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LOVE-FOR-VARIETY

Abstract

We study how love-for-variety, -- productivity (or utility) gains from increasing variety of differentiated inputs (or consumer goods) --, depends on the underlying demand structure. Under general symmetric homothetic demand systems, substitutability across goods and love-for-variety can be both expressed as functions of the variety of available goods, V, only. Since the homotheticity alone imposes little restrictions on the properties of these two functions, we turn to three classes of homothetic demand systems, H.S.A., HDIA, and HIIA, which are pairwise disjoint with the sole exception of CES. For each of these three classes, we establish the three main results. First, substitutability is increasing in V, if and only if Marshall's 2nd law of demand (the price elasticity of demand for each good is increasing in its price) holds. Second, increasing (decreasing) substitutability implies diminishing (increasing) love-for-variety, but the converse is not true. Third, love-for-variety is constant, if and only if substitutability is constant, which occurs only under CES within the three classes. These classes thus offer a tractable way of capturing the intuition that gains from increasing variety is diminishing, if different goods are more substitutable when more variety is available.

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We study how love-for-variety, -- productivity (or utility) gains from increasing variety of differentiated inputs (or consumer goods) --, depends on the underlying demand structure. Under general symmetric homothetic demand systems, substitutability across goods and love-for-variety can be both expressed as functions of the variety of available goods, V, only. Since the homotheticity alone imposes little restrictions on the properties of these two functions, we turn to three classes of homothetic demand systems, H.S.A., HDIA, and HIIA, which are pairwise disjoint with the sole exception of CES. For each of these three classes, we establish the three main results. First, substitutability is increasing in V, if and only if Marshall's 2nd law of demand (the price elasticity of demand for each good is increasing in its price) holds. Second, increasing (decreasing) substitutability implies diminishing (increasing) love-for-variety, but the converse is not true. Third, love-for-variety is constant, if and only if substitutability is constant, which occurs only under CES within the three classes. These classes thus offer a tractable way of capturing the intuition that gains from increasing variety is diminishing, if different goods are more substitutable when more variety is available.

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1. Introduction

Love-for-variety[1] captures the idea that producers (or consumers) can achieve higher level of productivity (or utility) when they have access to a wider variety of differentiated inputs (or consumer goods). It is a natural consequence of the convexity of the production technologies (or preferences). It represents productivity (or utility) gains from increasing variety of inputs (or consumer goods) and hence forms the basis for willingness to pay for new inputs (or consumer goods); e.g., Dixit and Stiglitz (1977), Krugman (1980), Ethier (1982), and Romer (1987). As such, love-for-variety plays a central role in many fields of economics, as Matsuyama (1995) pointed out, most prominently in economic growth (Grossman and Helpman 1993, Gancia and Zilibotti 2005, and Acemoglu 2008), international trade (Helpman and Krugman 1985), and economic geography (Fujita, Krugman, and Venables 1999). Even though commonly discussed in monopolistic competition settings, the concept of love-for-variety is also useful in other contexts, such as gains from trade in Armington-type competitive models, where goods produced in different countries are distinct from each other.

In spite of its importance, however, little is known about how love-for-variety depends on the underlying production (or utility) function, and, in particular, how love-for-variety changes as a wider variety of goods becomes available. In a standard treatment, e.g., Matsuyama (1995, Sec.3A), the analytical expression for love-for-variety is obtained under CES with gross substitutes.[2] It is equal to $1/(\sigma - 1)$, where $\sigma > 1$ represents both the (constant) elasticity of substitution across different goods and the (constant) price elasticity of demand for each good. Even though this expression exhibits the appealing property that love-for-variety is smaller when different goods are more substitutable and the price elasticity of demand for each good is higher (i.e., a larger σ), it also exhibits the property many find less appealing; that is, love-for-variety is constant. It seems implausible to think that productivity (or utility) gains enjoyed by producers (or

[1]Different authors called this concept differently; e.g., "the desirability of variety" (Dixit and Stiglitz 1977), "love of variety" (Helpman and Krugman 1985, sec. 6.2), "taste for variety" (Benassy 1996), etc. In the context of input variety, Ethier (1982) called it "gains from an increased division of labor" and Romer (1987) "increasing returns due to specialization." We call "love-for-variety," as in Parenti et.al. (2017) and Thisse and Ushchev (2020), since it seems most common in the recent literature.

[2]CES is also assumed in most empirical assessment of love-for-variety; see e.g., Feenstra (1994), Bils and Klenow (2001), and Broda and Weinstein (2006). A few exceptions include Feenstra and Weinstein (2017), which use translog.

consumers) from having access to more variety of inputs (or consumer goods) are independent of how much variety they have already access to.[3] Of course, constant love-for-variety may be an artifact of the CES demand system. But the question is then: under which non-CES demand systems should we expect love-for-variety to diminish as more variety becomes available? This is the question we address in this paper.

In Section 2, we first recall some general properties of symmetric CRS production functions and symmetric homothetic demand systems for different inputs that they generate.[4] Then, we show that both substitutability across differentiated inputs and love-for-variety can be expressed as functions of variety of available inputs V only, as $\sigma(V)$ and $\mathcal{L}(V)$, respectively. It turns out that the properties of these two functions, particularly their relation to each other, can be quite complex under general symmetric homothetic demand systems. Moreover, whether Marshall's 2nd law of demand holds or not (the price elasticity of demand for each input is increasing in its own price or not) tells us little about the properties of these two functions.

Since the homothetic restriction is not strong enough to offer much insight, we turn to three classes of homothetic demand systems in Section 3: H.S.A. (Section 3.1), HDIA (Section 3.2), and HIIA (Section 3.3).[5] These three subsections are written in a self-contained way so that they can be read independently in any order. The three classes are pairwise disjoint with the sole exception of CES, as depicted in Figure 1. For each of these three classes, we establish the three main results, also illustrated in Figure 2. First, the substitutability, $\sigma(V)$, is increasing in V, if and only if Marshall's 2nd law of demand holds. Second, increasing (decreasing) substitutability $\sigma'(V) > (<) 0$, implies

[3]Perhaps due to this unappealing feature of love-for-variety under CES, some may prefer "the ideal variety approach," e.g., Helpman and Krugman (1985, sec. 6.3), in which consumers are heterogenous in taste, and each consumer buys the only variety closest to his/her ideal variety. Despite each consumer buys only one variety, increasing variety is beneficial in that each consumer finds a variety closer to the ideal variety on average as product variety increases, and yet the benefit of adding variety is diminishing, as the product space becomes congested. In spite of such appealing feature, the ideal variety approach has not been used widely in applied general equilibrium models due to its intractability.

[4] It should be noted that homothetic and symmetric demand systems are not so restrictive as they may seem, because one can nest them into a nonhomothetic and/or asymmetric upper-tier demand system. In other words, homothetic symmetric non-CES can serve as building blocks to construct such nonhomothetic and/or asymmetric non-CES. Moreover, one of the messages of this paper is that homotheticity and symmetry are not strong enough that one needs to look for tighter restrictions.

[5]H.S.A., HDIA, and HIIA stand for *Homothetic Single Aggregator*, *Homothetic Direct Implicit Additivity*, and *Homothetic Indirect Implicit Additivity*. Matsuyama (2023; 2025) discuss the relation between these three and other classes of non-CES demand systems in detail.

diminishing (increasing) love-for-variety $\mathcal{L}'(V) < (>)0$, but the converse is not true. Third, love-for-variety is constant, $\mathcal{L}'(V) = 0$, if and only if substitutability is constant, $\sigma'(V) = 0$, which occurs only under CES within these three classes. These three classes thus offer a tractable way of capturing the intuition that gains from increasing variety is diminishing, if different goods are more substitutable when more variety is available.[6]

We offer some concluding remarks in Section 4. All technical materials, including the proofs of lemmas, are in Appendices.

Before proceeding, it should be pointed out that this paper is all about the demand side for expanding variety. As such, the results are relevant regardless of what is assumed on the supply side, whether it is modelled as the central planner's problem, perfect competition, oligopolistic competition, or monopolistic competition with or without heterogeneous firms and with or without multi-product firms. Thus, the three classes of demand systems should be a useful building block in many models across many different fields, particularly in international trade, economic growth, and economic geography, wherever gains from endogenous variety are of central importance.

2. Symmetric homothetic demand systems

In what follows, we discuss a general symmetric homothetic demand system in the context of the producer's demand for differentiated inputs. Consider a monotone, strictly quasi-concave, symmetric CRS production function, $X = X(\mathbf{x})$. Here, $\mathbf{x} = \{x_\omega; \omega \in \overline{\Omega}\}$ is the input quantity vector, defined over $\overline{\Omega}$, a continuum of the set of all potential inputs, which is divided into the set of available inputs, $\Omega \subset \overline{\Omega}$, and the set of unavailable inputs, $\overline{\Omega}\backslash\Omega$. That is, $x_\omega = 0$ for $\omega \in \overline{\Omega}\backslash\Omega$. We let the Lebesgue measure of Ω denoted by $V \equiv |\Omega|$. Our goal is to study the effect of changing V on productivity. To this end, it is necessary to assume that each input is inessential. That is, $x_\omega = 0$ for $\omega \in \overline{\Omega}\backslash\Omega$ does *not* imply $X(\mathbf{x}) = 0$, so that it is possible to produce a positive output, even when some potential inputs are unavailable.

[6]As such, they can be valuable alternatives to those who find "the ideal variety approach" more appealing than the love-for-love approach under CES despite that the former is less tractable. We thank Jim Markusen for pointing this out.

2.1. Duality Theory: A Refresher

Let us first recall some key results from the duality theory; see, e.g., Mas-Colell et al. (1995), and Jehle and Reny (2012). Let $\mathbf{p} = \{p_\omega; \omega \in \overline{\Omega}\}$ denote the input price vector, such that $p_\omega = \infty$ for $\omega \in \overline{\Omega}\backslash\Omega$ and $p_\omega < \infty$ for $\omega \in \Omega$. The non-essentiality of inputs ensures that the unit cost function corresponding to this production function,

$$P(\mathbf{p}) \equiv \min_{\mathbf{x}} \left\{ \mathbf{px} \equiv \int_\Omega p_\omega x_\omega d\omega \,\middle|\, X(\mathbf{x}) \geq 1 \right\}, \tag{1}$$

is well-defined, even though $p_\omega = \infty$ for $\omega \in \overline{\Omega}\backslash\Omega$. Furthermore, it also satisfies the monotonicity, strict quasi-concavity, linear homogeneity, and symmetry. The first-order condition of the minimization problem in eq.(1) yields the inverse demand curve for ω:

$$p_\omega = P(\mathbf{p}) \frac{\partial X(\mathbf{x})}{\partial x_\omega}; \quad \omega \in \Omega. \tag{2}$$

Alternatively, from the duality theorem, we can instead start from a monotonic, strict quasi-concave, linear homogeneous, and symmetric unit cost function, $P(\mathbf{p})$, from which the CRS production function $X(\mathbf{x})$ is derived as follows:

$$X(\mathbf{x}) \equiv \min_{\mathbf{p}} \left\{ \mathbf{px} \equiv \int_\Omega p_\omega x_\omega d\omega \,\middle|\, P(\mathbf{p}) \geq 1 \right\}, \tag{3}$$

which satisfies the monotonicity, strict quasi-concavity, and symmetry. The first-order condition of the minimization problem in eq.(3) yields the demand curve:

$$x_\omega = \frac{\partial P(\mathbf{p})}{\partial p_\omega} X(\mathbf{x}); \quad \omega \in \Omega. \tag{4}$$

This is also known as Shepherd's lemma. The duality theorem thus allows us to use either $X(\mathbf{x})$ or $P(\mathbf{p})$ as the primitive of the CRS technology.

From either eq.(3) or eq.(4), the Euler's theorem on linear homogeneity functions implies that

$$\mathbf{px} \equiv \int_\Omega p_\omega x_\omega d\omega = P(\mathbf{p}) \left[\int_\Omega \frac{\partial X(\mathbf{x})}{\partial x_\omega} x_\omega d\omega \right] = \left[\int_\Omega p_\omega \frac{\partial P(\mathbf{p})}{\partial p_\omega} d\omega \right] X(\mathbf{x}) = P(\mathbf{p}) X(\mathbf{x}). \tag{5}$$

This identity means that the total cost of inputs is equal to the total value of output.

2.2. Budget Share, Price Elasticity of Demand and the 2nd Law of Demand

From eq.(2), the budget share of ω can be written as a homogeneous function of degree 0 in $\mathbf{x}$ as:

$$s_\omega \equiv \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} \equiv s^*(x_\omega; \mathbf{x}) = s^*(1, \mathbf{x}/x_\omega); \;\; \omega \in \Omega. \tag{6}$$

From eq.(4), it can also be written as a homogeneous function of degree 0 in $\mathbf{p}$ as:

$$s_\omega \equiv \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} \equiv s(p_\omega; \mathbf{p}) = s(1, \mathbf{p}/p_\omega); \; \omega \in \Omega. \tag{7}$$

In what follows, we assume that different inputs are gross substitutes; that is, the budget share of each input goes down as its price goes up (and its quantity goes down) along its demand curve.

$$\frac{\partial \ln s(p_\omega; \mathbf{p})}{\partial \ln p_\omega} = \frac{\partial \ln s(1, \mathbf{p}/p_\omega)}{\partial \ln p_\omega} < 0 \Leftrightarrow \frac{\partial \ln s^*(x_\omega; \mathbf{x})}{\partial \ln x_\omega} = \frac{\partial \ln s^*(1, \mathbf{x}/x_\omega)}{\partial \ln x_\omega} > 0.$$

From the eq.(6) and eq.(7), the price elasticity of demand for ω can be written both as a function of the prices and as a function of the quantities as follows:

$$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = 1 - \frac{\partial \ln s(p_\omega; \mathbf{p})}{\partial \ln p_\omega} \equiv \zeta(p_\omega; \mathbf{p}) = \zeta(1, \mathbf{p}/p_\omega) > 1; \tag{8}$$

$$\zeta_\omega \equiv -\left[\frac{\partial \ln p_\omega}{\partial \ln x_\omega}\right]^{-1} = \left[1 - \frac{\partial \ln s^*(x_\omega; \; \mathbf{x})}{\partial \ln x_\omega}\right]^{-1} \equiv \zeta^*(x_\omega; \mathbf{x}) = \zeta^*(1, \mathbf{x}/x_\omega) \; > 1. \tag{9}$$

Both functions are homogeneous of degree 0 and gross substitutability implies that they are both greater than one.

