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NON-CES AGGREGATORS: A GUIDED TOUR

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Abstract

The CES aggregator and its demand system are ubiquitous in business cycles theory, macroeconomic growth and development, international trade and other general equilibrium fields, because it has many knife-edge properties, which help to keep the analysis tractable in the presence of many goods and factors. However, this also makes it hard to tell which properties of CES are responsible for certain results. Furthermore, it is necessary to relax some of them for certain applications. In this article, I review several classes of non-CES aggregators, each of which removes some properties of CES and keeps the rest to introduce some flexibility while retaining the tractability of CES as much as possible. These classes are named after the properties of CES they keep. I explain how these classes are related to each other and discuss their relative strengths and weaknesses to indicate which classes are suited for which applications.

JEL Classification: N/A

Keywords: Nonhomothetic preferences

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Abstract The CES aggregator and its demand system are ubiquitous in business cycles theory, macroeconomic growth and development, international trade and other general equilibrium fields, because it has many knife-edge properties, which help to keep the analysis tractable in the presence of many goods and factors. However, this also makes it hard to tell which properties of CES are responsible for certain results. Furthermore, it is necessary to relax some of them for certain applications. In this article, I review several classes of non-CES aggregators, each of which removes some properties of CES and keeps the rest to introduce some flexibility while retaining the tractability of CES as much as possible. These classes are named after the properties of CES they keep. I explain how these classes are related to each other and discuss their relative strengths and weaknesses to indicate which classes are suited for which applications.

Keywords: Aggregators; Nonhomothetic Preferences; CRS Production Functions; Implicit Additivity, Gross Substitutes vs Gross Complements, Essentials vs. Inessentials

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1. Introduction

The scope of this article can be summarized by Figure. It shows that the Constant-Elasticity-of-Substitution (CES) aggregator is an intersection of many different classes of aggregators, and that we could depart from CES in many different directions.

We all know CES and love using it. CES is ubiquitous in business cycles theory, macroeconomic growth and development, international trade and other general equilibrium fields. Most researchers in these fields use CES almost everywhere they need some kinds of aggregators (preferences, production functions, matching functions, externalities, etc.), since it has many knife-edge properties, which help to keep the analysis tractable even in the presence of many goods and factors. But precisely because it has many properties, it is hard to tell which properties are responsible for particular results. Moreover, it is desirable or even necessary to relax some of them for certain applications. Yet, we may want to relax just a few at a time, while keeping the rest. This is important not only to retain the tractability of CES as much as possible, but also to understand implications of departing from CES in different directions.

A large number of studies has already attempted to depart from CES. However, I find many of them problematic for several reasons. First, many people tend to use a particular alternative to CES repeatedly for all purposes. For example, Stone-Geary is a favorite of many whenever they need non-CES, even though Stone-Geary is just one of many and it has its own limitations. Translog is another example. Quite often, I can think of better options, depending on the goal of the analysis. Second, the relation between different classes of non-CES aggregators is poorly understood. For example, some studies use a demand system that belongs to the DEA class and claim that it is general enough to encompass all homothetic demand systems. In some other studies, the authors use CES and yet claim that the results are more general because they carry over to any demand system that satisfies a particular set of assumptions, despite that CES is the only demand system that satisfies that set of assumptions.

The fact that such claims are frequently made and repeated by others indicates the need for a Guided Tour, which collects in one place many results on different classes of non-CES, which are scattered in the literature over the past 60 years. However, my aim is not just to write a Guided Tour, by explaining how they are related. I also aim to highlight some key features of different classes, both their strengths and weaknesses and to indicate which classes are well-suited for what purposes, providing a sort of a User's Guide.

Before proceeding, three caveats should be mentioned. First, most demand systems reviewed have found many applications, but my main goal is to explain the relation between different demand systems and their relative merits. For this reason, several applications are cited but their findings are not discussed in detail, unless they shed lights on the relative merits of different demand systems. Furthermore, no applications to monopolistic competition are even cited for the reasons explained in the concluding section. Second, although the materials covered here are technical in nature, I try to keep the discussion as non-technical as possible. I offer some intuition behind the main results, but I provide no formal proofs and skip most derivations. Also, some regularity assumptions, such as continuity and differentiability, are not explicitly stated. This review should thus never be considered as a substitute for consulting the references cited. Finally, my goal is to clarify. Hence, I do not hesitate to drop some original terminologies in favor of alternatives, whenever I judge that they are so uninformative and/or misleading that they have become constant sources of confusion.[1]

2. Standard CES

Let us start with CES of the following form and its monotone transformation.[2]

$$U(\mathbf{x}) = \left[\sum_{i=1}^{n} (\beta_i)^{\frac{1}{\sigma}} (x_i)^{1-\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}. \tag{1}$$

For the moment, let us interpret $U(\mathbf{x})$ as the direct utility function. Thus, $x_i \geq 0$ is consumption of good $i \in I = \{1,2,\dots,n\}$ with $\mathbf{x} = (x_1,\dots,x_n) \in \mathbb{R}_+^n$ being the consumption vector; $\beta_i > 0$ is the share-shift parameter of $i \in I$, and $\sigma \in (0,1) \cup (1,\infty)$ is the (constant) elasticity of substitution.[3] Let $\mathbf{p} = (p_1,\dots,p_n) \in \mathbb{R}_+^n$ denote the price vector. Then, maximizing $U(\mathbf{x})$ subject to the budget constraint, $\mathbf{px} = \sum_{i\in I} p_i x_i \leq E$ yields the CES demand

$$x_i = \frac{\beta_i (p_i)^{-\sigma} E}{\sum_{k\in I} \beta_k (p_k)^{1-\sigma}} = \frac{\beta_i (p_i)^{-\sigma} E}{\left(P(\mathbf{p})\right)^{1-\sigma}} = \beta_i \left(\frac{p_i}{P(\mathbf{p})}\right)^{-\sigma} U(\mathbf{x}), \tag{2}$$

where $P(\mathbf{p})$ is the cost-of-living index given by

[1]Two such examples are “generalized CES” and “additive preferences.”

[2]Even though Arrow et al. (1961) proposed CES as the CRS production function with two factors, capital and labor, and many subsequent studies on non-CES, e.g., Sato (1975, 1977), restrict themselves on the two-factor cases, I focus on non-CES aggregators defined over an arbitrary number of factors or goods.

[3]Although Leontieff, Cobb-Douglas, and Linear preferences may be viewed as special cases of CES with $\sigma = 0, \sigma = 1$ and $\sigma = \infty$, I do not discuss them separately for the readability.

$$P(\mathbf{p}) \equiv \min_{\mathbf{x}\in\mathbb{R}_+^n}\{\mathbf{px}|U(\mathbf{x}) \geq 1\} = \left[\sum_{i=1}^{n} \beta_i (p_i)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}. \tag{3}$$

From these, the budget share of $i \in I$ and the indirect utility function are obtained as

$$m_i \equiv \frac{p_i x_i}{E} = \beta_i \left(\frac{p_i}{P(\mathbf{p})}\right)^{1-\sigma} = (\beta_i)^{\frac{1}{\sigma}} \left(\frac{x_i}{U(\mathbf{x})}\right)^{1-\frac{1}{\sigma}}; \tag{4}$$

$$U\left(\frac{\mathbf{p}}{E}\right) \equiv \frac{1}{P(\mathbf{p}/E)} = \left[\sum_{i=1}^{n} \beta_i \left(\frac{E}{p_i}\right)^{\sigma-1}\right]^{\frac{1}{\sigma-1}}. \tag{5}$$

Some notable properties of the standard CES are:

- Income elasticity of demand for each good is one. No good is neither a necessity nor a luxury. This is due to the *homotheticity* of CES.
- Marginal rate of substitution between any two goods, and hence their relative inverse demand, $p_i/p_j = \frac{\partial U(\mathbf{x})/\partial x_i}{\partial U(\mathbf{x})/\partial x_j} = \left[(x_i/x_j)/(\beta_i/\beta_j)\right]^{-1/\sigma}$, is *independent* of the quantity of a third good. This is due to the *directly explicit additivity* (DEA) of CES. Furthermore, x_i and x_j enter only as the ratio, x_i/x_j.
- Relative demand between any two goods, $x_i/x_j = (\beta_i/\beta_j)(p_i/p_j)^{-\sigma}$, is *independent* of the price of a third good. This is due to its *indirectly explicit additivity* (IEA) of CES, as defined later. Furthermore, p_i and p_j enter only as the ratio, p_i/p_j.
- The elasticities of substitution between all pairs of goods are *identical* across all pairs, and the price elasticity of demand for each good, holding $P(\mathbf{p})$ fixed, is constant and *identical*;[4]
- All goods are either *gross complements* (i.e., m_i is increasing in its relative price, $p_i/P(\mathbf{p})$), for $\sigma < 1$ or *gross substitutes* (i.e., m_i is decreasing in $p_i/P(\mathbf{p})$) for $\sigma > 1$.
- All goods are either *essential* (i.e., $p_i \to \infty$ implies $P(\mathbf{p}) \to \infty$) for $\sigma < 1$, or *inessential* (i.e., $p_i \to \infty$ implies $P(\mathbf{p}) < \infty$) for $\sigma > 1$.
- If the goods are gross substitutes, they cannot be essential under CES.

[4]As discussed by Uzawa (1962), McFadden (1963), and Blackorby and Russell (1981), the elasticities of substitution between every pair being constant *implies* the common elasticity of substitution, and that the price elasticity of demand for each good is constant (and common). The reverse is not true. The price elasticities of demand for each good being constant implies *neither* that all goods share the common price elasticity *nor* that the elasticity of substitution between every pair is constant.

- Demand for any good remains strictly positive when its relative price becomes arbitrarily high (*No choke price*).
- Demand for any good goes up unbounded when its relative price becomes arbitrarily low (*No satiation*).
- With $\sigma \neq 1$, one could set $\beta_i = 1$ by choosing the unit of measurement of each good appropriately; the standard CES can be assumed to be symmetric without loss of generality.

These features of CES make it highly tractable, which explains its popularity. CES possesses a high degree of symmetry. The impact on the relative demand and the relative price between the two goods can be studied independently of what happens to other goods. This feature makes CES tractable even when it is defined over an arbitrarily large number of goods. No choke price/no satiation means that we do not need to worry about a corner solution. Knowing the local properties of demand, say whether the goods are gross complements or gross substitutes, is enough to know its global properties, say, whether the goods are essentials or not. And being characterized effectively by one parameter, σ, simplifies the task of estimating and calibrating.

However, precisely because CES has so many properties, it is hard to tell which ones are responsible for certain results. Moreover, these features make CES restrictive and inflexible.[5] We certainly do not need all these features every time we need some types of aggregators somewhere in our models. Yet, we do not need to drop all of them. Instead, we may want to drop just a few at a time, not only to retain the tractability of CES as much as possible, but also because which features should be dropped and which features should be kept depend on the goal of the analysis. For some purposes, we may need goods to differ in their income elasticities, not in their price elasticities. For other purposes, we may need goods to differ in their price

[5]One often used quick fix to remove some restrictions of CES is nested CES, going back at least to Sato (1967). See, e.g., Caselli (2016) for an extensive use of nested CES. Indeed, any multi-sector models, where the intersectoral demand is given by the representative consumer with CES preferences, and each sector produces its output using a CES production function or a variety of goods aggregated by CES, such as multi-sector Eaton & Kortum (2002) models by Costinot et al. (2012) and Caliendo & Parro (2015), effectively use nested CES. And it works well for some purposes. For example, the relative demand for skilled vs unskilled labor depends on the price of capital, if capital and skilled labor is in the same nest and unskilled labor is not (Krusell et al. 2000). Nevertheless, nested CES inherits much of the restrictive features of CES since CES are its building blocks. And any flexibility of nested CES is entirely due to how goods are partitioned into different nests, not due to the flexible functional forms. For example, the elasticities of substitution between all pairs of goods within the same nest are identical; Relative demand between two goods in the same nest is independent of the prices of a third good; Some combinations of essential and inessential goods are ruled out; Essential goods cannot be gross substitutes, etc. Moreover, we can use any of homothetic non-CES aggregators discussed later as building blocks in a nested structure. Such "nested homothetic non-CES" can do everything nested CES can do and more.

elasticities, not in their income elasticities. For some applications, we may need a mixture of gross complements and gross substitutes, or a mixture of essentials and inessentials. For some other applications, we may want to avoid the local properties of demand systems dictating their global properties, etc.

The questions are then: how do we depart from CES and make it more flexible in some dimensions, while maintaining the restrictive features in the others to keep its tractability as much as possible? And how do we achieve this systematically? To this goal in mind, I organize this review by different classes of non-CES, each of which is defined and named by a particular property of the standard CES it maintains.

3. Direct Explicit Additivity (DEA) and Indirect Explicit Additivity (IEA)[6]

Let us start with the following three properties. (In what follows, $\mathcal{M}[\cdot]$ denotes a monotone transformation.)

Direct Explicit Additivity (DEA): Preferences are called *directly explicitly additive* (DEA), if the *direct* utility function, $U(\mathbf{x})$, is *explicitly additive*:

$$U(\mathbf{x}) = \mathcal{M}\left[\sum_{i\in I} \bar{u}_i(x_i)\right], \tag{6}$$

where $\bar{u}_i(\cdot), i \in I,$ satisfy some additional conditions to ensure that $U(\mathbf{x})$ is strictly increasing and quasi-concave.

