DISCUSSION PAPER SERIES

DP18184
(v. 4)

LOVE-FOR-VARIETY

Kiminori Matsuyama and Philip Ushchev

INTERNATIONAL TRADE AND REGIONAL ECONOMICS AND MACROECONOMICS AND GROWTH

CEPR

ISSN 0265-8003

LOVE-FOR-VARIETY

Kiminori Matsuyama and Philip Ushchev

Discussion Paper DP18184
First Published 31 May 2023
This Revision 12 August 2025

Centre for Economic Policy Research
187 boulevard Saint-Germain, 75007 Paris, France
2 Coldbath Square, London EC1R 5HL
Tel: +44 (0)20 7183 8801
www.cepr.org

LOVE-FOR-VARIETY

Abstract

We study how love-for-variety, -- utility (or productivity) gains from expanding variety of consumer goods (or inputs) --, depends on the underlying demand structure. To this end, we first define the substitutability measure and love-for-variety measure, which are both functions of the variety of available goods, V, only. Since homotheticity alone does not impose much restriction, we turn to five subclasses of homothetic demand systems, which are pairwise disjoint with the sole exception of CES. Under the two classes of GM-CES, both measures are constant, but the love-for-variety measure under CES underestimates the true love-for-variety under GM-CES, potentially by a wide margin. Under the three classes of H.S.A., HDIA, and HIIA, we establish, among others, the following. First, substitutability is increasing in V, if and only if the 2nd law of demand (the price elasticity of demand for each good is increasing in its price) holds. Second, increasing substitutability implies diminishing love-for-variety. Third, the love-for-variety measure under CES overestimates the true love-for-variety under diminishing love-for-variety. We illustrate the implications of these results by applying to a simple Armington-type competitive model of trade and show how biased the estimates of the gains from trade under CES can be.

JEL Classification:

Keywords:

Kiminori Matsuyama - k-matsuyama@northwestern.edu
Northwestern University and CEPR

Philip Ushchev - ph.ushchev@gmail.com
ECARES, Université Libre De Bruxelles and CEPR

Acknowledgements
We are grateful to the conference and seminar participants at Universities of Tokyo, Gakushuin, Peking, Chicago, Harvard, and Yale for their feedback. This project started during Matsuyama's visits to CREi and ECARES, whose hospitality he gratefully acknowledged. The usual disclaimer applies.

Love-for-Variety

Kiminori Matsuyama
Northwestern University

Philip Ushchev
ECARES, Université Libre de Bruxelles

First Version: May 30, 2023
Second Version: December 8, 2024
Third Version: March 11, 2025
This Version: August 12, 2025

Abstract

We study how love-for-variety, -- utility (or productivity) gains from expanding variety of consumer goods (or inputs) --, depends on the underlying demand structure. To this end, we first define the substitutability measure and love-for-variety measure, which are both functions of the variety of available goods, V, only. Since homotheticity alone does not impose much restriction, we turn to five subclasses of homothetic demand systems, which are pairwise disjoint with the sole exception of CES. Under the two classes of GM-CES, both measures are constant, but the love-for-variety measure under CES underestimates the true love-for-variety under GM-CES, potentially by a wide margin. Under the three classes of H.S.A., HDIA, and HIIA, we establish, among others, the following. First, substitutability is increasing in V, if and only if the 2nd law of demand (the price elasticity of demand for each good is increasing in its price) holds. Second, increasing substitutability implies diminishing love-for-variety. Third, the love-for-variety measure under CES overestimates the true love-for-variety under diminishing love-for-variety. We illustrate the implications of these results by applying to a simple Armington-type competitive model of trade and show how biased the estimates of the gains from trade under CES can be.

This paper is prepared for the inaugural edition of International Seminar on Trade (ISoT) at the World Bank in May 2025. We thank the organizers and Ahmad Lashkaripour for his discussion. It substantially expands an earlier version of the paper with the same title, presented at (chronologically) Universities of Tokyo, Gakushuin, Peking, Chicago, Harvard, Yale, and Chicago Fed. The project started during Matsuyama's March 2023 visits to CREi and ECARES, whose hospitality is gratefully acknowledged. The usual disclaimer applies.

1. Introduction

Love-for-variety[1] captures the idea that consumers (or producers) can achieve higher level of utility (or productivity) when they have access to a wider variety of consumer goods (or inputs), as emphasized by Dixit and Stiglitz (1977), Krugman (1980), Ethier (1982), and Romer (1987). It is a natural consequence of the convexity of preferences (or the production technologies). As it represents benefits from expanding variety of goods, love-for-variety plays a central role in many fields of economics, as surveyed in Matsuyama (1995), most prominently in economic growth (Grossman and Helpman 1993, Gancia and Zilibotti 2005, and Acemoglu 2008), international trade (Helpman and Krugman 1985), and economic geography (Fujita, Krugman, and Venables 1999). Even though commonly discussed in monopolistic competition settings, the concept of love-for-variety is also useful in other contexts, such as gains from trade in Armington-type competitive models, in which different countries produce different sets of goods.

In spite of its importance, however, little is known about how love-for-variety depends on the demand system for the goods generated by the underlying utility (or production) function. In a standard treatment, e.g., Matsuyama (1995, Sec.3A), the analytical expression for the love-for-variety (LV) measure, $\mathcal{L}$, is obtained under CES with gross substitutes.[2] It is equal to

$$\mathcal{L} = \frac{1}{\sigma - 1},$$

where $\sigma > 1$ represents both the (constant) elasticity of substitution (ES) across different goods and the (constant) price elasticity (PE) of demand for each good. This expression has some appealing properties. First, LV is inversely related to both ES and PE. Second, knowing PE gives you everything you need to know about ES and LV. However, this expression also exhibits some unappealing properties. For example, LV is constant. Many

[1]Different authors called this concept differently; e.g., “the desirability of variety” (Dixit and Stiglitz 1977), “love of variety” (Helpman and Krugman 1985, sec. 6.2), “taste for variety” (Benassy 1996), etc. In the context of input variety, Ethier (1982) called it “gains from an increased division of labor” and Romer (1987) “increasing returns due to specialization.” We call “love-for-variety,” as in Parenti et.al. (2017) and Thisse and Ushchev (2020), since it seems most common in the recent literature.

[2]CES is also assumed in most empirical assessment of love-for-variety; see e.g., Feenstra (1994), Bils and Klenow (2001), and Broda and Weinstein (2006). A few exceptions include Feenstra and Weinstein (2017), which use translog.

find it implausible that utility (or productivity) gains enjoyed by consumers (or producers) from having access to more variety of consumer goods (or inputs) would be independent of how much variety they have already access to.[3] Many also find that the relation between PE, ES, and LV under CES is so hard-wired that it gives no flexibility. For this reason, many need to introduce some sorts of direct externalities from the variety to TFP, as proposed by Benassy (1977), to fill the gap between the love-for-variety inferred from, say, productivity growth, and the love-for-variety implied by the CES demand.

Of course, these features may be an artifact of CES. In this paper, we investigate how LV changes when we depart from CES. How is LV related to ES and PE, which are distinct concepts outside of CES? How biased is our estimate of LV, if we incorrectly assume that the underlying demand system is CES when it is not? Under what conditions does LV decline as the variety of available goods increases? Does it help to depart from CES to introduce the Marshall's 2nd law of demand (i.e., PE becomes higher at a higher price along the demand curve for each good)? And can we develop "love-for-variety approach" with diminishing LV, which remains tractable? These are some of the questions we address.

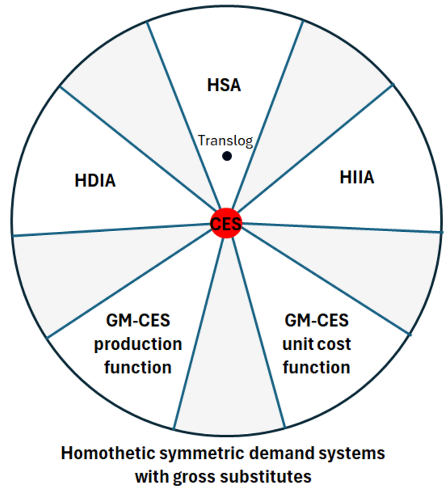
In Section 2, we first recall some general properties of homothetic symmetric demand systems.[4] Then, to help organize the investigation, we define both substitutability

---

[3]Indeed, several economists have expressed to us that, because to this unappealing feature of love-for-variety under CES, they prefer "the ideal variety approach," e.g., Helpman and Krugman (1985, sec. 6.3), in which consumers are heterogenous in taste, and each consumer buys the only variety closest to his/her ideal variety. Despite each consumer buys only one variety, increasing variety is beneficial in that each consumer finds a variety closer to the ideal variety on average as the variety increases, and yet the benefit of adding variety is diminishing, as the product space becomes congested. In spite of such an appealing feature, the ideal variety approach has not been used widely in applied general equilibrium models due to its intractability.

[4] It should be noted that homotheticity and symmetry are not so restrictive as they may seem, because one can nest homothetic symmetric demand systems into a nonhomothetic and/or asymmetric upper-tier demand system. In other words, homothetic symmetric non-CES can serve as building blocks to construct such nonhomothetic and/or asymmetric non-CES. Indeed, the homotheticity restriction is an advantage, as it makes our results applicable to a sector-level analysis in multi-sector models. Moreover, one of the messages of this paper is that even homotheticity and symmetry are not strong enough, "anything goes" so that one needs to look for tighter restrictions. Another message is that the standard measures of love-for-variety and gains from trade derived under CES require substantial modifications when we move away from CES, *even if we restrict ourselves to* homotheticity and symmetry. It should be obvious that these modifications will be necessary when we also allow for nonhomotheticity and asymmetry.

Figure 1: Five Classes



across different goods and love-for-variety, which can be expressed as functions of the available variety V only, as $\mathcal{S}(V)$ and $\mathcal{L}(V)$, respectively. We end Section 2 by discussing briefly generalized CES proposed by Benassy (1977) in an attempt to break the tight link between ES and LV while preserving the CES demand structure.

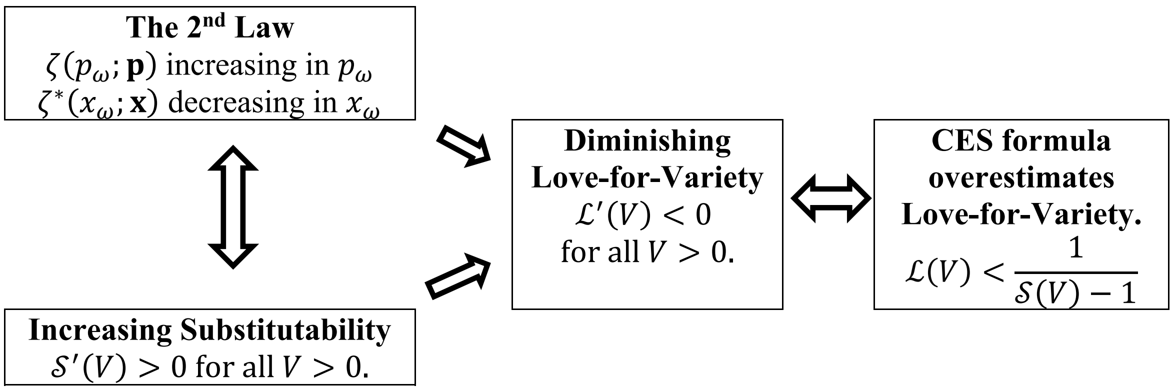
Because homotheticity is not strong enough to impose much restriction on the properties of the two functions, $\mathcal{S}(V)$ and $\mathcal{L}(V)$, we turn to five subclasses of homothetic demand systems, which are pair-wise disjoint with the sole exception of CES, as illustrated in Figure 1. The reader can also find the formulae for $\mathcal{S}(V)$ and $\mathcal{L}(V)$ under different classes in Table 1.

