.6.Main H.D.I.A. Theorem and Some Examples

Proposition 5 states that the sign of $r_{\phi}'(y^E)$ determines whether entry is procompetitive or anticompetitive, while Proposition 6 states the sign of $\mathcal{E}_{\phi}'(y^E)$ determines whether entry is excessive or insufficient. Hence, one might think, unlike under H.S.A, that these conditions are unrelated to each other, and that both procompetitive entry and anticompetitive entry can be either excessive or insufficient under H.D.I.A.. However, the next lemma shows that there exists a tight connection between the two conditions.

Lemma 5: *Assume that* $r_{\phi}'(\cdot)$ *does not change sign over* $(0, y_0)$*, where* $0 < y_0 \leq \infty$*. Then, for all* $y \in (0, y_0)$*,*

$$r_{\phi}'(\cdot) \gtreqless 0 \implies \mathcal{E}_{\phi}'(\cdot) \lesseqgtr 0.$$

Proof: See Appendix A. ■

Here is the implication of Lemma 5. Suppose that, for all $L/F > (L/F)_0$, entry is procompetitive at the unique symmetric equilibrium given by eq.(26) and eq.(27). That means that $r_{\phi}'(y) > 0$ for all $y < y_0$, where y_0 satisfies $r_{\phi}(y_0)\phi(y_0)(L/F)_0 = 1$. Then, Lemma 5 tells us $\mathcal{E}_{\phi}'(y) < 0$ for all $y < y_0$. Hence, for all $L/F > (L/F)_0$, entry is excessive at the unique symmetric equilibrium. Likewise, suppose that, for all $L/F > (L/F)_0$, entry is anticompetitive at the unique symmetric equilibrium given by eq.(26) and eq.(27). That means that $r_{\phi}'(y) < 0$ for all $y < y_0$. Then, Lemma 5 tells us $\mathcal{E}_{\phi}'(y) > 0$ for all $y < y_0$. Hence, for all $L/F > (L/F)_0$, entry is insufficient at the unique symmetric equilibrium.

We are now ready to summarize the main properties of H.D.I.A. in the next theorem, by consolidating Propositions 4, 5, and 6 and Lemma 5. In doing so, we take into account that y^E is strictly decreasing in L/F and takes any value in $(0, \infty)$, as L/F varies from zero to infinity, and that the existence of the choke price, $\phi'(0) < \infty$, implies $\lim_{y \to 0} r_\phi(y) = 0$ and $\lim_{y \to 0} \mathcal{E}_\phi(y) = 1$, and hence $r'_\phi(y) > 0$ and $\mathcal{E}'_\phi(y) < 0$ for a sufficiently small y, which means that entry is procompetitive and excessive for a sufficiently large $L/F > 0$.

Theorem 2: *Consider monopolistic competition under symmetric H.D.I.A. with gross substitutes. Assume **D1** to ensure the symmetry of equilibrium and **D2** to ensure the uniqueness of the symmetric equilibrium. Then, the unique symmetric equilibrium is given by eq.*(26) *and eq.*(27). *Assume **D3** to ensure that the planner's problem is well-defined. Then, at the unique symmetric equilibrium, entry is,*

- *procompetitive and excessive for any $L/F > 0$, if $r_\phi'(y) > 0$ for all $y \in (0, \infty)$;*
- *neutral and optimal for any $L/F > 0$, if $r_\phi'(y) = 0$ for all $y \in (0, \infty)$, that is, under CES;*
- *anticompetitive and insufficient for any $L/F > 0$, if $r_\phi'(y) < 0$ for all $y \in (0, \infty)$.*

Furthermore, in the presence of the choke price, $\phi'(0) < \infty$, entry is procompetitive and excessive for a sufficiently large $L/F > 0$.

We now turn to some examples to illustrate Theorem 2.

Example 4; Perturbed CES, H.D.I.A. with global monotonicity. Consider a family of H.D.I.A. technologies, such that

$$r_\phi(y) \equiv -\frac{y\phi''(y)}{\phi'(y)} = \frac{1}{\sigma} + \delta\left(1 - \frac{1}{\sigma}\right) g(y),$$

where $\sigma > 1$, and $g(y)$ satisfies $g'(y) > 0$ for all $y > 0$ with $g(0) = 0$ and $g(\infty) = 1$ and $\sup\{y g'(y) | y > 0\} \equiv \kappa < \infty$. For example,

$$g(y) = \frac{y}{\eta + y}, \eta > 0 \Rightarrow \kappa = \frac{1}{4} < \infty$$

$$g(y) = 1 - e^{-\mu y}, \mu > 0 \Rightarrow \kappa = e^{-1} < \infty$$

satisfy these conditions. In addition, we impose the following restrictions on σ, δ, and κ:

$$-\frac{1}{(1+\kappa)\sigma - 1} < \delta < 1,$$

so that $0 < r_\phi(y) < 1$, **D1, D2,** and **D3** hold.[18] Then, Theorem 2 can be applied. In this example, entry is procompetitive and excessive for all $L/F > 0$ when $0 < \delta < 1$, while it is anticompetitive and insufficient for all $L/F > 0$ when $-\frac{1}{(1+\kappa)\sigma-1} < \delta < 0$.

Example 5: H.D.I.A. with a choke price. Consider a family of H.D.I.A. technologies such that

$$0 < r_\phi(y) \equiv -\frac{y\phi''(y)}{\phi'(y)} < 1,$$

satisfies $r'_\phi(y) > 0$ for all $y > 0$ and

$$\lim_{y\to 0}\int_{y}^{y_0'} \frac{r_\phi(\xi)}{\xi}d\xi < \infty \iff \phi^{'}(0) = \lim_{y\to 0} exp\left[\int_{y}^{y_0'} \frac{r_\phi(\xi)}{\xi}d\xi\right] < \infty,$$

which implies the choke price. For example, $r_\phi(y) = \frac{y}{\eta+y},\ \eta > 0$ and $r_\phi(y) = 1 - e^{-\mu y},\ \mu > 0$, satisfy these conditions. Clearly, **D1**, **D2**, and **D3** are all satisfied, and from Lemma 5, $r'_\phi(y) > 0 \Longrightarrow \mathcal{E}'_\phi(y) < 0$ for all $y > 0$. Hence, entry is always procompetitive and excessive, not just for a sufficiently large L/F.

The assumption of the global monotonicity of $r_\phi(\cdot)$ in Theorem 2, which implies the global monotonicity of $\mathcal{E}_\phi(\cdot)$ by Lemma 5, is important. Otherwise, entry could be procompetitive and yet insufficient, or anticompetitive and yet excessive, as the next example illustrates.

Example 6. Perturbed CES, H.D.I.A. without global monotonicity. Consider a family of H.D.I.A technologies with

$$\mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)} = 1 - \frac{1}{\sigma} + \delta g(y),$$

[18]It is easy to verify $0 < r_\phi(y) < 1$ and **D3**. For **D1** and **D2,** if $\delta \geq 0$, $r'_\phi(y) \geq 0$, which implies both **D1** and **D2**. If $\delta < 0$, $r'_\phi(y) < 0$ for all $y > 0$. From Lemma 5, this implies $\mathcal{E}'_\phi(y) > 0$ for all $y > 0$, which means that **D2** implies **D1.** To verify that **D2** for $\delta < 0$, note that $r'_\phi(y) < 0$ and $\mathcal{E}'_\phi(y) > 0$ for all $y > 0$ implies

$$r_\phi(y)\mathcal{E}_\phi(y) > r_\phi(\infty)\mathcal{E}_\phi(0) = \left(\frac{1}{\sigma} + \delta\left(1 - \frac{1}{\sigma}\right)\right)\left(1 - \frac{1}{\sigma}\right),$$

while $\delta < 0$ and the definition of κ implies

$$yr'_\phi(y) = \delta\left(1 - \frac{1}{\sigma}\right)yg'(y) > \delta\left(1 - \frac{1}{\sigma}\right)\kappa$$