Marshall's 2nd law of demand states that the price elasticity of demand for each input goes up as its price goes up (and its quantity goes down) along its demand curve.

Definition: *Marshall's 2nd Law of demand* holds if and only if

$$\frac{\partial \zeta(p_\omega; \mathbf{p})}{\partial p_\omega} = \frac{\partial \zeta(1, \mathbf{p}/p_\omega)}{\partial p_\omega} > 0 \Leftrightarrow \frac{\partial \zeta^*(x_\omega; \mathbf{x})}{\partial x_\omega} = \frac{\partial \zeta^*(1, \mathbf{x}/x_\omega)}{\partial x_\omega} < 0.$$

If the opposite inequality holds, we shall say that anti-2nd law holds. CES is clearly the borderline case.

Note that eqs.(6)-(9) show that the budget share of $\omega \in \Omega$, s_ω, and its price elasticity of demand, ζ_ω, are both functions of $\mathbf{p}/p_\omega$ or $\mathbf{x}/x_\omega$. Even though symmetry implies that they are invariant of permutation, the budget share and the price elasticity still depend on the entire distribution of the prices (or the quantities) normalized by its own price (or its own quantity), which is infinite dimensional. This suggests that the cross-product interactions could be complicated under general homothetic symmetric demand systems.

2.3 Unit Quantity and Price Vectors

To further characterize homothetic symmetric demand systems, it is useful to define the unit quantity vector, $\mathbf{1}_\Omega \equiv \{(1_\Omega)_\omega; \omega \in \overline{\Omega}\}$ and the unit price vector, $\mathbf{1}_\Omega^{-1} \equiv \{(1_\Omega^{-1})_\omega; \omega \in \overline{\Omega}\}$, as follows:

$$(1_\Omega)_\omega \equiv \begin{cases} 1 & \text{for} \quad \omega \in \Omega \\ 0 & \text{for} \quad \omega \in \overline{\Omega}\backslash\Omega \end{cases}; \qquad (1_\Omega^{-1})_\omega \equiv \begin{cases} 1 & \text{for} \quad \omega \in \Omega \\ \infty & \text{for} \quad \omega \in \overline{\Omega}\backslash\Omega \end{cases},$$

which satisfies $\int_\Omega (1_\Omega)_\omega d\omega = \int_\Omega (1_\Omega^{-1})_\omega d\omega = |\Omega| \equiv V$. Then, symmetric quantity and price patterns among all the available inputs are expressed as:

$$\mathbf{x} = x\mathbf{1}_\Omega; \quad \mathbf{p} = p\mathbf{1}_\Omega^{-1},$$

where $x > 0$ and $p > 0$ are scalars.

2.4 Substitutability Measure Across Different Goods

Since $\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$ are both homogenous of degree zero, the price elasticity at symmetric patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$ or $\mathbf{x} = x\mathbf{1}_\Omega$, is $\zeta(1; \mathbf{1}_\Omega^{-1}) = \zeta^*(1; \mathbf{1}_\Omega)$, which is independent of p or x, and can be written as $\sigma(V)$, a function of V only. Moreover, Appendix A shows that $\sigma(V)$ is equal to the Allen-Uzawa elasticity of substitution between every pair of inputs at the symmetric patterns.[7] Thus, we use the following definition for the substitutability across different inputs in the presence of available variety of inputs V:

Definition: *The substitutability measure* is defined by

$$\sigma(V) \equiv \zeta(1; \mathbf{1}_\Omega^{-1}) = \zeta^*(1; \mathbf{1}_\Omega) > 1. \tag{10}$$

In general, $\sigma(V)$ can be nonmonotonic in V. We shall call the case of $\sigma'(V) > (<) 0$ for all $V > 0$, the case of ***increasing (decreasing) substitutability***.

2.5 Love-for-Variety Measure

[7] Since the set of available inputs is a continuum, there is no point of looking into the Morishima elasticity of substitution.

Love-for-variety is commonly measured as the productivity gains from a higher V under the symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$, while holding the total amounts of inputs, xV, constant. That is,

$$\left.\frac{d\ln X(\mathbf{x})}{d\ln V}\right|_{\mathbf{x}=x\mathbf{1}_\Omega, xV=const.} = \left.\frac{d\ln xX(\mathbf{1}_\Omega)}{d\ln V}\right|_{xV=const.} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0,$$

which is positive due to the strict quasi-concavity of $X(\mathbf{x})$ and could depend solely on V. This definition is essentially the same with the one proposed by Benassy (1996, eq.(2)) for what he called "taste for variety," even though he applied it only for CES demand systems with externalities.

Alternatively, we could also measure love-for-variety as the rate of decline in the unit cost under symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$, while holding the price of each input, p, constant. That is,

$$-\left.\frac{d\ln P(\mathbf{p})}{d\ln V}\right|_{\mathbf{p}=p\mathbf{1}_\Omega^{-1},\ p=const.} = -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} > 0,$$

which is also positive due to the monotonicity of $P(\mathbf{p})$ and could depend solely on V.

These two measures of love-for-variety are indeed identical. To see this, inserting $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$ and $\mathbf{x} = x\mathbf{1}_\Omega$ into eq.(5) yields $pxV = pP(\mathbf{1}_\Omega^{-1})xX(\mathbf{1}_\Omega)$, so that

$$P(\mathbf{1}_\Omega^{-1})X(\mathbf{1}_\Omega) = V \Rightarrow -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0.$$

Hence, we use them interchangeably as the love-for-variety measure.

Definition. *The love-for-variety measure* is defined by:

$$\mathcal{L}(V) \equiv -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0. \tag{11}$$

In general, $\mathcal{L}(V)$ can be nonmonotonic in V. We shall call the case of $\mathcal{L}'(V) < (>) 0$ for all $V > 0$, the case of ***diminishing (increasing) love-for-variety***.

2.6. Standard CES with gross substitutes

In the case of standard CES with gross substitutes,

$$X(\mathbf{x}) = Z\left[\int_\Omega x_\omega^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \Leftrightarrow P(\mathbf{p}) = \frac{1}{Z}\left[\int_\Omega p_\omega^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}},$$

where $\sigma > 1$ is the (constant) elasticity of substitution parameter and Z is the TFP parameter, it is easy to verify:

$$\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x}) = \sigma > 1;\ \sigma(V) = \sigma > 1;\ \mathcal{L}(V) = \frac{1}{\sigma - 1} > 0.$$

Thus, under CES, the price elasticity of demand for each input is *everywhere* constant and equal to σ. Obviously, this implies that our substitutability measure, which is equal to the price elasticity evaluated at the symmetric patterns, $\sigma(V)$, is also equal to σ, and independent of V. Moreover, the love-for-variety measure, $\mathcal{L}(V)$, is also independent of V, and depends solely on the single parameter, σ, with a one-to-one inverse relation between the two. Perhaps for these reasons, some authors have incorrectly claimed that $\sigma(V)$ is constant only under CES, and/or that $\sigma(V)$ is the inverse measure of love-for-variety, even under general homothetic demand systems.

2.7. Modified CES a la Benassy (1996): A Digression

Before proceeding to explore love-for-variety under non-CES demand systems, let us briefly digress to discuss an attempt to break the tight relation between $\sigma(V)$ and $\mathcal{L}(V)$ by modifying CES by Benassy (1996). He proposed to generalize CES as:

$$X(\mathbf{x}) = Z(V)\left[\int_\Omega x_\omega^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \Leftrightarrow P(\mathbf{p}) = \frac{1}{Z(V)}\left[\int_\Omega p_\omega^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}},$$

by making TFP a function of V as $Z(V)$, justified by some sorts of direct externalities from V to TFP. Under such modified CES,

$$\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x}) = \sigma > 1;\ \sigma(V) = \sigma > 1;\ \mathcal{L}(V) = \frac{1}{\sigma - 1} + \frac{\partial \ln Z(V)}{\partial \ln V}.$$

This demand system is still CES, as it features the constant price elasticity and the constant substitutability. Yet this modification allows the gap between the observed love-for-variety and the love-for-variety implied by CES demand to be "accounted for" by $\partial \ln Z(V)/\partial \ln V$, the term we would call 'the Benassy residual," in analogy with the Solow residual in the growth accounting. In addition, Benassy (1996) assumed that $\partial \ln Z(V)/\partial \ln V = \nu - 1/(\sigma - 1)$, so that $\mathcal{L}(V) = \nu$, which can be chosen independently from $\sigma(V) = \sigma$. If we assume instead $\partial \ln Z(V)/\partial \ln V$ is another parameter independent of $\sigma(V) = \sigma$, $\mathcal{L}(V)$ is still inversely related to $\sigma(V) = \sigma$. Even if one believed in the presence of such direct externalities from V to TFP, one should note that any estimate of the Benassy residual hinges on the assumption of CES. In any case, introducing the Benassy residual does not address our question; that is, how $\mathcal{L}(V)$ depends on V under homothetic non-CES demand systems, in particular, when we expect $\mathcal{L}(V)$ to be diminishing in V.

For the remainder of the paper, we assume that TFP is independent of V for simplicity, since adding the Benassy residual would be straightforward.

2.8. General Cases

The relation between the price elasticity, $\zeta(p_\omega;\mathbf{p}) = \zeta^*(x_\omega;\mathbf{x})$, substitutability, $\sigma(V)$, and love-for-variety, $\mathcal{L}(V)$, can be quite complex under general homothetic demand systems. First, whether Marshall's 2nd law holds or not in general tells us little about the sign of the derivative of $\sigma(V)$. This should not be surprising because the former is about how $\zeta(p_\omega; p\mathbf{1}_\Omega^{-1})$ or $\zeta^*(x_\omega; x\mathbf{1}_\Omega)$ responds to a change in p_ω or x_ω, while the latter is about how $\zeta(p; p\mathbf{1}_\Omega^{-1}) = \zeta(1;\mathbf{1}_\Omega^{-1})$ or $\zeta^*(x; x\mathbf{1}_\Omega) = \zeta^*(1;\mathbf{1}_\Omega)$ responds to a change in V through its effect on $\mathbf{1}_\Omega^{-1}$ or $\mathbf{1}_\Omega$. Second, as shown in Appendix B, there exists a parametric family of homothetic non-CES demand systems in which $\sigma(V)$ and $\mathcal{L}(V)$ are both independent of V, and they move in the same direction as one of the parameters changes. More generally, with both $\sigma(V)$ and $\mathcal{L}(V)$ being functions of V, one cannot expect $\sigma(V)$ and $\mathcal{L}(V)$ to always move in the opposite direction as V varies. In short, "almost anything goes" under general homothetic symmetric demand systems.

3. Three Classes of Homothetic Symmetric Demand Systems

Nevertheless, it is intuitive to think that, when different inputs are more substitutable, the price elasticity of demand for each input should be larger, and the love-for-variety measure should be smaller. Unfortunately, homotheticity (and symmetry) alone is not restrictive enough to capture this intuition: one need to find some additional restrictions to capture this intuition. To this end, we turn to three classes of homothetic symmetric demand systems, H.S.A., HDIA, and HIIA. These three classes are useful for two reasons. First, they are pairwise disjoint with the sole exception of CES, as seen in Figure 1. Thus, they offer three alternative ways of departing from CES, while keeping

CES as a special case. Second, each of the three classes generates the demand system with the property that the price elasticity of demand for each input can be expressed as $\zeta(p_\omega; \mathbf{p}) = \zeta(p_\omega/\mathcal{A}(\mathbf{p}))$ and $\zeta^*(x_\omega; \mathbf{x}) = \zeta^*(x_\omega/\mathcal{A}^*(\mathbf{x}))$. That is, the price elasticity is a function of a single variable, $p_\omega/\mathcal{A}(\mathbf{p})$ or $x_\omega/\mathcal{A}^*(\mathbf{x})$, where $\mathcal{A}(\mathbf{p})$ or $\mathcal{A}^*(\mathbf{x})$ is the linear homogeneous aggregator in $\mathbf{p}$ or in $\mathbf{x}$, whose value serves as a sufficient statistic that captures the interdependence of price elasticities across different inputs.[8] Thus, in these three classes, the price elasticity responds to an increase in p_ω and to a decline in $\mathcal{A}(\mathbf{p})$ in the same way, and hence also to an increase in V in the symmetric price patterns. Or equivalently, the price elasticity responds to a decline in x_ω and to an increase in $\mathcal{A}^*(\mathbf{x})$ in the same way, and hence also to an increase in V at the symmetric quantity patterns. This feature imposes the tight restriction between $\sigma(V)$ and $\mathcal{L}(V)$, as summarized in Table, which enables us to establish the following three results for each of the three classes. First, Marshall's 2nd law is equivalent to increasing substitutability, $\sigma'(V) > 0$. Second, both Marshall's 2nd law and increasing (decreasing) substitutability $\sigma'(V) > (<)0$ are sufficient but not necessary for diminishing (increasing) love-for-variety, $\mathcal{L}'(V) < (>)0$. Third, $\mathcal{L}(V)$ is constant if and only if it is CES. See also Figure 2.

We now explain each of the three classes in great detail in the next three sections. Because these three sections are written in a self-contained way, they may be read independently in any order.