Indirect Explicit Additivity (IEA): Preferences are called *indirectly explicitly additive* (IEA), if the *indirect* utility function, $U(\mathbf{p}/E),$ is *explicitly additive* as follows:

$$U\left(\frac{\mathbf{p}}{E}\right) \equiv \frac{1}{P(\mathbf{p}/E)} = \mathcal{M}\left[\sum_{i\in I} \bar{v}_i\left(\frac{p_i}{E}\right)\right], \tag{7}$$

[6]Following Hanoch (1975), I distinguish four types of additivity, DEA, IEA, DIA, and IIA. Being the first type of additivity introduced in the literature, DEA is often called simply as "additive" without any qualifier. This common practice unfortunately created the false impression that IEA, DIA, and IIA were special cases of DEA. Quite the contrary, DEA is a special case of DIA and disjoint with IEA and IIA with the sole exception of CES, as shown in Figure. Likewise, IEA is often called simply as "indirectly additive," which created the false impression that IIA were a special case of IEA. Again, quite the contrary, IEA is a special case of IIA. These common practices have become frequent sources of confusion. To avoid such confusion, I refer to these two classes of preferences only by *directly explicitly additive* (DEA) and *indirectly explicitly additive* (IEA).

where $\bar{v}_i(\cdot), i \in I$, satisfy some additional conditions to ensure that $U(\mathbf{p}/E)$ is strictly decreasing and quasi-convex, or equivalently that $P(\mathbf{p}/E)$ is strictly increasing and quasi-concave.

Homotheticity: Preferences are called *homothetic*, if the *direct* utility function $U(\mathbf{x})$ can be represented as a monotone transformation of a linear homogenous function of $\mathbf{x}$ as follows: $U(\mathbf{x}) = \mathcal{M}[X(\mathbf{x})]$, where $X(\mathbf{x})$ satisfies $X(\lambda\mathbf{x}) = \lambda X(\mathbf{x})$ for any $\lambda > 0$.

Clearly from eq.(1) and eq.(3), CES satisfies all three properties, and hence belongs to the three classes labelled as DEA, IEA and Homothetic in Figure. Furthermore, CES is the only intersection of DEA and Homothetic (Bergson's Law). Samuelson (1965) showed that CES is also the only intersection of DEA and IEA. Berndt & Christensen (1973, Theorem 6) showed that it is also the only intersection of IEA and Homothetic. These three classes are thus pairwise disjoint with the sole exception of CES, as shown in Figure, and hence offer three alternative ways of departing from CES.

In the remainder of this section, we discuss DEA and IEA in detail. We will turn to the Homothetic class in section 5.

3.1. Direct Explicit Additivity (DEA):

From eq.(6), it is easy to show that DEA satisfies the following properties.

- Marginal rate of substitution between any two goods and hence their relative inverse demand is *independent* of the quantity of a third good:

$$\frac{p_i}{p_j} = \frac{\partial U(\mathbf{x})/\partial x_i}{\partial U(\mathbf{x})/\partial x_j} = \frac{\bar{u}_i{}'(x_i)}{\bar{u}_j{}'(x_j)}.$$

From this expression, the inverse demand curve for good $i \in I$ can be derived as $p_i = \frac{\bar{u}_i{}'(x_i)E}{\sum_j \bar{u}_j{}'(x_j)x_j}$.

- The relative inverse demand is not a function of x_i/x_j, and hence a proportional increase in x_i and x_j changes p_i/p_j, unless $\bar{u}_i(\cdot), i \in I$, are all power functions with a common exponent, i.e., unless it is CES.

This in turn implies:

- DEA is homothetic if and only if CES (Bergson's Law), as indicated in Figure.

Many non-CES commonly used in the literature belong to DEA.

Example 1: Quasi-Linear

$$U(\mathbf{x}) = \mathcal{M}\left[x_k + \sum_{i \neq k}^{n} u_i(x_i)\right],$$

where $u_i(x_i), i \neq k$ are all strictly concave. The income elasticity of k is one, and those of $i \neq k$ are zero.

Example 2: Distance to the Bliss Points

$$U(\mathbf{x}) = -\sum_{i=1}^{n} \beta_i (b_i - x_i)^{1+\delta},$$

for $0 < x_i < b_i$ where $\delta > 0$. (This one does not satisfy strict monotonicity.) For $\delta = 1$, this is the negative of the quadratic loss function.

Example 3: Stone-Geary[7]

$$U(\mathbf{x}) = \left[\sum_{i=1}^{n} (\beta_i)^{\frac{1}{\sigma}} (x_i - \bar{x}_i)^{1-\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},$$

or equivalently,

$$U(\mathbf{x}) = \sum_{i=1}^{n} \frac{\left(\breve{\beta}_i\right)^{1/\sigma} (x_i - \bar{x}_i)^{1-1/\sigma}}{1 - 1/\sigma},$$

for $x_i \geq \min\{\bar{x}_i, 0\}$, where $\bar{x}_i > 0$ may be interpreted as the subsistence level of consumption of good i and $-\bar{x}_i > 0$ as the nontransferable endowment of good i. With the budget constraint, $\sum_{i=1}^{n} p_i x_i \leq E$, the demand takes form of:

$$m_i \equiv \frac{p_i x_i}{E} = B_i(\mathbf{p}) + \frac{\Gamma_i(\mathbf{p})}{E},$$

where $\sum_{i=1}^{n} B_i(\mathbf{p}) = 1$ and $\sum_{i=1}^{n} \Gamma_i(\mathbf{p}) = 0$ for E large enough to ensure $m_i > 0$ for all $i \in I$.

Until recently, Stone-Geary was by far the most commonly-used nonhomothetic preferences in the growth, trade and development fields; see, e.g., Caselli & Ventura (2000), Kongsamut et al. (2001), Markusen (1986, 2013), and Matsuyama (1992, 2009), just to name a few. In fact, it was so common that some people use "Stone-Geary" as synonymous with "nonhomothetic."

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[7]The original Stone-Geary, $U(\mathbf{x}) = \sum_{i=1}^{n} \beta_i \ln(x_i - \bar{x}_i)$, was proposed as a departure from Cobb-Douglas.

Here are some key properties of Stone-Geary:

- The budget share of i (its *average* propensity to consume) is decreasing in E (i.e., a necessity) for $\Gamma_i(\mathbf{p}) > 0$, and increasing in E (i.e., a luxury) for $\Gamma_i(\mathbf{p}) < 0$.
- The *marginal* propensity to consume, $\partial(p_i x_i)/\partial E = B_i(\mathbf{p})$, is independent of E, which allow for aggregation across households with different expenditure.
- *Asymptotically homothetic*; nonhomotheticity is quantitatively important only for poor households/countries. This feature is not only inconsistent with the evidence of stable slopes of Engel's curves (Comin et al. 2021), but also makes Stone-Geary difficult to fit the long-run data (Buera and Kaboski 2009).
- The price elasticity of demand for a necessity (a luxury) is increasing (decreasing) in E.
- The key parameters, $\bar{x}_i$, are defined in quantity of good i, hence not unit-free. One could thus choose the unit of each good so that $\bar{x}_i = 1, = 0$, or $= -1$, without loss of generality. In other words, Stone-Geary cannot meaningfully distinguish more than three goods in terms of their income elasticities.
- If two or more goods have a subsistence level of consumption, say, $\bar{x}_1 > 0$ and $\bar{x}_2 > 0$, its domain cannot be extended unambiguously to $0 \le x_1 < \bar{x}_1$ and $0 \le x_2 < \bar{x}_2$.

Example 4:

$$U(\mathbf{x}) = -\sum_{i=1}^{n} \tilde{\beta}_i \exp(-\alpha_i x_i),$$

which can be viewed as a limit of Stone-Geary as $\sigma \to 0$ and $\bar{x}_i = (1 - 1/\sigma)/\alpha_i \to -\infty$. It implies that $\alpha_i x_i - \alpha_j x_j$ is independent of E.

Examples 2, 3, and 4, are often called the Pollak (1971) family or Linear Expenditure Systems (LES).[8] They all imply the marginal propensity to consume each good is constant and hence have nice aggregation properties across households with different total expenditures and that they are all asymptotically homothetic.[9]

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[8] Not to be confused with Linear Demand Systems (LDS), derived from linear-quadratic $U(\mathbf{x}) = \sum_{i=1}^{n} \delta_i x_i - (1/2)\sum_{i,j=1}^{n} \gamma_{ij} x_i x_j$. LDS are not DEA, unless $\gamma_{ij} = \gamma_{ji} = 0$ for all $i \neq j$.

[9] LES is not the only demand system that allows for aggregation across households with different total expenditures. One example is price independent generalized linearity (PIGL) proposed by Muellbauer (1975, 1976) and recently

Example 5: Constant Ratios of Elasticities of Substitution (CRES)

$$U(\mathbf{x}) = \left[\sum_{i=1}^{n} (\beta_i)^{\frac{1}{\sigma_i}} (x_i)^{1-\frac{1}{\sigma_i}}\right]^{\frac{\sigma_0}{\sigma_0 - 1}}; \ \frac{\sigma_i - 1}{\sigma_0 - 1} > 0,$$

or equivalently,

$$U(\mathbf{x}) = \sum_{i=1}^{n} \frac{\check{\beta}_i (x_i)^{1-1/\sigma_i}}{1 - 1/\sigma_i}; \ \sigma_i \neq 1.$$

Houthakker (1960) called this "direct addilog." Let η_i be the income elasticity of i and σ_{ij} be the Allen/Uzawa[10] elasticity of substitution between i and j. Then, for any $i \neq j \neq k \in I$,

$$\sigma_{ij} = \frac{\sigma_i \sigma_j}{\bar{\sigma}}; \ \frac{\eta_i}{\eta_j} = \frac{\sigma_{ik}}{\sigma_{jk}} = \frac{\sigma_i}{\sigma_j},$$

where $\bar{\sigma} \equiv \sum_{l=1}^{n} m_l \sigma_l$ is the budget-share weighted average of $\{\sigma_l\}$. Notice that σ_{ij} is not constant, because $\bar{\sigma}$ is not. Yet, the ratio, σ_{ik}/σ_{jk}, is constant, σ_i/σ_j. For this reason, Mukerji (1963) called it Constant Ratios of Elasticity of Substitution (CRES). Note that η_i/η_j is also constant, σ_i/σ_j. For this reason, Caron et al. (2014) called it Constant Relative Income Elasticity (CRIE). Unlike the Pollak family, nonhomotheticity does not disappear as the expenditure goes up. However, the (constant) ratio of income elasticities between any two goods is always equal to the (constant) ratio of their price elasticities. This makes it unclear whether any results obtained by departing from CES within CRES = CRIE should be interpreted as due to the income elasticity differences, as Fieler (2011) and Caron et al. (2014, 2020) did, or due to the price elasticity differences.[11]

Indeed, this is a general feature of DEA, as shown by Houthakker (1960), Goldman & Uzawa (1964), Hanoch [1975; Eq.(2.11)], among others.

Pigou's Law: Under DEA, for any $i \neq j \neq k \in I$,

applied by Boppart (2014). Another is the hierarchical demand system discussed later. These demand systems are not asymptotically homothetic.

[10]Hicks originally defined the elasticity of substitution for $n = 2$. For $n > 2$, there are related but alternative definitions. Allen/Uzawa is one; Morishima is another; Blackorby & Russell (1981, 1989) on this issue.

[11]Although Fieler (2011) and Caron et al. (2014, 2020) performed some robustness checks using alternative classes of nonhomothetic preferences, income and price elasticities are still tightly linked under those alternatives with the exception of isoelastic nonhomothetic CES (Example 7) used in Caron et al. (2020).

$$\frac{\eta_i}{\eta_j} = \frac{\sigma_{ik}}{\sigma_{jk}}.$$

Clearly, Bergson's Law is a special case. Pigou's Law also explains why, with Quasi-linear preferences (Example 1), the income elasticity of k is one, and those of $i \neq k$ are zero, and why, with Stone-Geary (Example 3), the relative price elasticity of luxury goods must be decreasing in the total expenditure, because their income elasticities are also decreasing in the total expenditure, due to its asymptotic homotheticity. It is also the reason behind the (well-known but counter-intuitive) result that the optimal commodity taxation, which should taxes the goods with *lower price elasticity* more heavily, should tax the goods with *lower income elasticity* more heavily; see, e.g., Auerbach (1985) and Chari & Kehoe (1999).

Pigou's Law is not only rejected empirically (Deaton 1974). It also suggests a limitation of using DEA as an attempt to introduce more flexibility to CES. Under DEA, the effects of the income elasticity differences across goods cannot be disentangled from those of the price elasticity differences.

3.2 Indirect Explicit Additivity (IEA)

From eq.(7), it is easy to show that IEA satisfies the following properties.

- Relative demand for any two goods is *independent* of the price of any other goods, because

$$\frac{x_i}{x_j} = \frac{\partial U(\mathbf{p}/E)/\partial p_i}{\partial U(\mathbf{p}/E)/\partial p_j} = \frac{\bar{v}_i{}'(p_i/E)}{v_j{}'(p_j/E)}.$$

From this expression, the demand curve for good $i \in I$ can be derived as $x_i = \frac{\bar{v}_i{}'(p_i/E)}{\sum_j (p_j/E)\bar{v}_j{}'(p_j/E)}$.[12]

- Relative demand is neither independent of E, nor a function of p_i/p_j, hence a change in E and a proportional increase in p_i and p_j shift x_i/x_j, unless $\bar{v}_i(\cdot), i \in I$, are all power functions with a common exponent, i.e., unless it is CES.