Adding each side of the two inequalities above yields $r_\phi(y)\mathcal{E}_\phi(y) + yr'_\phi(y) > 0$, because $-\frac{1}{(1+\kappa)\sigma-1} < \delta < 0$, which is equivalent to **D2.**

$$r_\phi(y) \equiv -\frac{y\phi''(y)}{\phi'(y)} = 1 - \mathcal{E}_\phi(y) - \frac{y\mathcal{E}_\phi'(y)}{\mathcal{E}_\phi(y)} = \frac{1}{\sigma} + \delta g(y) - \frac{\delta y g'(y)}{1 - \frac{1}{\sigma} + \delta g(y)}.$$

where $\sigma > 1, \delta$ can be either positive or negative (but sufficiently small in absolute value to ensure **D1**, **D2** and **D3**), while $g(y)$ is twice-continuously differentiable, single-peaked, and satisfies $g(0) = g(\infty) = 0,\ \sup|g'(y)| < \infty$. Let $\tilde{y} > 0$ be the maximizer of $g(y)$. Hence, $g'(\tilde{y}) = 0 > g''(\tilde{y})$. For example,

$$g(y) = \frac{y}{\lambda + y^2}, \lambda > 0 \implies \tilde{y} = \sqrt{\lambda};$$

$$g(y) = ye^{-\mu y}, \mu > 0 \implies \tilde{y} = 1/\mu.$$

From Proposition 6, entry is excessive if and only if $\mathcal{E}_\phi'(y^E) = \delta g'(y^E) < 0$, while it is insufficient if and only if $\mathcal{E}_\phi'(y^E) = \delta g'(y^E) > 0$. Evaluating $r_\phi'(y)$ at $y = \tilde{y}$ yields:

$$r_\phi'(\tilde{y}) = -\frac{\tilde{y}\mathcal{E}_\phi''(\tilde{y})}{\mathcal{E}_\phi(\tilde{y})} = -\delta g''(\tilde{y})\frac{\tilde{y}}{\mathcal{E}_\phi(\tilde{y})} \gtreqless 0 \Leftrightarrow \delta \gtreqless 0,$$

Thus, from Proposition 5, entry is procompetitive in the vicinity of $\tilde{y}$, if $\delta > 0$, while it is anticompetitive in the vicinity of $\tilde{y}$, if $\delta < 0$.

Combining these two observations, we conclude that entry is procompetitive and yet insufficient for $\delta > 0$ and y^E slightly higher than $\tilde{y}$, or equivalently, F/L slightly higher than $\phi(\tilde{y})r_\phi(\tilde{y})$, while it is anticompetitive and yet excessive for $\delta < 0$ and y^E slightly lower than $\tilde{y}$, or equivalently F/L slightly lower than $\phi(\tilde{y})r_\phi(\tilde{y})$.

5. Dixit-Stiglitz under H.I.I.A. (Homothetic Indirect Implicit Additivity)

5.1.H.I.I.A. Demand System

We call CRS technology, $X = X(\mathbf{x})$ or $P = P(\mathbf{p})$, *homothetic with indirect implicit additivity* (H.I.I.A.)[19] if $P = P(\mathbf{p})$ can be defined implicitly by:

$$\int_\Omega \theta\left(\frac{p(\omega)}{P(\mathbf{p})}\right)d\omega = 1, \tag{28}$$

[19] More generally, $P = P(\mathbf{p})$ satisfies *indirect implicit additivity* (I.I.A.) if it is defined implicitly by $\int_\Omega \tilde{\theta}(p(\omega), P)d\omega = 1$. See Hanoch (1975; Section 3). Clearly, H.I.I.A. is a subclass of I.I.A., where $\tilde{\theta}(p(\omega), P) = \theta(p(\omega)/P)$. In contrast, $P = P(\mathbf{p})$ satisfies *indirect explicit additivity* (I.E.A.) if it can be written as $P = \mathcal{M}\left(\int_\Omega \theta(p(\omega))d\omega\right)$ where $\mathcal{M}(\cdot)$ is a monotone transformation. Although I.E.A. is another subclass of I.I.A., it cannot be homothetic unless it is CES.

where $\theta(\cdot): \mathbb{R}_+ \to \mathbb{R}_+$ is thrice continuously differentiable, strictly decreasing, and strictly convex, as long as $\theta(z) > 0$ with $\lim_{z\to 0} \theta(z) = \infty$ and $\lim_{z\to \bar{z}} \theta(z) = 0$, where $\bar{z} \equiv \inf\{z > 0 | \theta(z) = 0\}$. Again, we allow for the possibility of $\bar{z} < \infty$, the existence of the choke price, in which case, $\lim_{z\to\bar{z}} \theta'(z) = 0$. If $\bar{z} = \infty$, the choke price does not exist and demand for each input always remains positive for any positive price vector.

In the following analysis, both the elasticity of $\theta(\cdot)$ in its absolute value,

$$\mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} > 0. \tag{29}$$

and the elasticity of $\theta'(\cdot)$ in its absolute value,

$$r_\theta(z) \equiv -\frac{z\theta''(z)}{\theta'(z)} > 1 \tag{30}$$

both defined over $(0, \bar{z})$, play important roles. That $\theta(z)$ is strictly decreasing and strictly positive in $(0, \bar{z})$ ensures $\mathcal{E}_\theta(z) > 0$, and that $\theta(z)$ is strictly decreasing and strictly convex in $(0, \bar{z})$ ensures $r_\theta(z) > 0$. However, the convexity of $\theta(z)$ does not impose any upper bound on $\mathcal{E}_\theta(z)$. In addition, it is necessary to assume $r_\theta(z) > 1$ to ensure that inputs are gross substitutes, as will be seen below. Note that $\mathcal{E}_\theta(z) > 0$ is twice continuously differentiable in $(0, \bar{z})$ and satisfies $\lim_{z\to\bar{z}} \mathcal{E}_\theta(z) = \infty$ if $\bar{z} < \infty$, and that $r_\theta(z) > 1$ is continuously differentiable in $(0, \bar{z})$ and satisfies $\lim_{z\to\bar{z}} r_\theta(z) = \infty$ if $\bar{z} < \infty$. Conversely, either from any twice continuously differentiable $\mathcal{E}_\theta(z) > 0$, defined over $(0, \bar{z})$, satisfying $\lim_{z\to\bar{z}} \mathcal{E}_\theta(z) = \infty$ if $\bar{z} < \infty$ or from any continuously differentiable $r_\theta(z) > 1$, defined over $(0, \bar{z})$, satisfying $\lim_{z\to\bar{z}} r_\theta(z) = \infty$ if $\bar{z} < \infty$, one could recover $\theta(z)$ as follows:

$$\theta(z) = exp\left[-\int_{z_0}^{z} \mathcal{E}_\theta(\xi)\frac{d\xi}{\xi}\right];$$

$$\theta(z) = \int_{z}^{\infty} exp\left[-\int_{z_0'}^{\xi} r_\theta(\xi')\frac{d\xi'}{\xi'}\right] d\xi.$$

where $z_0 > 0$ and $z_0' > 0$ are both constants.[20] One could also verify from eq.(29) and eq.(30) that $\mathcal{E}_\theta(z)$ and $r_\theta(z)$ are related as follows:

$$\frac{z\mathcal{E}_\theta{}'(z)}{\mathcal{E}_\theta(z)} = 1 + \mathcal{E}_\theta(z) - r_\theta(z).$$

Clearly, CES with gross substitutes is a special case with $\theta(z) = Az^{1-\sigma}$ and $\mathcal{E}_\theta(z) + 1 = r_\theta(z) = \sigma > 1$.