In Section 3, we look at the two classes of Generalized Mean of CES (GM-CES), obtained by taking the weighted geometric mean of CES unit cost functions and of CES production functions. In Theorem 1, we show that both $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are constant, and their constant values, $\mathcal{S}^{GMCES}$and $\mathcal{L}^{GMCES}$, satisfy

$$\mathcal{L}^{GMCES} > \frac{1}{\mathcal{S}^{GMCES} - 1},$$

unless it is CES. Moreover, $\mathcal{L}^{GMCES}$ can be arbitrarily large, holding $\mathcal{S}^{GMCES}$ fixed. This suggests that the standard CES formula for LV underestimates the true value of LV under GM-CES, potentially by a wide margin.

Figure 2: The Relation between the Price Elasticity, Substitutability and Love-for Variety under H.S.A., HDIA and HIIA.



In Section 4, we look at the three remaining classes: H.S.A. (Section 4.1.); HDIA (Section 4.2.); and HIIA (Section 4.3.).[5] These three subsections are written in a self-contained way so that they can be read independently in any order. For each of these three classes, we establish the following results, some of which are also illustrated in Figure 2. First, substitutability, $\mathcal{S}(V)$, is increasing in V, if and only if the 2nd law of demand holds (Theorem 2-i). Second, increasing (decreasing) substitutability, $\mathcal{S}'(V) > (<) 0$, implies diminishing (increasing) love-for-variety $\mathcal{L}'(V) < (>) 0$, but the converse is not true (Theorem 2-ii). Third, love-for-variety is constant, $\mathcal{L}'(V) = 0$, if and only if substitutability is constant, $\mathcal{S}'(V) = 0$, which occurs only under CES within these three classes (Theorem 2-iii).[6] Fourth, $\mathcal{L}'(V) < (>)\ 0$ is sufficient and necessary for

$$\mathcal{L}(V) < (>) \frac{1}{\mathcal{S}(V) - 1}.$$

The standard CES formula for LV thus over(under)estimates the true value of LV in the case of diminishing (increasing) LV under the three classes (Theorem 3). We also show

[5]H.S.A., HDIA, and HIIA stand for *Homothetic Single Aggregator*, *Homothetic Direct Implicit Additivity*, and *Homothetic Indirect Implicit Additivity*. Matsuyama and Ushchev (2017) first introduced these three classes without symmetry and gross substitutability restrictions. Matsuyama (2023; 2025) discuss the relation between these three and other classes of non-CES demand systems in detail.

[6]These three results thus suggest that H.S.A., HDIA, and HIIA offer a tractable way of capturing the intuition that gains from increasing variety is diminishing, if different goods are more substitutable when more variety is available. As such, they can be valuable alternatives to those who find "the ideal variety approach" more appealing than the love-for-love approach under CES despite that the former is less tractable.

that the gap between $\mathcal{L}(V)$ and $1/[\mathcal{S}(V)-1]$ goes asymptotically to zero, as V goes to infinity, under these three classes (Theorem 4).

It should be noted that Theorems 1 through 4 are all about the demand side of expanding variety. As such, they hold regardless of what is assumed on the supply side of the variety change and hence appliable to a wide range of models. Nevertheless, in Section 5, we illustrate some implications of these results by applying them to a simple Armington-type competitive model of trade between the two countries, which produce different sets of goods. The reader can also find the formulae for the gains from trade, $GT,$ under different classes of demand systems of this model in Table 2. For example, under GM-CES, it can be written as:

$$GT = \left(\frac{1}{\lambda}\right)^{\mathcal{L}^{GMCES}} > \left(\frac{1}{\lambda}\right)^{1/\epsilon^{GMCES}},$$

where $0 < \lambda < 1$ is the country's domestic expenditure share, and $\epsilon^{GMCES} = \mathcal{S}^{GMCES} - 1 > 0$ is the trade elasticity, and the strict inequality holds unless it is CES. This formula preserves the key feature of the ACR formula[7] by Arkolakis, Costinot, and Rodríguez-Clare 2012), in the sense that it is a decreasing power function of λ. However, it differs in the sense that the exponent is $\mathcal{L}^{GMCES}$, not $1/\epsilon^{GMCES}$. Moreover, $\mathcal{L}^{GMCES}$ can be arbitrarily large, even after controlling for $\epsilon^{GMCES} = \mathcal{S}^{GMCES} - 1 > 0$. This suggests that the standard practice of estimating the gains from trade applying the ACR formula using the trade elasticity as a sufficient statistic underestimates the gains from trade under GM-CES, potentially by a wide margin. Under the three classes of H.S.A., HDIA, and HIIA, the ACR formula does not hold in general in the sense that it is no longer a power function of the domestic expenditure share, λ. In particular, even after controlling for λ, the gains from trade depend on the sizes of the two countries. For example, given the size of one country, an increase in the size of its trading partner leads to an increase in its gains from trade, but with the limit, if these three classes of demand systems feature the

[7] We define the gains from trade as the rate of increase in the country's welfare when the country moves from autarky to trade. ACR define the gains from trade as the rate of decline in the country's welfare when the country moves from trade to autarky. If we follow their ACR definition, the formula is $GT = 1 - \lambda^{\mathcal{L}^{GMCES}} > 1 - \lambda^{1/\epsilon^{GMCES}}$. Of course, nothing of substance depends on this choice, as there is the one-to-one correspondence between the two definitions. We have chosen our definition, as it helps to keep the formulae simpler under many different classes of the demand systems that we explore.

choke price. This suggests that the standard gains from trade formula overestimates the gains from trading with a large country.

In Section 6, we offer some concluding remarks. All technical materials, including many proofs, are in Appendices.

2. Symmetric homothetic demand systems

In what follows, we discuss a general symmetric homothetic demand system in the context of the producer's demand for differentiated inputs generated by a monotone, strictly quasi-concave, symmetric CRS production function, $X = X(\mathbf{x})$. Here, $\mathbf{x} = \{x_\omega; \omega \in \overline{\Omega}\}$ is the input quantity vector, defined over $\overline{\Omega}$, a continuum of the set of all potential inputs, which is divided into the set of available inputs, $\Omega \subset \overline{\Omega}$, and the set of unavailable inputs, $\overline{\Omega}\backslash\Omega$. That is, $x_\omega = 0$ for $\omega \in \overline{\Omega}\backslash\Omega$. We let the Lebesgue measure of Ω denoted by $V \equiv |\Omega|$. Our goal is to study how the effect of changing V on productivity is related to the demand system for inputs. To this end, it is necessary to assume that each input is inessential. That is, $x_\omega = 0$ for $\omega \in \overline{\Omega}\backslash\Omega$ does *not* imply $X(\mathbf{x}) = 0$, so that it is possible to produce a positive output, even when some potential inputs are unavailable.

2.1. Duality Theory: A Refresher

Let us first recall some key results from the duality theory; see, e.g., Mas-Colell et. al. (1995), and Jehle and Reny (2012), but, in doing so, we reformulate them for a continuum set of inputs. Let $\mathbf{p} = \{p_\omega; \omega \in \overline{\Omega}\}$ denote the input price vector, such that $p_\omega = \infty$ for $\omega \in \overline{\Omega}\backslash\Omega$ and $p_\omega < \infty$ for $\omega \in \Omega$. The non-essentiality of inputs ensures that the unit cost function corresponding to this production function,

$$P(\mathbf{p}) \equiv \min_{\mathbf{x}} \left\{ \mathbf{px} \equiv \int_\Omega p_\omega x_\omega d\omega \,\middle|\, X(\mathbf{x}) \geq 1 \right\}, \tag{1}$$

is finite, and hence well-defined, even though $p_\omega = \infty$ for $\omega \in \overline{\Omega}\backslash\Omega$. Furthermore, it also satisfies the monotonicity, strict quasi-concavity, linear homogeneity, and symmetry. The first-order condition of the minimization problem in eq.(1) yields the inverse demand curve for ω:

$$p_\omega = P(\mathbf{p})\frac{\partial X(\mathbf{x})}{\partial x_\omega}; \quad \omega \in \Omega. \tag{2}$$[8]

Alternatively, from the duality theorem, we can instead start from a monotonic, strict quasi-concave, linear homogeneous, and symmetric unit cost function, $P(\mathbf{p})$, from which the CRS production function $X(\mathbf{x})$ is derived as follows:

$$X(\mathbf{x}) \equiv \min_{\mathbf{p}} \left\{ \mathbf{px} \equiv \int_\Omega p_\omega x_\omega d\omega \,\middle|\, P(\mathbf{p}) \geq 1 \right\}, \tag{3}$$

which satisfies monotonicity, strict quasi-concavity, and symmetry. The first-order condition of the minimization problem in eq.(3) yields the demand curve for ω:

$$x_\omega = \frac{\partial P(\mathbf{p})}{\partial p_\omega} X(\mathbf{x}); \quad \omega \in \Omega. \tag{4}$$

This is also known as Shepherd's lemma. The duality theorem thus allows us to use either $X(\mathbf{x})$ or $P(\mathbf{p})$ as the primitive of the CRS technology.

From either eq.(3) or eq.(4), the Euler's theorem on linear homogeneity functions implies that

$$\mathbf{px} \equiv \int_\Omega p_\omega x_\omega d\omega = P(\mathbf{p}) \left[\int_\Omega \frac{\partial X(\mathbf{x})}{\partial x_\omega} x_\omega d\omega \right] = \left[\int_\Omega p_\omega \frac{\partial P(\mathbf{p})}{\partial p_\omega} d\omega \right] X(\mathbf{x}) = P(\mathbf{p}) X(\mathbf{x}). \tag{5}$$

This identity means that the total cost of inputs is equal to the total value of output.

2.2. Budget Share, Price Elasticity of Demand and the 2nd Law of Demand

From eq.(2), the budget share of ω can be written as a homogeneous function of degree 0 in $\mathbf{x}$ as:

$$s_\omega \equiv \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} \equiv s^*(x_\omega; \mathbf{x}) = s^*(1, \mathbf{x}/x_\omega); \;\; \omega \in \Omega. \tag{6}$$

From eq.(4), it can also be written as a homogeneous function of degree 0 in $\mathbf{p}$ as:

$$s_\omega \equiv \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} \equiv s(p_\omega; \mathbf{p}) = s(1, \mathbf{p}/p_\omega); \; \omega \in \Omega. \tag{7}$$

[8]We note two technical issues. First, we discuss the partial derivatives like $\partial X(\mathbf{x})/\partial x_\omega$ and $\partial P(\mathbf{p})/\partial p_\omega$ heuristically in this paper, even though they are defined formally as Fréchet derivatives in the L^2-norm defined over a continuum set of available inputs, as in Parenti et.al. (2017). Second, the first-order conditions in eq.(2) and eq.(4) assume the interior solution. This is ensured under CES or GM-CES introduced in Section 3. In general, however, it is important to address the possibility of the corner solution. That is, $x_\omega \geq 0$ may be binding, when p_ω exceeds "the choke price." Nevertheless, for the expositional reason, we postpone an explicit treatment of the choke price until Section 4, where it becomes relevant.