This in turn implies

- IEA is homothetic if and only if CES, as indicated in Figure.

Example 6: Constant Differences of Elasticities of Substitution (CDES)

[12]Note that, if $\bar{v}_i{}'(p_i/E) < 0$ for $0 < p_i/E < z_i < \infty$; $= 0$ for $p_i/E \geq z_i$, z_iE is the choke price. However, it is easy to see that $z_i < \infty$ for all i would violate the monotonicity of preferences. Thus, contrary to the claim often made, it is not possible for every good to have a choke price under IEA.

$$U\left(\frac{\mathbf{p}}{E}\right) = \left[\sum_{i=1}^{n} \beta_i \left(\frac{E}{p_i}\right)^{\sigma_i - 1}\right]^{\frac{1}{\sigma_0 - 1}}; \ \frac{\sigma_i - 1}{\sigma_0 - 1} > 0,$$

or equivalently,

$$U\left(\frac{\mathbf{p}}{E}\right) = \sum_{i=1}^{n} \frac{\tilde{\beta}_i (E/p_i)^{\sigma_i - 1}}{\sigma_i - 1}; \ \sigma_i \neq 1.$$

Houthakker (1960) called this "indirect addilog." Hanoch (1975) called it CDE. Jensen et al. (2011) discuss its properties and history of its use in detail. Analogously to CRES = CRIE, one may also call it Constant Difference of Income Elasticities (CDIE), due to the following relation between income and price elasticities imposed by IEA, which I call it in want of better name,

Indirect Pigou's Law. Houthakker (1960) and Hanoch (1975; Eq.(3.11)). Let η_i denote the income elasticity of i and σ_{ij} denote the Allen/Uzawa elasticity of substitution between i and j. Then, under IEA,

$$\sigma_{ik} - \sigma_{jk} = \eta_i - \eta_j$$

for any $i \neq j \neq k \in I$.

Again, like DEA, the effects of the income elasticity differences and those of the price elasticity differences cannot be disentangled under IEA.

Both (direct and indirect) Pigou's Laws show a limitation of *explicit additivity* and of the DEA and IEA classes of demand systems. Of course, there are many preferences that belong neither to DEA nor IEA. For example, the linear-quadratic direct utility function,

$$U(\mathbf{x}) = \sum_{i=1}^{n} \delta_i x_i - \frac{1}{2} \sum_{i,j=1}^{n} \gamma_{ij} x_i x_j$$

or CDES, augmented by the Stone-Geary subsistence consumption shifters,

$$U\left(\frac{\mathbf{p}}{E}\right) = \left[\sum_{i=1}^{n} \beta_i \left(\frac{E - \sum_{j=1}^{n} p_j \bar{x}_j}{p_i}\right)^{\sigma_i - 1}\right]^{\frac{1}{\sigma_0 - 1}},$$

used by Święcki (2017), are not explicitly additive due to the presence of the interactive terms. While these interactive terms add more flexibility, these functional forms still impose tight links between the income and price elasticities. So is the Almost Ideal Demand System (AIDS) proposed by Deaton & Muellbauer (1980) and applied recently by e.g., Fajgelbaum &

Khandelwal (2016), in which both income and price elasticities are controlled by the same parameters.

4. Direct Implicit Additivity (DIA), Indirect Implicit Additivity (IIA) and Implicit CES

The restrictive nature of explicit additivity motivated Hanoch (1975) to introduce the weaker notion of *implicit additivity*, which makes it possible to control for the income and price elasticity difference across goods separately.

4.1 Direct Implicit Additivity (DIA) and Indirect Implicit Additivity (IIA)

Let us now introduce two weaker properties, DIA and IIA, and the two classes of demand systems they define.[13]

Direct Implicit Additivity (DIA): Preferences are called *directly implicitly additive* (DIA), if the *direct* utility function, $U(\mathbf{x})$**,** is *implicitly additive*:

$$\mathcal{M}\left[\sum_{i=1}^{n}\tilde{u}_i(x_i,U)\right]=const. \tag{8}$$

where $\tilde{u}_i(\cdot,\cdot), i\in I$, satisfy some additional conditions for strict monotonicity and quasi-concavity of $U(\mathbf{x})$. DEA is a subclass of DIA, with $\tilde{u}_i(x_i,U)=\bar{u}_i(x_i)g(U)$.

Indirect Implicit Additivity (IIA): Preferences are called *indirectly implicitly additive* (IIA), if the *indirect* utility function, $U(\mathbf{p}/E)$**,** is *implicitly additive*:

$$\mathcal{M}\left[\sum_{i=1}^{n}\tilde{v}_i\left(\frac{p_i}{E},U\right)\right]=const. \tag{9}$$

where $\tilde{v}_i(\cdot,\cdot), i\in I$, satisfy some additional conditions for strict monotonicity and quasi-convexity of $U(\mathbf{p}/E)$. IEA is a subclass of IIA, where $\tilde{v}_i(p_i/E\,,U)=\bar{v}_i(p_i/E)h(U)$.

Implicit additivity has clear advantage relative to explicit additivity.[14] It allows us to control for the price elasticity difference and the income elasticity difference across goods

[13]Hanoch (1975) defines DIA as $\sum_{i=1}^{n}\tilde{u}_i(x_i,U)=1$ and IIA as $\sum_{i=1}^{n}\tilde{v}_i(p_i/E\,,U)=1$. Though the definitions, eq.(8) and eq.(9), are equivalent, they offer some flexibility, which turn out to be convenient for some applications.

[14]Some people seem to view that any implicitly-defined direct or indirect utility functions as in eq.(8) and eq.(9) are *illegitimate*. My response is that many commonly used functions are defined implicitly, for example, log is defined as an inverse of an exponential function, and arctangent is defined as an inverse of a tangent function.

separately. For DIA, the price elasticity depends on the curvature of $\tilde{u}_i(x_i, U)$ with respect to x_i and the income elasticity on the curvature of $\tilde{u}_i(x_i, U)$ with respect to U; in particular, the two elasticities can be controlled separately for $\tilde{u}_i(x_i, U) = \bar{u}_i(x_i)g_i(U)$. Similarly for IIA, the price elasticity depends on the curvature of $\tilde{v}_i(p_i/E\,, U)$ with respect to p_i, and the income elasticity on the curvature of $\tilde{v}_i(p_i/E\,, U)$ with respect to U; in particular, the two elasticities can be controlled separately for $\tilde{v}_i(p_i/E\,, U) = \bar{v}_i(p_i/E)h_i(U)$.

4.2 Nonhomothetic CES

Due to such flexibility of implicit additivity, the standard CES is not the sole member of the intersection of DIA and IIA. Indeed, Hanoch (1975) showed that implicit CES defined below satisfies both DIA and IIA. Furthermore, implicit CES is the only demand system that satisfies both DIA and IIA, as illustrated in Figure.[15] More formally,

Implicit CES: Preferences are called *implicit CES*, if the *direct* utility function, $U(\mathbf{x})$**,** is defined *implicitly* as follows:

$$\left[\sum_{i=1}^{n} (\beta_i(U))^{\frac{1}{\sigma(U)}} (x_i)^{1-\frac{1}{\sigma(U)}}\right]^{\frac{\sigma(U)}{\sigma(U)-1}} \equiv 1,$$

where $\sigma(U) > 0;\ \neq 1$, and $\beta_i(U) > 0, i \in I$, are functions of U and must satisfy some additional conditions to ensure that $U(\mathbf{x})$ is strict monotonic and quasi-concave; see Fally (2022; A4). Its indirect utility function, $U(\mathbf{p}/E)$, is written implicitly by:

$$\left[\sum_{i=1}^{n} \beta_i(U) \left(\frac{p_i}{E}\right)^{1-\sigma(U)}\right]^{\frac{1}{1-\sigma(U)}} \equiv 1;$$

and its cost-of-living index, $P(\mathbf{p}, U)$, is by:

$$\left[\sum_{i=1}^{n} \frac{\beta_i(U)}{U^{1-\sigma(U)}} \left(\frac{p_i}{P}\right)^{1-\sigma(U)}\right]^{\frac{1}{1-\sigma(U)}} \equiv 1$$

and $U = U(\mathbf{p}/E)$ and $P = P(\mathbf{p}, U)$ satisfy the identity, $PU = E$.

[15] I am not aware of any existing proof of this. However, it follows from the proof of Proposition 4(iii) in Matsuyama & Ushchev (2017); Though this proposition states that HDIA and HIIA (homothetic restrictions of DIA and IIA defined later) imply homothetic CES, homotheticity does not play any role in the proof.

This class of preferences is *nonhomothetic* whenever $\partial \ln \beta_i(U)/\partial \ln U$ depend on i and/or $\sigma(U)$ depend on U. Nevertheless, they are CES in that the Hicksian demand generated is indistinguishable with those generated by the standard CES, because the Hicksian demand is calculated for a fixed level of the utility.[16]

Among this class, the following parametric family found many applications in the structural transformation literature: see Bohr et al. (2021), Comin et al. (2021), Cravino & Sotelo (2019), Fujiwara & Matsuyama (2022), Lewis et al. Zhang (2022), Matsuyama (2019), and Sposi et al. (2021), among others.

Example 7: Isoelastic Nonhomothetic CES: $\sigma(U) = \sigma > 0; \neq 1$ and $\beta_i(U) = \beta_i(U)^{\varepsilon_i - \sigma}$, where $\varepsilon_i > 0$ are constants, so that $\partial \ln \beta_i(U)/\partial \ln U = \varepsilon_i - \sigma$. Then, $U(\mathbf{x})$ is given implicitly as:

$$\left[\sum_{i=1}^{n} (\beta_i)^{\frac{1}{\sigma}} \big(U(\mathbf{x})\big)^{\frac{\varepsilon_i - \sigma}{\sigma}} (x_i)^{1-\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \equiv 1, \tag{10}$$

where $\sigma > 0$ ensures global quasi-concavity, while global monotonicity requires $(\varepsilon_i - \sigma)/(1-\sigma) > 0$.[17] By maximizing $U(\mathbf{x})$ subject to $\sum_{i=1}^{n} p_i x_i \leq E$, the budget shares are:

$$m_i = \frac{\beta_i (U)^{\varepsilon_i - \sigma} (p_i)^{1-\sigma}}{\sum_{k=1}^{n} \beta_k (U)^{\varepsilon_k - \sigma} (p_k)^{1-\sigma}} = \beta_i (U)^{\varepsilon_i - \sigma} \left(\frac{p_i}{E}\right)^{1-\sigma} = \beta_i \left(\frac{E}{P}\right)^{\varepsilon_i - 1} \left(\frac{p_i}{P}\right)^{1-\sigma}, \tag{11}$$

where indirect utility, $U = U(\mathbf{p}/E)$, is implicitly given by

$$\left[\sum_{i=1}^{n} \beta_i (U)^{\varepsilon_i - \sigma} \left(\frac{p_i}{E}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \equiv 1,$$

where $(\varepsilon_i - \sigma)/(1-\sigma) > 0$, the condition for global monotonicity, ensures that $U = U(\mathbf{p}/E)$ is strictly increasing E, and the cost-of-living index, $P = P(\mathbf{p}, E)$ is implicitly given by

$$\left[\sum_{i=1}^{n} \beta_i \left(\frac{E}{P}\right)^{\varepsilon_i - 1} \left(\frac{p_i}{P}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \equiv 1,$$

[16] Another notable feature of implicit CES is that it is the only class in which Allen/Uzawa and Morishima elasticities of substitution are identical; see Blackorby & Russell (1981; Theorem 3).

[17] To capture the idea that the rich are less sensitive to price changes, Auer et al. (2022) extends eq.(10) with $\sigma(U) = \max\{\underline{\sigma}, \sigma + \sigma_1 \ln U\}$ with $\sigma_1 < 0 < \underline{\sigma}$. This requires tighter parameter restrictions for global monotonicity.

satisfying $P(\mathbf{p}, E)U(\mathbf{p}/E) = E$.

From eq.(11), we obtain the familiar double-log CES demand systems:

$$\ln\left(\frac{m_i}{m_j}\right) = \ln\left(\frac{\beta_i}{\beta_j}\right) - (\sigma - 1)\ln\left(\frac{p_i}{p_j}\right) + (\varepsilon_i - \varepsilon_j)\ln\left(\frac{E}{P}\right), \tag{12}$$

with an additional term representing the income effect, with the constant slope, $\varepsilon_i - \varepsilon_j$, which is, unlike Stone-Geary, consistent with the empirical evidence of the stable slopes of the Engel's curve (Comin et al. 2021).[18] One could also show that

$$\eta_i \equiv \frac{\partial \ln x_i}{\partial \ln E} = \frac{\partial \ln x_i}{\partial \ln(E/P)} = 1 + \frac{\partial \ln m_i}{\partial \ln(E/P)} = 1 + \varepsilon_i - \sum_{k=1}^{n} m_k \varepsilon_k, \tag{13}$$

which means that good $i \in I$ is a necessity if and only if $\varepsilon_i < \bar{\varepsilon}$ and a luxury if and only if $\varepsilon_i > \bar{\varepsilon}$, where $\bar{\varepsilon} \equiv \sum_{k=1}^{n} m_k \varepsilon_k$ is the budget-share weighted average of ε_i, $i \in I$. Thus, unlike DEA or IEA, the income elasticities of demand for different goods, $\eta_i, i \in I$, can be controlled by the parameters, ε_i, $i \in I$, separately from the constant elasticity of substitution parameter, σ, which governs the price elasticity.