The cost minimization problem, eq.(2) subject to eq. (28) implies that the demand curve for each $\omega \in \Omega$ can be written as:

$$x(\omega) = -B^*(\mathbf{x})\theta'\left(\frac{p(\omega)}{P(\mathbf{p})}\right) > 0, \tag{31}$$

where $B^*(\mathbf{x}) > 0$ is the Lagrange multiplier associated with eq.(28), and it is the linear homogenous function in **x**, given by

$$\int_\Omega \theta\left((-\theta')^{-1}\left(\frac{x(\omega)}{B^*(\mathbf{x})}\right)\right)d\omega \equiv 1.$$

From eq.(31), the market share of each input can be written either as a function of **p**, or as a function of **x**, as follows:

$$\frac{p(\omega)}{P(\mathbf{p})}\frac{x(\omega)}{X(\mathbf{x})} = -\theta'\left(\frac{p(\omega)}{P(\mathbf{p})}\right)\frac{p(\omega)}{C(\mathbf{p})} = (-\theta')^{-1}\left(\frac{x(\omega)}{B^*(\mathbf{x})}\right)\frac{x(\omega)}{X(\mathbf{x})},$$

where the production function is given by:

$$X = X(\mathbf{x}) = \int_\Omega (-\theta')^{-1}\left(\frac{x(\omega)}{B^*(\mathbf{x})}\right)x(\omega)d\omega \tag{32}$$

and $C(\mathbf{p})$ is a linear homogenous function of **p**, given by

$$C(\mathbf{p}) \equiv -\int_\Omega \theta'\left(\frac{p(\omega)}{P(\mathbf{p})}\right)p(\omega)d\omega > 0,$$

and it satisfies

$$\frac{C(\mathbf{p})}{P(\mathbf{p})} = -\int_\Omega \theta'\left(\frac{p(\omega)}{P(\mathbf{p})}\right)\frac{p(\omega)}{P(\mathbf{p})}d\omega = \int_\Omega (-\theta')^{-1}\left(\frac{x(\omega)}{B^*(\mathbf{x})}\right)\frac{x(\omega)}{B^*(\mathbf{x})}d\omega = \frac{X(\mathbf{x})}{B^*(\mathbf{x})}.$$

[20] These constants imply that $\theta(z)$is determined up to a positive scalar multiplier. However, $\gamma\theta(z)$ with $\gamma > 0$ generate the same CRS technology. All we need is to renormalize the indices of varieties, as $\int_\Omega \gamma\phi\theta(p(\omega)/P)d\omega = \int_\Omega \gamma\theta(p(\omega')/P)d\omega' = 1$, with $\omega' = \gamma\omega$.

These two expressions for the market share under H.I.I.A. show that it is either a function of the two relative prices, $p(\omega)/P(\mathbf{p})$ and $p(\omega)/C(\mathbf{p})$, or a function of the two relative quantities, $x(\omega)/X(\mathbf{x})$ and $x(\omega)/B^*(\mathbf{x})$, unless $P(\mathbf{p})/C(\mathbf{p}) = B^*(\mathbf{x})/X(\mathbf{x}) = c > 0$ for a constant c, which occurs if and only if it is CES. Thus, H.I.I.A. and H.S.A. do not overlap with the sole exception of CES. [21]

From the demand curve, eq.(31), the elasticity of substitution between a pair of inputs, ω_1 and ω_2, evaluated at the same price (and hence at the same quantity) can be expressed as:

$$-\left.\frac{\partial \ln(x(\omega_1)/x(\omega_2))}{\partial \ln(p(\omega_1)/p(\omega_2))}\right|_{p(\omega_1)=p(\omega_2)=p} = r_\theta\left(\frac{p}{P}\right) > 1,$$

hence $r_\theta(z) > 1$ to ensure that inputs are gross substitutes.

5.2. Profit Maximization by Input Producing Firms under H.I.I.A.

From the demand curve, eq.(31), the profit of firm $\omega \in \Omega$ is given by:

$$\pi(\omega) = -(p(\omega) - \psi)B^*(\mathbf{x})\theta'\left(\frac{p(\omega)}{P(\mathbf{p})}\right) - F.$$

Firm ω chooses its price, $p(\omega)$, to maximize its profit $\pi(\omega)$, taking the aggregate variables, $P = P(\mathbf{p})$ and $B^*(\mathbf{x})$ as given. Or equivalently, it chooses $z(\omega) \equiv p(\omega)/P(\mathbf{p})$ to *minimize*

$$\left(z(\omega) - \frac{\psi}{P(\mathbf{p})}\right)\theta'(z(\omega)).$$

The FOC is:

$$\theta'(z(\omega)) + \left(z(\omega) - \frac{\psi}{P(\mathbf{p})}\right)\theta''(z(\omega)) = z(\omega)\theta''(z(\omega))\left[1 - \frac{\psi}{p(\omega)} - \frac{1}{r_\theta(z(\omega))}\right] = 0. \tag{33}$$

In what follows, we keep it simple by imposing the following assumption to ensure that FOC is sufficient for the global optimum.

Assumption I1: *For all* $z \in (0, \bar{z})$,

$$\frac{\theta'''(z)}{\theta''(z)} - 2\frac{\theta''(z)}{\theta'(z)} > 0 \iff \frac{z r_\theta'(z)}{r_\theta(z)} + r_\theta(z) - 1 > 0.$$

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[21] See Proposition 3-(ii) in Matsuyama and Ushchev (2017). Its Proposition 4-(iii) also shows that H.D.I.A. and H.I.I.A. do not overlap with the sole exception of CES.

I1 is equivalent to the strict concavity of $1/\theta'(\cdot)$. It is readily verified that the LHS of the FOC, eq.(33) increases in the neighborhood of every solution to eq.(33) if and only if **I1** holds. Hence, eq.(33) gives the unique profit-maximizing price for each firm. Thus, all the firms set the same price, $p(\omega) = p$, and produce the same amount, $x(\omega) = x$. Hence, under **I1**, asymmetric equilibria do not exist. Unlike in the case of H.S.A., but as in the case of H.D.I.A., the condition that rules out asymmetric equilibria does not ensure the uniqueness of a symmetric equilibrium under H.I.I.A., which needs to be introduced separately; see **I2** below.

5.3. Symmetric Free-Entry Equilibrium under H.I.I.A.

A *symmetric free-entry equilibrium* under H.I.I.A. satisfies the following conditions:

H.I.I.A. integral condition, eq.(28) under symmetry:

$$V\theta\left(\frac{p}{P}\right) = 1; \tag{34}$$

Firm's pricing formula, given by FOC eq.(33) under symmetry:

$$1 - \frac{\psi}{p} = \frac{1}{r_\theta\left(\frac{p}{P}\right)}, \tag{35}$$

in addition to the zero-profit (free-entry) condition, (9) and the resource constraint, (10).

For the uniqueness of a symmetric equilibrium, we introduce the following condition:

Assumption I2: *For all* $z \in (0, \bar{z})$,

$$\frac{z\theta'''(z)}{\theta''(z)} + 1 + r_\theta(z) + \mathcal{E}_\theta(z) > 0 \iff \frac{z r_\theta{}'(z)}{r_\theta(z)} + \mathcal{E}_\theta(z) > 0.$$

Clearly, **I1** implies **I2** if

$$\frac{z\mathcal{E}_\theta{}'(z)}{\mathcal{E}_\theta(z)} = 1 + \mathcal{E}_\theta(z) - r_\theta(z) > 0,$$

and **I2** implies **I1**, if

$$\frac{z\mathcal{E}_\theta{}'(z)}{\mathcal{E}_\theta(z)} = 1 + \mathcal{E}_\theta(z) - r_\theta(z) < 0,$$

and **I2** and **I1** are equivalent if and only if

$$\frac{z\mathcal{E}_\theta{}'(z)}{\mathcal{E}_\theta(z)} = 1 + \mathcal{E}_\theta(z) - r_\theta(z) = 0,$$

that is, under and only under CES.