3.1. The HSA class.

A homothetic symmetric demand system belongs to H.S.A. (*Homothetic Single Aggregator*) if there exists a function of a single variable, $s: \mathbb{R}_{++} \to \mathbb{R}_+$, which is C^2 and strictly decreasing as long as $s(z) > 0$, with $\lim_{z\to 0} s(z) = \infty$ and $\lim_{z\to \bar{z}} s(z) = 0$, where $\bar{z} \equiv \inf\{z > 0 | s(z) = 0\}$, such that the budget share of $\omega \in \Omega$ can be written as:

$$s_\omega = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(\mathbf{p})}\right), \tag{12}$$

where $A(\mathbf{p})$ is defined implicitly by the adding-up constraint,

[8]Recall that, in general, the price elasticity of each input depends on $\mathbf{p}/p_\omega$ or $\mathbf{x}/x_\omega$, the entire distribution of the prices (or quantities) normalized by its own price (or quantity) as shown in eq.(8) and eq.(9).

$$\int_{\Omega} s\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega \equiv 1. \tag{13}$$

By construction, $A(\mathbf{p})$ is linear homogenous in $\mathbf{p}$ for any fixed Ω and that the budget shares of all inputs are added up to one. CES with gross substitutes is a special case where $s(z) = \gamma z^{1-\sigma}$ $(\sigma > 1)$. Translog unit cost function is another special case, where $s(z) = \gamma \max\{-\ln(z/\bar{z}), 0\}$, where $\bar{z} < \infty$.[9] The CoPaTh family[10] of H.S.A. is given by

$$s(z) = \gamma \max\left\{\left[\sigma - (\sigma - 1)z^{\frac{1-\rho}{\rho}}\right]^{\frac{\rho}{1-\rho}}, 0\right\},$$

where $0 < \rho < 1$, with $\bar{z} = (1 - 1/\sigma)^{-\frac{\rho}{1-\rho}} \to \infty$ and $s(z) \to \gamma z^{1-\sigma}$, as $\rho \nearrow 1$. Symmetric H.S.A. have been recently applied to a variety of monopolistic competition models.[11]

Eqs.(12)-(13) state that the budget share of $\omega \in \Omega$ is decreasing in its *normalized price,* $z_\omega \equiv p_\omega / A(\mathbf{p})$, which is defined as its own price, p_ω, divided by the *common price aggregator,* $A(\mathbf{p})$. Note that the budget share function, $s(\cdot)$, is the primitive of H.S.A., while $A(\mathbf{p})$ is not, as it is derived from $s(\cdot)$, using eq.(13). The monotonicity of $s(\cdot)$, combined with the assumptions, $\lim_{z\to 0} s(z) = \infty$ and $\lim_{z\to\bar{z}} s(z) = 0$, ensures that $A(\mathbf{p})$ is defined uniquely by eq.(13) for any $V \equiv |\Omega| > 0$. $A(\mathbf{p})$ is independent of ω, and thus captures "the average price" against which the prices of *all* inputs in Ω are measured. In other words, one could keep track of all the cross-price effects in the demand system by looking at a single aggregator, $A(\mathbf{p})$, which is the key feature of H.S.A.[12] Note also

[9]For $s: \mathbb{R}_{++} \to \mathbb{R}_{+}$, satisfying the above conditions, a class of the budget share functions, $s_\gamma(z) \equiv \gamma s(z)$ for $\gamma > 0$, generate the same demand system with the same common price aggregator. This can be seen by reindexing the inputs, as $\omega' = \gamma\omega$, so that $\int_{\Omega} s_\gamma(p_\omega/A(\mathbf{p}))d\omega = \int_{\Omega} s(p_{\omega'}/A(\mathbf{p}))d\omega' = 1$. In this sense, $s_\gamma(z) \equiv \gamma s(z)$ for $\gamma > 0$ are all equivalent. Note also that a class of the budget share functions, $s_\lambda(z) \equiv s(\lambda z)$ for $\lambda > 0$, generate the same demand system, with $A_\lambda(\mathbf{p}) = \lambda A(\mathbf{p})$, because $s_\lambda(p_\omega/A_\lambda(\mathbf{p})) = s(\lambda p_\omega/A_\lambda(\mathbf{p})) = s(p_\omega/A(\mathbf{p}))$. In this sense, $s_\lambda(z) \equiv s(\lambda z)$ for $\lambda > 0$ are all equivalent.

[10]CoPaTh stands for Constant Pass-Through; it is so named, since, when a monopolistic competitive firm faces the demand curve generated by this family, its pricing behavior features a constant pass-through rate, $0 < \rho < 1$, and it converges to CES, as $\rho \nearrow 1$. Matsuyama and Ushchev (2020b) developed the CoPaTh family of demand systems within H.S.A., HDIA, and HIIA.

[11]See, e.g., Baqaee, Farhi, and Sangani (2024), Fujiwara and Matsuyama (2022), Grossman, Helpman, and Luillier (2023), Matsuyama and Ushchev (2020a, 2020b, 2022a, 2022b). A large literature on monopolistic competition models under translog demand systems, which follows Feenstra (2003), may be also added to this list, because a symmetric translog unit cost function is a special case of symmetric H.S.A. with gross substitutes.

[12]In contrast, that $s(\cdot)$ is independent of ω is not a defining feature of H.S.A., but due to the assumption that the underlying production function is symmetric. Generally, the H.S.A. class of the production functions is defined by the

that we allow for the possibility of $\bar{z} < \infty$, in which case $\bar{z}A(\mathbf{p})$ is the choke price, at which demand for a variety goes to zero. If $\bar{z} = \infty$, the choke price does not exist and demand for each input always remains positive for any positive price vector.

The price elasticity of $\omega \in \Omega$ can be written as a function of $z_\omega \equiv p_\omega / A(\mathbf{p})$ as

$$\zeta_\omega = \zeta(p_\omega; \mathbf{p}) = 1 - \frac{z_\omega s'(z_\omega)}{s(z_\omega)} \equiv \zeta^S(z_\omega) = \zeta^S\left(\frac{p_\omega}{A(\mathbf{p})}\right) > 1,$$

where $\zeta^S: (0, \bar{z}) \to (1, \infty)$ is C^1 for $z \in (0, \bar{z})$, and $\lim_{z \to \bar{z}} \zeta^S(z) = \infty$ if $\bar{z} < \infty$.[13] It turns out to be convenient to introduce another function, $H: (0, \bar{z}) \to \mathbb{R}_+$,

$$H(z) \equiv \int_z^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi > 0,$$

so that

$$\zeta^S(z) \equiv 1 - \frac{zs'(z)}{s(z)} \equiv -\frac{zH''(z)}{H'(z)} > 1. \tag{14}$$

In general, $\zeta^S(\cdot)$ can be nonmonotonic. Under CES, it is constant, $\zeta^{S\prime}(\cdot) = 0$. Marshall's 2nd law, $\partial\zeta(p_\omega; \mathbf{p})/\partial p_\omega > 0$, holds if and only if $\zeta^{S\prime}(\cdot) > 0$. This condition is satisfied both by translog with $\zeta^S(z_\omega) = 1 - [\ln(z_\omega/\bar{z})]^{-1}$ and by CoPaTh with $\zeta^S(z_\omega) = \left[1 - (1 - 1/\sigma){z_\omega}^{(1-\rho)/\rho}\right]^{-1} = \left[1 - (z_\omega/\bar{z})^{(1-\rho)/\rho}\right]^{-1}$.

After deriving $A(\mathbf{p})$ from $s(\cdot)$, the unit cost function, $P(\mathbf{p})$, can be obtained by integrating eq.(12), which yields

$$\ln\left[\frac{A(\mathbf{p})}{cP(\mathbf{p})}\right] = \int_\Omega \left[\int_{p_\omega/A(\mathbf{p})}^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi\right] d\omega \equiv \int_\Omega H\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega \tag{15}$$

$$\equiv \int_\Omega s\left(\frac{p_\omega}{A(\mathbf{p})}\right) \Phi\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega$$

where

property that the budget share of ω is given by $s_\omega(p_\omega/A(\mathbf{p}))$, where $A(\mathbf{p})$ is the unique solution to $\int_\Omega s_\omega(p_\omega/A(\mathbf{p}))d\omega = 1$. Note that $s_\omega(\cdot)$ may depend on ω but $A(\cdot)$ does not.

[13] Conversely, from any C^1 function $\zeta^S: (0, \bar{z}) \to (1 + \varepsilon, \infty)$, satisfying $\lim_{z \to \bar{z}} \zeta^S(z) = \infty$ if $\bar{z} < \infty$, one could reverse-engineer as $s(z) = \gamma \exp\left[\int_{z_0}^{z}[1 - \zeta^S(\xi)]\, d\xi/\xi\right] > 0;\ z_0, z \in (0, \bar{z})$, where $\gamma = s(z_0)$ is a positive constant. One could thus use $\zeta^S(\cdot)$ instead of $s(\cdot)$, as a primitive of symmetric H.S.A. with gross substitutes.

$$\Phi(z) \equiv \frac{1}{s(z)} \int_{z}^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi \equiv -\frac{H(z)}{zH'(z)} > 0$$

and c is a positive constant, which is proportional to TFP.[14] The unit cost function, $P(\mathbf{p})$, satisfies the linear homogeneity, monotonicity, and strict quasi-concavity, and so does the corresponding production function, $X(\mathbf{x})$. This follows from Matsuyama and Ushchev (2017; Proposition 1-i)) and guarantees the integrability (in the sense of Samuelson 1950 and Hurwicz and Uzawa 1971) of H.S.A. demand systems. It is important to note that, with the sole exception of CES, $A(\mathbf{p})/P(\mathbf{p})$ is not constant and depends on $\mathbf{p}$.[15] This can be verified by differentiating eq.(13) to yield

$$\frac{\partial \ln A(\mathbf{p})}{\partial \ln p_\omega} = \frac{z_\omega s'(z_\omega)}{\int_\Omega s'(z_{\omega'}) z_{\omega'} d\omega'} = \frac{[\zeta^S(z_\omega) - 1] s(z_\omega)}{\int_\Omega [\zeta^S(z_{\omega'}) - 1] s(z_{\omega'}) d\omega'}$$

which differs from

$$\frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s(z_\omega),$$

unless $\zeta^S(z)$ is independent of z, or equivalently, $s(z) = \gamma z^{1-\sigma}$ with $\zeta^S(z) = \sigma > 1$. This should not come as a surprise. After all, $A(\mathbf{p})$ is the "average input price", which captures the *cross-product effects* in the demand system, while $P(\mathbf{p})$ is the inverse measure of TFP, which captures the *productivity (or welfare) effects* of price changes. There is no reason to think *a priori* that they should move together.

We are now ready to derive the substitutability and love-for-variety measures under H.S.A. For symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$, eq.(13) is simplified to

$$s\left(\frac{1}{A(\mathbf{1}_\Omega^{-1})}\right) V = 1 \Longrightarrow \frac{1}{A(\mathbf{1}_\Omega^{-1})} = s^{-1}\left(\frac{1}{V}\right)$$

Hence, from eq.(14), the substitutability measure is given by:

[14]The constant term in eq.(15), which appears by integrating eq.(12), cannot be pinned down. First, $A(\mathbf{p})$, the "average input price", depends on the unit of measurement of inputs, but not on the unit of measurement of the final good. In contrast, $P(\mathbf{p})$ is the cost of producing one unit of the final good, when the input prices are given by $\mathbf{p}$. Hence, it depends not only on the unit of measurement of inputs but also on that of the final good. Second, a change in TFP, though it affects $P(\mathbf{p})$, leaves the budget share of each input, and hence $A(\mathbf{p})$, unaffected.

[15]This holds more generally, that is, for asymmetric H.S.A., as well as H.S.A. with gross complements, as shown in Matsuyama and Ushchev (2017; Proposition 1-iii).

$$\sigma(V) \equiv \zeta(1; \mathbf{1}_\Omega^{-1}) = \zeta^S\left(s^{-1}\left(\frac{1}{V}\right)\right) = -\frac{zH''(z)}{H'(z)}\bigg|_{z=s^{-1}(1/V)} > 1. \tag{16}$$

For the love-for-variety measure, from eq. (15),

$$-\ln[P(\mathbf{1}_\Omega^{-1})] = \ln\left[cs^{-1}\left(\frac{1}{V}\right)\right] + \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) \Rightarrow$$

$$-\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} = -\left[\frac{d[\ln z + \Phi(z)]}{d\ln z}\Big/\frac{d\ln s(z)}{d\ln z}\right]\bigg|_{z=s^{-1}(1/V)} = \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right)$$

so that

$$\mathcal{L}(V) \equiv -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} = \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) = -\frac{H(z)}{zH'(z)}\bigg|_{z=s^{-1}(1/V)}. \tag{17}$$ [16]

Since $s^{-1}(1/V)$ is increasing in V, eqs.(16)-(17) imply

$$\zeta^{S\prime}(\cdot) \gtreqless 0 \Leftrightarrow \sigma'(\cdot) \gtreqless 0; \qquad \Phi'(\cdot) \gtreqless 0 \Leftrightarrow \mathcal{L}'(\cdot) \gtreqless 0.$$

The next lemma shows the relation between the following two functions:

$$\zeta^S(z) \equiv 1 - \frac{zs'(z)}{s(z)} \equiv -\frac{zH''(z)}{H'(z)} > 1; \text{ and } \Phi(z) \equiv \frac{1}{s(z)}\int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi \equiv -\frac{H(z)}{zH'(z)} > 0.$$

Lemma S:

$$\frac{z\Phi'(z)}{\Phi(z)} = \zeta^S(z) - 1 - \frac{1}{\Phi(z)} = \zeta^S(z) - \int_z^{\bar{z}} \zeta^S(\xi) w(\xi; z) d\xi.$$

where $w^S(\xi; z) \equiv -H'(\xi)/H(z)$, which satisfies $\int_z^{\bar{z}} w^S(\xi; z) d\xi = 1$. Hence,

$$\zeta^{S\prime}(z) \gtreqless 0, \forall z \in (z_0, \bar{z}) \Rightarrow \Phi'(z) \lesseqgtr 0, \forall z \in (z_0, \bar{z}).$$

The opposite is not true in general. However,

$$\zeta^{S\prime}(z) = 0, \forall z \in (z_0, \bar{z}) \Leftrightarrow \Phi'(z) = 0, \forall z \in (z_0, \bar{z}).$$

The proof of Lemma S is in Appendix D. By combining Lemma S, eq.(16) and eq.(17),

Proposition S. For $s(z_0)V_0 = 1$,

$$\zeta^{S\prime}(z) \gtreqless 0, \forall z \in (z_0, \bar{z}) \Leftrightarrow \sigma'(V) \gtreqless 0, \forall V \in (V_0, \infty);$$

$$\Phi'(z) \lesseqgtr 0, \forall z \in (z_0, \bar{z}) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

Moreover,

[16] Moreover, by evaluating eq.(15) at the symmetric price patterns, we can show that $\mathcal{L}(V) = \ln\left[\frac{A(\mathbf{1}_\Omega^{-1})}{cP(\mathbf{1}_\Omega^{-1})}\right]$.

$$\sigma'(V) \gtreqless 0, \forall V \in (V_0, \infty) \Longrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

The opposite is not true in general. However,

$$\sigma'(V) = 0, \forall V \in (V_0, \infty) \Leftrightarrow \mathcal{L}'(V) = 0, \forall V \in (V_0, \infty).$$

In particular, if $z_0 \to 0$ implies $s(z_0) \to \infty, V_0 \to 0$ so that Marshall's 2nd Law, $\zeta^{S\prime}(\cdot) > 0$ for all $z < \bar{z}$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$ for all $V > 0$, both of which imply diminishing love-for-variety, $\mathcal{L}'(\cdot) < 0$ for all $V > 0$.[17] The converse is not true. Diminishing love-for-variety for all $V > 0$ does not necessarily imply increasing substitutability or Marshall's 2nd Law globally. However, constant love-for-variety, $\mathcal{L}'(\cdot) = 0$ for all $V > 0$, implies both constant substitutability, $\sigma'(\cdot) = 0$ for all $V > 0$, and constant price elasticity $\zeta'(\cdot) = 0$ for all $z < \bar{z} = \infty$, under H.S.A., which occurs only under CES.