To explore further, let us index the goods such that $\varepsilon_1 < \cdots < \varepsilon_n$, which implies that $\eta_1 < \cdots < \eta_n$. That is, the goods are ordered such that higher-indexed goods have higher income elasticities. Then, from eq.(12) and eq.(13),

- A larger $U = E/P$ shifts the budget shares, $m_i, i \in I$, towards more income-elastic, higher indexed goods in a monotone likelihood way.
- The income elasticity of $i \in I$, η_i, declines monotonically in $U = E/P$ and hence in E with
  - $\eta_1 < 1$; $\eta_n > 1$ for any $E > 0$.
  - For $2 \le i \le n-1$ (with $n \ge 3$), $\eta_i > 1$ for a small E and $\eta_i < 1$ for a large E, since

$$\eta_i \lesseqgtr 1 \Leftrightarrow \varepsilon_i \lesseqgtr \bar{\varepsilon} \equiv \sum_{k=1}^{n} m_k \varepsilon_k.$$

Thus, for $2 \le i \le n-1$, good i is a luxury ($\eta_i > 1$) for the poor but a necessity ($\eta_i < 1$), for the rich. Even though the ratio of the budget shares of two goods is monotonic in $U = E/P$ and

[18]Note that (ε_i) and (ε_i') related by $\varepsilon_i = \sigma + \mu(\varepsilon_i' - \sigma)$ with $\mu > 0$ imply $\mu \ln(E/P) = \ln(E/P')$, so that they are isomorphic. Comin et al. (2021; sec 2.1.) propose to normalize $\mu = (1-\sigma)/(\varepsilon_b' - \sigma)$ so that $\varepsilon_b = 1$ for some base good $b \in I$ to identify the parameters and to interpret $U = E/P$ as the real expenditure. Of course, this measure of the real expenditure depends on the normalization. To evaluate the welfare impact of shocks, it is preferable to use equivalent or compensating variations. See Deaton & Muellbauer (1980, ch.7), Luttmer (2017), Redding & Weinstein (2020) and Baqaee & Burstein (2022) on this issue.

hence in E, the budget share of good i is *hump-shaped*. This means that isoelastic nonhomothetic CES can capture the situations like, a private jet may be a luxury for most people but a necessity for the billionaire, or air-conditioners or smart phones may be necessities for most, but luxuries for the poor.[19] This feature makes Example 7 well-suited for explaining the rise and fall of industry, more generally structural transformation, where sectoral shares exhibit hump-shaped paths over the course of development. see Bohr et al. (2021), Comin et al. (2021), Fujiwara & Matsuyama (2022), and Matsuyama (2019). In contrast, Stone-Geary and other LES, CRES = CRIE, and AIDS, cannot capture such situations, because whether a good is a necessity or a luxury is independent of the household expenditure.[20] A downside of this feature is that nonhomothetic CES does not aggregate easily across households with different expenditures, unlike LES or PIGL.[21]

5. Homothetic and Linear Homogeneous Functions: A Quick Refresher

We now turn to homothetic non-CES. Departing from the standard CES without giving up homotheticity is important for several reasons. First, when we model a competitive industry, we often need to assume that its production technologies satisfy constant-returns-to-scale (CRS).[22] This means that we need to have a linear homogenous (hence homothetic) function.

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[19]Banks et al. (1997) showed the evidence that the budget shares of alcohol and clothing are hump-shaped in the total expenditure. This motivated them to propose Quadratic AIDS, an extension of AIDS, in which the budget shares are quadratic in log total expenditure, which violates global monotonicity. In contrast, Example 7 generates hump-shapes without violating global monotonicity.

[20]This explains why Kongsamut et al. (2001), which use Stone-Geary, were unable to generate the hump-shaped path of the manufacturing share in spite of having three sectors.

[21]Nonhomothetic demand systems, in which some goods are luxuries for the poor and necessities for the rich, with nice aggregation properties, exist in the form of *Hierarchical Demand Systems:* see Matsuyama (2000, 2002**),** Foellmi & Zweimueller (2006)**,** Buera & Kaboski (2012a, 2012b), among others. In these demand systems, goods are ranked according to priority, and as the income goes up, the household expands the range of goods by going down on the shopping list. For example, let $U(\mathbf{x}) = \sum_{j=1}^{\infty} \beta_j \min\{x_j, \hat{x}_j\}$, where $\hat{x}_j$ is the saturation level of good j. If β_j/p_j is monotone decreasing, households buy goods $j \in \{1, \dots, J\}$ up to the saturation levels, and some of good $J + 1$, where J is determined by $\sum_{j=1}^{J} p_j\hat{x}_j \le E < \sum_{j=1}^{J+1} p_j\hat{x}_j$. Thus, as E rises, J goes up. This means that each good is a luxury for poor households, and a necessity for rich households. Alternatively, for $\beta < 1$, $u(x_j) = \beta \min\{x_j, 1\}$, let $U(\mathbf{x}) = \sum_{j=1}^{n}\left(\prod_{k=1}^{j} u(x_k)\right) = u(x_1) + u(x_1)u(x_2) + \cdots$. Then, if $x_j = 0$, $\partial U(\mathbf{x})/\partial x_k = 0$, for any $j < k$. Then, demand is hierarchical for any prices, and each good is a luxury for the poor and a necessity for the rich. The hierarchical systems have easy aggregation properties, but with their own limitation (most goods are either consumed at their saturation levels or not at all).

[22]Recall that the CRS technology of a competitive industry is consistent with the firm-level technologies subject to increasing returns due to some fixed costs and decreasing returns due to some managerial constraints. As any introductory textbook shows, the U-shaped average cost curve of a firm leads to the constant average cost of an industry as industry size changes with the number of firms in the industry.

Second, think of any level of aggregation that defines "a composite good." For example, "food" is not a physical object. Instead, it is a category of goods, say, bread, fish, fruits, meat, vegetable, etc. Most of these "goods" are in turn a composite of finer categories of goods. For example, "fruits" consists of apples, bananas, oranges, etc., and "vegetable" consists of carrots, cucumbers, onions, potatoes, tomatoes, etc. In order to give a cardinal (i.e., quantity) interpretation to any composite of goods, so that the statement like "a 10% increase in food consumption" makes sense, an aggregator that maps a quantity vector of component goods into a quantity of the composite must be linear homogeneous (hence homothetic). Third, we often write down a general equilibrium model in which an overall demand system of the economy is given by multi-layers of the demand systems with nested structures. Then, assuming demand systems to be nonhomothetic anywhere except the highest tier would create a technical problem, because that would prevent us from solving an overall demand system by breaking it down to smaller problems and solving them sequentially using a multi-stage budgeting procedure.[23] Fourth, we often abstract from nonhomotheticity for the tractability. For example, the homotheticity assumption may be necessary for ensuring the existence of a steady state in dynamic general equilibrium. Moreover, homothetic functions are used not only for utility and production functions, but also used often for matching functions and externality terms to keep the model scale-free. For all these reasons, it is useful to have linear homogenous aggregators, for which we may not want our choice to be restricted to the standard CES.

First, let us recall the definitions of homothetic and linear homogenous functions and their basic properties, which can be found in any graduate level micro textbooks: see, e.g., Mas-Colell et al. (1995), and Jehle & Reny (2012).

5.1 Homothetic and Linear Homogeneous Functions: A general case

An aggregator, $X(\mathbf{x})\colon \mathbb{R}_+^n \to \mathbb{R}_+$ is *linear homogeneous* if $X(\lambda\mathbf{x}) = \lambda X(\mathbf{x})$ for all $\lambda > 0$. An aggregator, $H(\mathbf{x})$ is *homothetic in* $\mathbf{x} \in \mathbb{R}_+^n$ if $H(\mathbf{x}) = \mathcal{M}[X(\mathbf{x})]$, where $\mathcal{M}[\cdot]$ is a *monotone* transformation, with *linear homogeneous* $X(\mathbf{x})$. Conversely, any *homothetic* $H(\mathbf{x})$ can be expressed as $H(\mathbf{x}) = \mathcal{M}[X(\mathbf{x})]$, where $X(\mathbf{x})$ is determined up to a positive scalar.

[23] Of course, for some applications, e.g. Fajgelbaum et al. (2011) and Flam & Helpman (1987), it is essential to have sector-level nonhomothetic demand, but this needs to be combined with some specific assumptions on intersectoral demand to keep the model tractable.

In what follows, for concreteness, let us interpret $\mathbf{x} \in \mathbb{R}_+^n$ as a quantity vector of the factors of production and $X(\mathbf{x})$, as a *CRS production function*. Then, with a factor price vector, $\mathbf{p} \in \mathbb{R}_+^n$, we define *the unit cost function*, as

$$P(\mathbf{p}) \equiv \min_{\mathbf{x} \in \mathbb{R}_+^n} \{\mathbf{px} | X(\mathbf{x}) \geq 1\},$$

which is *linear homogeneous, monotone, quasi-concave in* $\mathbf{p} \in \mathbb{R}_+^n$. Furthermore, if $X(\mathbf{x})$ is *monotone and quasi-concave*, it can be recovered from $P(\mathbf{p})$ as:

$$X(\mathbf{x}) \equiv \min_{\mathbf{p} \in \mathbb{R}_+^n} \{\mathbf{px} | P(\mathbf{p}) \geq 1\}.$$

Due to this duality, either $X(\mathbf{x})$ or $P(\mathbf{p})$ can be used as a primitive of the CRS technology.

5.2 Homothetic demands and budget shares: A general case

Let us denote the factor demand by the competitive producers by

$$\mathbf{x}(\mathbf{p}) \equiv \underset{\mathbf{x} \in \mathbb{R}_+^n}{Argmin} \{\mathbf{px} | X(\mathbf{x}) \geq X\}.$$

For a strictly quasi-concave $X(\mathbf{x})$, Shepherd's lemma tells us

$$x_i(\mathbf{p}) = \frac{\partial P(\mathbf{p})}{\partial p_i} X$$

from which the budget share of factor i can be written as a function of $\mathbf{p} \in \mathbb{R}_+^n$.

$$m_i = \frac{p_i x_i(\mathbf{p})}{P(\mathbf{p}) X(\mathbf{x})} = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_i}. \tag{14}$$

From Euler's theorem on linear homogenous functions, these shares are added up to one, $\sum_{i=1}^n m_i = 1$, and each of them is homogenous of degree zero in $\mathbf{p}$.

The inverse factor demand can be given by:

$$\mathbf{p}(\mathbf{x}) \equiv \underset{\mathbf{p} \in \mathbb{R}_+^n}{Argmin} \{\mathbf{px} | P(\mathbf{p}) \geq P\}.$$

For a strictly quasi-concave $P(\mathbf{p})$,

$$p_i(\mathbf{x}) = P \frac{\partial X(\mathbf{x})}{\partial x_i}$$

from which the budget share of factor i can be written as a function of $\mathbf{x} \in \mathbb{R}_+^n$.

$$m_i = \frac{p_i(\mathbf{x}) x_i}{P X(\mathbf{x})} = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_i} \tag{15}$$

Again, from Euler's theorem on linear homogenous functions, these shares are added up to one, $\sum_{i=1}^n m_i = 1$, and each of them is homogenous of degree zero in $\mathbf{x}$.

5.3 Three Classes of Linear Homogenous Functions: An Overview

As shown in eq.(14) and eq.(15), the budget shares can be written as functions of homogeneity of degree zero in $\mathbf{p} \in \mathbb{R}_+^n$, or in $\mathbf{x} \in \mathbb{R}_+^n$. This also means that they could generally depend up to $(n-1)$-relative prices or $(n-1)$-relative quantities.[24] For the tractability, some restrictions may be imposed so that the budget shares depend on a few relative prices or quantities.

To this end, Matsuyama & Ushchev (2017) consider three properties of demand systems called HSA, HDIA, and HIIA, each of which is used to define a class of homothetic functions, because of the following advantages:

- For $n > 2$, HSA, HDIA, and HIIA are pairwise disjoint with the sole exception of CES, as shown in Figure. Thus, they offer three alternative ways of departing from CES without giving up the homotheticity.
- They contain some existing families of homothetic functions.
- Each is *tractable* because the budget share of each factor is a function of one relative price (for HSA) or of two relative prices (for HDIA and HIIA) for *any number of factors,* which drastically reduces the dimensionality of the problem.
- The price elasticity of each factor is a function of one relative price in each class. This allows for a natural extension of the definition of "gross substitutes," and "gross complements".
- Each is defined nonparametrically, and hence *flexible*. It provides a template to construct many different types of homothetic functions that relax some features of CES. For example,
    - Different factors have different but constant price elasticities.
    - Factors can be gross substitutes and yet essential.[25]
    - Any combination of essential and of inessential factors are possible.