To see why **I2** ensures the existence and the uniqueness of a symmetric free-entry equilibrium under H.I.I.A., note first that the pricing formula, eq.(35), and the free entry condition eq.(9) can be combined to yield:

$$px = r_\theta(p/P)F. \tag{36}$$

From eq.(9) and eq.(10), $pVx = L$, which can be combined with eq.(36) to obtain:

$$\frac{L}{V} = px = r_\theta(p/P)F,$$

which becomes after using the H.I.I.A. condition, eq.(34):

$$\frac{\theta(p/P)}{r_\theta(p/P)} = \frac{F}{L}.$$

The LHS of this equation is decreasing in p/P, because **I2** implies

$$\frac{d\log\,[\theta(z)/r_\theta(z)]}{d\log z} = \frac{z\theta'''(z)}{\theta''(z)} + 1 + r_\theta(z) + \mathcal{E}_\theta(z) > 0 \text{ for all } z \in (0, \bar{z})$$

Furthermore, $\lim_{z\to 0}\theta(z)/r_\theta(z) = \infty$ and $\lim_{z\to \bar{z}}\theta(z)/r_\theta(z) = 0$. Hence, *for each* $L/F > 0$, the equilibrium value of z, z^E, is pinned down uniquely by

$$\frac{\theta(z^E)}{r_\theta(z^E)} = \frac{F}{L} \tag{37}$$

and z^E is increasing in L/F with the range $(0, \bar{z})$. By inserting this value into eq.(34), eq.(35), and eq.(9),

$$V^E = \frac{1}{\theta(z^E)};$$

$$P^E = \frac{p^E}{z^E} = \frac{\psi/z^E}{1 - 1/r_\theta(z^E)} > 0; \tag{38}$$

$$x^E = \frac{(r_\theta(z^E) - 1)F}{\psi} = \frac{\theta(z^E)L - F}{\psi} > 0.$$

Thus, we have shown:

Proposition 7. *Under **I1**, no asymmetric equilibria exist. Furthermore, under **I1** and **I2**, there exists a unique symmetric free-entry equilibrium under H.I.I.A. for each* $L/F > 0$, *given by eq.*(37) *and eq.*(38).

5.4. Comparative Statics under H.I.I.A.: Procompetitive versus Anticompetitive

Let us now turn to the comparative statics to study the market size effect.

Proposition 8. *Assume* **I1** and **I2.** *At the unique symmetric equilibrium in monopolistic competition under H.I.I.A., given by eq.*(37) *and eq.*(38)*,*

Procompetitive:
$$r_\theta'(z^E) > 0 \Rightarrow \frac{\partial p^E}{\partial L} < 0; \ \frac{\partial \log V^E}{\partial \log L} < 1; \ \frac{\partial x^E}{\partial L} > 0$$

Neutral (CES):
$$r_\theta'(z^E) = 0 \Rightarrow \frac{\partial p^E}{\partial L} = 0; \ \frac{\partial \log V^E}{\partial \log L} = 1; \ \frac{\partial x^E}{\partial L} = 0$$

Anticompetitive:
$$r_\theta'(z^E) < 0 \Rightarrow \frac{\partial p^E}{\partial L} > 0; \ \frac{\partial \log V^E}{\partial \log L} > 1; \ \frac{\partial x^E}{\partial L} < 0.$$

Proof: Since eq.(37) implies $\partial z^E/\partial L > 0$ under **I2**, this follows from eq.(38). ■

The conditions for the procompetitive vs. anticompetitive cases under H.I.I.A. are analogous to those under H.S.A and H.D.I.A. For example, the condition for the procompetitive case is $\zeta'(z^E) = \zeta'(p^E/A^E) > 0$ under H.S.A., while it is $r_\theta'(z^E) = \zeta'(p^E/P^E) > 0$ under H.I.I.A. That is, the price elasticity of demand for an input goes up as its price goes up, holding the aggregates fixed. This is nothing but *Marshall's 2nd law of demand.* Note also that, if the condition for the procompetitive case holds globally, $r_\theta'(\cdot) > 0$, **I1** and **I2** hold automatically. However, neither **I1** nor **I2** necessarily implies $r_\theta'(z) > 0$. This means that **I1** and **I2** do not rule out the anticompetitive case, $r_\theta'(z) < 0$.

5.5. Welfare Analysis under H.I.I.A.: Excessive versus Insufficient

We now turn to the welfare analysis under H.I.I.A.. The social planner maximizes the output, given by eq.(32), subject to the resource constraint,

$$VF + \psi \int_\Omega x(\omega) d\omega = L.$$

Because of the symmetry and the convexity of this problem, the solution has to be symmetric, $x(\omega) = x$. By denoting

$$z = (-\theta')^{-1}\left(\frac{x}{B(\mathbf{x})}\right),$$

the problem is hence reduced to maximize $X = Vxz$ subject to $V\theta(z) = 1$ and $(F + \psi x)V = L$, or equivalently,

$$\max_{z} X = \max_{0 \le z \le \hat{z}} W(z) \equiv z\left[1 - \frac{F/L}{\theta(z)}\right],$$

where $\hat{z} \equiv \theta^{-1}(F/L) \in (0, \overline{z})$. Clearly, $W(0) = W(\hat{z}) = 0$, and $W(z) > 0$ when $0 < z < \hat{z}$.

Lemma 6. *Assume* **I2**. *Then, $W(z)$ is unimodal, with*

$$W'(z) \gtreqless 0 \Leftrightarrow \frac{L}{F} \gtreqless \frac{1 + \mathcal{E}_\theta(z)}{\theta(z)} \Leftrightarrow z \lesseqgtr z^O$$

where $z^o \in (0, \hat{z})$ is the socially optimal value of z, uniquely given by

$$\frac{L}{F} = \frac{1 + \mathcal{E}_\theta(z^O)}{\theta(z^O)}$$

and z^O is strictly increasing in L/F.

Proof. Differentiating $W(z)$ yields

$$W'(z) = 1 - \frac{F}{L}\frac{1 + \mathcal{E}_\theta(z)}{\theta(z)}.$$

$$W''(z) = \frac{\mathcal{E}_\theta(z)}{z}\left[\frac{r_\theta(z)}{\theta(z)}\frac{F}{L} - 2\big(1 - W'(z)\big)\right].$$

To show that $W(z)$ is unimodal with the unique global optimizer, $z^o \in (0, \hat{z})$ satisfying $W'(z^o) = 0$, suppose the contrary. Then, there exist $0 < z_1 < z_2 < z_3 < \hat{z}$, such that z_1 and z_3 are local maxima satisfying $W'(z_1) = 0 > W''(z_1)$ and $W'(z_3) = 0 > W''(z_3)$ and z_2 is a local minimum satisfying $W'(z_2) = 0 < W''(z_2)$. This implies

$$\frac{r_\theta(z)}{\theta(z)}\frac{F}{L} - 2\big(1 - W'(z)\big) = \frac{r_\theta(z)}{\theta(z)}\frac{F}{L} - 2$$

is negative at z_1 and z_3 and positive at z_2, contradicting the monotonicity of $r_\theta(z)/\theta(z)$, hence **I2.** That z^O is strictly increasing in L/F follows from $W''(z^O) < 0$ and $\partial W'(z^O)/\partial(L/F) > 0$.■

Proposition 9. *Assume* **I1** and **I2**. *Then, at the unique symmetric equilibrium in monopolistic competition under H.I.I.A., given by eq.*(37) *and eq.*(38), *V^E, the equilibrium mass of firms that enter = the equilibrium mass of varieties produced and V^O, the mass of the optimal mass of firms that enter = the optimal mass of varieties produced,, satisfy*

Excessive Entry: $\dfrac{z^E \mathcal{E}_\theta{}'(z^E)}{\mathcal{E}_\theta(z^E)} = 1 + \mathcal{E}_\theta(z^E) - r_\theta(z^E) > 0 \Leftrightarrow V^E > V^O$

Optimal Entry (CES): $\dfrac{z^E \mathcal{E}_\theta{}'(z^E)}{\mathcal{E}_\theta(z^E)} = 1 + \mathcal{E}_\theta(z^E) - r_\theta(z^E) = 0 \Leftrightarrow V^E = V^O$