In what follows, we assume that inputs are gross substitutes; that is, the budget share of each input goes down as its price goes up (and its quantity goes down) along its demand curve:

Definition: *Gross substitutability of inputs*

$$\frac{\partial \ln s(p_\omega; \mathbf{p})}{\partial \ln p_\omega} = \frac{\partial \ln s(1, \mathbf{p}/p_\omega)}{\partial \ln p_\omega} < 0 \Leftrightarrow \frac{\partial \ln s^*(x_\omega; \mathbf{x})}{\partial \ln x_\omega} = \frac{\partial \ln s^*(1, \mathbf{x}/x_\omega)}{\partial \ln x_\omega} > 0.$$

From eq.(6) and eq.(7), the price elasticity of demand for ω can be written both as a function of the prices and as a function of the quantities as follows:

$$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = 1 - \frac{\partial \ln s(p_\omega; \mathbf{p})}{\partial \ln p_\omega} \equiv \zeta(p_\omega; \mathbf{p}) = \zeta(1, \mathbf{p}/p_\omega) > 1; \tag{8}$$

$$\zeta_\omega \equiv -\left[\frac{\partial \ln p_\omega}{\partial \ln x_\omega}\right]^{-1} = \left[1 - \frac{\partial \ln s^*(x_\omega;\ \mathbf{x})}{\partial \ln x_\omega}\right]^{-1} \equiv \zeta^*(x_\omega; \mathbf{x}) = \zeta^*(1, \mathbf{x}/x_\omega)\ > 1. \tag{9}$$

Both functions are homogeneous of degree 0. Moreover, gross substitutability implies that they are both greater than one.

The literature on non-CES demand system often focuses on the case where Marshall's 2^{nd} law of demand (or the 2^{nd} law for short) holds. That is, the price elasticity of demand for each input goes up as its price goes up (and its quantity goes down) along its demand curve.

Definition: *The 2^{nd} law* holds if and only if

$$\frac{\partial \zeta(p_\omega; \mathbf{p})}{\partial p_\omega} = \frac{\partial \zeta(1, \mathbf{p}/p_\omega)}{\partial p_\omega} > 0 \Leftrightarrow \frac{\partial \zeta^*(x_\omega; \mathbf{x})}{\partial x_\omega} = \frac{\partial \zeta^*(1, \mathbf{x}/x_\omega)}{\partial x_\omega} < 0.$$

If the opposite inequality holds, we shall say that the anti-2^{nd} law holds. CES is clearly a borderline case.

Note that eqs.(6)-(9) show that the budget share of $\omega \in \Omega$, s_ω, and its price elasticity of demand, ζ_ω, are both functions of $\mathbf{p}/p_\omega$ or $\mathbf{x}/x_\omega$. Even though symmetry implies that they are invariant of permutation, the budget share and the price elasticity still depend on the entire distribution of the prices (or the quantities) normalized by its own price (or its own quantity), which is infinite dimensional. For this reason, the cross-product interactions could be complicated under general homothetic symmetric demand systems.

2.3 Unit Quantity and Price Vectors

To further characterize homothetic symmetric demand systems, it is useful to define the unit quantity vector, $\mathbf{1}_\Omega \equiv \{(1_\Omega)_\omega; \omega \in \overline{\Omega}\}$ and the unit price vector, $\mathbf{1}_\Omega^{-1} \equiv \{(1_\Omega^{-1})_\omega; \omega \in \overline{\Omega}\}$, as follows:

$$(1_\Omega)_\omega \equiv \begin{cases} 1 & \text{for} \quad \omega \in \Omega \\ 0 & \text{for} \quad \omega \in \overline{\Omega}\backslash\Omega \end{cases}; \qquad (1_\Omega^{-1})_\omega \equiv \begin{cases} 1 & \text{for} \quad \omega \in \Omega \\ \infty & \text{for} \quad \omega \in \overline{\Omega}\backslash\Omega \end{cases},$$

which satisfies $\int_\Omega (1_\Omega)_\omega d\omega = \int_\Omega (1_\Omega^{-1})_\omega d\omega = |\Omega| \equiv V.$ Then, symmetric quantity and price patterns among all the available inputs are expressed as:

$$\mathbf{x} = x\mathbf{1}_\Omega; \quad \mathbf{p} = p\mathbf{1}_\Omega^{-1},$$

where $x > 0$ and $p > 0$ are scalars.

2.4 Substitutability Measure Across Different Goods

Since $\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$ are both homogenous of degree zero, the price elasticity at symmetric patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$ or $\mathbf{x} = x\mathbf{1}_\Omega$, is $\zeta(1; \mathbf{1}_\Omega^{-1}) = \zeta^*(1; \mathbf{1}_\Omega)$, which is independent of p or x, and can be written as $\mathcal{S}(V)$, a function of V only. Moreover, Appendix A shows that $\mathcal{S}(V) = \zeta(1; \mathbf{1}_\Omega^{-1}) = \zeta^*(1; \mathbf{1}_\Omega)$ is equal to the Allen-Uzawa elasticity of substitution between every pair of inputs at the symmetric patterns.[9] Thus, we use the following definition for the substitutability across different inputs in the presence of available variety of inputs V:

Definition: *The substitutability measure* is defined by

$$\mathcal{S}(V) \equiv \zeta(1; \mathbf{1}_\Omega^{-1}) = \zeta^*(1; \mathbf{1}_\Omega) > 1. \tag{10}$$

In general, $\mathcal{S}(V)$ can be nonmonotonic in V. We shall call the case of $\mathcal{S}'(V) > (<) 0$ for all $V > 0$, the case of ***increasing (decreasing) substitutability***.

2.5 Love-for-Variety Measure

Love-for-variety is commonly measured as the productivity gains from a higher V under the symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$, while holding the total amounts of inputs, xV, constant. That is,

[9] Since the set of available inputs is a continuum, there is no point of looking into the Morishima elasticity of substitution.

$$\left.\frac{d\ln X(\mathbf{x})}{d\ln V}\right|_{\mathbf{x}=x\mathbf{1}_\Omega, xV=const.} = \left.\frac{d\ln xX(\mathbf{1}_\Omega)}{d\ln V}\right|_{xV=const.} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0,$$

which is positive due to the strict quasi-concavity of $X(\mathbf{x})$ and could depend solely on V. This definition is essentially the same with the one proposed by Benassy (1996, eq.(2)) for what he called "taste for variety," even though he applied it only for generalized CES, which will be discussed in Section 2.7.

Alternatively, we could also measure love-for-variety as the rate of decline in the unit cost under symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$, while holding the price of each input, p, constant. That is,

$$-\left.\frac{d\ln P(\mathbf{p})}{d\ln V}\right|_{\mathbf{p}=p\mathbf{1}_\Omega^{-1},\ p=const.} = -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} > 0,$$

which is also positive due to the monotonicity of $P(\mathbf{p})$ and could depend solely on V.

These two measures of love-for-variety are indeed identical. To see this, inserting $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$ and $\mathbf{x} = x\mathbf{1}_\Omega$ into eq.(5) yields $pxV = pP(\mathbf{1}_\Omega^{-1})xX(\mathbf{1}_\Omega)$, so that

$$P(\mathbf{1}_\Omega^{-1})X(\mathbf{1}_\Omega) = V \Longrightarrow -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0.$$

Hence, we use the two definitions interchangeably as the love-for-variety measure.

Definition. *The love-for-variety measure* is defined by:

$$\mathcal{L}(V) \equiv -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0. \tag{11}$$

In general, $\mathcal{L}(V)$ can be nonmonotonic in V. We shall call the case of $\mathcal{L}'(V) < (>) 0$ for all $V > 0$, the case of ***diminishing (increasing) love-for-variety***.

2.6. Standard CES

In the case of standard CES with gross substitutes,

$$X(\mathbf{x}) = Z\left[\int_\Omega x_\omega{}^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \Leftrightarrow P(\mathbf{p}) = \frac{1}{Z}\left[\int_\Omega p_\omega{}^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}},$$

where $\sigma > 1$ is the (constant) elasticity of substitution parameter and Z is the TFP parameter. (In the context of spatial economics, Z is often assumed to vary across locations and interpreted as the locational affinity). It is easy to verify:

$$\zeta(p_\omega;\mathbf{p}) = \zeta^*(x_\omega;\mathbf{x}) = \sigma > 1;\ \mathcal{S}(V) = \sigma > 1;\ \mathcal{L}(V) = \frac{1}{\sigma - 1} > 0.$$

Thus, under CES, the price elasticity of demand for each input is everywhere constant and equal to σ. Obviously, this implies that our substitutability measure, which is equal to the price elasticity evaluated at the symmetric patterns, $\mathcal{S}(V)$, is also equal to σ, and independent of V. Moreover, the love-for-variety measure, $\mathcal{L}(V)$, is also independent of V, and depends solely on the single parameter, σ, with a one-to-one inverse relation between the two.

Perhaps for these reasons, some authors claimed without proof that $\mathcal{S}(V)$ is constant only under CES, and/or that $\mathcal{S}(V)$ is the inverse measure of love-for-variety, even under general homothetic demand systems. Neither claim is correct. Indeed, the relation between the price elasticity, $\zeta(p_\omega;\mathbf{p}) = \zeta^*(x_\omega;\mathbf{x})$, substitutability, $\mathcal{S}(V)$, and love-for-variety, $\mathcal{L}(V)$, can be quite complex in general. First, whether the 2$^{\text{nd}}$ law holds or not tells us little about the sign of the derivative of $\mathcal{S}(V)$. This should not be surprising because the former is about how $\zeta(p_\omega; p\mathbf{1}_\Omega^{-1})$ responds to a change in p_ω, while the latter is about how $\zeta(1;\mathbf{1}_\Omega^{-1})$ responds to a change in V through its effect on $\mathbf{1}_\Omega^{-1}$. Second, as explained in Section 3, there exist any number of families of homothetic non-CES in which $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are both independent of V, and they move in the same direction as one moves within each family. Indeed, knowing $\mathcal{S}(V)$ tells us little about $\mathcal{L}(V)$ in general. Again, this should not be surprising because the former measures how $\zeta(1;\mathbf{1}_\Omega^{-1})$ responds to a change in V through its effect on $\mathbf{1}_\Omega^{-1}$, and hence is related to how the curvature of the demand system, hence that of the marginal utility (productivity) function, responds to a change in V, while $\mathcal{L}(V)$ measures how $P(\mathbf{1}_\Omega^{-1})$ responds to a change in V through its effect on $\mathbf{1}_\Omega^{-1}$, and hence is related to how the curvature of the utility (production) function responds to a change in V. In short, "almost anything goes" under general homothetic symmetric demand systems. Therefore, to better understand the relation between $\mathcal{S}(V)$ and $\mathcal{L}(V)$ under non-CES, we turn to some subclasses of homothetic symmetric demand systems in Section 3 and in Section 4.

2.7. Generalized CES a la Benassy (1996): A Digression

Before proceeding, however, let us digress to discuss a previous attempt to break the tight relation between $\mathcal{S}(V)$ and $\mathcal{L}(V)$ while preserving the CES demand systems. Benassy (1996) proposed to generalize CES as:

$$X(\mathbf{x}) = Z(V)\left[\int_{\Omega} x_{\omega}{}^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \Leftrightarrow P(\mathbf{p}) = \frac{1}{Z(V)}\left[\int_{\Omega} p_{\omega}{}^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}},$$

by making TFP a function of V as $Z(V)$, justified by some sorts of direct externalities from V to TFP. Under such modified CES,

$$\zeta(p_{\omega};\mathbf{p}) = \zeta^{*}(x_{\omega};\mathbf{x}) = \sigma > 1;\ \mathcal{S}(V) = \sigma > 1;\ \mathcal{L}(V) = \frac{1}{\sigma-1} + \frac{\partial \ln Z(V)}{\partial \ln V}.$$

This demand system is still CES because it features the constant price elasticity and the constant substitutability. Yet this extension allows the gap between the "revealed" love-for-variety, $\mathcal{L}(V)$, which could be obtained from productivity growth data, and the love-for-variety implied by CES demand, $1/(\sigma-1)$, to be "accounted for" by $\partial \ln Z(V)/\partial \ln V$, the term we shall call 'the Benassy residual," in analogy with the Solow residual in the growth accounting.[10]

Even if one believes in the presence of such direct externalities from V to TFP, one should note that any estimate of the Benassy residual hinges on the assumption of CES. In any case, adding the Benassy residual to the standard CES does not address our question; that is, how $\mathcal{L}(V)$ depends on V under homothetic non-CES demand systems, particularly, the question of when $\mathcal{L}(V)$ is diminishing in V. Moreover, as will be shown below, $\mathcal{L}(V) \neq 1/(\mathcal{S}(V)-1)$ in general under non-CES, even if TFP is independent of V, so that one could interpret non-CES as a way of microfounding the gap or the Benassy residual.