Before proceeding, it should be pointed out that there exists an alternative (but equivalent) definition of H.S.A. For the sake of completeness, we discuss this alternative in Appendix C.

3.2. The HDIA class

A homothetic symmetric demand system for differentiated inputs belongs to HDIA (*Homothetic Direct Implicit Additivity*) with gross substitutes if it is generated by the cost minimization of the competitive industry whose CRS production function, $X = X(\mathbf{x}) \equiv Z\hat{X}(\mathbf{x})$ can be defined implicitly by:

$$\int_{\Omega} \phi\left(\frac{Zx_\omega}{X(\mathbf{x})}\right) d\omega = \int_{\Omega} \phi\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right) d\omega \equiv 1. \tag{18}$$

here $\phi(\cdot): \mathbb{R}_+ \to \mathbb{R}_+$ is C^3, with $\phi'(y) > 0 > \phi''(y)$, $-y\phi''(y)/\phi'(y) < 1$ for $\forall y \in (0, \infty)$ and $\phi(0) = 0$ and $\phi(\infty) = \infty$, and independent of $Z > 0$, the TFP parameter. Note that, unlike eq.(13), the adding-up constraint of the H.S.A, eq.(18) defines the production function $X(\mathbf{x})$ directly.[18] CES with gross substitutes is a special case where $\phi(y) = (y)^{1-1/\sigma}$ $(\sigma > 1)$. The CoPaTh family of HDIA is given by

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[17] This generalizes the case of translog, $s(z) = \gamma \max\{-\ln(z/\bar{z}), 0\}$, where $\sigma(V) = 1 + \gamma V$ and $\mathcal{L}(V) = 1/(2\gamma V)$.

[18] This means that, unlike H.S.A. but similar to HIIA defined in the next section, we do not need to worry about the integrability of HDIA. Note also that $\hat{X}(\mathbf{x}) = X(\mathbf{x})/Z$ defined by eq.(18) is invariant of TFP, $Z > 0$, by construction. Thus, an increase in Z causes a proportionate increase in $X(\mathbf{x})$. This allows us to examine the effect of TFP without shifting $\phi(\cdot)$. Alternatively, we could have defined $X(\mathbf{x})$ by $\int_{\Omega} \phi(x_\omega/X(\mathbf{x}))d\omega = 1$, as in Matsuyama and Ushchev

$$\phi(y) = \int_0^{y} \left(1 + \frac{1}{\sigma - 1}(\xi)^{\frac{1-\rho}{\rho}}\right)^{\frac{\rho}{\rho-1}} d\xi,$$

for $0 < \rho < 1$, which converges to CES with $\rho \nearrow 1$. Symmetric HDIA defined as above may be viewed as an extension of the Kimball (1995) aggregator in that the set of available inputs Ω is not fixed, and in particular, its Lebesgue measure, $V \equiv |\Omega|$, is a variable.

From the cost minimization problem, eq.(1), subject to eq.(18), we obtain the inverse demand curve,

$$p_\omega = B(\mathbf{p})\phi'\left(\frac{Zx_\omega}{X(\mathbf{x})}\right) = B(\mathbf{p})\phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right), \tag{19}$$

and hence the demand curve,

$$x_\omega = (\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\hat{X}(\mathbf{x}) = (\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\frac{X(\mathbf{x})}{Z},$$

where $B(\mathbf{p})$ is defined by:

$$\int_\Omega \phi\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right) d\omega \equiv 1.$$

This shows that the choke price is equal to $B(\mathbf{p})\phi'(0)$ if $\phi'(0) < \infty$, and that there is no choke price if $\phi'(0) = \infty$. The unit cost function is:

$$P(\mathbf{p}) = \frac{\hat{P}(\mathbf{p})}{Z} \equiv \frac{1}{Z}\int_\Omega p_\omega (\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) d\omega.$$

Clearly, both $B(\mathbf{p})$ and $\hat{P}(\mathbf{p})$ are linear homogenous in $\mathbf{p}$, and independent of $Z > 0$. Hence, an increase in TFP, Z, causes a proportional decline in the unit cost function, $P(\mathbf{p}) = \hat{P}(\mathbf{p})/Z$.

The budget share, $s_\omega = s(p_\omega; \mathbf{p}) = s^*(x_\omega; \mathbf{x})$, is:

$$\frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{p_\omega}{\hat{P}(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) = \phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right)\frac{x_\omega}{C^*(\mathbf{x})}. \tag{20}$$

where

$$C^*(\mathbf{x}) \equiv \int_\Omega \phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right) x_\omega d\omega$$

(2017). Though mathematically equivalent, this definition requires that $\phi(\cdot)$ would no longer be independent of TFP, which would make it harder to show that $\sigma(V)$ and $\mathcal{L}(V)$ are independent of TFP.

is linear homogenous in $\mathbf{x}$ and independent of $Z > 0$, and satisfies the identity

$$\frac{\hat{P}(\mathbf{p})}{B(\mathbf{p})} = \int_{\Omega} \frac{p_\omega}{B(\mathbf{p})} (\phi')^{-1} \left(\frac{p_\omega}{B(\mathbf{p})}\right) d\omega = \int_{\Omega} \phi' \left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right) \frac{x_\omega}{\hat{X}(\mathbf{x})} d\omega = \frac{C^*(\mathbf{x})}{\hat{X}(\mathbf{x})}. \tag{21}$$

Eqs.(20)-(21) show that the budget share under HDIA is a function of the two relative prices, $p_\omega/\hat{P}(\mathbf{p})$ and $p_\omega/B(\mathbf{p})$, or a function of the two relative quantities, $x_\omega/\hat{X}(\mathbf{x})$ and $x_\omega/C^*(\mathbf{x})$, unless $\hat{P}(\mathbf{p})/B(\mathbf{p}) = C^*(\mathbf{x})/\hat{X}(\mathbf{x})$ is a positive constant, which occurs if and only if it is CES. Thus, HDIA and H.S.A. do not overlap with the sole exception of CES.[19]

From the inverse demand curve, eq.(19), the price elasticity of demand can be written as a function of a single variable, $y_\omega \equiv x_\omega/\hat{X}(\mathbf{x})$ as:

$$\zeta_\omega = \zeta^*(x_\omega; \boldsymbol{x}) = -\frac{\phi'(y_\omega)}{y\phi''(y_\omega)} \equiv \zeta^D(y_\omega) = \zeta^D\left(\frac{x_\omega}{\hat{X}(\boldsymbol{x})}\right) > 1 \tag{22}$$

where $\zeta^D(y) > 1$ ensures gross substitutability. Using eq.(19), it can also be written as a function of $p_\omega/B(\mathbf{p}) = \phi'(y_\omega)$ as:

$$\zeta_\omega = \zeta(p_\omega; \mathbf{p}) = \zeta^D\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right) > 1.$$

Under CES, $\zeta^{D\prime}(\cdot) = 0$. Marshall's 2nd law, $\partial\zeta^*(x_\omega; \mathbf{x})/\partial x_\omega < 0$, holds if and only if $\zeta^{D\prime}(\cdot) < 0$, the condition satisfied by CoPaTh, with $\zeta^D(y) = 1 + (\sigma - 1)(y)^{\frac{\rho-1}{\rho}}$.

We are now ready to derive the substitutability and love-for-variety measures under HDIA. For symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$, eq.(18) is simplified to

$$\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right)V = 1 \Rightarrow \frac{1}{\hat{X}(\mathbf{1}_\Omega)} = \phi^{-1}\left(\frac{1}{V}\right).$$

Hence, from eq.(22), the substitutability measure is given by:

$$\sigma(V) \equiv \zeta^*(1; \mathbf{1}_\Omega) = \zeta^D\left(\phi^{-1}\left(\frac{1}{V}\right)\right) = -\left.\frac{\phi'(y)}{y\phi''(y)}\right|_{y=\phi^{-1}(1/V)} > 1. \tag{23}$$

The love-for-variety measure under HDIA is given by:

$$\mathcal{L}(V) \equiv \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 = \frac{1}{\mathcal{E}_\phi\left(\phi^{-1}(1/V)\right)} - 1 \equiv \left.\frac{\phi(y)}{y\phi'(y)}\right|_{y=\phi^{-1}(1/V)} - 1 > 0 \tag{24}$$

[19]This statement is a special case of Proposition 2-(ii) in Matsuyama and Ushchev (2017).

where

$$0 < \mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)} < 1.$$ [20]

Since $\phi^{-1}(1/V)$ is decreasing in V, eqs.(23)-(24) imply

$$\zeta^{D\prime}(\cdot) \lesseqgtr 0 \Leftrightarrow \sigma'(\cdot) \gtreqless 0; \quad \mathcal{E}'_\phi(\cdot) \gtreqless 0 \Leftrightarrow \mathcal{L}'(\cdot) \gtreqless 0,$$

The next lemma shows the relation between the following two functions:

$$\zeta^D(y) \equiv -\frac{\phi'(y)}{y\phi''(y)} > 1 \quad \text{and} \quad 0 < \mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)} < 1.$$

Lemma D:

$$\frac{y\mathcal{E}'_\phi(y)}{\mathcal{E}_\phi(y)} = 1 - \frac{1}{\zeta^D(y)} - \mathcal{E}_\phi(y) = \int_0^y \left[\frac{1}{\zeta^D(\xi)}\right] w^D(\xi; y) d\xi - \frac{1}{\zeta^D(y)}.$$

where $w^D(\xi; y) \equiv \phi'(\xi)/\phi(y) > 0$, which satisfies $\int_0^y w^D(\xi;\, y)\, d\xi = 1$. Hence,

$$\zeta^{D\prime}(y) \lesseqgtr 0, \forall y \in (0, y_0) \Rightarrow \mathcal{E}'_\phi(y) \lesseqgtr 0, \forall y \in (0, y_0).$$

The opposite is not true in general. However,

$$\zeta^{D\prime}(y) = 0, \forall y \in (0, y_0) \Leftrightarrow \mathcal{E}'_\phi(y) = 0, \forall y \in (0, y_0).$$

The proof of Lemma D is in Appendix D. By combining Lemma D, eq.(23) and eq.(24),

Proposition D: For $\phi(y_0)V_0 = 1$,

$$\zeta^{D\prime}(y) \lesseqgtr 0\ \forall y \in (0, y_0) \Leftrightarrow \sigma'(V) \gtreqless 0, \forall V \in (V_0, \infty);$$

$$\mathcal{E}'_\phi(y) \lesseqgtr 0, \forall y \in (0, y_0) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

Moreover,

$$\sigma'(V) \gtreqless 0, \forall V \in (V_0, \infty) \Rightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

The opposite is not true in general. However,

$$\sigma'(V) = 0, \forall V \in (V_0, \infty) \Leftrightarrow \mathcal{L}'(V) = 0, \forall V \in (V_0, \infty).$$

In particular, as $y_0 \to \infty$, $V_0 \to 0$ so that Marshall's 2nd Law, $\zeta^{D\prime}(\cdot) < 0$ for all $y > 0$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$ for all $V > 0$, both of which imply diminishing love-for-variety, $\mathcal{L}'(\cdot) < 0$ for all $V > 0$. The converse is not true. Diminishing love-for-variety for all $V > 0$ does not necessarily imply increasing

[20] Moreover, by evaluating eq.(21) at the symmetric price and quantity patterns, one can show that $\frac{\hat{P}(\mathbf{1}_\Omega^{-1})}{B(\mathbf{1}_\Omega^{-1})} = \frac{C^*(\mathbf{1}_\Omega)}{\hat{X}(\mathbf{1}_\Omega)} = \int_\Omega \mathcal{E}_\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right)\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right) d\omega = \mathcal{E}_\phi\left(\phi^{-1}\left(\frac{1}{V}\right)\right) \Rightarrow \mathcal{L}(V) = \frac{B(\mathbf{1}_\Omega^{-1})}{\hat{P}(\mathbf{1}_\Omega^{-1})} - 1 = \frac{\hat{X}(\mathbf{1}_\Omega)}{C^*(\mathbf{1}_\Omega)} - 1.$

substitutability or Marshall's 2nd Law globally. However, constant love-for-variety, $\mathcal{L}'(\cdot) = 0$ for all $V > 0$, implies both constant substitutability, $\sigma'(\cdot) = 0$ for all $V > 0$, and constant price elasticity $\zeta^{D\prime}(\cdot) = 0$ for all $y > 0$ under HDIA, which occurs only under CES.