Insufficient Entry: $\dfrac{z^E \mathcal{E}_\theta{}'(z^E)}{\mathcal{E}_\theta(z^E)} = 1 + \mathcal{E}_\theta(z^E) - r_\theta(z^E) < 0 \Leftrightarrow V^E < V^O$

Proof. Since eq.(37) implies

$$W'(z^E) = 1 - \frac{F}{L}\frac{1 + \mathcal{E}_\theta(z^E)}{\theta(z^E)} = 1 - \frac{1 + \mathcal{E}_\theta(z^E)}{r_\theta(z^E)},$$

and Lemma 6 implies $W'(z^E) \gtreqless 0 \Leftrightarrow z^E \lesseqgtr z^O \Leftrightarrow V^E \lesseqgtr V^O$, we have

$$r_\theta(z^E) - 1 - \mathcal{E}_\theta(z^E) \gtreqless 0 \Leftrightarrow V^E \lesseqgtr V^O.$$

This completes the proof. ■

Note that, in order for the equilibrium entry to be optimal for a range of the parameter values under H.I.I.A, $\frac{z\mathcal{E}_\theta'(z)}{\mathcal{E}_\theta(z)} = 1 + \mathcal{E}_\theta(z) - r_\theta(z) = 0$ must hold for the relevant range of z, that is, under and only under CES. Thus, CES offers the borderline case between the cases of excessive entry and insufficient entry within H.I.I.A.

5.6.Main H.I.I.A. Theorem and Some Examples

Proposition 8 states that the sign of $r_\theta'(z^E)$ determines whether entry is procompetitive or anticompetitive, while Proposition 9 states the sign of $\mathcal{E}_\theta'(z^E)$ determines whether entry is excessive or insufficient. Hence, one might think, unlike under H.S.A, that these conditions are unrelated to each other, and that both procompetitive entry and anticompetitive entry can be either excessive or insufficient under H.I.I.A.. However, similar to the case of H.D.I.A., the next lemma shows that there exists a tight connection between the two conditions.

Lemma 7: *Assume that* $r_\theta'(\cdot)$ *does not change sign over* $(z_0, \overline{z})$*, where* $0 < z_0 < \overline{z}$*. Then, for all* $z \in (z_0, \overline{z})$*,*

$$r_\theta'(\cdot) \gtreqless 0 \;\Rightarrow\; \mathcal{E}_\theta'(\cdot) \gtreqless 0.$$

Proof: See Appendix B. ■

Here is the implication of Lemma 7. Suppose that, for all $L/F > (L/F)_0$, entry is procompetitive at the unique symmetric equilibrium given by eq.(37) and eq.(38). That means that $r_\theta'(z) > 0$ for all $z \in (z_0, \overline{z})$, where z_0 satisfies $(\theta(z_0)/r_\theta(z_0))(L/F)_0 = 1$. Then, Lemma 7 tells us $\mathcal{E}_\theta'(z) > 0$ for all $z \in (z_0, \overline{z})$. Hence, for all $L/F > (L/F)_0$, entry is excessive at the unique symmetric equilibrium. Likewise, suppose that, for all $L/F > (L/F)_0$, entry is anticompetitive at the unique symmetric equilibrium given by eq.(37) and eq.(38). That means $r_\theta'(z) < 0$ for all $z \in (z_0, \overline{z})$. Then, Lemma 7 tells us $\mathcal{E}_\theta'(z) < 0$ for all $z \in (z_0, \overline{z})$. Hence, for all $L/F > (L/F)_0$, entry is insufficient at the unique symmetric equilibrium.

We are now ready to summarize the main properties of H.I.I.A. in the next theorem, by consolidating Propositions 7, 8, and 9 and Lemma 7. In doing so, we take into account that z^E is strictly increasing in L/F and takes any value in $(0, \overline{z})$, as L/F varies from zero to infinity, and that the existence of the choke price, $\overline{z} < \infty$, implies $\lim_{z \to \overline{z}} r_\theta(z) = \infty$ and $\lim_{z \to \overline{z}} \mathcal{E}_\theta(z) = \infty$, and hence $r_\theta{}'(z) > 0$ and $\mathcal{E}_\theta{}'(z) > 0$ for z sufficiently close to $\overline{z}$, which means that entry is procompetitive and excessive for a sufficiently large $L/F > 0$.

Theorem 3: *Consider monopolistic competition under symmetric H.I.I.A. with gross substitutes. Assume **I1** to ensure the symmetry of equilibrium and **I2** to ensure the uniqueness of the symmetric equilibrium. Then, the unique symmetric equilibrium is given by eq.*(37) *and eq.*(38)*. At the unique symmetric equilibrium, entry is,*

- *procompetitive and excessive for any $L/F > 0$, if $r_\theta{}'(z) > 0$ for all $z \in (0, \overline{z})$;*
- *neutral and optimal for any $L/F > 0$, if $r_\theta{}'(z) = 0$ for all $z \in (0, \infty)$; that is, under CES;*
- *anticompetitive and insufficient for any $L/F > 0$, if $r_\theta{}'(z) < 0$ for all $z \in (0, \infty)$.*

Furthermore, in the presence of the choke price, $\overline{z} < \infty$, entry is procompetitive and excessive for a sufficiently large $L/F > 0$.

We now turn to some examples to illustrate Theorem 3.

Example 7: Perturbed CES, H.I.I.A. with global monotonicity. Consider a family of H.I.I.A. technologies, given by

$$r_\theta(z) \equiv -\frac{z\theta''(z)}{\theta'(z)} = \sigma + \delta(\sigma - 1)g(z),$$

where $\sigma > 1$ and $g(z)$ satisfies $g'(z) > 0$ for all $z > 0$ with $g(0) = -1$, $g(\infty) = 0$ and $\sup\{zg'(z)|z > 0\} \equiv \nu < \infty$. For example,

$$g(z) = -\frac{\eta}{\eta + z}, \eta > 0 \implies \nu = \frac{1}{4} < \infty,$$

$$g(z) = -e^{-\mu z}, \mu > 0 \implies \nu = e^{-1} < \infty,$$

satisfy these conditions. In addition, we impose the following restriction on σ, δ, and ν:

$$-\frac{\sigma}{\nu} < \delta < 1,$$

so that $r_\theta(z) > 1$, **I1**, and **I2** hold.[22] Then, Theorem 3 can be applied. In this example, entry is procompetitive and excessive for all $L/F > 0$, when $0 < \delta < 1$, while it is anticompetitive and insufficient for all $L/F > 0$, when $-\frac{\sigma}{\nu} < \delta < 0$.

Example 8: H.I.I.A. with a choke price. Consider an H.I.I.A. technology, given by

$$\theta(z) = \begin{cases} \dfrac{(\log(\overline{z}/z))^{1+\delta}}{1+\delta}, 0 < z < \overline{z}, \\ 0, \qquad\qquad\qquad z \geq \overline{z}, \end{cases}$$

with $0 < \overline{z} < \infty$ and $\delta > 0$. For all z such that $0 < z < \overline{z}$, we have

$$\theta'(z) = -\frac{(\log(\overline{z}/z))^{\delta}}{z} < 0; \quad \theta''(z) = \frac{(\log(\overline{z}/z))^{\delta}}{z^2}\left[1 + \frac{\delta}{\log(\overline{z}/z)}\right] > 0;$$