For the remainder of the paper, we present $\mathcal{L}(V)$ without the Benassy residual by assuming that Z is independent of V, i.e., $\partial \ln Z(V)/\partial \ln V = 0$. This is for the expositional and notational simplicity. Adding the Benassy residual to any homothetic non-CES demand systems explored below would be straightforward. One would just

[10] In addition, Benassy (1996) assumed that $\partial \ln Z(V)/\partial \ln V = \nu - 1/(\sigma-1)$, so that $\mathcal{L}(V) = \nu$, which can be chosen independently from $\mathcal{S}(V) = \sigma$. If we assume instead $\partial \ln Z(V)/\partial \ln V$ is another parameter independent of $\mathcal{S}(V) = \sigma$, $\mathcal{L}(V)$ is still inversely related to $\mathcal{S}(V) = \sigma$.

need to append $\partial \ln Z(V)/\partial \ln V$ to the expression for $\mathcal{L}(V)$; it would not affect the expression for $\mathcal{S}(V)$.

3. Geometric Means of CES (GM-CES)

We now look at two classes of homothetic symmetric demand systems, obtained by taking weighted geometric means of symmetric CES, thus containing symmetric CES as a special case. In what follows, we shall call these two classes "GM-CES" for short. As shown, under GM-CES, $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are both independent of V, just like CES. Unlike CES, however, $\mathcal{L}(V)$ is no longer tightly linked to $\mathcal{S}(V)$, which only affects the lower bound as $\mathcal{L}(V) \geq 1/(\mathcal{S}(V) - 1)$, where the equality holds iff it is CES.

3.1. GM-CES Unit Cost Function

The first class of GM-CES is generated by the unit cost function, $P(\mathbf{p})$, defined by the weighted geometric mean of symmetric CES unit cost functions:

$$\ln P(\mathbf{p}) \equiv \int_1^{\infty} \ln P(\mathbf{p}; \sigma)\, dG(\sigma) \equiv \mathbb{E}_G[\ln P(\mathbf{p}; \sigma)],$$

where

$$P(\mathbf{p}; \sigma) \equiv \left[\int_{\Omega} p_{\omega}{}^{1-\sigma}\, d\omega\right]^{1/(1-\sigma)}$$

is the unit cost function of the symmetric CES with the constant elasticity of substitution, $\sigma > 1$, and $G(\cdot)$ is a cdf of $\sigma \in (1, \infty)$, satisfying $G(1) = 0$ and $G(\infty) = 1$, with $\mathbb{E}_G[\cdot]$ denoting the expectation operator with respect to $G(\cdot)$. Being the weighted geometric means of linear homogenous, symmetric and strictly quasi-concave functions, $P(\mathbf{p})$ is also linear homogenous, symmetric, and strictly quasi-concave. One possible interpretation is that there are many final goods, all of which generate CES demand but with different σ, and these final goods are aggregated by the Cobb-Douglas with $G(\sigma)$ being the cumulative share of the final goods whose demand for inputs have price elasticity less than or equal to σ.[11]

[11] GM-CES should not be confused with nested-CES. GM-CES is *not* nested-CES, because the domain of $P(\mathbf{p}; \sigma)$, Ω, is common across all $\sigma \in (1, \infty)$ in the definition of GM-CES, meaning that each input is used in the production of every final good. In nested CES, the domain of $P(\mathbf{p}; \sigma)$ depends on σ, as $\Omega(\sigma)$, such that $\Omega(\sigma) \cap \Omega(\sigma') = \emptyset$ if $\sigma \neq \sigma'$, meaning that the set of all inputs is partitioned into nests and that each input belongs to a particular nest and used in the production of one type of the final good. GM-CES is related to mixed CES, used in Adao, Costinot, and Donaldson (2017), in that the demand systems under the

3.2. GM-CES Production Function

The second class of GM-CES is generated by the CRS production function, $X(\mathbf{x})$, defined by the weighted geometric mean of symmetric CES production functions:

$$\ln X(\mathbf{x}) \equiv \int_1^\infty \ln X(\mathbf{x};\sigma)\, dG(\sigma) \equiv \mathbb{E}_G[\ln X(\mathbf{x};\sigma)]$$

where

$$X(\mathbf{p};\sigma) \equiv \left[\int_\Omega x_\omega{}^{1-1/\sigma}\, d\omega\right]^{\sigma/(\sigma-1)}$$

is the symmetric CES production function with the constant elasticity of substitution, $\sigma > 1$. Again, $G(\cdot)$ is a cdf of $\sigma \in (1,\infty)$, satisfying $G(1) = 0$ and $G(\infty) = 1$, with $\mathbb{E}_G[\cdot]$ denoting the expectation operator with respect to $G(\cdot)$. Again, being the weighted geometric means of linear homogenous, symmetric, and strictly quasi-concave functions, $X(\mathbf{x})$ is also linear homogenous, symmetric and strictly quasi-concave.

Appendix B derives the key properties of these two classes of GM-CES, which are summarized as:

Theorem 1 (GM-CES): Under the two classes of GM-CES demand systems,

1-i): $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are both constant. They are given by

$$\mathcal{S}(V) = \mathbb{E}_G[\sigma] > 1;\ \ \mathcal{L}(V) = \mathbb{E}_G\left[\frac{1}{\sigma-1}\right] > 0.$$

for the GM-CES unit cost function, and by:

$$\mathcal{S}(V) = \frac{1}{\mathbb{E}_G[1/\sigma]} > 1;\ \ \mathcal{L}(V) = \mathbb{E}_G\left[\frac{1}{\sigma-1}\right] > 0.$$

for the GM-CES production function.

1-ii): $\mathcal{L}(V)$ can be arbitrarily large, without any upper bound, while its lower bound is given by:

$$\mathcal{L}(V) \geq \frac{1}{\mathcal{S}(V)-1} > 0.$$

where the equality holds if and only if $G(\cdot)$ is non-degenerate, i.e., only under CES.

GM-CES unit cost function are special cases of mixed CES demand systems. We use GM-CES, partly because $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are both independent of V under GM-CES, which helps to point out some key issues in a crystallized manner, and partly because we are unable to ensure the integrability of mixed-CES in general. That is, we do not know of the existence of the utility (production) functions that generate mixed CES demand systems.

First, Theorem 1-i), also shown in the top half of Table 1, states that love-for-variety of both GM-CES unit cost and production functions is the arithmetic mean of love-for-variety of the component CES, while the substitutability of GM-CES unit cost (production) function is the arithmetic (harmonic) mean of the substitutability of the component CES. Since the arithmetic mean of σ and the harmonic mean of σ are equal to each other only when $G(\cdot)$ is non-degenerate, this implies that the two classes of GM-CES do not overlap with the sole exception of CES, as illustrated in Figure 1. Theorem 1-i) also means that it would be false to claim that $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are constant if and only if CES. Second, Theorem 1-ii) suggests that, if one mistakenly interprets $\mathcal{S}(V)$ being

Table 1: Substitutability and Love-for-Variety under CES and the Five Classes.

		Substitutability: $\mathcal{S}(V)$	Love-for-Variety: $\mathcal{L}(V)$
CES		σ	$\frac{1}{\sigma-1}$
GM-CES unit cost function		$\mathbb{E}_G[\sigma]$	$\mathbb{E}_G\left[\frac{1}{\sigma-1}\right]$
GM-CES production function		$\frac{1}{\mathbb{E}_G[1/\sigma]}$	$\mathbb{E}_G\left[\frac{1}{\sigma-1}\right]$
H.S.A.		$\zeta^S\left(s^{-1}\left(\frac{1}{V}\right)\right)$	$\Phi\left(s^{-1}\left(\frac{1}{V}\right)\right)=\frac{1}{\mathcal{E}_H\left(s^{-1}(1/V)\right)}$
$\zeta^S(z)\equiv-\frac{zH''(z)}{H'(z)}>1;\ \frac{1}{\Phi(z)}=\mathcal{E}_H(z)\equiv-\frac{zH'(z)}{H(z)}>0;\ H(z)>0$ decreasing convex.			
Special Case of H.S.A.	Translog	$1+\gamma V$	$\frac{1}{2\gamma V}$
	Generalized Translog	$1+(\sigma-1)(\gamma V)^{1/\eta}$	$\frac{1}{\sigma-1}\left(\frac{\eta}{1+\eta}\right)(\gamma V)^{-1/\eta}$
	Constant Path-Through	$\sigma(\gamma V)^{\frac{1-\rho}{\rho}}$	$\sum_{n=0}^{\infty}\frac{\rho}{1+(1-\rho)n}\left[\frac{1}{\sigma(\gamma V)^{\frac{1-\rho}{\rho}}}\right]^{n+1}$
HDIA		$\zeta^D\left(\phi^{-1}\left(\frac{1}{V}\right)\right)$	$\frac{1}{\mathcal{E}_\phi\left(\phi^{-1}(1/V)\right)}-1$
$\zeta^D(y)\equiv-\frac{\phi'(y)}{y\phi''(y)}>1;0<\mathcal{E}_\phi(y)\equiv\frac{y\phi'(y)}{\phi(y)}<1;\ \phi(y)>0$ increasing and concave.			
HIIA		$\zeta^I\left(\theta^{-1}\left(\frac{1}{V}\right)\right)$	$\frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/V)\right)}$
$\zeta^I(z)\equiv-\frac{z\theta''(z)}{\theta'(z)}>1;\ \mathcal{E}_\theta(z)\equiv-\frac{z\theta'(z)}{\theta(z)}>0;\ \theta(z)>0$ decreasing and convex.			

constant as the evidence for CES, one underestimates love-for-variety under GM-CES. Or if one obtains an estimate of $\mathcal{L}(V)$ directly from productivity growth data, one overestimates the Benassy residual by attributing all the gap between the revealed $\mathcal{L}(V)$ and the CES-implied love-for-variety, $[\mathcal{S}(V) - 1]^{-1}$ to it. Indeed, the GM-CES could potentially account for the gap. Third, because $\mathcal{S}(V)$ imposes only the lower bound on $\mathcal{L}(V)$, one could come up with any number of families of the cdf's, $G(\cdot)$, such that $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are positively related to each other within each family. This suggests that it would be false to claim that $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are inversely related to each other in general.

4. H.S.A., HDIA, and HIIA.

Under CES and GM-CES, $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are both independent of V. We now look for the classes of homothetic symmetric demand systems, in which both $\mathcal{S}(V)$ and $\mathcal{L}(V)$ respond to a change in V, with the special attention on the cases of increasing substitutability $\mathcal{S}'(V) > 0$ and diminishing love-for-variety, $\mathcal{L}'(V) < 0$. Intuitively, one might think that, if adding more variety makes different inputs more substitutable, the price elasticity of demand for each input would become larger, and love-for-variety would become smaller. As pointed out earlier, however, homotheticity (and symmetry) alone is not restrictive enough to capture this intuition: one need to find some additional restrictions to capture this intuition.