3.3. The HIIA class.

A homothetic symmetric demand system for differentiated inputs belongs to HIIA (*Homothetic Indirect Implicit Additivity*) with gross substitutes if it is generated by the cost minimization of the competitive industry whose unit cost function, $P = P(\mathbf{p}) = \hat{P}(\mathbf{p})/Z$, can be defined implicitly by:

$$\int_{\Omega} \theta\left(\frac{p_\omega}{ZP(\mathbf{p})}\right) d\omega = \int_{\Omega} \theta\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) d\omega = 1, \tag{25}$$

where $\theta: \mathbb{R}_{++} \to \mathbb{R}_+$ is C^3, with $\theta'(z) < \theta''(z)$, and $-z\theta''(z)/\theta'(z) > 1$, for $\theta(z) > 0$ with $\lim_{z\to 0} \theta(z) = \infty$ and $\lim_{z\to \bar{z}} \theta(z) = 0$, where $\bar{z} \equiv \inf\{z > 0 | \theta(z) = 0\}$, and it is independent of $Z > 0$. If $\bar{z} < \infty$, the choke price is equal to $\hat{P}(\mathbf{p})\bar{z} = ZP(\mathbf{p})\bar{z}$, and $\lim_{z\to\bar{z}} \theta'(z) = 0$. If $\bar{z} = \infty$, the choke price does not exist and demand for each input always remains positive for any positive price vector. Note that, unlike eq.(13), the adding-up constraint of the H.S.A, eq.(25) defines the unit cost function $P(\mathbf{p})$ directly.[21] CES with gross substitutes is a special case where $\theta(z) = (z)^{1-\sigma}$ $(\sigma > 1)$. The CoPaTh family of HIIA is given by

$$\theta(z) = \sigma^{\frac{\rho}{1-\rho}} \int_{z/\bar{z}}^{1} \left((\xi)^{\frac{\rho-1}{\rho}} - 1\right)^{\frac{\rho}{1-\rho}} d\xi$$

for $z < \bar{z} = (1 - 1/\sigma)^{-\frac{\rho}{1-\rho}}$; $0 < \rho < 1$, which converges to CES as $\rho \nearrow 1$.

The minimization problem, eq.(3), subject to eq.(25) leads to the demand curve

$$x_\omega = -B^*(\mathbf{x})\theta'\left(\frac{p_\omega}{ZP(\mathbf{p})}\right) = -B^*(\mathbf{x})\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) > 0 \tag{26}$$

[21]This means that, unlike H.S.A. but similar to HDIA defined in the previous section, we do not need to worry about the integrability of HIIA. Note also that $\hat{P}(\mathbf{p}) = ZP(\mathbf{p})$ defined by eq.(25) is invariant of TFP, $Z > 0$, by construction. Thus, an increase in Z causes a proportionate decline in $P(\mathbf{p})$. This allows us to examine the effect of TFP without shifting $\theta(\cdot)$. Alternatively, we could define $P(\mathbf{p})$ by $\int_{\Omega} \theta(p_\omega/P(\mathbf{p}))d\omega = 1$, as in Matsuyama and Ushchev (2017). Though mathematically equivalent, this definition requires that $\theta(\cdot)$ would no longer be independent of TFP, which would make it harder to show that $\sigma(V)$ and $\mathcal{L}(V)$ are independent of TFP.

for $z < \bar{z}$, and hence the inverse demand curve,

$$p_\omega = \hat{P}(\mathbf{p})(-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right) = ZP(\mathbf{p})(-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right),$$

where $B^*(\mathbf{x}) > 0$ is defined by

$$\int_\Omega \theta\left((-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\right) d\omega \equiv 1.$$

Thus, the choke price is $\hat{P}(\mathbf{p})\bar{z} = ZP(\mathbf{p})\bar{z}$, if $\bar{z} < \infty$. The production function is

$$X = X(\mathbf{x}) = Z\hat{X}(\mathbf{x}) \equiv Z\int_\Omega (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right) x_\omega d\omega.$$

Clearly, both $B^*(\mathbf{x})$ and $\hat{X}(\mathbf{x})$ are linear homogeneous in **x** and independent of $Z > 0$, by construction. Thus, an increase in TFP, Z, causes a proportional increase in $X(\mathbf{x}) = Z\hat{X}(\mathbf{x})$.

The budget share is

$$\frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{p_\omega}{C(\mathbf{p})}\left[-\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)\right] = (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\frac{x_\omega}{\hat{X}(\mathbf{x})}, \tag{27}$$

where

$$C(\mathbf{p}) \equiv \int_\Omega p_\omega\left[-\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)\right] d\omega > 0$$

is linear homogenous in **p**, and independent of $Z > 0$ and satisfies the identity,

$$\frac{C(\mathbf{p})}{\hat{P}(\mathbf{p})} = \int_\Omega \frac{p_\omega}{\hat{P}(\mathbf{p})}\left[-\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)\right] d\omega = \int_\Omega (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\frac{x_\omega}{B^*(\mathbf{x})} d\omega = \frac{\hat{X}(\mathbf{x})}{B^*(\mathbf{x})}. \tag{28}$$

Eqs.(27)-(28) show that the budget share under HIIA is a function of the two relative prices, $p_\omega/\hat{P}(\mathbf{p})$ and $p_\omega/C(\mathbf{p})$, or a function of the two relative quantities, $x_\omega/\hat{X}(\mathbf{x})$ and $x_\omega/B^*(\mathbf{x})$, unless $C(\mathbf{p})/\hat{P}(\mathbf{p}) = \hat{X}(\mathbf{x})/B^*(\mathbf{x})$ is a positive constant, which occurs if and only if it is CES. Thus, HIIA and H.S.A. do not overlap with the sole exception of CES.[22] Furthermore, by comparing the expressions for the budget share under HDIA and the budget share under HIIA, one could see that HDIA and HIIA do not overlap with the sole exception of CES.[23]

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[22]This statement is a special case of Proposition 3-(ii) in Matsuyama and Ushchev (2017).
[23]This statement is a special case of Proposition 4-(iii) in Matsuyama and Ushchev (2017).

From the demand curve, eq.(26), the price elasticity of demand can be written as a function of a single variable, $z_\omega \equiv p_\omega / P(\mathbf{p})$as:

$$\zeta_\omega = \zeta(p_\omega; \boldsymbol{p}) = -\frac{z_\omega \theta''(z_\omega)}{\theta'(z_\omega)} \equiv \zeta^I(z_\omega) = \zeta^I\left(\frac{p_\omega}{P(\boldsymbol{p})}\right) > 1, \tag{29}$$

where $\zeta^I(z) > 1$ ensures gross substitutability. Using eq.(26), it can also be written as a function of $x_\omega / B^*(\mathbf{x}) = -\theta'(z_\omega)$ as:

$$\zeta_\omega \equiv \zeta^*(x_\omega;\, \mathbf{x}) = \zeta^I\left((-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\right) > 1.$$

Under CES, $\zeta^{I'}(\cdot) = 0$. Marshall's 2nd law, $\partial\zeta(p_\omega; \mathbf{p})/\partial p_\omega > 0$, holds if and only if $\zeta^{I'}(\cdot) > 0$, the condition satisfied by CoPaTh with $\zeta^I(z_\omega) = \left[1 - (1 - 1/\sigma)(z_\omega)^{(1-\rho)/\rho}\right]^{-1} = \left[1 - (z_\omega/\bar{z})^{(1-\rho)/\rho}\right]^{-1}$.

We are now ready to derive the substitutability and love-for-variety measures under HIIA. For symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$, eq.(25) is simplified to

$$\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)V = 1 \Rightarrow \frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})} = \theta^{-1}\left(\frac{1}{V}\right).$$

Hence, from eq.(29), the substitutability measure under HIIA is given by:

$$\sigma(V) \equiv \zeta(1; \mathbf{1}_\Omega^{-1}) = \zeta^I\left(\theta^{-1}\left(\frac{1}{V}\right)\right) = -\left.\frac{z\theta''(z)}{\theta'(z)}\right|_{z=\theta^{-1}(1/V)} > 1. \tag{30}$$

The love-for-variety measure under HIIA is given by:

$$\mathcal{L}(V) \equiv -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} = \frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/V)\right)} \equiv -\left.\frac{\theta(z)}{z\theta'(z)}\right|_{z=\theta^{-1}(1/V)} > 0. \tag{31}$$

where

$$\mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} > 0.$$ [24]

Since $\theta^{-1}(1/V)$ is increasing in V, eqs.(30)-(31) imply

$$\zeta^{I'}(\cdot) \gtreqless 0 \Leftrightarrow \sigma'(\cdot) \gtreqless 0; \quad \mathcal{E}_\theta'(\cdot) \lesseqgtr 0 \Leftrightarrow \mathcal{L}'(\cdot) \gtreqless 0.$$

The next lemma shows the relation between the following two functions:

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[24]Moreover, by evaluating eq.(28) at the symmetric price and quantity patterns, one can show that
$$\frac{C(\mathbf{1}_\Omega^{-1})}{\hat{P}(\mathbf{1}_\Omega^{-1})} = \frac{\hat{X}(\mathbf{1}_\Omega)}{B^*(\mathbf{1}_\Omega)} = \int_\Omega \mathcal{E}_\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)d\omega = \mathcal{E}_\theta\left(\theta^{-1}\left(\frac{1}{V}\right)\right) \Rightarrow \mathcal{L}(V) = \frac{\hat{P}(\mathbf{1}_\Omega^{-1})}{C(\mathbf{1}_\Omega^{-1})} = \frac{B^*(\mathbf{1}_\Omega)}{\hat{X}(\mathbf{1}_\Omega)}.$$

$$\zeta^I(z) \equiv -\frac{z\theta''(z)}{\theta'(z)} > 1 \quad \text{and} \quad \mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} > 0.$$

Lemma I:

$$\frac{z\mathcal{E}'_\theta(z)}{\mathcal{E}_\theta(z)} = \mathcal{E}_\theta(z) + 1 - \zeta^I(z) = \int_z^{\overline{z}} \zeta^I(\xi) w^I(\xi; z) d\xi - \zeta^I(z).$$

where $w^I(\xi; z) \equiv -\theta'(\xi)/\theta(z) > 0$, which satisfies $\int_z^{\overline{z}} w^I(\xi; z)\, d\xi = 1$.

$$\zeta^{I'}(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \quad \Longrightarrow \quad \mathcal{E}'_\theta(z) \gtreqless 0, \forall z \in (z_0, \overline{z}).$$

The opposite is not true in general. However,

$$\zeta^{I'}(z) = 0, \forall z \in (z_0, \overline{z}) \quad \Leftrightarrow \quad \mathcal{E}'_\theta(z) = 0, \forall z \in (z_0, \overline{z}).$$

The proof of Lemma I is in Appendix D. By combining Lemma I, eq.(30), and eq.(31),

Proposition I: For $\theta(z_0)V_0 = 1$,

$$\zeta^{I'}(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \Leftrightarrow \sigma'(V) \gtreqless 0, \forall V \in (V_0, \infty);$$

$$\mathcal{E}'_\theta(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

Moreover,

$$\sigma'(V) \gtreqless 0, \forall V \in (V_0, \infty) \Longrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

The opposite is not true in general. However,

$$\sigma'(V) = 0, \forall V \in (V_0, \infty) \Leftrightarrow \mathcal{L}'(V) = 0, \forall V \in (V_0, \infty).$$

In particular, as $z_0 \to 0, V_0 \to 0$ so that Marshall's 2nd Law, $\zeta^{I\prime}(\cdot) < 0$ for all $z < \overline{z}$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$ for all $V > 0$, both of which imply diminishing love-for-variety, $\mathcal{L}'(\cdot) < 0$ for all $V > 0$. The converse is not true. Diminishing love-for-variety for all V does not necessarily imply increasing substitutability or Marshall's 2nd Law globally. However, constant love-for-variety, $\mathcal{L}'(\cdot) = 0$ for all $V > 0$, implies both constant substitutability, $\sigma'(\cdot) = 0$ for all $V > 0$, and constant price elasticity $\zeta^{I\prime}(\cdot) = 0$ for all $z < \overline{z} = \infty$, under HIIA, which occurs only under CES.

4. Concluding Remarks

In this paper, we studied how love-for-variety is determined by the underlying demand structure. Under general symmetric homothetic demand systems, both substitutability across different goods and love-for-variety are expressed as functions of

the variety of available goods V only, as $\sigma(V)$ and $\mathcal{L}(V)$. Since the homotheticity alone imposes little restrictions on their properties of these two functions, we turn to three classes of homothetic demand systems, H.S.A., HDIA, and HIIA, which are pairwise disjoint with the sole exception of CES. For each of these three classes, we establish the three main results. First, substitutability is increasing in V, if and only if Marshall's 2nd law of demand holds. Second, increasing (decreasing) substitutability implies diminishing (increasing) love-for-variety, but the converse is not true. Third, love-for-variety is constant, if and only if substitutability is constant, which occurs only under CES within these three classes. The key results are also illustrated in Figure 2.
These three classes thus offer a tractable way of capturing the intuition that gains from increasing variety is diminishing, if different goods are more substitutable when a wider variety of goods are available.

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Figure 1:
Three classes of homothetic symmetric demand systems with gross substitutes

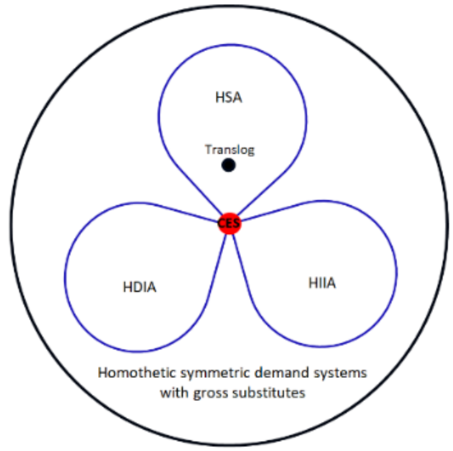
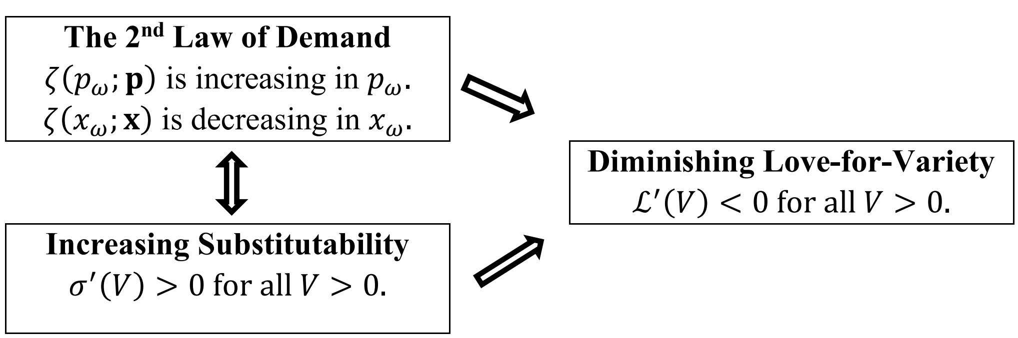


Table: Price elasticity, substitutability and love-for-variety under the three classes.