Hence, $\lim_{z \to \overline{z}} \theta(z) = \lim_{z \to \overline{z}} \theta'(z) = 0$. Also,

$$\mathcal{E}_\theta(z) = \frac{1+\delta}{\log(\overline{z}/z)} > 0; \quad r_\theta(z) = 1 + \frac{\delta}{\log(\overline{z}/z)} > 1,$$

which implies $\lim_{z \uparrow \overline{z}} \mathcal{E}_\theta(z) = \infty = \lim_{z \uparrow \overline{z}} r_\theta(z)$. Furthermore, for all z such that $0 < z < \overline{z}$, we have:

$$\mathcal{E}_\theta'(z) > 0, \qquad r_\theta'(z) > 0,$$

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[22] It is easy to verify $r_\theta(z) > 1$ holds. For **I1** and **I2**, if $\delta \geq 0$, $r_\theta'(z) \geq 0$, which implies both **I1** and **I2**. If $\delta < 0$, $r_\theta'(z) < 0$ for all $z > 0$. From Lemma 7, this implies $\mathcal{E}_\theta'(z) < 0$ for all $z > 0$, which means that **I2** implies **I1**. To verify **I2** for $\delta < 0$, note that $r_\theta'(z) < 0$ and $\mathcal{E}_\theta'(z) < 0$ for all $z > 0$ implies

$$r_\theta(z)\mathcal{E}_\theta(z) > r_\theta(\infty)\mathcal{E}_\theta(\infty) = \sigma(\sigma - 1),$$

while $\delta < 0$ and the definition of ν imply

$$zr_\theta{}'(z) = \delta(\sigma - 1)zg'(z) > \nu\delta(\sigma - 1).$$

Adding each side of these two inequalities yields $r_\theta(z)\mathcal{E}_\theta(z) + zr_\theta{}'(z) > (\sigma + \nu\delta)(\sigma - 1) > 0$, which is equivalent to **I2.**

and hence entry is always procompetitive and excessive, not just for a sufficient large L/F.

The assumption of the global monotonicity of $r_\theta(\cdot)$ in Theorem 3, which implies the global monotonicity of $\mathcal{E}_\theta(\cdot)$ by Lemma 7, is important. Otherwise, entry could be procompetitive and yet insufficient, or anticompetitive and yet excessive, as the next example illustrates.

Example 9. Perturbed CES, H.I.I.A. without global monotonicity. Consider a family of H.I.I.A technologies with

$$\mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} = \sigma - 1 + \delta g(z),$$

$$r_\theta(z) \equiv -\frac{z\theta''(z)}{\theta'(z)} = 1 + \mathcal{E}_\theta(z) - \frac{z\mathcal{E}_\theta'(z)}{\mathcal{E}_\theta(z)} = r_\theta(z) = \sigma + \delta g(z) - \frac{\delta z g'(z)}{\sigma - 1 + \delta g(z)},$$

where $\sigma > 1$, δ can be either positive or negative (but sufficiently small in absolute value to ensure **I1** and **I2**), while $g(z)$ is twice-continuously differentiable, single-peaked, and satisfies $g(0) = g(\infty) = 0,\ \sup|g'(z)| < \infty$. Let $\tilde{z} > 0$ be the maximizer of $g(z)$ Hence, $g'(\tilde{z}) = 0 > g''(\tilde{z})$. For example,

$$g(z) = \frac{z}{\lambda + z^2}, \lambda > 0 \Longrightarrow \tilde{z} = \sqrt{\lambda};$$

$$g(z) = ze^{-\mu z}, \mu > 0 \Longrightarrow \tilde{z} = 1/\mu.$$

From Proposition 9, entry is excessive if and only if $\mathcal{E}_\theta'(z^E) = \delta g'(z^E) > 0$, while it is insufficient if and only if $\mathcal{E}_\theta'(z^E) = \delta g'(z^E) < 0$. Evaluating $r_\theta'(z)$ at $z = \tilde{z}$ yields:

$$r_\phi'(\tilde{z}) = -\frac{\tilde{z}\mathcal{E}_\theta''(\tilde{z})}{\mathcal{E}_\theta(\tilde{z})} = -\delta g''(\tilde{z})\frac{\tilde{z}}{\mathcal{E}_\theta(\tilde{z})} \gtreqless 0 \Leftrightarrow \delta \gtreqless 0.$$

Thus, from Proposition 8, entry is procompetitive in the vicinity of $\tilde{z}$, if $\delta > 0$, while it is anticompetitive in the vicinity of $\tilde{z}$, if $\delta < 0$.

Combining these two observations, we conclude that entry is procompetitive and yet insufficient for $\delta > 0$ and z^E slightly higher than $\tilde{z}$, or equivalently, L/F slightly higher than $r_\theta(\tilde{z})/\theta(\tilde{z})$, while it is anticompetitive and yet excessive for $\delta < 0$ and z^E slightly lower than $\tilde{z}$, or equivalently, L/F slightly higher than $r_\theta(\tilde{z})/\theta(\tilde{z})$.

6. Conclusions

In this paper, we extended the canonical model of monopolistic competition with symmetric homothetic CES demand system with gross substitutes by Dixit and Stiglitz (1977, Section I) to three classes of homothetic demand systems, H.S.A., H.D.I.A., and H.I.I.A, which are mutually exclusive except that each class contains CES as a knife-edge case. These extensions allowed us to identify the condition for procompetitive vs. anticompetitive entry, as well as the condition for excessive vs. insufficient entry. Among the main findings are that entry is excessive (insufficient) if it is globally procompetitive (anticompetitive) and that, in the presence of the choke price, entry is procompetitive and excessive at least for a sufficiently large market size. One implication is that, if procompetitive entry is the empirically relevant case, entry is excessive, which means that (small) regulation of entry is welfare-improving, at least in the absence of other forms of distortion.[23]

One natural next step would be to follow the footsteps of Dhingra and Morrow (2019) to address the nature of inefficiency in the free-entry equilibrium under these three classes in the presence of heterogenous firms. It should be noted that the sources of heterogeneity may matter. Whether firms are heterogenous in productivity *a la* Melitz or in weights in the demand systems, though they are isomorphic to each other under CES, could make differences under non-CES.

The monopolistic competition models under these three classes of homothetic demand systems developed in this paper, being so tractable, should also find many applications. In particular, homotheticity makes it easier to use it as a building block in dynamic general equilibrium settings. Indeed, in Matsuyama and Ushchev (2020), we develop a dynamic monopolistic competition model under H.S.A. to investigate how market size affects the dynamics of innovation through the procompetitive effect and find it to be as tractable as the CES case, with much richer implications.

[23]An open question is whether these results can be extended to general symmetric homothetic demand systems with gross substitutes. The difficulty for proving them more generally is to find the conditions that ensure the existence and uniqueness of symmetric free-entry equilibrium, as well as the conditions that ensure global monotonicity of the markup rate response. Unaware of any counter example, we conjecture that these results hold more generally. This is why we indicate the possibility of procompetitive and yet insufficient entry and anticompetitive and yet excessive entry by small gray zones in Figure 2.

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Appendix A: Proof of Lemma 5

We first prove the two preliminary lemmas, Lemma A1 and Lemma A2.

Lemma A1. *For any $\phi(\cdot)$ which is strictly increasing, strictly concave, and satisfies $\phi(0) = 0$,*

$$\mathcal{E}_\phi(0) = 1 - r_\phi(0).$$

Proof. From $\phi(0) = 0$ and $0 < y\phi'(y) < \phi(y)$ for all $y > 0$, $\lim_{y\to 0} y\phi'(y) = 0$. It is thus legitimate to use the l'Hospital's rule in computing the following limit:

$$\mathcal{E}_\phi(0) \equiv \lim_{y\downarrow 0} \mathcal{E}_\phi(y) = \lim_{y\downarrow 0} \frac{y\phi'(y)}{\phi(y)} = \lim_{y\downarrow 0} \frac{\phi'(y) + y\phi''(y)}{\phi'(y)} = 1 - r_\phi(0).$$

This completes the proof. ■

Lemma A2. *Assume that $r'_\phi(\cdot)$ does not change sign over $(0, y_0)$, where $0 < y_0 \leq \infty$. Then, for all $y \in (0, y_0)$,*

$$\mathcal{E}_\phi(y) \lesseqqgtr \mathcal{E}_\phi(0) \Leftrightarrow r'_\phi(y) \gtreqqless 0.$$

Proof. Two cases may arise.