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[24]Indeed, it is easy to construct homothetic demand systems with n-factors, which depend on $(n-1)$-relative prices or quantities. For example, the nested CES of n-factors, $X^n(\mathbf{x^n})$, $\mathbf{x^n} \in \mathbb{R}_+^n$, given recursively by $X^1(\mathbf{x^1}) = x_1$, $[X^j(\mathbf{x^j})]^{1-1/\sigma_j} = [X^{j-1}(\mathbf{x^{j-1}})]^{1-1/\sigma_j} + (x_j)^{1-1/\sigma_j}$, for $j = ,2, \dots, n$, if $\sigma_j \neq 1$ are all different.

[25]For any $X(\mathbf{x})$ and $P(\mathbf{p})$, factor i is *essential* (or *indispensable*) if $x_i = 0$ implies $X(\mathbf{x}) = 0$ (or equivalently, if $p_i \to \infty$ implies $P(\mathbf{p}) \to \infty$), and *inessential* (or *dispensable*), otherwise. The notion of *essentials* should not be confused with *necessities*, which are defined as the goods whose income elasticities are less than one.

- For HDIA and HIIA, any combination of gross substitutes and gross complements are possible. and a factor can be a gross substitute or a gross complement, depending on the relative prices.

etc.

We now formally define each of the three and explain their properties in some detail.

6. Homothetic with a Single Aggregator (HSA)

CRS production function $X(\mathbf{x})$ and its unit cost function, $P(\mathbf{p})$, are called *homothetic with a single aggregator* (HSA), if the budget share of each factor as a function of $\mathbf{p} \in \mathbb{R}_+^n$ can be written as

$$m_i \equiv \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_i} = s_i\left(\frac{p_i}{A(\mathbf{p})}\right), \tag{16}$$

where $s_i\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a function of a single variable, and $A(\mathbf{p})$ is linear homogenous in $\mathbf{p}$, defined implicitly and uniquely[26] by the adding-up constraint,

$$\sum_{i=1}^{n} s_i\left(\frac{p_i}{A(\mathbf{p})}\right) \equiv 1, \tag{17}$$

which ensures, by construction, that the budget shares of all factors are added up to one. Eq.(16) and eq.(17) state that the budget share of a factor is a function of its *relative price*, $z_i \equiv p_i/A(\mathbf{p})$, defined as its own price, p_i, divided by the *common price aggregator,* $A(\mathbf{p})$. Notice that $A(\mathbf{p})$ is independent of i; it is "the average factor price" against which the relative price of *every* factor is measured. In other words, one could keep track of all the cross-price effects in the demand system by looking at a single aggregator, $A(\mathbf{p})$, which is the key feature of HSA.[27] The unit cost function, $P(\mathbf{p})$, behind this HSA demand system can be obtained by integrating eq.(16), which yields

$$\ln P(\mathbf{p}) = \ln A(\mathbf{p}) + \sum_{i=1}^{n} \int_{c_1}^{p_i/A(\mathbf{p})} \frac{s_i(\xi)}{\xi} \mathrm{d}\xi,$$

[26]The unique solution requires that s_i is either non-increasing in all i with $\sum_i s_i(0) > 1 > \sum_i s_i(\infty)$ or non-decreasing in all i with $\sum_i s_i(0) < 1 < \sum_i s_i(\infty)$.

[27] The HSA class is the homothetic restriction of what Pollak (1972) refers to as *generalized additively separable* demand systems. However, we prefer to call it HSA instead of *homothetic with generalized additivity,* because it does *not* contain *any* demand systems with additivity (whether direct or indirect or explicit or implicit) with the exception of CES, as seen in Figure.

where c_1 is an integral constant.[28] By applying *Antonelli's integrability theorem* (Antonelli, 1886[1971]; Hurwicz and Uzawa, 1971; see also Ch. 3 in Mas-Colell et al., 1995, and Ch. 2 in Jehle and Reny, 2012), Matsuyama & Ushchev (2017; Proposition 1-i)) shows that the demand system is well-defined by eq.(16) and eq.(17) and that the unit cost function, $P(\mathbf{p})$, satisfies the linear homogeneity, monotonicity, and strict quasi-concavity, if $z_i s_i'(z_i) < s_i(z_i)$ and $s_i'(z_i) s_j'(z_j) \geq 0$. By defining the price elasticity function,

$$-\frac{\partial \ln x_i}{\partial \ln z_i} = 1 - \frac{z_i s_i'(z_i)}{s_i(z_i)} \equiv \zeta_i(z_i),$$

these conditions can be further rewritten as

$$\zeta_i(z_i) > 0;\ [1 - \zeta_i(z_i)][1 - \zeta_j(z_j)] \geq 0.$$

This guarantees the integrability of the HSA demand system; that is, the existence of the underlying CRS technology, $X(\mathbf{x})$ or $P(\mathbf{p})$, that generates this HSA demand system.

It is also important to note that, for $n > 2$, $P(\mathbf{p}) \neq cA(\mathbf{p})$ for any constant $c > 0$ with the sole exception of CES.[29] This can be verified by differentiating eq.(17) to obtain

$$\frac{\partial \ln A(\mathbf{p})}{\partial \ln p_i} = \frac{z_i s_i'(z_i)}{\sum_k z_k s_k'(z_k)} = \frac{[1 - \zeta_i(z_i)] s_i(z_i)}{\sum_k [1 - \zeta_k(z_k)] s_k(z_k)} \neq s_i(z_i) = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_i},$$

unless $\zeta_i(z_i) = \zeta_j(z_j)$ for all $i \neq j \in I$, which requires, for $n > 2$, $\zeta_i(z_i) = \sigma > 0$ or $s_i(z_i) = \beta_i z_i^{1-\sigma}$ for all $i \in I$. This should not come as a surprise. After all, $A(\mathbf{p})$ is the "average factor price", capturing the *cross-price effects* in the demand system, while $P(\mathbf{p})$ is the unit cost of production, capturing the *productivity (or welfare) effects* of factor price changes. There is no reason to think *a priori* that they should move together.

Because the budget share of $i \in I$, is a function of a single relative price, z_i, $s_i(z_i)$, the notion of gross substitutes and gross complements under CES can extended naturally. That is, we call factor $i \in I$ is a *gross substitute (complement)* when $s_i(z_i)$ is strictly decreasing (strictly increasing) in z_i. In other words, factor $i \in I$ is a *gross substitute* when $\zeta_i(z_i) \equiv 1 - z_i s_i'(z_i)/s_i(z_i) > 1$, and factor $i \in I$ is a *gross complement* when $0 < \zeta_i(z_i) < 1$. Notice that

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[28] Note that this constant *cannot* be pinned down. First, $A(\mathbf{p})$, the "average factor price", does not depend on the unit of measurement of the final good. In contrast, $P(\mathbf{p})$ is the cost of producing one unit of the final good, when the factors prices are given by $\mathbf{p}$. Hence, it depends not only on the units of measurement of factors but also on that of the final good. Second, a change in TFP, though it affects $P(\mathbf{p})$, leaves the relative factor demand, hence $A(\mathbf{p})$, unaffected.

[29] The condition, $n > 2$, is necessary. If $n = 2$, the budget share of both factors is always a function of one relative price. Hence, all homothetic functions are HSA In other words, HSA are restrictive only for $n > 2$.

one of the integrability conditions, $[1-\zeta_i(z_i)][1-\zeta_j(z_j)] \geq 0$, implies that HSA does not allow for a mixture of gross substitutes and gross complements. However, a factor with $\zeta_i(z_i) = 1$ can co-exist either with gross substitutes or with gross complements.[30]

Before proceeding to some examples, let us point out that there exists an alternative and yet equivalent definition of HSA That is, CRS production function $X(\mathbf{x})$ and its unit cost function, $P(\mathbf{p})$, are called *homothetic with a single aggregator* (HSA), if the budget share of factor i as a function of $\mathbf{x} \in \mathbb{R}_+^n$ can be written as

$$m_i \equiv \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_i} = s_i^*\left(\frac{x_i}{A^*(\mathbf{x})}\right), \tag{18}$$

where $s_i^*: \mathbb{R}_+ \to \mathbb{R}_+$ is a function of a single variable, its relative quantity, $y_i \equiv x_i/A^*(\mathbf{x})$, and $A^*(\mathbf{x})$ is *the common quantity aggregator* defined implicitly and uniquely[31] by

$$\sum_{i=1}^{n} s_i^*\left(\frac{x_i}{A^*(\mathbf{x})}\right) \equiv 1. \tag{19}$$

The CRS production function, $X(\mathbf{x})$, behind this HSA demand system can be obtained by integrating eq.(18), which yields

$$\ln X(\mathbf{x}) = \ln A^*(\mathbf{x}) + \sum_{i=1}^{n} \int_{c_1^*}^{x_i/A^*(\mathbf{x})} \frac{s_i^*(\xi)}{\xi} d\xi.$$

Matsuyama & Ushchev (2017) shows that the two definitions define the same class of homothetic functions, with the one-to-one correspondence between $s_i(z_i)$ and $s_i^*(y_i)$, defined by

$$s_i^*(y_i) \equiv s_i\left(\frac{s_i^*(y_i)}{y_i}\right) \Leftrightarrow s_i(z_i) \equiv s_i^*\left(\frac{s_i(z_i)}{z_i}\right).$$

Note that differentiating either of these equalities yields

$$\zeta_i(z_i) \equiv 1 - \frac{d\ln s_i(z_i)}{d\ln z_i} = \left[1 - \frac{d\ln s_i^*(y_i)}{d\ln y_i}\right]^{-1} \equiv \zeta_i^*(y_i),$$

so that factor $i \in I$ is a *gross substitute (complement)* if and only if $s_i(z_i)$ is strictly decreasing (increasing), which is equivalent to $\zeta_i(z_i) = \zeta_i^*(y_i) > 1$ $(0 < \zeta_i(z_i) = \zeta_i^*(y_i) < 1)$, which is

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[30] One could also show that both Allen/Uzawa and Morishina elasticities of substitution between i and j are greater than one if $\zeta_i(z_i), \zeta_j(z_j) > 1$ and smaller than one if $\zeta_i(z_i), \zeta_j(z_j) < 1$.

[31] The unique solution requires that s_i^* is either non-increasing in all i with $\sum_i s_i^*(0) > 1 > \sum_i s_i^*(\infty)$ or non-decreasing in all i with $\sum_i s_i^*(0) < 1 < \sum_i s_i^*(\infty)$.

equivalent to $s_i^*(y_i)$ is strictly increasing (decreasing). Furthermore, from $p_i x_i / P(\mathbf{p}) X(\mathbf{x}) = s_i(z_i) = s_i^*(y_i) = z_i y_i = p_i x_i / A(\mathbf{p}) A^*(\mathbf{x})$, $A^*(\mathbf{x})/X(\mathbf{x}) = P(\mathbf{p})/A(\mathbf{p})$, which cannot be a constant with the sole exception of CES for $n > 2$.

We now turn to several examples of HSA.

Example 8: CES as a Special Case of HSA

Let $s_i(z_i) = \beta_i z_i^{1-\sigma} \Leftrightarrow s_i^*(y_i) = \beta_i^{\frac{1}{\sigma}} y_i^{1-\frac{1}{\sigma}}$ with $\sigma > 0; \neq 1,\ \beta_i > 0$. Then, $\zeta_i(z_i) = \zeta_i^*(y_i) = \sigma > 0$, and from eqs.(16)-(17)

$$A(\mathbf{p}) = \left(\sum_{i=1}^{n} \beta_i p_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}} = ZP(\mathbf{p})$$

and from eqs.(18)-(19),

$$A^*(\mathbf{x}) = \left(\sum_{i=1}^{n} \beta_i^{\frac{1}{\sigma}} x_i^{1-\frac{1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} = \frac{X(\mathbf{x})}{Z},$$

where $Z > 0$ is an integral constant, which can be interpreted as TFP. Note that both $A(\mathbf{p})$ and $A^*(\mathbf{x})$ are independent of TFP, which is true for any HSA demand systems. Indeed, TFP shocks to the CRS production function does not affect its relative factor demand. Note also that $A(\mathbf{p})/P(\mathbf{p}) = X(\mathbf{x})/A^*(\mathbf{x}) = Z$ is constant. This is true only for CES, as already pointed out.

For $\sigma > 1$, $s_i(z_i)$ is globally strictly decreasing for $i \in I$, which means that every factor is always a gross substitute, and yet, $s_i(z_i) > 0$ for any $z_i < \infty$, meaning that it has no choke price. Moreover, the generic condition for factor- i being inessential, which can be expressed as;

$$s_i(\infty) + \sum_{k \neq i} s_k(0) > 1 \quad \& \quad \int_{c_1}^{\infty} \frac{s_i(\xi)}{\xi} \mathrm{d}\xi < \infty,$$

automatically holds so that every factor is inessential. For $\sigma < 1$, $s_i(z_i)$ is globally strictly increasing for $i \in I$, so that $\int_{c_1}^{\infty} (s_i(\xi)/\xi) \mathrm{d}\xi = \infty$, which means that every factor i is always a gross complement and essential.