	Price Elasticity: $\zeta(p_\omega; \mathbf{p})$	Substitutability: $\sigma(V)$	Love-for-Variety: $\mathcal{L}(V)$
H.S.A.	$\zeta_\omega = \zeta^S\left(\frac{p_\omega}{A(\mathbf{p})}\right)$	$\sigma(V) = \zeta^S\left(s^{-1}\left(\frac{1}{V}\right)\right)$	$\mathcal{L}(V) = \frac{1}{\mathcal{E}_H\left(s^{-1}(1/V)\right)}$
$\zeta^S(z) \equiv -\frac{zH''(z)}{H'(z)} > 1, \mathcal{E}_H(z) \equiv -\frac{zH'(z)}{H(z)} > 0$ with $H(z) > 0$ decreasing and convex.			
HDIA	$\zeta_\omega = \zeta^D\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right)$	$\sigma(V) = \zeta^D\left(\phi^{-1}\left(\frac{1}{V}\right)\right)$	$\mathcal{L}(V) = \frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/V))} - 1$
$\zeta^D(y) \equiv -\frac{\phi'(y)}{y\phi''(y)} > 1; 0 < \mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)} < 1$ with $\phi(y) > 0$ increasing and concave.			
HIIA	$\zeta_\omega = \zeta^I\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)$	$\sigma(V) = \zeta^I\left(\theta^{-1}\left(\frac{1}{V}\right)\right)$	$\mathcal{L}(V) = \frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/V)\right)}$
$\zeta^I(z) \equiv -\frac{z\theta''(z)}{\theta'(z)} > 1;\ \mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} > 0$ with $\theta(z) > 0$ decreasing and convex.			

Figure 2: The 2nd law, increasing substitutability, and diminishing love-for-variety under the three classes.



Appendix A: Allen-Uzawa elasticity of substation at the symmetric patterns under general symmetric homothetic demand systems.

The Allen-Uzawa elasticity of substitution between two inputs, $\omega, \omega' \in \Omega$, are given by:

$$AES(p_\omega, p_{\omega'}, \mathbf{p}) = \frac{P(\mathbf{p})P_{\omega\omega'}(p_\omega, p_{\omega'}, \mathbf{p})}{x(p_\omega, \mathbf{p})x(p_{\omega'}, \mathbf{p})},$$

where $x(p_\omega, \mathbf{p})$ is the demand for ω per unit of output, while the functions $P_{\omega\omega'}(p_\omega, p_{\omega'}, \mathbf{p})$ are the ``second cross-derivatives'' of $P(\mathbf{p})$. The second-order Taylor approximation of $P(\mathbf{p})$ is

$$P(\mathbf{p} + \alpha\mathbf{h}) = P(\mathbf{p}) + \alpha \int_\Omega x(p_\omega, \mathbf{p})h_\omega d\omega + \frac{\alpha^2}{2}\int_\Omega \frac{\partial x(p_\omega, \mathbf{p})}{\partial p_\omega} h_\omega^2 d\omega$$
$$+ \frac{\alpha^2}{2}\int_\Omega \int_\Omega P_{\omega\omega'}(p_\omega, p_{\omega'}, \mathbf{p})h_\omega h_{\omega'}\, d\omega d\omega' + o(\alpha^2),$$

where $\mathbf{h}$ is a function over Ω, and α is a scalar. The linear homogeneity of $P(\mathbf{p})$ implies the following identity:

$$\int_\Omega \frac{\partial x(p_\omega, \mathbf{p})}{\partial p_\omega} p_\omega^2 d\omega + \int_\Omega \int_\Omega P_{\omega\omega'}(p_\omega, p_{\omega'}, \mathbf{p})p_\omega p_{\omega'}\, d\omega d\omega' = 0.$$

By setting $(p_\omega, \mathbf{p}) = (1, \mathbf{1}_\Omega^{-1})$ and $(p_\omega, p_{\omega'}, \mathbf{p}) = (1,1, \mathbf{1}_\Omega^{-1})$ in the identity, we obtain:

$$\left[\frac{\partial x(p_\omega, \mathbf{1}_\Omega^{-1})}{\partial p_\omega}\right]\Bigg|_{p_\omega=1} \underbrace{\left[\int_\Omega d\omega\right]}_{=V} + P_{\omega\omega'}(1,1, \mathbf{1}_\Omega^{-1})\left[\underbrace{\int_\Omega \int_\Omega d\omega d\omega'}_{=V^2}\right] = 0.$$

Using the definition of $\sigma(V)$,

$$\sigma(V) \equiv \zeta(1; \mathbf{1}_\Omega^{-1}) = -\left[\frac{\partial \ln x(p_\omega, p\mathbf{1}_\Omega^{-1})}{\partial \ln p_\omega}\right]\Bigg|_{p_\omega=p} \Rightarrow \frac{\partial x(p_\omega, \mathbf{1}_\Omega^{-1})}{\partial p_\omega}\Bigg|_{p_\omega=1}$$
$$= -\sigma(V)x(1, \mathbf{1}_\Omega^{-1}),$$

the above identity can be further rewritten as:

$$P_{\omega\omega'}(1,1, \mathbf{1}_\Omega^{-1}) = \frac{\sigma(V)}{V}x(1, \mathbf{1}_\Omega^{-1}).$$

Moreover, by setting $\mathbf{p} = P(\mathbf{1}_\Omega^{-1})$ in $P(\mathbf{p}) = \int_\Omega x(p_\omega, \mathbf{p}) p_\omega d\omega$,

$$P(\mathbf{1}_\Omega^{-1}) = Vx(1, \mathbf{1}_\Omega^{-1}).$$

Thus, the Allen-Uzawa elasticity of substitution evaluated at a symmetric outcome:

$$AES_{\omega\omega'}(1,1,\mathbf{1}_\Omega^{-1}) = \frac{P(\mathbf{1}_\Omega^{-1}) P_{\omega\omega'}(1,1,\mathbf{1}_\Omega^{-1})}{[x(1,\mathbf{1}_\Omega^{-1})]^2} = \frac{Vx(1,\mathbf{1}_\Omega^{-1}) \frac{\sigma(V)}{V} x(1,\mathbf{1}_\Omega^{-1})}{[x(1,\mathbf{1}_\Omega^{-1})]^2} = \sigma(V).$$

Appendix B: $\boldsymbol{\sigma(V)}$ and $\boldsymbol{\mathcal{L}(V)}$ under Geometric Means of CES

This appendix shows that there exists a class of homothetic non-CES demand systems in which $\sigma(V)$ and $\mathcal{L}(V)$ are independent of V. Moreover, within this class, they exist a parametric family in which that $\sigma(V)$ and $\mathcal{L}(V)$ move in the same direction as one of the parameters changes.

Consider the symmetric CRS production function, $X(\mathbf{x})$, defined by a weighted geometric mean of symmetric CES production functions with different $\sigma \in (1, \infty)$:

$$\ln X(\mathbf{x}) \equiv \int_1^\infty \ln X(\mathbf{x}; \sigma)\, dF(\sigma),$$

where

$$[X(\mathbf{x}; \sigma)]^{1-\frac{1}{\sigma}} \equiv \int_\Omega {x_\omega}^{1-\frac{1}{\sigma}} d\omega$$

and $F(\cdot)$ is a c.d.f. of $\sigma \in (1, \infty)$,

$$\int_1^\infty dF(\sigma) = 1.$$

Proposition B: Consider the homothetic demand system generated by a weighted geometric mean of symmetric CES production functions. Then,

i) The substitutability measure, $\sigma(V)$, is independent of V and given by:

$$\sigma(V) = \frac{1}{E_F(1/\sigma)} > 1;$$

ii) The love-for-variety measure, $\mathcal{L}(V)$, is independent of V and given by

$$\mathcal{L}(V) = E_F\left(\frac{1}{\sigma - 1}\right) > 0;$$

iii) The range of $\sigma(V)$ and $\mathcal{L}(V)$ is given by:

$$0 < \frac{1}{\sigma(V)-1} \le \mathcal{L}(V) < \infty,$$

where the equality holds if and only if F is degenerate.

Proof. The inverse demand for variety $\omega \in \Omega$ is

$$p_\omega = P(\mathbf{p})\frac{\partial X(\mathbf{x})}{\partial x_\omega} = P(\mathbf{p})X(\mathbf{x})\int_1^\infty \frac{x_\omega^{-\frac{1}{\sigma}}}{[X(\mathbf{x};\sigma)]^{1-\frac{1}{\sigma}}} dF(\sigma)$$

$$= P(\mathbf{p})X(\mathbf{x})\int_1^\infty x_\omega^{-\frac{1}{\sigma}} dF^*(\mathbf{x};\sigma),$$

where $dF^*(\mathbf{x};\sigma) \equiv dF(\sigma)/[X(\mathbf{x};\sigma)]^{1-\frac{1}{\sigma}}$. Thus, the price elasticity of demand, $\zeta^*(x_\omega;\mathbf{x})$, as a function of the quantities, satisfies

$$\frac{1}{\zeta^*(x_\omega;\mathbf{x})} \equiv -\frac{\partial \ln p_\omega}{\partial \ln x_\omega} = \frac{\int_1^\infty \frac{1}{\sigma} x_\omega^{-\frac{1}{\sigma}} dF^*(\mathbf{x};\sigma)}{\int_1^\infty x_\omega^{-\frac{1}{\sigma}} dF^*(\mathbf{x};\sigma)} < 1.$$

By evaluating this at the symmetric quantity patterns $\mathbf{x} = x\mathbf{1}_\Omega$,

$$\frac{1}{\sigma(V)} \equiv \frac{1}{\zeta^*(x;x\mathbf{1}_\Omega)} \equiv \frac{\int_1^\infty \frac{1}{\sigma} x^{-\frac{1}{\sigma}} \frac{dF(\sigma)}{[X(x\mathbf{1}_\Omega;\sigma)]^{1-\frac{1}{\sigma}}}}{\int_1^\infty x^{-\frac{1}{\sigma}} \frac{dF(\sigma)}{[X(x\mathbf{1}_\Omega;\sigma)]^{1-\frac{1}{\sigma}}}} = \frac{\int_1^\infty \frac{x^{-\frac{1}{\sigma}}}{Vx^{1-\frac{1}{\sigma}}} \frac{dF(\sigma)}{\sigma}}{\int_1^\infty \frac{x^{-\frac{1}{\sigma}}}{Vx^{1-\frac{1}{\sigma}}} dF(\sigma)} = \int_1^\infty \frac{dF(\sigma)}{\sigma}$$

$$= E_F\left(\frac{1}{\sigma}\right) < 1.$$

This proves i).

Next, from $\ln X(\mathbf{1}_\Omega) \equiv \int_1^\infty \ln X(\mathbf{1}_\Omega;\sigma)\, dF(\sigma) = \int_1^\infty \ln V \frac{\sigma}{\sigma-1} dF(\sigma) = E_F\left(\frac{\sigma}{\sigma-1}\right)\ln V$,

$$\mathcal{L}(V) \equiv -\frac{d\ln \mathcal{Y}(V)}{d\ln V} - 1 = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 = E_F\left(\frac{\sigma}{\sigma-1}\right) - 1 = E_F\left(\frac{1}{\sigma-1}\right) > 0.$$

This proves ii).

For iii), Jensen's inequality implies

$$\mathcal{L}(V) = E_F\left(\frac{1/\sigma}{1-1/\sigma}\right) \ge \frac{E_F(1/\sigma)}{1-E_F(1/\sigma)} = \frac{1/\sigma(V)}{1-1/\sigma(V)} = \frac{1}{\sigma(V)-1},$$

where the lower bound is reached if and only if F is degenerate. Next, consider the Pareto distribution of σ:

$$F(\sigma) = 1 - \left(\frac{\sigma_{min}}{\sigma}\right)^{\alpha}, \qquad \sigma \geq \sigma_{min} \equiv \frac{\alpha\sigma_0}{\alpha+1} > 1,$$

where $\sigma_0 > 1$ and $\alpha > 1/(\sigma_0 - 1)$. The distribution and density of $x = 1/\sigma$ are given by:

$$G(x) = (\sigma_{min}x)^{\alpha};\; g(x) = \alpha(\sigma_{min})^{\alpha}x^{\alpha-1}, \qquad x \in \left(0, \frac{1}{\sigma_{min}}\right).$$

Thus,

$$\frac{1}{\sigma(V)} = \mathbb{E}_F\left(\frac{1}{\sigma}\right) = \mathbb{E}_G(x) = \alpha(\sigma_{min})^{\alpha}\int_0^{1/\sigma_{min}} x^{\alpha}\,dx = \frac{1}{\sigma_{min}}\frac{\alpha}{\alpha+1} = \frac{1}{\sigma_0} < 1;$$

$$\mathcal{L}(V) = \mathbb{E}_F\left(\frac{1}{\sigma-1}\right) = \mathbb{E}_G\left(\frac{x}{1-x}\right) = \mathbb{E}_G\left(\sum_{k=1}^{\infty} x^k\right) = \sum_{k=1}^{\infty}\mathbb{E}_G(x^k)$$

$$= \sum_{k=1}^{\infty}\alpha(\sigma_{min})^{\alpha}\int_0^{1/\sigma_{min}} x^{\alpha+k-1}\,dx$$

$$= \sum_{k=1}^{\infty}\frac{\alpha}{\alpha+k}\left(\frac{1}{\sigma_{min}}\right)^k = \sum_{k=1}^{\infty}\frac{\alpha}{\alpha+k}\left(\frac{\alpha+1}{\alpha}\right)^k\left(\frac{1}{\sigma_0}\right)^k = \sum_{k=1}^{\infty}\frac{(1+1/\alpha)^k}{1+k/\alpha}\left(\frac{1}{\sigma_0}\right)^k$$