Case 1: $r_\phi(0) < 1$. First, define:

$$\Delta r_\phi(y) \equiv r_\phi(y) - r_\phi(0)$$

to obtain the identity,

$$\frac{d\log\phi'(y)}{d\log y} \equiv -r_\phi(y) = -r_\phi(0) - \Delta r_\phi(y),$$

integrating which yields

$$\phi'(y) = f(y)y^{-r_\phi(0)}, \qquad f(y) \equiv \exp\left\{-\int_{y_0}^{y} \frac{\Delta r_\phi(\xi)}{\xi} d\xi\right\} > 0;$$

$$\phi(y) = \int_0^y \phi'(\xi)d\xi = \int_0^y f(\xi)\xi^{-r_\phi(0)} d\xi.$$

Hence,

$$\mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)} = \frac{f(y)y^{1-r_\phi(0)}}{\int_0^y f(\xi)\xi^{-r_\phi(0)} d\xi}.$$

By the mean value theorem, there exists $\alpha(y) \in (0, y)$, such that

$$\int_0^y f(\xi)\xi^{-r_\phi(0)} d\xi = f(\alpha(y)) \int_0^y \xi^{-r_\phi(0)} d\xi = \frac{1}{1 - r_\phi(0)} f(\alpha(y)) y^{1-r_\phi(0)}.$$

Hence, using the definition of $f(y)$ and Lemma A1,

$$\mathcal{E}_\phi(y) = \left(1 - r_\phi(0)\right)\frac{f(y)}{f(\alpha(y))} = \mathcal{E}_\phi(0)\exp\left\{-\int_{\alpha(y)}^{y}\frac{\Delta r_\phi(\xi)}{\xi}d\xi\right\}.$$

Then, for all $y \in (0, y_0)$, $0 < \alpha(y) < y < y_0$ implies

$$\mathcal{E}_\phi(y) \lesseqgtr \mathcal{E}_\phi(0) \Leftrightarrow \int_{\alpha y}^{y}\frac{\Delta r_\phi(\xi)}{\xi}d\xi \gtreqless 0 \Leftrightarrow r'_\phi(y) \gtreqless 0.$$

Case 2: $r_\phi(0) = 1$. This happens only when $r'_\phi(y) < 0$ for all $y \in (0, y_0)$, because $r_\phi(y) < 1 = r_\phi(0)$ for all $y \in (0, y_0)$. And from Lemma A1, $\mathcal{E}_\phi(y) > 0 = \mathcal{E}_\phi(0)$ for all $y \in (0, y_0)$.

This completes the proof. ■

We are now ready to prove Lemma 5.

Lemma 5. *Assume that* $r'_\phi(\cdot)$ *does not change sign over* $(0, y_0)$*, where* $0 < y_0 \le \infty$. *Then, for all* $y \in (0, y_0)$

$$r'_\phi(\cdot) \gtreqless 0 \implies \mathcal{E}'_\phi(\cdot) \lesseqgtr 0.$$

Proof. Three cases may arise.

Case 1: $r'_\phi(y) > 0$ for all $y \in (0, y_0)$. To prove by contradiction, suppose to the contrary that that there is $y_1 \in (0, y_0)$, such that $\mathcal{E}'_\phi(y_1) \ge 0$. Two sub-cases may arise.

Case 1-1: $\mathcal{E}'_\phi(y_1) > 0$. Because Lemma A2 implies $\mathcal{E}_\phi(y_1) < \mathcal{E}_\phi(0)$, $\mathcal{E}_\phi(y)$ must have an interior local minimizer $y_2 \in (0, y_1)$, which satisfies

$$\mathcal{E}'_\phi(y_2) = 0, \qquad \mathcal{E}''_\phi(y_2) \ge 0.$$

Differentiating the identity

$$\mathcal{E}'_\phi(y) = \frac{\mathcal{E}_\phi(y)}{y}\left(1 - \mathcal{E}_\phi(y) - r_\phi(y)\right),$$

at $y = y_2$ and using $\mathcal{E}'_\phi(y_2) = 0$, we obtain:

$$\mathcal{E}''_\phi(y_2) = -\frac{\mathcal{E}_\phi(y_2)}{y_2}r'_\phi(y_2) < 0,$$

which clearly contradicts $\mathcal{E}''_\phi(y_2) \ge 0$.

Case 1-2: $\mathcal{E}'_\phi(y_1) = 0$. In this case, differentiating the identity

$$\mathcal{E}'_\phi(y) = \frac{\mathcal{E}_\phi(y)}{y}\left(1 - \mathcal{E}_\phi(y) - r_\phi(y)\right),$$

at $y = y_1$ yields

$$\mathcal{E}''_\phi(y_1) = -\frac{\mathcal{E}_\phi(y_1)}{y_1} r'_\phi(y_1) < 0.$$

Therefore, for a small $h > 0$ we have: $\mathcal{E}'_\phi(y_1 - h) > \mathcal{E}'_\phi(y_1) = 0$. By replacing y_1 with $y_1 - h$, we use the same argument as in case 1-1.

Thus, we have:

$$r'_\phi(y) > 0 \ \ for\ all\ y \in (0, y_0) \Rightarrow \ \mathcal{E}'_\phi(y) < 0 \ \ for\ all\ y \in (0, y_0).$$

Case 2: $r'_\phi(y) = 0$ for all $y \in (0, y_0)$ This is the CES case, which is straightforward.

Case 3: $r'_\phi(y) < 0$ for all $y \in (0, y_0)$. One can handle this case along the same lines as case 1.

This completes the proof. ■

Appendix B: Proof of Lemma 7

We first prove the three preliminary lemmas, Lemma B1, Lemma B2, and Lemma B3.

Lemma B1. $\bar{z}\theta'(\bar{z}) = 0$.

Proof. For $\bar{z} < \infty$, this follows from $\theta'(\bar{z}) = 0$. For $\bar{z} = \infty$,

$$\theta(z) = -\int_z^\infty \theta'(\xi)d\xi = -\int_z^\infty \frac{\xi\theta'(\xi)}{\xi} d\xi = -\lim_{x\to\infty} \int_z^x \frac{\xi\theta'(\xi)}{\xi} d\xi$$

Suppose that there is $z_0 > 0$ such that, for all $z > z_0$, $-z\theta'(z) > c > 0$. Then,

$$\theta(z_0) = -\lim_{x\to\infty} \int_{z_0}^x \frac{\xi\theta'(\xi)}{\xi} d\xi > \lim_{x\to\infty} \int_{z_0}^x \frac{c}{\xi} d\xi = \infty,$$

a contradiction. Hence, $\bar{z}\theta'(\bar{z}) = \lim_{z\to\infty} z\theta'(z) = 0$. This completes the proof. ■

Lemma B2. *For any* $\theta(\cdot)$ *which defines an H.I.I.A. technology, we have*:

$$\mathcal{E}_\theta(\bar{z}) = r_\theta(\bar{z}) - 1,$$

where $0 < \bar{z} \equiv \inf\{z > 0 \mid \theta(z) = 0\} \leq \infty$.

Proof. Since $\theta(\bar{z}) = 0 = \bar{z}\theta'(\bar{z})$ by Lemma B1, it is legitimate to use the l'Hospital's rule in computing the following limit:

$$\mathcal{E}_\theta(\bar{z}) \equiv \lim_{z\uparrow\bar{z}} \mathcal{E}_\theta(z) = \lim_{z\uparrow\bar{z}} \frac{-z\theta'(z)}{\theta(z)} = \lim_{z\uparrow\bar{z}} \frac{-z\theta''(z) - \theta'(z)}{\theta'(z)} = r_\theta(\bar{z}) - 1.$$

This completes the proof. ■

Lemma B3. *Assume that* $r'_\theta(\cdot)$ *does not change sign over* $(z_0, \bar{z})$, *where* $0 < z_0 < \bar{z}$. *Then, for all* $z \in (z_0, \bar{z})$,

$$\mathcal{E}_\theta(z) \lesseqgtr \mathcal{E}_\theta(\bar{z}) \Leftrightarrow r'_\theta(z) \gtreqless 0.$$

Proof. If $r_\theta(\overline{z}) = \infty$, then the only possibility is that $r_\theta'(\cdot) > 0$. In this case, by Lemma B2, we have: $\mathcal{E}_\theta(\overline{z}) = \infty > \mathcal{E}_\theta(z)$ for all $z \in (0, \overline{z})$. Consider now the case when $1 < r_\theta(\overline{z}) < \infty$, hence $\overline{z} = \infty$. Two cases may arise.