Under generic HSA, it is easy to verify that, when $s_i(z_i)$ is globally strictly increasing for all $i \in I$ (the case of gross complements), all factors must be essential. On the other hand, when $s_i(z_i)$ is globally strictly decreasing for all $i \in I$ (the case of gross substitutes), there are four possibilities:

- $\lim_{z_i \to \infty} s_i(z_i) = s_i(\infty) > 0$, so that $\int_{c_1}^{\infty}(s_i(\xi)/\xi)\mathrm{d}\xi = \infty$, hence essential.
- $s_i(z_i) > 0$ for $z_i < \infty$; $\lim_{z_i \to \infty} s_i(z_i) = 0$, $\int_{c_1}^{\infty}(s_i(\xi)/\xi)\mathrm{d}\xi = \infty$, hence essential.
- $s_i(z_i) > 0$ for $z_i < \infty$; $\lim_{z_i \to \infty} s_i(z_i) = 0$, $\int_{c_1}^{\infty}(s_i(\xi)/\xi)\mathrm{d}\xi < \infty$, which means inessential if $s_i(\infty) + \sum_{k\neq i} s_k(0) > 1$.
- $s_i(z_i) = 0$ for $z_i \geq \bar{z}_i$ for a finite $\bar{z}_i$ (zero demand for $p_i \geq \bar{z}_i A(\mathbf{p})$, where $\bar{z}_i A(\mathbf{p})$ is the choke price) and $\int_{c_1}^{\infty}(s_i(\xi)/\xi)\mathrm{d}\xi < \infty$, which means inessential if $s_i(\infty) + \sum_{k\neq i} s_k(0) > 1$.

Under CES with $\sigma > 1$, only the third case is allowed. We now turn to examples for the first case, i.e., some gross substitutes factors can be essential.

Example 9: Hybrids of Cobb-Douglas and CES under HSA

Consider the HSA demand system, eqs.(16)-(17), given by:

$$s_i(z_i) = \varepsilon\alpha_i + (1-\varepsilon)\beta_i {z_i}^{1-\sigma};\;\; 0 < \varepsilon < 1,\; \alpha_i \geq 0,\;\; \beta_i > 0,\;\; \sum_{k=1}^{n}\alpha_k = \sum_{k=1}^{n}\beta_k = 1.$$

Then,

$$\zeta_i(z_i) = \frac{\varepsilon\alpha_i + \sigma(1-\varepsilon)\beta_i {z_i}^{1-\sigma}}{\varepsilon\alpha_i + (1-\varepsilon)\beta_i {z_i}^{1-\sigma}};$$

$$A(\mathbf{p}) = \left(\sum_{i=1}^{n}\beta_i p_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}};\; P(\mathbf{p}) = \frac{1}{Z}\left(\prod_{i=1}^{n} p_i^{\alpha_i}\right)^{\varepsilon}\left(A(\mathbf{p})\right)^{1-\varepsilon}.$$

This is a convex combination of Cobb-Douglas and CES since it is Cobb-Douglas for $\varepsilon = 1$ and CES for $\varepsilon = 0$. Similarly, consider the HSA inverse demand system, eqs.(18)-(19), given by:

$$s_i^*(y_i) = \varepsilon\alpha_i + (1-\varepsilon)\beta_i^{1/\sigma} {y_i}^{1-1/\sigma};\;\; 0 < \varepsilon < 1,\; \alpha_i \geq 0,\;\; \beta_i > 0,\; \sum_{k=1}^{n}\alpha_k = \sum_{k=1}^{n}\beta_k = 1$$

Then,

$$\zeta_i^*(y_i) = \frac{\varepsilon\alpha_i + (1-\varepsilon)\beta_i^{1/\sigma} y^{1-1/\sigma}}{\varepsilon\alpha_i + (1/\sigma)(1-\varepsilon)\beta_i^{1/\sigma} y^{1-1/\sigma}};$$

$$A^*(\mathbf{x}) = \left(\sum_{i=1}^{n}\beta_i^{1/\sigma} x_i^{1-1/\sigma}\right)^{\frac{\sigma}{\sigma-1}};\; X(\mathbf{x}) = Z\left(\prod_{i=1}^{n} x_i^{\alpha_i}\right)^{\varepsilon}\left(A^*(\mathbf{x})\right)^{1-\varepsilon}.$$

This is another convex combination of Cobb-Douglas and CES.[32] In both cases, for all factors are gross substitutes for $\sigma > 1$, and yet, factor i is *essential* if $\alpha_i > 0$ and *inessential* if $\alpha_i = 0$. Thus, some gross substitutes are essential. Furthermore, any combination of essential and inessential factors can coexist.

To see implications, consider a model of international trade, where each country produces the single nontradeable consumption good using tradeable factors under the HSA technologies described above. With a small ε, the demand system can be approximated by CES with trade elasticity, σ. If CES ($\varepsilon = 0$), autarky would lead to a small welfare loss with a moderately large $\sigma > 1$. Yet, for an arbitrarily small but positive $\varepsilon > 0$, the welfare loss of autarky, measured in the cost-of-living index, is *infinity* if a country has no domestic supply of an essential factor.[33]

More broadly, when gross substitutes are essential with their price elasticities converging to one as they become increasingly scarcer, the welfare impacts of large shocks, say, sanctions or pandemics-induced lockdowns, would be large. This offers a caution against assessing the impacts of large changes by using the empirical evidence obtained by local changes as "disciplines," under the straitjacket of CES.

The next example features gross substitutes with the choke prices.

Example 10: "Separable" Translog

Two often-used non-CES are translog unit cost functions and translog production functions (Christensen et al., 1973, 1975). They are isolated from CES and not an extension of CES. Nevertheless, it is worth discussing them here, because they have two subfamilies that belong to HSA

First, consider the translog unit cost function,

$$P(\mathbf{p}) = \frac{1}{Z}\exp\left[\sum_{i=1}^{n}\delta_i \ln p_i - \frac{1}{2}\sum_{i,j=1}^{n}\gamma_{ij}\ln p_i \ln p_j\right],$$

[32]These two convex combinations are not equivalent, because $s_i^*(y_i) \neq s_i(s_i^*(y_i)/y_i)$ and $s_i(z_i) \neq s_i^*(s_i(z_i)/z_i)$. Note also that neither of them is a nested CES, because $\alpha_i\beta_i \neq 0$ for some i.

[33]Using nested CES, Ossa (2016) offered a similar caution against Arkolakis et al. (2012) and a large number of subsequent studies, which have used models with CES demand systems and concluded that the gains from trade (or the loss from autarky) are rather small, given relatively large estimated values of the trade elasticities.

where $\delta_i > 0$; (γ_{ij}) is symmetric and non-negative semi-definite, which can be normalized as $\sum_{j=1}^{n}\delta_j = 1$, and $\sum_{j=1}^{n}\gamma_{ij} = 0$. In general, the translog unit cost function is not HSA. However, under the following "separability" condition, satisfied by symmetric translog,

$$\gamma_{ij} = \begin{cases}\gamma\beta_i(1-\beta_i), i = j\\ -\gamma\beta_i\beta_j, i \neq j\end{cases}, \qquad \gamma \geq 0;\ \beta_i > 0;\ \sum_{i=1}^{n}\beta_i = 1,$$

it is HSA with

$$s_i\left(\frac{p_i}{A(\mathbf{p})}\right) = \max\left\{\delta_i - \gamma\beta_i \ln\frac{p_i}{A(\mathbf{p})}, 0\right\}.$$

If $\gamma = 0$, this is Cobb-Douglas. If $\gamma > 0$, all factors are *gross substitutes* with *the choke prices*, $\bar{z}_i A(\mathbf{p})$, where $\bar{z}_i = \exp(\delta_i/\gamma\beta_i)$, and inessential. For $p_i < \bar{z}_i A(\mathbf{p})$ for all i,

$$\ln A(\mathbf{p}) = \sum_{i=1}^{n}\beta_i \ln p_i\,;$$

$$P(\mathbf{p}) = \frac{1}{Z}\exp\left\{\sum_{i=1}^{n}\delta_i \ln p_i - \frac{\gamma}{2}\left[\sum_{i=1}^{n}\beta_i(\ln p_i)^2 - \left(\sum_{i=1}^{n}\beta_i \ln p_i\right)^2\right]\right\} \neq A(\mathbf{p}).$$

Next, consider the translog production function:

$$X(\mathbf{x}) = Z\exp\left[\sum_{i=1}^{n}\delta_i \ln x_i - \frac{1}{2}\sum_{i,j=1}^{n}\gamma_{ij}\ln x_i \ln x_j\right]$$

where $\delta_i > 0$; (γ_{ij}) is symmetric and non-negative semi-definite, which can be normalized as $\sum_{j=1}^{n}\delta_j = 1$, and $\sum_{j=1}^{n}\gamma_{ij} = 0$. In general, the translog production function is not HSA. However, under the following "separability" condition, satisfied by symmetric translog,

$$\gamma_{ij} = \begin{cases}\gamma\beta_i(1-\beta_i), i = j\\ -\gamma\beta_i\beta_j, i \neq j\end{cases}, \qquad \gamma \geq 0;\ \beta_i > 0;\ \sum_{i=1}^{n}\beta_i = 1,$$

it is HSA with

$$s_i^*\left(\frac{x_i}{A^*(\mathbf{x})}\right) = \max\left\{\delta_i - \gamma\beta_i \ln\frac{x_i}{A^*(\mathbf{x})}, 0\right\}.$$

If $\gamma = 0$, this is Cobb-Douglas. If $\gamma > 0$, all factors are *gross complements* with *the saturation points*, $\bar{y}_i A^*(\mathbf{x})$, where $\bar{y}_i = \exp(\delta_i/\gamma\beta_i)$, and essential. For $x_i < \bar{y}_i A^*(\mathbf{x})$, for all i,

$$\ln A^*(\mathbf{x}) = \sum_{i=1}^{n}\beta_i \ln x_i$$

$$X(\mathbf{x}) = Z \cdot \exp\left\{\sum_{i=1}^{n} \delta_i \ln x_i - \frac{\gamma}{2}\left[\sum_{i=1}^{n} \beta_i (\ln x_i)^2 - \left(\sum_{i=1}^{n} \beta_i \ln x_i\right)^2\right]\right\} \neq A^*(\mathbf{x}).$$

These calculations reveal the restrictive nature of the translog aggregators, which seems unnoticed by many in spite of their popularity as an alternative to CES. In the case of the translog unit cost function, it allows only for gross substitutes and inessential factors with the choke prices. In the case of the translog production function, it allows only for gross complements and essential factors with the saturation point.[34]

Example 11: HSA Demand Systems with Constant but Different Price Elasticities

$$s_i(z_i) = \beta_i (z_i)^{1-\sigma_i} \Leftrightarrow s_i^*(y_i) = \beta_i^{\frac{1}{\sigma_i}} (y_i)^{1-\frac{1}{\sigma_i}},$$

where either $\sigma_i \leq 1$ for all i, or $\sigma_i \geq 1$ for all i. Then, $A(\mathbf{p})$ and $A^*(\mathbf{x})$ are given implicitly by

$$\sum_{i=1}^{n} \beta_i \left(\frac{p_i}{A(\mathbf{p})}\right)^{1-\sigma_i} = \sum_{i=1}^{n} \beta_i^{1/\sigma_i} \left(\frac{x_i}{A^*(\mathbf{x})}\right)^{1-1/\sigma_i} = 1,$$

and

$$\ln P(\mathbf{p}) = \ln A(\mathbf{p}) + \sum_{i=1}^{n} \int_{c_1}^{p_i/A(\mathbf{p})} \beta_i (\xi)^{-\sigma_i} d\xi,$$

$$\ln X(\mathbf{x}) = \ln A^*(\mathbf{x}) + \sum_{i=1}^{n} \int_{c_1^*}^{x_i/A^*(\mathbf{x})} \beta_i^{\frac{1}{\sigma_i}} (\xi)^{-\frac{1}{\sigma_i}} d\xi.$$

Both Allen/Uzawa and Morishima elasticities of substitution between each pair are *variable* unless $\sigma_i = \sigma$ for all $i \in I$. However, holding $A(\mathbf{p})$ or $A^*(\mathbf{x})$ fixed, the own price elasticity of each factor is constant but different, because $\zeta_i(p_i/A(\mathbf{p})) = \zeta_i^*(x_i/A^*(\mathbf{x})) = \sigma_i$. Furthermore, for a large n, the impact of a change in p_i on $A(\mathbf{p})$ and the impact of a change in x_i on $A^*(\mathbf{x})$ are negligible. Hence, the own price elasticity when all other prices are fixed, or when all other

[34]Both translog unit cost and production functions, as well as their nonhomothetic counterparts, AIDS, are often touted as "flexible." But they are flexible only in the sense that they offer local approximations to any aggregators up to their second derivatives. Such approximations may be good enough for studying the impacts of small shocks to a competitive economy, where all firms are price takers. However, they should be used with great caution when studying the impacts of large shocks or even those of small shocks if some firms have the price-setting powers, since the results would then depend on the global properties and/or the third or higher derivatives of the aggregators.

quantities are fixed, is approximately constant, and converging to σ_i, as $n \to \infty$.[35] Thus, this example, along with Examples 13 and 15, shown later, can isolate the role of price elasticity differences across factors without giving up homotheticity, unlike Example 5 under DEA, which are subject to Pigou's Law and Example 6 under IEA, which are subject to indirect Pigou's Law.