To this end, we look at three different classes of homothetic symmetric demand systems, each named by its defining property. They are Homothetic Single Aggregator (H.S.A.), Homothetic Direct Implicit Additivity (HDIA), and Homothetic Indirect Implicit Additivity (HIIA). These three classes are useful for two reasons. First, they are pairwise disjoint with the sole exception of CES, as seen in Figure 1. Thus, they offer three alternative ways of departing from CES, while keeping CES as a special case. Second, each of the three classes generates the demand system with the property that the price elasticity of demand for ω can be expressed as $\zeta(p_\omega; \mathbf{p}) = \zeta(p_\omega/\mathcal{A}(\mathbf{p}))$. That is, the price elasticity is a function of a single variable, $p_\omega/\mathcal{A}(\mathbf{p})$, where $\mathcal{A}(\mathbf{p})$ is the linear homogeneous aggregator in $\mathbf{p}$, whose value serves as a sufficient statistic that captures the interdependence of price elasticities across different inputs.[12] Thus, in these three classes, the price elasticity responds to an increase in p_ω and to a decline in $\mathcal{A}(\mathbf{p})$ in the same way, and hence also to an increase in V in the symmetric price patterns. This property also allows for a simple characterization of the existence of the choke price.

These features impose the functional relation between $\mathcal{S}(V)$ and $\mathcal{L}(V)$ in these three classes, as summarized in Table 1. This functional relation enables us to establish the following results for each of the three classes.

Theorem 2: Under H.S.A., HDIA, and HIIA,

2-i) $\mathcal{S}'(V) > 0$ if and only if the 2nd law holds.

2-ii) $\mathcal{S}'(V) \gtreqqless 0$ for all $V \in (V_0, \infty) \Longrightarrow \mathcal{L}'(V) \lesseqqgtr 0$ for all $V \in (V_0, \infty)$.

2-iii) $\mathcal{L}'(V) = 0$ for all $V \in (V_0, \infty) \Leftrightarrow \mathcal{S}'(V) = 0$ for all $V \in (V_0, \infty)$.

In particular, $\mathcal{L}'(V) = 0$ for all $V > 0 \Leftrightarrow \mathcal{S}'(V) = 0$ for all $V > 0 \Leftrightarrow$ CES.

Theorem 3: Under H.S.A., HDIA, and HIIA,

$$\mathcal{L}'(V) \lesseqqgtr 0 \Leftrightarrow \mathcal{L}(V) \lesseqqgtr \frac{1}{\mathcal{S}(V) - 1}.$$

Theorem 4: Under H.S.A., HDIA, and HIIA,

$$\lim_{V \to \infty} \mathcal{L}(V) = \lim_{V \to \infty} \frac{1}{\mathcal{S}(V) - 1}.$$

In particular, $\lim_{V \to \infty} \mathcal{S}(V) = \infty \Leftrightarrow \lim_{V \to \infty} \mathcal{L}(V) = 0.$

Theorem 2-i) suggests that the 2nd law is equivalent to increasing substitutability. Theorem 2-ii), for $V_0 = 0$, suggests that, if the 2nd law (or equivalently, increasing substitutability) holds everywhere, love-for-variety is diminishing everywhere. The converse is not true. Substitutability may not be increasing everywhere, even if love-for-variety is diminishing everywhere. However, Theorem 2-iii) states that constant love-for-variety is equivalent to constant substitutability, which occurs only under CES under the three classes.[13]

Theorem 3 suggests that, if one infers love-for-variety from the observed price elasticity under the (false) CES assumption, one overestimates (underestimates) love-for-

[12]Recall that, in general, the price elasticity of each input depends on $\mathbf{p}/p_\omega$ or $\mathbf{x}/x_\omega$, the entire distribution of the prices (or quantities) normalized by its own price (or quantity) as shown in eq.(8) and eq.(9).

[13] This also implies that the two classes of GM-CES and these three classes do not overlap with the sole exception of CES, as shown in Figure 1.

variety in the case of diminishing (increasing) love-for-variety. Or, if one obtains an estimate of $\mathcal{L}(V)$ directly from productivity growth data, one underestimates (overestimates) the Benassy residual by attributing all the gap between the revealed $\mathcal{L}(V)$ and the CES-implied love-for-variety, $[\mathcal{S}(V)-1]^{-1}$ to it, when love-for-variety is diminishing (increasing). Figure 2 illustrates schematically the results of Theorem 2 and Theorem 3. Theorem 4 states that this gap asymptotically disappears, as V goes to infinity.

Even though all three theorems hold under these three classes, there are many subtle differences across the three, so that it is more efficient to prove these theorems separately for each class. Therefore, in the next three subsections, we offer the formal definition, review the key properties, and prove these theorems for each of the three classes. These three subsections are deliberately written in a self-contained way so that they may be read in any order. The reader may also skip the following subsections and go directly to Section 5 for an application of the above theorems with no loss of continuity.

4.1. The H.S.A. class.

A homothetic symmetric demand system for inputs with gross substitutes belongs to H.S.A. (*Homothetic Single Aggregator*) if there exists a function of a single variable, $s: \mathbb{R}_{++} \to \mathbb{R}_{+}$, which is C^2 and strictly decreasing as long as $s(z) > 0$, with $\lim_{z\to 0} s(z) = \infty$ and $\lim_{z\to \bar{z}} s(z) = 0$, where $\bar{z} \equiv \inf\{z > 0 | s(z) = 0\}$, such that the budget share of $\omega \in \Omega$ can be written as:

$$s_\omega = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(\mathbf{p})}\right), \tag{12}$$

where $A(\mathbf{p})$ is defined implicitly by the adding-up constraint,

$$\int_\Omega s\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega \equiv 1. \tag{13}$$

By construction, $A(\mathbf{p})$ is linear homogenous in $\mathbf{p}$ for any fixed Ω and that the budget shares of all inputs are added up to one. [14]

[14]For $s: \mathbb{R}_{++} \to \mathbb{R}_{+}$, satisfying the above conditions, a class of the budget share functions, $s_\gamma(z) \equiv \gamma s(z)$ for $\gamma > 0$, generate the same demand system with the same common price aggregator. This can be seen by reindexing the inputs, as $\omega' = \gamma\omega$, so that $\int_\Omega s_\gamma(p_\omega/A(\mathbf{p}))d\omega = \int_\Omega s(p_{\omega'}/A(\mathbf{p}))d\omega' = 1$. In this sense, $s_\gamma(z) \equiv \gamma s(z)$ for $\gamma > 0$ are all equivalent. Note also that a class of the budget share functions, $s_\lambda(z) \equiv s(\lambda z)$ for $\lambda > 0$, generate the same demand system, with $A_\lambda(\mathbf{p}) = \lambda A(\mathbf{p})$, because $s_\lambda(p_\omega/A_\lambda(\mathbf{p})) = s(\lambda p_\omega/A_\lambda(\mathbf{p})) = s(p_\omega/A(\mathbf{p}))$. In this sense, $s_\lambda(z) \equiv s(\lambda z)$ for $\lambda > 0$ are all equivalent.

CES with gross substitutes is a special case where $s(z) = \gamma z^{1-\sigma}$ $(\sigma > 1)$. Translog unit cost function is another special case, where $s(z) = -\gamma \ln(z/\bar{z})$, for $\bar{z} < \infty$. Generalized Translog is defined by $s(z) = \gamma\left(1 - \frac{\sigma-1}{\eta}\ln(z)\right)^{\eta} = \gamma\left(-\frac{\sigma-1}{\eta}\ln(z/\bar{z})\right)^{\eta}$, $(\eta > 0)$, for $z < \bar{z} \equiv e^{\frac{\eta}{\sigma-1}}$, which is equivalent to translog for $\eta = 1$ and contains CES as the limit case, as $\eta \to \infty$. The CoPaTh family[15] of H.S.A. is defined by

$$s(z) = \gamma\left[\sigma - (\sigma-1)z^{\frac{1-\rho}{\rho}}\right]^{\frac{\rho}{1-\rho}} = \gamma\sigma^{\frac{\rho}{1-\rho}}\left[1 - \left(\frac{z}{\bar{z}}\right)^{\frac{1-\rho}{\rho}}\right]^{\frac{\rho}{1-\rho}}$$

where $0 < \rho < 1$, for $z < \bar{z} = (1 - 1/\sigma)^{-\frac{\rho}{1-\rho}}$, which contains CES as the limit case, since $\bar{z} \to \infty$ and $s(z) \to \gamma z^{1-\sigma}$, as $\rho \nearrow 1$. Various families of H.S.A. have been recently applied to a variety of monopolistic competition models.[16]

Eqs. (12)-(13) state that the budget share of $\omega \in \Omega$ is decreasing in its *normalized price,* $z_\omega \equiv p_\omega/A(\mathbf{p})$, which is defined as its own price, p_ω, divided by the *common price aggregator,* $A(\mathbf{p})$. Note that the budget share function, $s(\cdot)$, is the primitive of H.S.A., while $A(\mathbf{p})$ is not, as it is derived from $s(\cdot)$, using eq.(13). The monotonicity of $s(\cdot)$, combined with the assumptions, $\lim_{z\to 0} s(z) = \infty$ and $\lim_{z\to\bar{z}} s(z) = 0$, ensures that $A(\mathbf{p})$ is defined uniquely by eq.(13) for any $V \equiv |\Omega| > 0$. $A(\mathbf{p})$ is independent of ω, and thus captures "the average price" against which the prices of *all* inputs in Ω are measured. In other words, one could keep track of all the cross-price effects in the demand system by looking at a single aggregator, $A(\mathbf{p})$, which is the key feature of H.S.A.[17] Note also that we allow for the possibility of $\bar{z} < \infty$, so that the budget share of ω is zero for $p_\omega \geq \bar{z}A(\mathbf{p})$. That is, $\bar{z}A(\mathbf{p})$ is the choke price. If $\bar{z} = \infty$, there is no choke price and the budget share for each input remains strictly positive for any positive price vector.

[15]CoPaTh stands for Constant Pass-Through; it is so named, since, when a monopolistic competitive firm faces the demand curve generated by this family, its pricing behavior features a constant pass-through rate, $0 < \rho < 1$, and it converges to CES, as $\rho \nearrow 1$. Matsuyama and Ushchev (2020b) developed the CoPaTh family of demand systems within H.S.A., HDIA, and HIIA.

[16]See, e.g., Baqaee, Farhi, and Sangani (2024), Fujiwara and Matsuyama (2022), Grossman, Helpman, and Luillier (2023), Matsuyama and Ushchev (2020a, 2020b, 2022a, 2022b), Ren and Zhang (2025). A large literature on monopolistic competition models under translog demand systems, which follows Feenstra (2003), may be also added to this list, because a symmetric translog unit cost function is a special case of symmetric H.S.A. with gross substitutes.

[17]In contrast, that $s(\cdot)$ is independent of ω is not a defining feature of H.S.A., but due to the assumption that the underlying production function is symmetric. Generally, the H.S.A. class of the production functions is defined by the property that the budget share of ω is given by $s_\omega(p_\omega/A(\mathbf{p}))$, where $A(\mathbf{p})$ is the unique solution to $\int_\Omega s_\omega(p_\omega/A(\mathbf{p}))d\omega = 1$. Note that $s_\omega(\cdot)$ may depend on ω but $A(\cdot)$ may not.

The price elasticity of $\omega \in \Omega$ can be written as a function of $z_\omega \equiv p_\omega/A(\mathbf{p}) < \bar{z}$ as

$$\zeta_\omega = \zeta(p_\omega; \mathbf{p}) = 1 - \frac{z_\omega s'(z_\omega)}{s(z_\omega)} \equiv \zeta^S(z_\omega) = \zeta^S\left(\frac{p_\omega}{A(\mathbf{p})}\right) > 1,$$

where $\zeta^S: (0, \bar{z}) \to (1, \infty)$ is C^1 for $z \in (0, \bar{z})$, and $\lim_{z \to \bar{z}} \zeta^S(z) = \infty$ if $\bar{z} < \infty$.[18] It turns out to be convenient to introduce another function, $H: (0, \bar{z}) \to \mathbb{R}_+$,

$$H(z) \equiv \int_z^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi > 0,$$

which satisfies $H'(z) < 0$ and $H''(z) > 0$, so that

$$\zeta^S(z) \equiv 1 - \frac{zs'(z)}{s(z)} \equiv -\frac{zH''(z)}{H'(z)} > 1. \tag{14}$$

In general, $\zeta^S(\cdot)$ can be nonmonotonic. Under CES, it is constant, $\zeta^{S\prime}(\cdot) = 0$. The 2nd law, $\partial\zeta(p_\omega; \mathbf{p})/\partial p_\omega > 0$, holds if and only if $\zeta^{S\prime}(\cdot) > 0$. The 2nd law holds for Generalized Translog with $\zeta^S(z_\omega) = 1 - \eta[\ln(z_\omega/\bar{z})]^{-1}$ and for CoPaTh with $\zeta^S(z_\omega) = \left[1 - (1 - 1/\sigma){z_\omega}^{(1-\rho)/\rho}\right]^{-1} = \left[1 - (z_\omega/\bar{z})^{(1-\rho)/\rho}\right]^{-1}$.