Case 1: $r_\theta(\infty) > 1$. First, define:

$$\Delta r_\theta(z) \equiv r_\theta(\infty) - r_\theta(z),$$

to obtain the identity,

$$\frac{d\log[-\theta'(z)]}{d\log z} = \Delta r_\theta(z) - r_\theta(\infty),$$

integrating which yields

$$-\theta'(z) = f(z) z^{-r_\theta(\infty)}, \qquad f(z) \equiv \exp\left\{\int_{z_0}^{z} \frac{\Delta r_\theta(\xi)}{\xi} d\xi\right\} > 0;$$

$$\theta(z) = -\int_z^\infty \theta'(\xi) d\xi = \int_z^\infty f(\xi)\xi^{-r_\theta(\infty)} d\xi.$$

Hence,

$$\mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} = \frac{f(z) z^{1-r_\theta(\infty)}}{\int_z^\infty f(\xi)\xi^{-r_\theta(\infty)} d\xi}.$$

By the mean value theorem, there exists $\beta(z) > z$, such that

$$\int_z^\infty f(\xi)\xi^{-r_\theta(\infty)} d\xi = f(\beta(z)) \int_z^\infty \xi^{-r_\theta(\infty)} d\xi = \frac{1}{r_\theta(\infty) - 1} f(\beta(z)) z^{1-r_\theta(\infty)}.$$

Hence, using the definition of $f(z)$ and Lemma B2 for $\overline{z} = \infty$,

$$\mathcal{E}_\theta(z) = (r_\theta(\infty) - 1)\frac{f(z)}{f(\beta(z))} = \mathcal{E}_\theta(\infty) \exp\left\{-\int_z^{\beta(z)} \frac{\Delta r_\theta(\xi)}{\xi} d\xi\right\}.$$

Then, for all $z > z_0$, $\beta(z) > z > z_0$ implies

$$\mathcal{E}_\theta(z) \lesseqgtr \mathcal{E}_\theta(\infty) \Leftrightarrow \int_z^{\beta(z)} \frac{\Delta r_\theta(\xi)}{\xi} d\xi \gtreqless 0 \Leftrightarrow r_\theta'(\cdot) \gtreqless 0.$$

Case 2: $r_\theta(\infty) = 1$. This happens only when $r_\theta'(z) < 0$ for all $z \in (z_0, \infty)$, because $r_\theta(z) > 1$ for all $z \in (z_0, \infty)$. From Lemma B2, $\mathcal{E}_\theta(z) > 0 = \mathcal{E}_\theta(\infty)$ for all $z \in (z_0, \infty)$.

This completes the proof. ■

We are now ready to prove Lemma 7.

Lemma 7. *Assume that* $r_\theta'(\cdot)$ *does not change sign over* $(z_0, \overline{z})$, *where* $0 < z_0 < \overline{z}$. *Then, for all* $z \in (z_0, \overline{z})$,

$$r'_\theta(\cdot) \gtreqless 0 \;\Rightarrow\; \mathcal{E}'_\phi(\cdot) \gtreqless 0.$$

Proof. Three cases may arise.

Case 1: $r'_\theta(z) > 0$ for all $z \in (z_0, \overline{z})$. To prove by contradiction, suppose to the contrary that there is $z_1 \in (z_0, \overline{z})$, such that $\mathcal{E}'_\theta(z_1) \leq 0$. Two sub-cases may arise.

Case 1-1: $\mathcal{E}'_\theta(z_1) < 0$. Because Lemma B3 implies $\mathcal{E}_\theta(z_1) < \mathcal{E}_\theta(\overline{z})$, $\mathcal{E}_\theta(\cdot)$ must have an interior local minimizer $z_2 \in (z_1, \overline{z})$, which satisfies

$$\mathcal{E}'_\theta(z_2) = 0, \qquad \mathcal{E}''_\theta(z_2) \geq 0.$$

Differentiating the identity

$$\mathcal{E}'_\theta(z) = \frac{\mathcal{E}_\theta(z)}{z}\big(1 + \mathcal{E}_\theta(z) - r_\theta(z)\big),$$

at $z = z_2$ and using $\mathcal{E}'_\theta(z_2) = 0$, we obtain:

$$\mathcal{E}''_\theta(z_2) = -\frac{\mathcal{E}_\theta(z_2)}{z_2} r'_\theta(z_2) < 0,$$

which clearly contradicts $\mathcal{E}''_\theta(z_2) \geq 0$.

Case 1-2: $\mathcal{E}'_\theta(z_1) = 0$. In this case, differentiating the identity

$$\mathcal{E}'_\theta(z) = \frac{\mathcal{E}_\theta(z)}{z}\big(1 + \mathcal{E}_\theta(z) - r_\theta(z)\big),$$

at $z = z_1$ yields

$$\mathcal{E}''_\theta(z_1) = -\frac{\mathcal{E}_\theta(z_1)}{z_1} r'_\theta(z_1) < 0.$$

Therefore, for a small $h > 0$ we have: $\mathcal{E}'_\theta(z_1 + h) < \mathcal{E}'_\phi(z_1) = 0$. By replacing z_1 with $z_1 + h$, we use the same argument as in case 1-1.

Thus, we have:

$$r'_\theta(z) > 0 \;\; for\ all\ z \in (z_0, \overline{z}) \Rightarrow \; \mathcal{E}'_\theta(z) > 0 \;\; for\ all\ z \in (z_0, \overline{z}).$$

Case 2: $r'_\theta(z) = 0$ for all $z \in (z_0, \overline{z})$. This is the CES case, which is straightforward.

Case 3: $r'_\theta(z) < 0$ for all $z \in (z_0, \overline{z})$. One can handle this case along the same lines as case 1.

This completes the proof. ■

Figure 1.

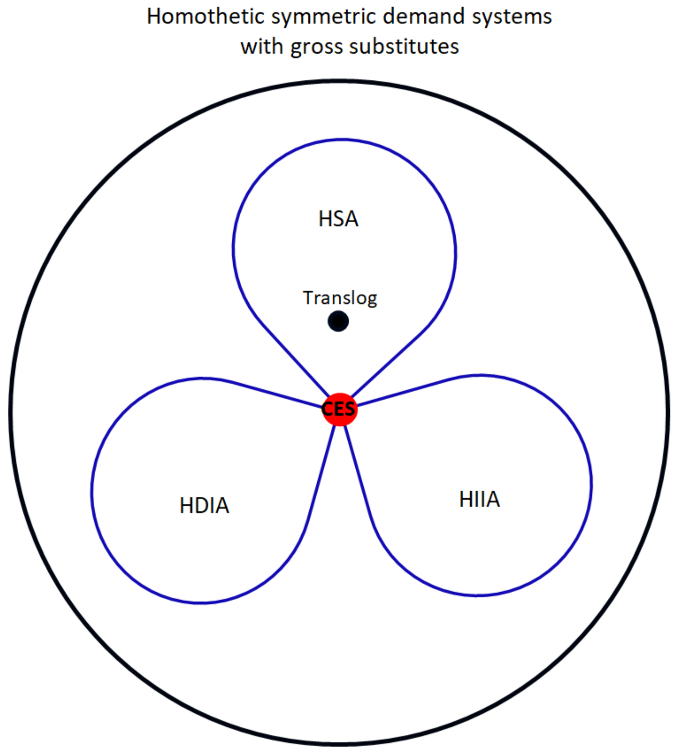

Figure 2.