Holding $\sigma(V) = \sigma_0$ constant, $\mathcal{L}(V)$ is monotonically decreasing in α because

$$\frac{d\ln\left[\frac{(1+1/\alpha)^k}{1+k/\alpha}\right]}{d\ln\alpha} = -\frac{\alpha k(k-1)}{(\alpha+1)(\alpha+k)} \leq 0.$$

Moreover,

$$\lim_{\alpha\to\infty}\sum_{k=1}^{\infty}\frac{(1+1/\alpha)^k}{1+k/\alpha}\left(\frac{1}{\sigma_0}\right)^k = \sum_{k=1}^{\infty}\left(\frac{1}{\sigma_0}\right)^k = \frac{1}{\sigma_0-1} = \frac{1}{\sigma(V)-1};$$

$$\lim_{\alpha \to 1/(\sigma_0 - 1)} \sum_{k=1}^{\infty} \frac{(1+1/\alpha)^k}{1+k/\alpha} \left(\frac{1}{\sigma_0}\right)^k = \sum_{k=1}^{\infty} \frac{1}{1+(\sigma_0 - 1)k} > \int_1^{\infty} \frac{dz}{1+(\sigma_0 - 1)z}$$

$$= \frac{\ln[1+(\sigma_0 - 1)z]}{(\sigma_0 - 1)}\Bigg]_1^{\infty} = \infty,$$

from which

$$\frac{1}{\sigma(V) - 1} \leq \mathcal{L}(V) < \infty.$$

This completes the proof. ■

We are now ready to construct a parametric family of the distribution, F_α, in which $\sigma(V)$ and $\mathcal{L}(V)$ are independent of V and move in the same direction as α varies.

$$F_\alpha(\sigma) = 1 - \left(\frac{\sigma_{min}}{\sigma}\right)^\alpha, \qquad \frac{1}{\sigma} \leq \frac{1}{\sigma_{min}} \equiv \frac{1+\alpha}{\alpha} h(\alpha) < 1.$$

Then, following the same step in the proof of Part iii) of Proposition,

$$\frac{1}{\sigma(V)} = \mathbb{E}_{F_\alpha}\left(\frac{1}{\sigma}\right) = \frac{1}{\sigma_{min}} \frac{\alpha}{\alpha+1} = h(\alpha);\ \mathcal{L}(V) = \mathbb{E}_{F_\alpha}\left(\frac{1}{\sigma - 1}\right)$$

$$= \sum_{k=1}^{\infty} \frac{(1+1/\alpha)^k}{1+k/\alpha} \big(h(\alpha)\big)^k$$

with

$$\frac{d \ln \left[\frac{(1+1/\alpha)^k}{1+k/\alpha} \big(h(\alpha)\big)^k\right]}{d \ln \alpha} = \alpha k \left[\frac{h'(\alpha)}{h(\alpha)} - \frac{(k-1)}{(\alpha+1)(\alpha+k)}\right].$$

Fix $h_0(\alpha)$, which is increasing in α, and satisfies $0 < h_0(\alpha) < 1$, and whose derivative is bounded. Then, for $0 < c < 1$, consider $h(\alpha, \varepsilon) \equiv (1-\varepsilon)c + \varepsilon h_0(\alpha)$. When $\varepsilon = 0$, $\sigma(V)$ is constant, while $\mathcal{L}(V)$ is decreasing in α. But, for a positive but sufficiently small ε, $\sigma(V)$ is decreasing, while $\mathcal{L}(V)$ continues to be decreasing in α by continuity.

Appendix C. An Alternative (and Equivalent) specification of the HSA class.

There exists an alternative (but equivalent) definition of H.S.A.. That is, a homothetic symmetric demand system belongs to H.S.A. (*Homothetic Single Aggregator*) if there exists a function of a single variable, $s^*: \mathbb{R}_{++} \to \mathbb{R}_{+}$ which is C^2 with $0 < \mathcal{E}_{s^*}(y) \equiv ys^{*\prime}(y)/s^*(y) < 1, s^*(0) = 0$ and $s^*(\infty) = \infty$, such that the budget share of $\omega \in \Omega$ can be written as:

$$s_\omega = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} = s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right), \tag{32}$$

where $A^*(\mathbf{x})$ is defined implicitly and uniquely by the adding-up constraint:

$$\int_\Omega s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega \equiv 1 \tag{33}$$

By construction, $A^*(\mathbf{x})$ is linear homogenous in $\mathbf{x}$ for any fixed Ω and The budget share of each input is a function of its *normalized quantity*, $y_\omega \equiv x_\omega/A^*(\mathbf{x})$, defined as its own quantity x_ω divided by the *common quantity aggregator* $A^*(\mathbf{x})$. The budget shares of all inputs are added to up to one.

The price elasticity of $\omega \in \Omega$ can be written as a function of $y_\omega \equiv x_\omega/A^*(\mathbf{x})$ as:

$$\zeta_\omega = \zeta^*(x_\omega; \mathbf{x}) = \left[1 - \frac{y_\omega s^{*\prime}(y_\omega)}{s^*(y_\omega)}\right]^{-1} \equiv \zeta^{S*}(y_\omega) = \zeta^{S*}\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) > 1,$$

where $\zeta^{S*}: (0, \infty) \to (1, \infty)$ is C^1. Note that the assumption, $0 < \mathcal{E}_{s^*}(y) \equiv ys^{*\prime}(y)/s^*(y) < 1$ ensures $\zeta^{S*}(y_\omega) > 1$, that is, *gross substitutability.*[25] It turns out to be convenient to introduce another function, $H^*: \mathbb{R}_{++} \to \mathbb{R}_{+}$,

$$H^*(y) \equiv \int_0^y \frac{s^*(\xi^*)}{\xi^*} d\xi^*,$$

so that

$$\zeta^{S*}(y) \equiv \left[1 - \frac{ys^{*\prime}(y)}{s^*(y)}\right]^{-1} \equiv -\frac{H^{*\prime}(y)}{yH^{*\prime\prime}(y)} > 1. \tag{34}$$

[25]Conversely, from any continuously differentiable $\zeta^*: (0, \infty) \to (1, \infty)$, one could reverse-engineer as $s^*(y) = \gamma^* \exp\left[\int_{y_0}^y \left[1 - \frac{1}{\zeta^*(\xi^*)}\right] \frac{d\xi^*}{\xi^*}\right] > 0$, where $\gamma^* = s^*(y_0)$ is a positive constant. Thus, we could also use $\zeta^*(\cdot)$ instead of $s^*(\cdot)$ as a primitive of symmetric H.S.A. with gross substitutes.

In general, $\zeta^{S*}(\cdot)$ can be nonmonotonic. Under CES, given by $s^*(y) = \gamma^{1/\sigma}(y)^{1-1/\sigma}$, it is constant, $\zeta^{S*\prime}(y) = 0$. Marshall's 2nd law, $\partial\zeta(x_\omega; \mathbf{x})/\partial x_\omega < 0$, holds if and only if $\zeta^{S*\prime}(\cdot) < 0$. The choke price exists if and only if $\lim_{y\to 0} s^{*\prime}(y) < \infty$, which implies $\lim_{y\to 0} ys^{*\prime}(y)/s^*(y) = 1$ and hence $\lim_{y\to 0} \zeta^{S*}(y) = \infty$. Translog corresponds to $s^*(y)$, defined implicitly by $s^* \exp(s^*/\gamma) \equiv \bar{z}y$, for $\bar{z} < \infty$. CoPaTh corresponds to $s^*(y) = \left[\frac{1}{\sigma} + \left(1 - \frac{1}{\sigma}\right) y^{-\frac{1-\rho}{\rho}}\right]^{-\frac{\rho}{1-\rho}} = \left[1 - \bar{z}^{-\frac{1-\rho}{\rho}} + (y\bar{z})^{-\frac{1-\rho}{\rho}}\right]^{-\frac{\rho}{1-\rho}}$ with $\bar{z} = s^{*\prime}(0) = \left(1 - \frac{1}{\sigma}\right)^{-\frac{\rho}{1-\rho}}$.

After deriving $A^*(\mathbf{x})$ from $s^*(\cdot)$, the production function, $X(\mathbf{x})$, can be obtained by integrating eq.(32), which yields

$$\ln\left[\frac{X(\mathbf{x})}{c^* A^*(\mathbf{x})}\right] = \int_\Omega \left[\int_0^{\frac{x_\omega}{A^*(\mathbf{x})}} \frac{s^*(\xi^*)}{\xi^*} d\xi^*\right] d\omega \equiv \int_\Omega H^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega \tag{35}$$

$$\equiv \int_\Omega s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) \Phi^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega$$

where c^* is a positive constant, which is proportional to TFP and

$$\Phi^*(y) \equiv \frac{1}{s^*(y)} \int_0^y \frac{s^*(\xi^*)}{\xi^*} d\xi^* \equiv \frac{H^*(y)}{yH^{*\prime}(y)} > 1,$$

where the inequality follows from $\mathcal{E}_{s^*}(y) \equiv ys^{*\prime}(y)/s^*(y) < 1$, which implies that $s^*(y)/y$ is decreasing in y, and hence $H^*(y)$ is concave.

Note that $X(\mathbf{x})/A^*(\mathbf{x})$ is constant, if and only if it is CES. To see this, differentiating eq.(33) yields,

$$\frac{\partial \ln A^*(\mathbf{x})}{\partial \ln x_\omega} = \frac{y_\omega s^{*\prime}(y_\omega)}{\int_\Omega s^{*\prime}(y_{\omega'}) y_{\omega'} d\omega'} = \frac{\left[1 - \frac{1}{\zeta^*(y_\omega)}\right] s^*(y_\omega)}{\int_\Omega \left[1 - \frac{1}{\zeta^*(y_{\omega'})}\right] s^*(y_{\omega'}) d\omega'},$$

which differs from

$$\frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} = s^*(y_\omega),$$

unless $\zeta^*(y_\omega)$ is constant.

For symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$, eq.(33) is simplified to

$$s^*\left(\frac{1}{A^*(\mathbf{1}_\Omega)}\right)V = 1 \Rightarrow \frac{1}{A^*(\mathbf{1}_\Omega)} = s^{*-1}\left(\frac{1}{V}\right).$$

Hence, from eq.(34), the substitutability measure is given by:

$$\sigma(V) \equiv \zeta^*(1;\mathbf{1}_\Omega) = \zeta^{S*}\left(s^{*-1}\left(\frac{1}{V}\right)\right) = -\frac{H^{*\prime}(y)}{yH^{*\prime\prime}(y)}\bigg|_{y=s^{*-1}(1/V)} > 1 \tag{36}$$

For the love-for-variety measure, from eq.(35),

$$\ln X(\mathbf{1}_\Omega) = \ln c^* + \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right) - \ln s^{*-1}\left(\frac{1}{V}\right) \Rightarrow$$

$$\frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} = \left[\frac{d[\ln y - \Phi^*(y)]}{d\ln y}\Big/\frac{d\ln s^*(y)}{d\ln y}\right]_{y=s^{*-1}(1/V)} = \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right)$$

so that

$$\mathcal{L}(V) \equiv \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 = \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right) - 1 = \frac{H^*(y)}{yH^{*\prime}(y)}\bigg|_{y=s^{*-1}(1/V)} - 1$$ (37)[26]

Since $s^{*-1}(1/V)$ is decreasing in V, eqs.(36)-(37) imply

$$\zeta^{S*\prime}(\cdot) \lesseqgtr 0 \Leftrightarrow \sigma'(\cdot) \gtreqless 0;\ \Phi^{*\prime}(\cdot) \gtreqless 0 \Leftrightarrow \mathcal{L}'(\cdot) \lesseqgtr 0,$$

The next lemma shows the relation between the following two functions:

$$\zeta^{S*}(y) \equiv \left[1 - \frac{ys^{*\prime}(y)}{s^*(y)}\right]^{-1} \equiv -\frac{H^{*\prime}(y)}{yH^{*\prime\prime}(y)} > 1;\ \Phi^*(y) \equiv \frac{1}{s^*(y)}\int_0^y \frac{s^*(\xi^*)}{\xi^*}d\xi^* \equiv \frac{H^*(y)}{yH^{*\prime}(y)} > 1.$$

Lemma S*

$$\frac{y\Phi^{*\prime}(y)}{\Phi^*(y)} = \frac{1}{\Phi^*(y)} - 1 + \frac{1}{\zeta^{S*}(y)} = \frac{1}{\zeta^{S*}(y)} - \int_0^y \left[\frac{1}{\zeta^{S*}(\xi^*)}\right] w^*(\xi^*;y)d\xi^*.$$

where $w^{S*}(\xi^*;y) \equiv H^{*\prime}(\xi^*)/H^*(y) > 0$, which satisfies $\int_0^y w^{S*}(\xi^*;y)d\xi^* = 1$. Hence,

$$\zeta^{S*\prime}(y) \lesseqgtr 0, \forall y \in (0,y_0) \Rightarrow \Phi^{*\prime}(y) \gtreqless 0, \forall y \in (0,y_0).$$

The opposite is not true in general. However,

$$\zeta^{S*\prime}(y) = 0, \forall y \in (0,y_0) \Leftrightarrow \Phi^{*\prime}(y) = 0, \forall y \in (0,y_0).$$

The proof of Lemma S* is in Appendix D. By combining Lemma S*, eq.(36) and eq.(37)

Proposition S*: For $s^*(y_0)V_0 = 1$,

[26] Moreover, by evaluating eq.(35) at the symmetric quantity patterns, $\mathcal{L}(V) = \ln\left(\frac{X(\mathbf{1}_\Omega)}{c^*A^*(\mathbf{1}_\Omega)}\right) - 1$.

$$\zeta^{S*\prime}(y) \lesseqgtr 0, \forall y \in (0, y_0) \Leftrightarrow \sigma'(V) \gtreqless 0, \forall V \in (V_0, \infty);$$

$$\Phi^{*\prime}(y) \gtreqless 0, \forall y \in (0, y_0) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

Moreover,

$$\sigma'(V) \gtreqless 0, \forall V \in (V_0, \infty) \Longrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

The opposite is not true in general. However,

$$\sigma'(V) = 0, \forall V \in (V_0, \infty) \Leftrightarrow \mathcal{L}'(V) = 0, \forall V \in (V_0, \infty).$$

In particular, if $y_0 \to \infty$ implies $s^*(y_0) \to \infty, V_0 \to 0$ so that Marshall's 2nd Law, $\zeta^{S*\prime}(\cdot) < 0$ for all $y > 0$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$ for all $V > 0$, both of which imply diminishing love-for-variety, $\mathcal{L}'(\cdot) < 0$ for all $V > 0$. The converse is not true. Diminishing love-for-variety for all $V > 0$ does not necessarily imply increasing substitutability or Marshall's 2nd Law globally. However, constant love-for-variety, $\mathcal{L}'(\cdot) = 0$ for all $V > 0$, implies both constant substitutability, $\sigma'(\cdot) = 0$ for all $V > 0$, and constant price elasticity $\zeta^{S*\prime}(\cdot) = 0$ for all $y > 0$ under H.S.A., which occurs only under CES.