After deriving $A(\mathbf{p})$ from $s(\cdot)$, the unit cost function, $P(\mathbf{p})$, can be obtained by integrating eq.(12), which yields

$$\ln\left[\frac{A(\mathbf{p})}{cP(\mathbf{p})}\right] = \int_\Omega \left[\int_{p_\omega/A(\mathbf{p})}^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi\right] d\omega \equiv \int_\Omega H\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega \tag{15}$$

$$\equiv \int_\Omega s\left(\frac{p_\omega}{A(\mathbf{p})}\right) \Phi\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega$$

where

$$\Phi(z) \equiv \frac{1}{s(z)} \int_z^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi \equiv -\frac{H(z)}{zH'(z)} > 0$$

[18] Conversely, from any C^1 function $\zeta^S: (0, \bar{z}) \to (1 + \varepsilon, \infty)$, satisfying $\lim_{z \to \bar{z}} \zeta^S(z) = \infty$ if $\bar{z} < \infty$, one could reverse-engineer as $s(z) = \gamma \exp\left[\int_{z_0}^{z} [1 - \zeta^S(\xi)]\, d\xi/\xi\right] > 0;\ z_0, z \in (0, \bar{z})$, where $\gamma = s(z_0)$ is a positive constant. One could thus use $\zeta^S(\cdot)$ instead of $s(\cdot)$, as a primitive of symmetric H.S.A. with gross substitutes.

and c is a positive constant, which is proportional to TFP.[19] The unit cost function, $P(\mathbf{p})$, satisfies the linear homogeneity, monotonicity, and strict quasi-concavity, and so does the corresponding production function, $X(\mathbf{x})$. This follows from Matsuyama and Ushchev (2017; Proposition 1-i)) and guarantees the integrability (in the sense of Samuelson 1950 and Hurwicz and Uzawa 1971) of H.S.A. demand systems. It is important to note that, with the sole exception of CES, $A(\mathbf{p})/P(\mathbf{p})$ is not constant and depends on $\mathbf{p}$.[20] This can be verified by differentiating eq.(13) to yield

$$\frac{\partial \ln A(\mathbf{p})}{\partial \ln p_\omega} = \frac{z_\omega s'(z_\omega)}{\int_\Omega s'(z_{\omega'}) z_{\omega'} d\omega'} = \frac{[\zeta^S(z_\omega) - 1]s(z_\omega)}{\int_\Omega [\zeta^S(z_{\omega'}) - 1]s(z_{\omega'}) d\omega'},$$

which differs from

$$\frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s(z_\omega),$$

unless $\zeta^S(z)$ is independent of z, or equivalently, $s(z) = \gamma z^{1-\sigma}$ with $\zeta^S(z) = \sigma > 1$. This should not come as a surprise. After all, $A(\mathbf{p})$ is the "average input price", which captures the *cross-product effects* in the demand system, while $P(\mathbf{p})$ is the inverse measure of TFP, which captures the *productivity effects* of price changes. There is no reason to think *a priori* that they should move together.

We are now ready to derive the substitutability and love-for-variety measures under H.S.A. For symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$, eq.(13) is simplified to

$$s\left(\frac{1}{A(\mathbf{1}_\Omega^{-1})}\right)V = 1 \Rightarrow \frac{1}{A(\mathbf{1}_\Omega^{-1})} = s^{-1}\left(\frac{1}{V}\right)$$

Hence, from eq.(14), the substitutability measure is given by:

$$\mathcal{S}(V) \equiv \zeta(1; \mathbf{1}_\Omega^{-1}) = \zeta^S\left(s^{-1}\left(\frac{1}{V}\right)\right) = -\left.\frac{zH''(z)}{H'(z)}\right|_{z=s^{-1}(1/V)} > 1. \tag{16}$$

For the love-for-variety measure, from eq. (15),

[19]The constant term in eq.(15), which appears by integrating eq.(12), cannot be pinned down. First, $A(\mathbf{p})$, the "average input price", depends on the unit of measurement of inputs, but not on the unit of measurement of the final good. In contrast, $P(\mathbf{p})$ is the cost of producing one unit of the final good, when the input prices are given by $\mathbf{p}$. Hence, it depends not only on the unit of measurement of inputs but also on that of the final good. Second, a change in TFP, though it affects $P(\mathbf{p})$, leaves the budget share of each input, and hence $A(\mathbf{p})$, unaffected.

[20]This holds more generally, that is, for asymmetric H.S.A., as well as H.S.A. with gross complements, as shown in Matsuyama and Ushchev (2017; Proposition 1-iii).

$$-\ln[P(\mathbf{1}_\Omega^{-1})] = \ln\left[cs^{-1}\left(\frac{1}{V}\right)\right] + \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) \Rightarrow$$

$$-\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} = -\left[\frac{d[\ln z + \Phi(z)]}{d\ln z}\Big/\frac{d\ln s(z)}{d\ln z}\right] \text{eqs.}\Big|_{z=s^{-1}(1/V)} = \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right)$$

so that

$$\mathcal{L}(V) \equiv -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} = \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) = -\frac{H(z)}{zH'(z)}\bigg|_{z=s^{-1}(1/V)}. \qquad (17)^{21}$$

The expressions for $\mathcal{S}(V)$ and $\mathcal{L}(V)$ under H.S.A. in eq.(16) and eq. (17) are also displayed in Table 1.[22] Since $s^{-1}(1/V)$ is monotonically increasing in V,

$$\zeta^{S\prime}(\cdot) \gtreqless 0 \Leftrightarrow \mathcal{S}'(\cdot) \gtreqless 0; \qquad \Phi'(\cdot) \gtreqless 0 \Leftrightarrow \mathcal{L}'(\cdot) \gtreqless 0.$$

Proposition S-1 shows the relation between $\zeta^S(z)$ and $\Phi(z)$.

Proposition S-1:

$$\frac{z\Phi'(z)}{\Phi(z)} = \zeta^S(z) - 1 - \frac{1}{\Phi(z)} = \zeta^S(z) - \int_z^{\bar{z}} \zeta^S(\xi)w(\xi;z)d\xi.$$

where $w^S(\xi;z) \equiv -H'(\xi)/H(z)$, which satisfies $\int_z^{\bar{z}} w^S(\xi;z)d\xi = 1$. Hence,

$$\zeta^{S\prime}(z) \gtreqless 0, \forall z \in (z_0,\bar{z}) \Rightarrow \Phi'(z) \lesseqgtr 0, \forall z \in (z_0,\bar{z}).$$

The opposite is not true in general. However,

$$\zeta^{S\prime}(z) = 0, \forall z \in (z_0,\bar{z}) \Leftrightarrow \Phi'(z) = 0, \forall z \in (z_0,\bar{z}).$$

The proof of Proposition S-1 is in Appendix D.

By combining Proposition S-1, eq.(16) and eq.(17),

Proposition S-2. For $s(z_0)V_0 = 1$,

$$\zeta^{S\prime}(z) \gtreqless 0, \forall z \in (z_0,\bar{z}) \Leftrightarrow \mathcal{S}'(V) \gtreqless 0, \forall V \in (V_0,\infty);$$

$$\Phi'(z) \lesseqgtr 0, \forall z \in (z_0,\bar{z}) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0,\infty).$$

Moreover,

$$\mathcal{S}'(V) \gtreqless 0, \forall V \in (V_0,\infty) \Rightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0,\infty).$$

[21] Moreover, by evaluating eq.(15) at the symmetric price patterns, we can show $\mathcal{L}(V) = \ln[A(\mathbf{1}_\Omega^{-1})/cP(\mathbf{1}_\Omega^{-1})]$.

[22] Table 1 also displays $\mathcal{S}(V)$ and $\mathcal{L}(V)$ under the three families of H.S.A., Translog, Generalized Translog and CoPaTh. Deriving these expressions for Tranlog and Generalized Translog is straightforward. For CoPaTh, see Appendix F.

The opposite is not true in general. However,

$$\mathcal{S}'(V) = 0, \forall V \in (V_0, \infty) \Leftrightarrow \mathcal{L}'(V) = 0, \forall V \in (V_0, \infty).$$

Proposition S-3:

$$\mathcal{L}'(V) \lesseqgtr 0 \Leftrightarrow \frac{z\Phi'(z)}{\Phi(z)} = \zeta^S(z) - 1 - \frac{1}{\Phi(z)} \lesseqgtr 0 \Leftrightarrow \mathcal{L}(V) \lesseqgtr \frac{1}{\mathcal{S}(V) - 1}.$$

The H.S.A. portions of Theorems 2 and 3 follow from Proposition S-2 and S-3, respectively. For the H.S.A. portion of Theorem 4, note

$$\lim_{V\to\infty} \mathcal{L}(V) = \lim_{V\to\infty} \Phi\left(s^{-1}(1/V)\right) = \lim_{z\to\bar{z}} \Phi(z) = \lim_{z\to\bar{z}} \frac{1}{s(z)} \int_z^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi.$$

Hence, by applying L'Hopital's rule,

$$\lim_{V\to\infty} \mathcal{L}(V) = \lim_{z\to\bar{z}} \left(-\frac{s(z)}{zs'(z)}\right) = \lim_{z\to\bar{z}} \frac{1}{\zeta^S(z) - 1} = \lim_{V\to\infty} \frac{1}{\mathcal{S}(V) - 1}.$$

Before proceeding, it should be pointed out that there exists an alternative (but equivalent) definition of H.S.A. For the sake of completeness, we discuss this alternative in Appendix C.

4.2. The HDIA class

A homothetic symmetric demand system for inputs with gross substitutes belongs to HDIA (*Homothetic Direct Implicit Additivity*) if it is generated by the cost minimization of the competitive industry whose CRS production function, $X = X(\mathbf{x}) \equiv Z\hat{X}(\mathbf{x})$ can be defined implicitly by:

$$\int_\Omega \phi\left(\frac{Zx_\omega}{X(\mathbf{x})}\right) d\omega = \int_\Omega \phi\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right) d\omega \equiv 1. \tag{18}$$

here $\phi(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+$ is C^3, with $\phi'(y) > 0 > \phi''(y), -y\phi''(y)/\phi'(y) < 1$ for $\forall y \in (0, \infty)$ and $\phi(0) = 0$, $\phi(\infty) = \infty$, $\phi'(\infty) = 0$, and independent of $Z > 0$, the TFP parameter. Note that, unlike eq.(13), the adding-up constraint of the H.S.A, eq.(18) defines the production function $X(\mathbf{x})$ directly.[23] CES with gross substitutes is a special case where $\phi(y) = (y)^{1-1/\sigma}$ $(\sigma > 1)$. The CoPaTh family of HDIA is given by

[23]This means that, unlike H.S.A. but similar to HIIA defined in Section 4.3, we do not need to worry about the integrability of HDIA. Note also that $\hat{X}(\mathbf{x}) = X(\mathbf{x})/Z$ defined by eq.(18) is invariant of TFP, $Z > 0$, by construction. Thus, an increase in Z causes a proportionate increase in $X(\mathbf{x})$. This allows us to examine the effect of TFP without shifting $\phi(\cdot)$. Alternatively, we could have defined $X(\mathbf{x})$ by $\int_\Omega \phi(x_\omega/X(\mathbf{x}))d\omega = 1$, as in Matsuyama and Ushchev (2017). Though mathematically equivalent, this definition requires that $\phi(\cdot)$ would no longer be independent of TFP, which would make it harder to show that $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are independent of TFP.

$$\phi(y) = \int_0^{y} \left(1 + \frac{1}{\sigma - 1}(\xi)^{\frac{1-\rho}{\rho}}\right)^{\frac{\rho}{\rho-1}} d\xi,$$

for $0 < \rho < 1$, which converges to CES with $\rho \nearrow 1$. Symmetric HDIA defined as above may be viewed as an extension of the Kimball (1995) aggregator in that the set of available inputs Ω, and its Lebesgue measure, $V \equiv |\Omega|$, can vary.