7. Homothetic Direct Implicit Additivity (HDIA)

7.1 Definition

CRS production function, $X(\mathbf{x})$, and its unit cost function, $P(\mathbf{p})$, are called *homothetic with direct implicit additivity* **(HDIA)** if $X(\mathbf{x})$ can be written as

$$\sum_{i=1}^{n} \phi_i\left(\frac{x_i}{X(\mathbf{x})}\right) = 0, \tag{20}$$

where $\phi_i: \mathbb{R}_+ \to \mathbb{R}$; $i \in I$, are strictly increasing, and strictly concave, and satisfy

$$\sum_{i=1}^{n} \phi_i(0) < 0 < \sum_{i=1}^{n} \phi_i(\infty).$$

This ensures the unique existence of monotonic and quasi-concave $X(\mathbf{x})$. Clearly, HDIA is the homothetic restriction of DIA, as shown in Figure.[36]

Solving $P(\mathbf{p}) \equiv \min_{\mathbf{x}\in\mathbb{R}_+^n}\{\mathbf{px}|X(\mathbf{x}) \geq 1\}$ subject to eq.(20) yields the HDIA demand system,

$$\frac{\partial \ln P(\mathbf{p})}{\partial \ln p_i} = \frac{p_i}{P(\mathbf{p})}(\phi_i{}')^{-1}\left(\frac{p_i}{B(\mathbf{p})}\right), \tag{21}$$

where $B(\mathbf{p})$ is the Lagrange multiplier associated with the minimization problem; it is a linear homogenous function in $\mathbf{p} \in \mathbb{R}_+^n$, implicitly defined by

$$\sum_{k=1}^{n} \phi_k\left((\phi_k{}')^{-1}\left(\frac{p_k}{B(\mathbf{p})}\right)\right) \equiv 0,$$

and $P(\mathbf{p})$ is the unit cost function, and related to $B(\mathbf{p})$ as follows:

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[35]The own price elasticity of each factor is constant without these qualifications for the case of a continuum of factors, with the summations in eq.(16) and eq.(17) or those in eq.(18) and eq.(19) being replaced by the integrals.
[36]According to the original definition of DIA by Hanoch (1975), its homothetic restriction HDIA should be written as $\sum_{i=1}^{n} \phi_i(x_i/X(\mathbf{x})) = 1$. However, the RHS of eq.(20) can be any constant, and we set it equal to zero, which has two advantages. First, one could restrict all ϕ_i to be strictly increasing and concave without loss of generality. Second, multiplying all ϕ_i by a positive constant would not change the function defined. For example, the Kimball aggregator, a special case of HDIA, is typically defined as $\sum_{i=1}^{n} \phi(x_i/X(\mathbf{x})) = 1$, where ϕ is strictly increasing and concave. This definition imposes not only symmetry, but also gross substitutability. Furthermore, the function defined changes if ϕ is multiplied by a positive constant.

$$P(\mathbf{p}) = \sum_{k=1}^{n} p_k (\phi_k')^{-1}\left(\frac{p_k}{B(\mathbf{p})}\right).$$

The HDIA inverse demand system can be obtained by differentiating eq.(20) as follows:

$$\frac{\partial \ln X(\mathbf{x})}{\partial \ln x_i} = \frac{x_i}{C^*(\mathbf{x})}\phi_i'\left(\frac{x_i}{X(\mathbf{x})}\right) \tag{22}$$

where $C^*(\mathbf{x})$ is a linear homogenous function in $\mathbf{x} \in \mathbb{R}^n_+$, defined by:

$$C^*(\mathbf{x}) \equiv \sum_{k=1}^{n} x_k \phi_k'\left(\frac{x_k}{X(\mathbf{x})}\right).$$

From these, we also obtain

$$\frac{p_k}{B(\mathbf{p})} = \phi_k'\left(\frac{x_k}{X(\mathbf{x})}\right) \Leftrightarrow \frac{x_k}{X(\mathbf{x})} = (\phi_k')^{-1}\left(\frac{p_k}{B(\mathbf{p})}\right)$$

and

$$\frac{P(\mathbf{p})}{B(\mathbf{p})} = \sum_{k=1}^{n} \frac{p_k}{B(\mathbf{p})}(\phi_k')^{-1}\left(\frac{p_k}{B(\mathbf{p})}\right) = \sum_{k=1}^{n} \phi_k'\left(\frac{x_k}{X(\mathbf{x})}\right)\frac{x_k}{X(\mathbf{x})} = \frac{C^*(\mathbf{x})}{X(\mathbf{x})}.$$

Notice that eq.(21) and eq.(22) suggest that the budget share of i under HDIA depends on the two different relative prices, $p_i/B(\mathbf{p})$ and $p_i/P(\mathbf{p})$, or on the two relative quantities, $x_i/X(\mathbf{x})$ and $x_i/C^*(\mathbf{x})$, unless $P(\mathbf{p})/B(\mathbf{p}) = C^*(\mathbf{x})/X(\mathbf{x}) = c$ for a constant $c > 0$. In other words, HDIA belongs to HSA if and only if the budget share of i can be written as a function of $p_i/P(\mathbf{p})$ or $x_i/X(\mathbf{x})$ only. This means that HDIA and HSA do not overlap with the sole exception of CES, for $n > 2$, as shown in Figure.

7.2 Price Elasticity Function under HDIA

Even though the budget share of i under HDIA depends on two different relative quantities, one of them, $x_i/C^*(\mathbf{x})$, enters proportionately. Thus, the price elasticity depends solely on $y_i \equiv x_i/X(\mathbf{x})$, as follows.

$$-\frac{\partial \ln(x_i/X(\mathbf{x}))}{\partial \ln(p_i/B(\mathbf{p}))} = -\frac{\phi_i'(y_i)}{y_i\phi_i''(y_i)} \equiv \zeta_i^D(y_i) > 0.$$

Cobb-Douglas is a special case, $\zeta_i^D(y_i) = 1,$where

$$\phi_i(y_i) = \alpha_i \ln\left(\frac{Z y_i}{\alpha_i}\right) \Rightarrow X(\mathbf{x}) = Z\prod_{i=1}^{n}\left(\frac{x_i}{\alpha_i}\right)^{\alpha_i}.$$

CES is a special case, $\zeta_i^D(y_i) = \sigma,$where

$$\phi_i(\mathcal{y}_i) = \beta_i \frac{(Z\mathcal{y}_i/\beta_i)^{1-1/\sigma} - 1}{1 - 1/\sigma} \Longrightarrow X(\mathbf{x}) = Z\left(\sum_{i=1}^{n} \beta_i^{1/\sigma} x_i^{1-1/\sigma}\right)^{\frac{1}{1-1/\sigma}}$$

Note that, under CES with gross substitutes, $\zeta_i^D(\mathcal{y}_i) = \sigma > 1, \phi_i(\mathcal{y}_i)$ is unbounded from above and bounded from below, while under CES with gross complements, $\zeta_i^D(\mathcal{y}_i) = \sigma < 1$, $\phi_i(\mathcal{y}_i)$ is unbounded from below and bounded from above. Thus, even though the price elasticity function, $\zeta_i^D(\mathcal{y}_i)$, is defined locally, the assumption that it is globally constant imposes a strong restriction on its global property. Cobb-Douglas, $\zeta_i^D(\mathcal{y}_i) = 1$, is the borderline case, where $\phi_i(\mathcal{y}_i)$ is unbounded both from below and from above.

In what follows, we call factor-i a *gross substitute* if $\zeta_i^D(\mathcal{y}_i) > 1$ and a *gross complement* if $0 < \zeta_i^D(\mathcal{y}_i) < 1$. Recall that, for HDIA to be well-defined by eq.(20), ϕ_i; $i \in I$, only need to be strictly increasing, and strictly concave, and satisfy $\sum_{i=1}^n \phi_i(0) < 0 < \sum_{i=1}^n \phi_i(\infty)$. Hence, unlike HSA, HDIA does not impose any restriction on the price elasticity functions, $\zeta_i^D(\mathcal{y}_i)$; $i \in I$, except that they all need to be positive. In particular, it is possible to have $\zeta_i^D(\mathcal{y}_i) > 1 > \zeta_j^D(\mathcal{y}_j)$, and hence gross substitutes and gross complements can co-exist. Indeed, $\zeta_i^D(\mathcal{y}_i) - 1$ may switch signs, and hence factor-i could switch from being a gross substitute to a gross complement, as $\mathcal{y}_i$ changes.

7.3. Essential vs Inessential Factors under HDIA

Recall that factor i is *essential* if $x_i = 0$ implies $X(\mathrm{x}) = 0$, and *inessential*, otherwise. Under HDIA, this means that factor i is *essential* if and only if $\phi_i(0) + \sum_{k\neq i}^n \phi_k(\mathcal{y}_k) < 0$ for all $\mathcal{y}_k > 0$. This condition is *always* satisfied under CES with gross complements or under Cobb-Douglas, because $\phi_i(\mathcal{y}_i)$ is unbounded from below. On the other hand, this condition is *never* satisfied under CES with gross substitutes, because $\phi_i(\mathcal{y}_i)$ is bounded from below and $\phi_k(\mathcal{y}_k)$ is unbounded from above. This is the reason why factors are inessential if and only if they are gross substitutes under CES.

However, gross substitutes can be essential under HDIA. To see this, let $\phi_i(\mathcal{y}_i) = \beta_i g(\mathcal{y}_i)$ where $\beta_i > 0$ is decreasing in i and $\sum_{i=1}^n \beta_i = 1$, and $g(\mathcal{y}_i)$ is strictly increasing and strictly concave, and $-\infty < g(0) < 0 < g(\infty) < \infty$. Then, factors $i = 1, \dots, j$ are *essential* and Factors $i = j + 1, \dots, n$ are *inessential* for

$$\frac{\beta_j}{1-\beta_j} > -\frac{g(\infty)}{g(0)} > \frac{\beta_{j+1}}{1-\beta_{j+1}} > 0.$$

This example suggests that HDIA can have j essential factors and $n-j$ inessential factors, where $j = 0,1,\ldots,n$. Furthermore, the price elasticity function, $\zeta_i^D(y_i) = -g'(y_i)/y_i g''(y_i)$ can be arbitrary, and hence the factors could be gross substitutes or gross complements, except asymptotically, as $y_i \to 0$ or as $y_i \to \infty$.

It is also easy to construct an example using a convex combination of Cobb-Douglas and CES, as follows.

Example 12: A Hybrid of Cobb-Douglas and CES under HDIA

$$\phi_i(y_i) = \varepsilon\alpha_i \log\left(\frac{Zy_i}{\alpha_i}\right) + (1-\varepsilon)\beta_i \frac{(Zy_i/\beta_i)^{1-1/\sigma}-1}{1-1/\sigma}$$

$$\Rightarrow \zeta_i^D(y_i) = \frac{\varepsilon\alpha_i + (1-\varepsilon)\beta_i(Zy_i/\beta_i)^{1-1/\sigma}}{\varepsilon\alpha_i + (1/\sigma)(1-\varepsilon)\beta_i(Zy_i/\beta_i)^{1-1/\sigma}}$$

where $0 < \varepsilon < 1$, $\alpha_i \geq 0$, $\beta_i > 0$, $\sum_{k=1}^n \alpha_k = \sum_{k=1}^n \beta_k = 1$. The implications are similar with Example 9 under HSA and Example 14 under HIIA.

Example 13: HDIA Demand System with Constant but Different Elasticities

$$\phi_i(y_i) = \beta_i \frac{(Zy_i/\beta_i)^{1-1/\sigma_i}-1}{1-1/\sigma_i} \Rightarrow \zeta_i^D(y_i) = \sigma_i.$$

CRESH class proposed by Hanoch (1971) is a special case of this example, where $\sigma_i > 1$ for at least some i. The properties are similar to Example 11 under HSA, except that there is no need to impose the restriction that either $\sigma_i \leq 1$ for all i, or $\sigma_i \geq 1$ for all i.