Indeed, these two definitions of H.S.A. are equivalent.[27] The isomorphism between the two is given by the one-to-one mapping between $s(z) \leftrightarrow s^*(y)$, defined by:

$$s^*(y) = s\left(\frac{s^*(y)}{y}\right); \qquad s(z) = s^*\left(\frac{s(z)}{z}\right).$$

Differentiating either of these two equalities yields the identity,

$$\zeta^{S*}(y) \equiv \left[1 - \frac{d\ln s^*(y)}{d\ln y}\right]^{-1} = \zeta^S(z) \equiv 1 - \frac{d\ln s(z)}{d\ln z} > 1,$$

which shows that $0 < \mathcal{E}_{s^*}(y) \equiv \frac{d\ln s^*(y)}{d\ln y} < 1$ is equivalent to $\mathcal{E}_s(z) \equiv \frac{d\ln s(z)}{d\ln z} < 0$.

Furthermore, the normalized quantity, $y_\omega \equiv x_\omega/A^*(\mathbf{x})$, and the normalized price, $z_\omega \equiv p_\omega/A(\mathbf{p})$, are negatively related as

$$z_\omega = \frac{s^*(y_\omega)}{y_\omega} \Leftrightarrow y_\omega = \frac{s(z_\omega)}{z_\omega},$$

$$\frac{dy_\omega}{y_\omega} = -\zeta(z_\omega)\frac{dz_\omega}{z_\omega} \Leftrightarrow \frac{dz_\omega}{z_\omega} = -\frac{1}{\zeta^*(y_\omega)}\frac{dy_\omega}{y_\omega}$$

and

[27] This isomorphism has been shown for the broader class of H.S.A., which allows for asymmetry as well as gross complements; see Matsuyama and Ushchev (2017, sec. 3, Remark 3).

$$\frac{z_\omega \zeta^{S\prime}(z_\omega)}{y_\omega \zeta^{S*\prime}(y_\omega)} = -\zeta^S(z_\omega) = -\zeta^{S*}(y_\omega) < 0.$$

In addition, if $\lim_{y\to 0} s^{*\prime}(y) < \infty$, then $\lim_{y\to 0} \zeta^{S*}(y) = \infty$ and

$$\lim_{y\to 0} \frac{s^*(y)}{y} = \lim_{y\to 0} s^{*\prime}(y) = \bar{z} \equiv \inf\{z > 0 | s(z) = 0\} < \infty$$

is the (normalized) choke price.

Moreover,

$$\frac{p_\omega x_\omega}{A(\mathbf{p})A^*(\mathbf{x})} = y_\omega z_\omega = s(z_\omega) = s^*(y_\omega) = \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})}$$

hence we have the identity,

$$c \exp\left[\int_\Omega s(z_\omega)\Phi(z_\omega)d\omega\right] = c \exp\left[\int_\Omega H(z_\omega)d\omega\right] = \frac{A(\mathbf{p})}{P(\mathbf{p})} = \frac{X(\mathbf{x})}{A^*(\mathbf{x})}$$

$$= c^* \exp\left[\int_\Omega s^*(y_\omega)\Phi^*(y_\omega)\, d\omega\right] = c^* \exp\left[\int_\Omega H^*(y_\omega)\, d\omega\right]$$

which is a positive constant if and only if it is CES. Furthermore, using

$$s(\xi) = s^*(\xi^*) = \xi\xi^* \to \frac{d\xi^*}{\xi^*} = \left[\frac{\xi s'(\xi)}{s(\xi)} - 1\right]\frac{d\xi}{\xi} \to s^*(\xi^*)\frac{d\xi^*}{\xi^*} = \left[s'(\xi) - \frac{s(\xi)}{\xi}\right]d\xi$$

$$\xi = z \leftrightarrow \xi^* = y;\ \xi = \bar{z} \leftrightarrow \xi^* = 0,$$

we have

$$\Phi^*(y) - \Phi(z) \equiv \frac{1}{s^*(y)}\int_0^y \frac{s^*(\xi^*)}{\xi^*}d\xi^* - \frac{1}{s(z)}\int_z^{\bar{z}} \frac{s(\xi)}{\xi}d\xi$$

$$= \frac{1}{s(z)}\int_{\bar{z}}^{z}\left[s'(\xi) - \frac{s(\xi)}{\xi}\right]d\xi - \frac{1}{s(z)}\int_z^{\bar{z}}\frac{s(\xi)}{\xi}d\xi = 1.$$

Since $w^S(\xi; z) = \frac{s(\xi)/\xi}{\int_z^{\bar{z}}[s(\xi')/\xi']d\xi'} \Leftrightarrow s(z)\Phi(z)w^S(\xi; z) = \frac{s(\xi)}{\xi}$ and $w^{S*}(\xi^*; y) = \frac{s^*(\xi^*)/\xi^*}{\int_0^y\left[s^*\left(\xi^{*\prime}\right)/\xi^{*\prime}\right]d\xi^{*\prime}} \Leftrightarrow s^*(y)\Phi^*(y)w^{S*}(\xi^*; y) = \frac{s^*(\xi^*)}{\xi^*}$, this implies

$$\frac{\xi w^S(\xi; z)}{\xi^* w^{S*}(\xi^*; y)} = \frac{\Phi^*(y)}{\Phi(z)} = 1 + \frac{1}{\Phi(z)} = \frac{\Phi^*(y)}{\Phi^*(y) - 1},$$

$$\ln\left(\frac{C}{C^*}\right) = \int_{\Omega} [s^*(y_\omega)\Phi^*(y_\omega) - s(z_\omega)\Phi(z_\omega)]d\omega = \int_{\Omega} [H^*(y_\omega) - H(z_\omega)]d\omega$$

$$= \int_{\Omega} s(z_\omega)d\omega = 1.$$

and

$$\mathcal{L}(V) = \Phi\left(s^{-1}(1/V)\right) = \Phi^*\left(s^{*-1}(1/V)\right) - 1.$$

Appendix D. Proofs of Lemmas S, D, I, and S*.

Proof of Lemma S: Let $w^S(\xi; z) \equiv -\frac{H'(\xi)}{H(z)} > 0$, which satisfies $\int_z^{\overline{z}} w^S(\xi; z)\, d\xi = 1$,

because $H(\overline{z}) = 0$. Then, because $\overline{z}H'(\overline{z}) = -s(\overline{z}) = 0$,

$$\int_z^{\overline{z}} [\zeta^S(\xi) - 1] w^S(\xi; z) d\xi = \frac{\int_z^{\overline{z}} [\xi H''(\xi) + H'(\xi)] d\xi}{H(z)} = \frac{\int_z^{\overline{z}} d[\xi H'(\xi)]}{H(z)} = -\frac{zH'(z)}{H(z)} = \frac{1}{\Phi(z)}.$$

Thus,

$$\frac{z\Phi'(z)}{\Phi(z)} = \frac{zH'(z)}{H(z)} - 1 - \frac{zH''(z)}{H'(z)} = \zeta^S(z) - 1 - \frac{1}{\Phi(z)} = \zeta^S(z) - \int_z^{\overline{z}} \zeta^S(\xi) w^S(\xi; z) d\xi,$$

from which

$$\zeta^{S\prime}(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \implies \Phi'(z) \lesseqgtr 0, \forall z \in (z_0, \overline{z}).$$

Furthermore, $\Phi'(z) = 0$ for $z \in (z_0, \overline{z})$ implies $\zeta^S(z) = 1 + 1/\Phi(z)$, which is hence constant and thus $\zeta^{S\prime}(z) = 0$ for $z \in (z_0, \overline{z})$. This completes the proof. ■

Proof of Lemma D. Let $w^D(\xi; y) \equiv \phi'(\xi)/\phi(y) > 0$, satisfying $\int_0^{y} w^D(\xi;\, y)\, d\xi = 1$.

Then,

$$\int_0^{y} \left[1 - \frac{1}{\zeta^D(\xi)}\right] w^D(\xi; y) d\xi = \frac{\int_0^{y} [\xi\phi''(\xi) + \phi'(\xi)] d\xi}{\phi(y)} = \frac{\int_0^{y} d[\xi\phi'(\xi)]}{\phi(y)} = \frac{y\phi'(y)}{\phi(y)} \equiv \mathcal{E}_\phi(y).$$

Thus,

$$\frac{y\mathcal{E}_\phi'(y)}{\mathcal{E}_\phi(y)} = 1 - \frac{1}{\zeta^D(y)} - \mathcal{E}_\phi(y) = \int_0^{y} \left[\frac{1}{\zeta^D(\xi)}\right] w^D(\xi; y) d\xi - \frac{1}{\zeta^D(y)},$$

from which

$$\zeta^{D\prime}(y) \lesseqgtr 0, \forall y \in (0, y_0) \implies \mathcal{E}_\phi'(y) \lesseqgtr 0, \forall y \in (0, y_0).$$

Furthermore, $\mathcal{E}_\phi'(\,y) = 0$ for $y \in (0, y_0)$ implies $\zeta^D(y) = 1/\left[1 - \mathcal{E}_\phi(y)\right]$, which is hence constant, and thus $\zeta^{D\prime}(y) = 0$ for $y \in (0, y_0)$. This completes the proof. ■

Proof of Lemma I: The proof is analogous to that of Lemma S. Let $w^I(\xi;z) \equiv -\theta'(\xi)/\theta(z) > 0$, which satisfies $\int_z^{\bar{z}} w^I(\xi;z)\,d\xi = 1$, because $\theta(\bar{z}) = 0$. Then, because $\bar{z}\theta'(\bar{z}) = 0$,[28]

$$\int_z^{\bar{z}} [\zeta^I(\xi) - 1] w^I(\xi;z) d\xi = \frac{\int_z^{\bar{z}}[\theta'(\xi) + \xi\theta''(\xi)]d\xi}{\theta(z)} = \frac{\int_z^{\bar{z}} d[\xi\theta'(\xi)]}{\theta(z)} = -\frac{z\theta'(z)}{\theta(z)} \equiv \mathcal{E}_\theta(z).$$

Thus,

$$\frac{z\mathcal{E}_\theta'(z)}{\mathcal{E}_\theta(z)} = \mathcal{E}_\theta(z) + 1 - \zeta^I(z) = \int_z^{\bar{z}} \zeta^I(\xi) w^I(\xi;z) d\xi - \zeta^I(z),$$

from which

$$\zeta^{I'}(z) \gtreqless 0, \forall z \in (z_0, \bar{z}) \quad \Rightarrow \quad \mathcal{E}_\theta'(z) \gtreqless 0, \forall z \in (z_0, \bar{z}).$$

Furthermore, $\mathcal{E}_\theta'(z) = 0$ for $z \in (z_0, \bar{z})$ implies $\zeta^I(z) = 1 + \mathcal{E}_\theta(z)$, which is hence constant and thus $\zeta^{I'}(z) = 0$ for $z \in (z_0, \bar{z})$. This completes the proof. ∎

Proof of Lemma S*: The proof is analogous to that of Lemma D. Let $w^{S*}(\xi^*;y) \equiv H^{*\prime}(\xi^*)/H^*(y) > 0$, satisfying $\int_0^y w^{S*}(\xi^*;y)d\xi^* = 1$. Then,

$$\int_0^y \left[1 - \frac{1}{\zeta^{S*}(\xi^*)}\right] w^{S*}(\xi^*;y)d\xi^* = \frac{\int_0^y [\xi^* H^{*\prime\prime}(\xi^*) + H^{*\prime}(\xi)]d\xi^*}{H^*(y)} = \frac{\int_0^y d[\xi^* H^{*\prime}(\xi^*)]}{H^*(y)} = \frac{yH^{*\prime}(y)}{H^*(y)}$$

$$= \frac{1}{\Phi^*(y)}.$$

Thus,

$$\frac{y\Phi^{*\prime}(y)}{\Phi^*(y)} = \frac{yH^{*\prime}(y)}{H^*(y)} - 1 - \frac{yH^{*''}(y)}{H^{*'}(y)} = \frac{1}{\Phi^*(y)} - 1 + \frac{1}{\zeta^{S*}(y)} = \frac{1}{\zeta^{S*}(y)} - \int_0^y \frac{w^{S*}(\xi^*;y)}{\zeta^{S*}(\xi^*)}\,d\xi^*,$$

from which

$$\zeta^{S*\prime}(y) \lesseqgtr 0, \forall y \in (0, y_0) \Rightarrow \Phi^{*'}(y) \gtreqless 0, \forall y \in (0, y_0).$$

Furthermore, $\Phi^{*'}(y) = 0$, for $y \in (0, y_0)$ implies $\zeta^{S*}(y) = \Phi^*(y)/[\Phi^*(y) - 1]$, which is hence constant and thus $\zeta^{S*\prime}(y) = 0$ for $y \in (0, y_0)$. This completes the proof. ∎

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[28]For $\bar{z} < \infty$, this follows from $\theta'(\bar{z}) = 0$. For $\bar{z} = \infty$, suppose the contrary, so that there exists $z_0 > 0$ such that, for all $z > z_0$, $-z\theta'(z) > c > 0$. Then, $\theta(z_0) = -\lim_{x\to\infty} \int_{z_0}^x \theta'(\xi)d\xi = -\lim_{x\to\infty} \int_{z_0}^x \xi\theta'(\xi)d\xi/\xi > \lim_{x\to\infty} \int_{z_0}^x c\,d\xi/\xi = \infty$, a contradiction.