From the cost minimization problem, eq.(1), subject to eq.(18), we obtain the inverse demand curve,

$$p_\omega = B(\mathbf{p})\phi'\left(\frac{Zx_\omega}{X(\mathbf{x})}\right) = B(\mathbf{p})\phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right), \tag{19}$$

and hence the demand curve,

$$x_\omega = (\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\hat{X}(\mathbf{x}) = (\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\frac{X(\mathbf{x})}{Z},$$

where $B(\mathbf{p})$ is defined by:

$$\int_\Omega \phi\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right) d\omega \equiv 1.$$

This shows that the choke price is equal to $B(\mathbf{p})\phi'(0)$ if $\phi'(0) < \infty$, and that there is no choke price if $\phi'(0) = \infty$. The unit cost function is:

$$P(\mathbf{p}) = \frac{\hat{P}(\mathbf{p})}{Z} \equiv \frac{1}{Z}\int_\Omega p_\omega (\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) d\omega.$$

Clearly, both $B(\mathbf{p})$ and $\hat{P}(\mathbf{p})$ are linear homogenous in $\mathbf{p}$, and independent of $Z > 0$. Hence, an increase in TFP, Z, causes a proportional decline in the unit cost function, $P(\mathbf{p}) = \hat{P}(\mathbf{p})/Z$.

The budget share, $s_\omega = s(p_\omega; \mathbf{p}) = s^*(x_\omega; \mathbf{x})$, is:

$$\frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{p_\omega}{\hat{P}(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) = \phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right)\frac{x_\omega}{C^*(\mathbf{x})}. \tag{20}$$

where

$$C^*(\mathbf{x}) \equiv \int_\Omega \phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right) x_\omega d\omega$$

is linear homogenous in $\mathbf{x}$ and independent of $Z > 0$, and satisfies the identity

$$\frac{\hat{P}(\mathbf{p})}{B(\mathbf{p})} = \int_\Omega \frac{p_\omega}{B(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) d\omega = \int_\Omega \phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right)\frac{x_\omega}{\hat{X}(\mathbf{x})} d\omega = \frac{C^*(\mathbf{x})}{\hat{X}(\mathbf{x})}. \tag{21}$$

Eqs.(20)-(21) show that the budget share under HDIA is a function of the two relative prices, $p_\omega/\hat{P}(\mathbf{p})$ and $p_\omega/B(\mathbf{p})$, or a function of the two relative quantities, $x_\omega/\hat{X}(\mathbf{x})$ and $x_\omega/C^*(\mathbf{x})$, unless $\hat{P}(\mathbf{p})/B(\mathbf{p}) = C^*(\mathbf{x})/\hat{X}(\mathbf{x})$ is a positive constant, which occurs if and only if it is CES. Thus, HDIA and H.S.A. do not overlap with the sole exception of CES.[24]

From the inverse demand curve, eq.(19), the price elasticity of demand can be written as a function of a single variable, $y_\omega \equiv x_\omega/\hat{X}(\mathbf{x})$ as:

$$\zeta_\omega = \zeta^*(x_\omega; \boldsymbol{x}) = -\frac{\phi'(y_\omega)}{y\phi''(y_\omega)} \equiv \zeta^D(y_\omega) = \zeta^D\left(\frac{x_\omega}{\hat{X}(\boldsymbol{x})}\right) > 1 \tag{22}$$

where $\zeta^D(y) > 1$ ensures gross substitutability. Using eq.(19), it can also be written as a function of $p_\omega/B(\mathbf{p}) = \phi'(y_\omega)$ as:

$$\zeta_\omega = \zeta(p_\omega; \mathbf{p}) = \zeta^D\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right) > 1.$$

Under CES, $\zeta^{D'}(\cdot) = 0$. The 2nd law, $\partial\zeta^*(x_\omega; \mathbf{x})/\partial x_\omega < 0$, holds if and only if $\zeta^{D'}(\cdot) < 0$, the condition satisfied by CoPaTh, with $\zeta^D(y) = 1 + (\sigma - 1)(y)^{\frac{\rho-1}{\rho}}$.

We are now ready to derive the substitutability and love-for-variety measures under HDIA. For symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$, eq.(18) is simplified to

$$\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right)V = 1 \implies \frac{1}{\hat{X}(\mathbf{1}_\Omega)} = \phi^{-1}\left(\frac{1}{V}\right).$$

Hence, from eq.(22), the substitutability measure is given by:

$$\mathcal{S}(V) \equiv \zeta^*(1; \mathbf{1}_\Omega) = \zeta^D\left(\phi^{-1}\left(\frac{1}{V}\right)\right) = -\left.\frac{\phi'(y)}{y\phi''(y)}\right|_{y=\phi^{-1}(1/V)} > 1. \tag{23}$$

The love-for-variety measure under HDIA is given by:

$$\mathcal{L}(V) \equiv \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 = \frac{1}{\mathcal{E}_\phi\left(\phi^{-1}(1/V)\right)} - 1 \equiv \left.\frac{\phi(y)}{y\phi'(y)}\right|_{y=\phi^{-1}(1/V)} - 1 > 0 \tag{24}$$

where

[24] This statement is a special case of Proposition 2-(ii) in Matsuyama and Ushchev (2017).

$$0 < \mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)} < 1.$$[25]

The expressions for $\mathcal{S}(V)$ and $\mathcal{L}(V)$ under HDIA in eq.(23) and eq.(24) are also displayed in Table 1. Since $\phi^{-1}(1/V)$ is monotonically decreasing in V,

$$\zeta^{D\prime}(\cdot) \lesseqqgtr 0 \Leftrightarrow \mathcal{S}'(\cdot) \gtreqqless 0; \quad \mathcal{E}_\phi'(\cdot) \gtreqqless 0 \Leftrightarrow \mathcal{L}'(\cdot) \gtreqqless 0,$$

Proposition D-1 shows the relation between $\zeta^D(y)$ and $\mathcal{E}_\phi(y)$.

Proposition D-1:

$$\frac{y\mathcal{E}_\phi'(y)}{\mathcal{E}_\phi(y)} = 1 - \frac{1}{\zeta^D(y)} - \mathcal{E}_\phi(y) = \int_0^{y} \left[\frac{1}{\zeta^D(\xi)}\right] w^D(\xi; y) d\xi - \frac{1}{\zeta^D(y)}.$$

where $w^D(\xi; y) \equiv \phi'(\xi)/\phi(y) > 0$, which satisfies $\int_0^{y} w^D(\xi; y)\, d\xi = 1$. Hence,

$$\zeta^{D\prime}(y) \lesseqqgtr 0, \forall y \in (0, y_0) \Rightarrow \mathcal{E}_\phi'(y) \lesseqqgtr 0, \forall y \in (0, y_0).$$

The opposite is not true in general. However,

$$\zeta^{D\prime}(y) = 0, \forall y \in (0, y_0) \Leftrightarrow \mathcal{E}_\phi'(y) = 0, \forall y \in (0, y_0).$$

The proof of Proposition D-1 is in Appendix D.

By combining Proposition D-1, eq.(23) and eq.(24),

Proposition D-2: For $\phi(y_0)V_0 = 1$,

$$\zeta^{D\prime}(y) \lesseqqgtr 0\ \forall y \in (0, y_0) \Leftrightarrow \mathcal{S}'(V) \gtreqqless 0, \forall V \in (V_0, \infty);$$

$$\mathcal{E}_\phi'(y) \lesseqqgtr 0, \forall y \in (0, y_0) \Leftrightarrow \mathcal{L}'(V) \lesseqqgtr 0, \forall V \in (V_0, \infty).$$

Moreover,

$$\mathcal{S}'(V) \gtreqqless 0, \forall V \in (V_0, \infty) \Rightarrow \mathcal{L}'(V) \lesseqqgtr 0, \forall V \in (V_0, \infty).$$

The opposite is not true in general. However,

$$\mathcal{S}'(V) = 0, \forall V \in (V_0, \infty) \Leftrightarrow \mathcal{L}'(V) = 0, \forall V \in (V_0, \infty).$$

Proposition D-3:

$$\mathcal{L}'(V) \lesseqqgtr 0 \Leftrightarrow \frac{y\mathcal{E}_\phi'(y)}{\mathcal{E}_\phi(y)} = 1 - \frac{1}{\zeta^D(y)} - \mathcal{E}_\phi(y) \lesseqqgtr 0 \Leftrightarrow \mathcal{L}(V) \lesseqqgtr \frac{1}{\mathcal{S}(V) - 1}.$$

The HDIA portions of Theorems 2 and 3 follow from Proposition D-2 and D-3, respectively. For the HDIA portion of Theorem 4, note

[25]Moreover, by evaluating eq.(21) at the symmetric price and quantity patterns, one can show that

$$\frac{\hat{P}(\mathbf{1}_\Omega^{-1})}{B(\mathbf{1}_\Omega^{-1})} = \frac{C^*(\mathbf{1}_\Omega)}{\hat{X}(\mathbf{1}_\Omega)} = \int_\Omega \mathcal{E}_\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right)\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right) d\omega = \mathcal{E}_\phi\left(\phi^{-1}\left(\frac{1}{V}\right)\right) \Rightarrow \mathcal{L}(V) = \frac{B(\mathbf{1}_\Omega^{-1})}{\hat{P}(\mathbf{1}_\Omega^{-1})} - 1 = \frac{\hat{X}(\mathbf{1}_\Omega)}{C^*(\mathbf{1}_\Omega)} - 1.$$

$$\lim_{V\to\infty}\mathcal{L}(V)=\lim_{V\to\infty}\frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/V))}-1=\lim_{y\to 0}\frac{\phi(y)}{y\phi'(y)}-1.$$

Hence, by applying L'Hopital's rule,

$$\lim_{V\to\infty}\mathcal{L}(V)=\lim_{y\to 0}\frac{\phi'(y)}{\phi'(y)+y\phi''(y)}-1=\lim_{y\to 0}\frac{\zeta^D(y)}{\zeta^D(y)-1}-1=\lim_{V\to\infty}\frac{1}{\mathcal{S}(V)-1}.$$

4.3. The HIIA class.

A homothetic symmetric demand system for inputs with gross substitutes belongs to HIIA (*Homothetic Indirect Implicit Additivity*) if it is generated by the cost minimization of the competitive industry whose unit cost function, $P=P(\mathbf{p})=\hat{P}(\mathbf{p})/Z$, can be defined implicitly by:

$$\int_\Omega \theta\left(\frac{p_\omega}{ZP(\mathbf{p})}\right)d\omega=\int_\Omega \theta\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)d\omega=1, \tag{25}$$

where $\theta(\cdot)$: $\mathbb{R}_{++}\to\mathbb{R}_{+}$ is C^3, with $\theta'(z)<\theta''(z)$, and $-z\theta''(z)/\theta'(z)>1$, for $\theta(z)>0$ with $\lim_{z\to 0}\theta(z)=\infty$ and $\lim_{z\to\bar{z}}\theta(z)=0$, where $\bar{z}\equiv$ $\inf\{z>0|\theta(z)=\theta'(z)=0\}$, and it is independent of $Z>0$, the TFP parameter. If $\bar{z}<\infty$, the choke price is equal to $\hat{P}(\mathbf{p})\bar{z}=ZP(\mathbf{p})\bar{z}$, and it satisfies $\lim_{z\to\bar{z}}[-z\theta''(z)/\theta'(z)]=\infty$. If $\bar{z}=\infty$, the choke price does not exist and demand for each input always remains positive for any positive price vector. Note that, unlike eq.(13), the adding-up constraint of the H.S.A, eq.(25) defines the unit cost function $P(\mathbf{p})$ directly.[26] CES with gross substitutes is a special case where $\theta(z)=(z)^{1-\sigma}$ $(\sigma>1)$. The CoPaTh family of HIIA is given by

$$\theta(z)=\sigma^{\frac{\rho}{1-\rho}}\int_{z/\bar{z}}^{1}\left((\xi)^{\frac{\rho-1}{\rho}}-1\right)^{\frac{\rho}{1-\rho}}d\xi$$

for $z<\bar{z}=(1-1/\sigma)^{-\frac{\rho}{1-\rho}}$; $0<\rho<1$, which converges to CES as $\rho\nearrow 1$.