8. Homothetic Indirect Implicit Additivity (HIIA)

8.1 Definition

CRS production function, $X(\mathbf{x})$, and its unit cost function, $P(\mathbf{p})$, are called *homothetically with indirect implicit additivity* **(HIIA)** if $P(\mathbf{p})$ can be written as**:**

$$\sum_{i=1}^n \theta_i\left(\frac{p_i}{P(\mathbf{p})}\right) = 0, \tag{23}$$

where $\theta_i: \mathbb{R}_+ \to \mathbb{R}$, $i \in I$, are strictly increasing and concave, and satisfy

$$\sum_{i=1}^{n}\theta_i(0) < 0 < \sum_{i=1}^{n}\theta_i(\infty).$$

This ensures the unique existence of monotonic and quasi-concave $P(\mathbf{p})$. Clearly, HIIA is the homothetic restriction of IIA, as shown in Figure.[37]

The HIIA Inverse Demand System can be obtained by differentiating eq.(23)

$$\frac{\partial \ln P(\mathbf{p})}{\partial \ln p_i} = \frac{p_i}{C(\mathbf{p})}\theta_i'\left(\frac{p_i}{P(\mathbf{p})}\right), \tag{24}$$

where $C(\mathbf{p})$ is a linear homogenous function in $\mathbf{p} \in \mathbb{R}_+^n$, defined by:

$$C(\mathbf{p}) \equiv \sum_{k=1}^{n} p_k\theta_k'\left(\frac{p_k}{P(\mathbf{p})}\right).$$

Solving $X(\mathbf{x}) \equiv \min_{\mathbf{p}\in\mathbb{R}_+^n}\{\mathbf{px}|P(\mathbf{p}) \geq 1\}$ subject to eq.(23) yields the HIIA inverse demand system,

$$\frac{\partial \ln X(\mathbf{x})}{\partial \ln x_i} = \frac{x_i}{X(\mathbf{x})}({\theta_i}')^{-1}\left(\frac{x_i}{B^*(\mathbf{x})}\right), \tag{25}$$

where $B^*(\mathbf{x})$ is the Lagrange multiplier associated with the above minimization problem; it is a linear homogenous function in $\mathbf{x} \in \mathbb{R}_+^n$, implicitly defined by

$$\sum_{k=1}^{n}\theta_k\left(({\theta_k}')^{-1}\left(\frac{x_k}{B^*(\mathbf{x})}\right)\right) = 0,$$

and $X(\mathbf{x})$, the production function, is related to $B^*(\mathbf{x})$ as follows:

$$X(\mathbf{x}) = \sum_{k=1}^{n} x_k(\theta_k')^{-1}\left(\frac{x_k}{B^*(\mathbf{x})}\right).$$

From these, we also obtain

$$\frac{p_i}{P(\mathbf{p})} = (\theta_k')^{-1}\left(\frac{x_k}{B^*(\mathbf{x})}\right) \Leftrightarrow \frac{x_k}{B^*(\mathbf{x})} = {\theta_i}'\left(\frac{p_i}{P(\mathbf{p})}\right)$$

and

$$\frac{X(\mathbf{x})}{B^*(\mathbf{x})} = \sum_{k=1}^{n}\frac{x_k}{B^*(\mathbf{x})}(\theta_k')^{-1}\left(\frac{x_k}{B^*(\mathbf{x})}\right) = \sum_{k=1}^{n}{\theta_i}'\left(\frac{p_i}{P(\mathbf{p})}\right)\frac{p_i}{P(\mathbf{p})} = \frac{C(\mathbf{p})}{P(\mathbf{p})}.$$

Notice that eq.(24) and eq.(25) suggest that the budget share of i under HIIA depends on the two different relative prices, $p_i/C(\mathbf{p})$ and $p_i/P(\mathbf{p})$ or on the two relative quantities, $x_i/B^*(\mathbf{x})$ and

[37]According to the original definition of IIA by Hanoch's (1975), its homothetic restriction HIIA should be defined as $\sum_{i=1}^{n}\theta_i(p_i/P(\mathbf{p})) = 1$. However, the RHS of eq.(23) can be any constant and we set it equal to zero, which has two advantages. First, one could restrict all θ_i to be strictly increasing and strictly concave without loss of generality. Second, multiplying all θ_i by a positive constant would not change the function defined.

$x_i/X(\mathbf{x})$, unless $C(\mathbf{p})/P(\mathbf{p}) = X(\mathbf{x})/B^*(\mathbf{x}) = c$ for a constant $c > 0$. In other words, HIIA belongs to HSA if and only if the budget share of i can be written as a function of $p_i/P(\mathbf{p})$ or $x_i/X(\mathbf{x})$ only. This means that HIIA and HSA do not overlap with the sole exception of CES, for $n > 2$, as shown in Figure. Comparing eq.(21) and eq.(24) or eq. (22) and eq.(25) also suggests that HDIA and HIIA can overlap if and only if both $P(\mathbf{p})/B(\mathbf{p}) = C^*(\mathbf{x})/X(\mathbf{x})$ and $C(\mathbf{p})/P(\mathbf{p}) = X(\mathbf{x})/B^*(\mathbf{x})$ are positive constants, which implies HDIA and HIIA do not overlap with the sole exception of CES, for $n > 2$, as shown in Figure.

8.2 Price Elasticity Function under HIIA

Even though the budget share of i under HIIA depends on two different relative prices, one of them, $p_i/C(\mathbf{p})$, enters proportionately. Thus, the price elasticity depends solely on $z_i \equiv p_i/P(\mathbf{p})$, as follows.

$$-\frac{\partial \ln(x_i/B^*(\mathbf{x}))}{\partial \ln(p_i/P(\mathbf{p}))} = -\frac{z_i \theta_i''(z_i)}{\theta_i'(z_i)} \equiv \zeta_i^I(z_i) > 0.$$

Cobb-Douglas is a special case, $\zeta_i^I(z_i) = 1$, where

$$\theta_i(z_i) = \alpha_i \log\left(\frac{z_i}{Z}\right) \Longrightarrow P(\mathbf{p}) = \frac{1}{Z}\prod_{i=1}^{n} p_i^{\alpha_i}.$$

CES is a special case, $\zeta_i^I(z_i) = \sigma$, where

$$\theta_i(z_i) = \beta_i \frac{(z_i/Z)^{1-\sigma} - 1}{1-\sigma} \Longrightarrow P(\mathbf{p}) = \frac{1}{Z}\left(\sum_{i=1}^{n} \beta_i p_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}}.$$

Note that, under CES with gross substitutes, $\zeta_i^I(z_i) = \sigma > 1, \theta_i(z_i)$ is unbounded from below and bounded from above, while under CES with gross complements, $\zeta_i^I(z_i) = \sigma < 1$, $\theta_i(z_i)$ is unbounded from above and bounded from below. Thus, even though the price elasticity function, $\zeta_i^I(z_i)$, is defined locally, the fact that it is constant imposes a strong restriction on its global property. Cobb-Douglas, $\zeta_i^I(z_i) = 1$, is the borderline case, where $\theta_i(z_i)$ is unbounded both from below and from above.

In what follows, we call factor-i a *gross substitute* if $\zeta_i^I(z_i) > 1$ and a *gross complement* if $\zeta_i^I(z_i) < 1$. Recall that, for HIIA to be well-defined by eq.(23), θ_i; $i \in I$, only need to be strictly increasing, and strictly concave, and satisfy $\sum_{i=1}^{n} \theta_i(0) < 0 < \sum_{i=1}^{n} \theta_i(\infty)$. Hence, unlike HSA but similar to HDIA, HIIA does not impose any restriction on the price elasticity functions, $\zeta_i^I(z_i);\ i \in I$, except that they all need to be positive. In particular, it is possible to have $\zeta_i^I(z_i) > 1 > \zeta_j^I(z_j)$, and hence gross substitutes and gross complements can co-exist. Indeed, $\zeta_i^I(z_i) - 1$ may switch signs, and hence factor-i could switch from being a gross substitute to a gross complement, as z_i changes.

8.3 Essential vs. Inessential Factors under HIIA

Recall that factor i is *essential* if $p_i \to \infty$ implies $P(\mathbf{p}) \to \infty$, and *inessential*, otherwise. Under HIIA, this means that factor i is *essential* if and only if $\theta_i(\infty) + \sum_{k\neq i}^{n} \theta_k(z_k) > 0$ for all $z_k > 0$. This condition is *always* satisfied under CES with gross complements or under Cobb-Douglas, because $\theta_i(z_i)$ is unbounded from above. On the other hand, this condition is *never* satisfied under CES with gross substitutes, because $\theta_i(z_i)$ is bounded from above and $\theta_k(z_k)$ is unbounded from below. This is the reason why factors are inessential if and only if they are gross substitutes under CES.

However, gross substitutes can be essential under HIIA. To see this, $\theta_i(z_i) = \beta_i g(z_i)$ where $\beta_i > 0$ is decreasing in i and $\sum_{i=1}^{n} \beta_i = 1$, and $g(z_i)$ is strictly increasing and strictly concave, and $-\infty < g(0) < 0 < g(\infty) < \infty$. Then, Factors $i = 1, \dots, j$ are *essential* and Factors $i = j+1, \dots, n$ are *inessential,* for

$$\frac{\beta_j}{1-\beta_j} > -\frac{g(0)}{g(\infty)} > \frac{\beta_{j+1}}{1-\beta_{j+1}} > 0.$$

This example suggests that HIIA can have j essential factors and $n-j$ inessential factors, where $j = 0,1, \dots, n$. Furthermore, the price elasticity function, $\zeta_i^I(z_i) = -z_i g''(z_i)/g'(z_i) > 0$ can be arbitrary, and hence the factors could be gross substitutes or gross complements, except asymptotically, as $z_i \to 0$ or as $z_i \to \infty$.

It is also easy to construct an example using a convex combination of Cobb-Douglas and CES, as follows.

Example 14: A Hybrid of Cobb-Douglas and CES under HIIA

$$\theta_i(z_i) = \varepsilon\alpha_i \ln\left(\frac{z_i}{Z}\right) + (1-\varepsilon)\beta_i \frac{(z_i/Z)^{1-\sigma} - 1}{1-\sigma}$$

$$\Rightarrow \zeta_i^I(z_i) = \frac{\varepsilon\alpha_i + \sigma(1-\varepsilon)\beta_i (z_i/Z)^{1-\sigma}}{\varepsilon\alpha_i + (1-\varepsilon)\beta_i (z_i/Z)^{1-\sigma}},$$

where $0 < \varepsilon < 1$, $\alpha_i \geq 0$, $\beta_i > 0$, $\sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} \beta_k = 1$. The implications are similar with Example 9 under HSA and Example 12 under HDIA.

Example 15: HDIA Demand System with Constant but Different Elasticities

$$\theta_i(z_i) = \beta_i \frac{(z_i/Z)^{1-\sigma_i} - 1}{1 - \sigma_i} \Rightarrow \zeta_i^I(z_i) = \sigma_i.$$

This corresponds to what Hanoch (1975) called homothetic CDE, but we prefer calling it CDESH instead, to make it parallel to his terminology of CRESH. The properties of CDESH are similar to Example 11 under HSA and Example 13 under HDIA, except that, *unlike in Example 11 but like Example 13,* there is no need to impose the restriction that either $\sigma_i \leq 1$ for all i, or $\sigma_i \geq 1$ for all i.

9. Concluding Remarks

Instead of recapitulating what has been covered, let me mention briefly one important topic I was unable to cover in this article due to the space limitation.

Following Dixit & Stiglitz (1977; Section I) and Melitz (2003), most monopolistic competition models assume the CES demand system, which imply that all firms face demand curves with constant and common price elasticity and hence charge the exogenous and common markup rate. One of the most active areas of research today is to allow for endogenous and/or heterogenous markup rates by replacing CES with non-CES, most of which belong to the classes of non-CES reviewed in this article.

To apply non-CES to monopolistic competition models, one must confront a whole set of additional issues. To ensure that no firm has the power to affect the aggregate price indices through its monopoly power over its own variety, we need to redefine the demand systems over a continuum of product varieties. To ensure that marginal revenue for each firm is positive, we need to assume that all products must be gross substitutes. Furthermore, the marginal revenue for each firm needs to be monotonically decreasing in its output (or increasing in its price) along its demand curve to ensure that the profit function is well-behaved. To allow for entry and exit and for endogenous product variety, all products must be inessential. It may also be necessary to impose additional restrictions on the demand systems to ensure the existence and uniqueness of free entry equilibrium. These are just some of the additional considerations that affect the pros and cons of using different classes of non-CES.[38] Addressing all these issues adequately and reviewing this rapidly growing literature on monopolistic competition under non-CES calls for an entirely separate treatment, which I hope to do in the near future.

[38] For example, one advantage of HDIA and HIIA relative to HSA, the ability to have a mixture of gross complements and gross substitutes, is irrelevant when applying them to monopolistic competition, because we need to assume that all products are gross substitutes. On the other hand, one advantage of HSA relative to HDIA and HIIA, the budget share is a function of only relative price under HSA, becomes significant to ensure the existence and uniqueness of free-entry equilibrium and facilitate comparative statics exercises.

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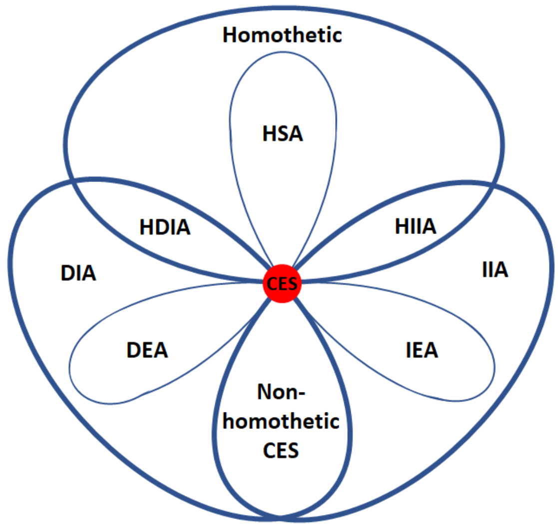
List of Abbreviations Used

AIDS	Almost Ideal Demand System
CDES	Constant Differences of Elasticities of Substitution
CDESH	Constant Differences of Elasticities of Substitution Homothetic
CDIE	Constant Differences of Income Elasticities
CES	Constant Elasticity of Substitution
CRES	Constant Ratios of Elasticities of Substitution
CRESH	Constant Ratios of Elasticities of Substitution Homothetic
CRIE	Constant Relative Income Elasticity
CRS	Constant Returns to Scale
DEA	Direct Explicit Additivity
DIA	Direct Implicit Additivity
HDIA	Homothetic Direct Implicit Additivity
HIIA	Homothetic Indirect Implicit Additivity
HSA	Homothetic with a Single Aggregator
IEA	Indirect Explicit Additivity
IIA	Indirect Implicit Additivity
LDS	Linear Demand System
LES	Linear Expenditure System
PIGL	Price Independent Generalized Linearity
TFP	Total Factor Productivity

Figure: Landscape of the non-CES world



For the acronyms, see List of Abbreviations Used.