[26]This means that, unlike H.S.A. but similar to HDIA defined in the previous section, we do not need to worry about the integrability of HIIA. Note also that $\hat{P}(\mathbf{p})=ZP(\mathbf{p})$ defined by eq.(25) is invariant of TFP, $Z>0$, by construction. Thus, an increase in Z causes a proportionate decline in $P(\mathbf{p})$. This allows us to examine the effect of TFP without shifting $\theta(\cdot)$. Alternatively, we could define $P(\mathbf{p})$ by $\int_\Omega \theta(p_\omega/P(\mathbf{p}))d\omega=1$, as in Matsuyama and Ushchev (2017). Though mathematically equivalent, this definition requires that $\theta(\cdot)$ would no longer be independent of TFP, which would make it harder to show that $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are independent of TFP.

From the minimization problem, eq.(3), subject to eq.(25), we obtain the demand curve,

$$x_\omega = -B^*(\mathbf{x})\theta'\left(\frac{p_\omega}{ZP(\mathbf{p})}\right) = -B^*(\mathbf{x})\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) > 0 \tag{26}$$

for $z < \bar{z}$, and hence the inverse demand curve,

$$p_\omega = \hat{P}(\mathbf{p})(-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right) = ZP(\mathbf{p})(-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right),$$

where $B^*(\mathbf{x}) > 0$ is defined by

$$\int_\Omega \theta\left((-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\right) d\omega \equiv 1.$$

Thus, the choke price is $\hat{P}(\mathbf{p})\bar{z} = ZP(\mathbf{p})\bar{z}$, if $\bar{z} < \infty$. The production function is

$$X(\mathbf{x}) = Z\hat{X}(\mathbf{x}) \equiv Z\int_\Omega (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right) x_\omega d\omega.$$

Clearly, both $B^*(\mathbf{x})$ and $\hat{X}(\mathbf{x})$ are linear homogeneous in **x** and independent of $Z > 0$, by construction. Thus, an increase in TFP, Z, causes a proportional increase in $X(\mathbf{x}) = Z\hat{X}(\mathbf{x})$.

The budget share, $s_\omega = s(p_\omega; \boldsymbol{p}) = s^*(x_\omega; \mathbf{x})$, is

$$\frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{p_\omega}{C(\mathbf{p})}\left[-\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)\right] = (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\frac{x_\omega}{\hat{X}(\mathbf{x})}, \tag{27}$$

where

$$C(\mathbf{p}) \equiv \int_\Omega p_\omega\left[-\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)\right] d\omega > 0$$

is linear homogenous in **p**, and independent of $Z > 0$ and satisfies the identity,

$$\frac{C(\mathbf{p})}{\hat{P}(\mathbf{p})} = \int_\Omega \frac{p_\omega}{\hat{P}(\mathbf{p})}\left[-\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)\right] d\omega = \int_\Omega (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\frac{x_\omega}{B^*(\mathbf{x})} d\omega = \frac{\hat{X}(\mathbf{x})}{B^*(\mathbf{x})}. \tag{28}$$

Eqs.(27)-(28) show that the budget share under HIIA is a function of the two relative prices, $p_\omega/\hat{P}(\mathbf{p})$ and $p_\omega/C(\mathbf{p})$, or a function of the two relative quantities, $x_\omega/\hat{X}(\mathbf{x})$ and $x_\omega/B^*(\mathbf{x})$, unless $C(\mathbf{p})/\hat{P}(\mathbf{p}) = \hat{X}(\mathbf{x})/B^*(\mathbf{x})$ is a positive constant, which occurs if and only if it is CES. Thus, HIIA and H.S.A. do not overlap with the sole exception of CES.[27] Furthermore, by comparing the expressions for the budget share under HDIA and the

[27]This statement is a special case of Proposition 3-(ii) in Matsuyama and Ushchev (2017).

budget share under HIIA, one could see that HDIA and HIIA do not overlap with the sole exception of CES.[28]

From the demand curve, eq.(26), the price elasticity of demand can be written as a function of a single variable, $z_\omega \equiv p_\omega/P(\mathbf{p})$, as:

$$\zeta_\omega = \zeta(p_\omega;\boldsymbol{p}) = -\frac{z_\omega \theta''(z_\omega)}{\theta'(z_\omega)} \equiv \zeta^I(z_\omega) = \zeta^I\left(\frac{p_\omega}{P(\boldsymbol{p})}\right) > 1, \tag{29}$$

where $\zeta^I(z) > 1$ ensures gross substitutability. Using eq.(26), it can also be written as a function of $x_\omega/B^*(\mathbf{x}) = -\theta'(z_\omega)$ as:

$$\zeta_\omega \equiv \zeta^*(x_\omega;\, \mathbf{x}) = \zeta^I\left((-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\right) > 1.$$

Under CES, $\zeta^{I'}(\cdot) = 0$. The 2nd law, $\partial\zeta(p_\omega;\mathbf{p})/\partial p_\omega > 0$, holds if and only if $\zeta^{I'}(\cdot) > 0$, the condition satisfied by CoPaTh with $\zeta^I(z_\omega) = \left[1-(1-1/\sigma)(z_\omega)^{(1-\rho)/\rho}\right]^{-1} = \left[1-(z_\omega/\bar{z})^{(1-\rho)/\rho}\right]^{-1}$.

We are now ready to derive the substitutability and love-for-variety measures under HIIA. For symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$, eq.(25) is simplified to

$$\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)V = 1 \Rightarrow \frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})} = \theta^{-1}\left(\frac{1}{V}\right).$$

Hence, from eq.(29), the substitutability measure under HIIA is given by:

$$\mathcal{S}(V) \equiv \zeta(1;\mathbf{1}_\Omega^{-1}) = \zeta^I\left(\theta^{-1}\left(\frac{1}{V}\right)\right) = -\frac{z\theta''(z)}{\theta'(z)}\bigg|_{z=\theta^{-1}(1/V)} > 1. \tag{30}$$

The love-for-variety measure under HIIA is given by:

$$\mathcal{L}(V) \equiv -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} = \frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/V)\right)} \equiv -\frac{\theta(z)}{z\theta'(z)}\bigg|_{z=\theta^{-1}(1/V)} > 0. \tag{31}$$

where

$$\mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} > 0.$$ [29]

---

[28]This statement is a special case of Proposition 4-(iii) in Matsuyama and Ushchev (2017).

[29]Moreover, by evaluating eq.(28) at the symmetric price and quantity patterns, one can show that

$$\frac{C(\mathbf{1}_\Omega^{-1})}{\hat{P}(\mathbf{1}_\Omega^{-1})} = \frac{\hat{X}(\mathbf{1}_\Omega)}{B^*(\mathbf{1}_\Omega)} = \int_\Omega \mathcal{E}_\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)d\omega = \mathcal{E}_\theta\left(\theta^{-1}\left(\frac{1}{V}\right)\right) \Rightarrow \mathcal{L}(V) = \frac{\hat{P}(\mathbf{1}_\Omega^{-1})}{C(\mathbf{1}_\Omega^{-1})} = \frac{B^*(\mathbf{1}_\Omega)}{\hat{X}(\mathbf{1}_\Omega)}.$$

The expressions for $\mathcal{S}(V)$ and $\mathcal{L}(V)$ under HIIA in eq.(30) and eq.(31) are also displayed in Table 1. Since $\theta^{-1}(1/V)$ is monotonically increasing in V,

$$\zeta^{I'}(\cdot) \gtreqless 0 \Leftrightarrow \mathcal{S}'(\cdot) \gtreqless 0; \quad \mathcal{E}_\theta'(\cdot) \lesseqgtr 0 \Leftrightarrow \mathcal{L}'(\cdot) \gtreqless 0.$$

Proposition I-1 shows the relation between $\zeta^I(z)$ and $\mathcal{E}_\theta(z)$.

Proposition I-1:

$$\frac{z\mathcal{E}_\theta'(z)}{\mathcal{E}_\theta(z)} = \mathcal{E}_\theta(z) + 1 - \zeta^I(z) = \int_z^{\overline{z}} \zeta^I(\xi) w^I(\xi; z) d\xi - \zeta^I(z).$$

where $w^I(\xi; z) \equiv -\theta'(\xi)/\theta(z) > 0$, which satisfies $\int_z^{\overline{z}} w^I(\xi; z)\, d\xi = 1$.

$$\zeta^{I'}(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \quad \Rightarrow \quad \mathcal{E}_\theta'(z) \gtreqless 0, \forall z \in (z_0, \overline{z}).$$

The opposite is not true in general. However,

$$\zeta^{I'}(z) = 0, \forall z \in (z_0, \overline{z}) \quad \Leftrightarrow \quad \mathcal{E}_\theta'(z) = 0, \forall z \in (z_0, \overline{z}).$$

The proof of Proposition I-1 is in Appendix D.

By combining Proposition I-1, eq.(30) and eq.(31),

Proposition I-2: For $\theta(z_0)V_0 = 1$,

$$\zeta^{I'}(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \Leftrightarrow \mathcal{S}'(V) \gtreqless 0, \forall V \in (V_0, \infty);$$

$$\mathcal{E}_\theta'(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

Moreover,

$$\mathcal{S}'(V) \gtreqless 0, \forall V \in (V_0, \infty) \Rightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

The opposite is not true in general. However,

$$\mathcal{S}'(V) = 0, \forall V \in (V_0, \infty) \Leftrightarrow \mathcal{L}'(V) = 0, \forall V \in (V_0, \infty).$$

Proposition I-3:

$$\mathcal{L}'(V) \lesseqgtr 0 \Leftrightarrow \frac{z\mathcal{E}_\theta'(z)}{\mathcal{E}_\theta(z)} = \mathcal{E}_\theta(z) + 1 - \zeta^I(z) \gtreqless 0 \Leftrightarrow \mathcal{L}(V) \lesseqgtr \frac{1}{\mathcal{S}(V) - 1}.$$

The HIIA portions of Theorems 2 and 3 follow from Proposition I-2 and I-3, respectively. For the HIIA portion of Theorem 4, note that $\overline{z}\theta'(\overline{z}) = 0$ holds.[30] Thus, by applying L'Hopital's Rule,

[30]For $\overline{z} < \infty$, this follows from $\theta'(\overline{z}) = 0$. For $\overline{z} = \infty$, suppose the contrary, so that there exists $z_0 > 0$ such that, for all $z > z_0$, $-z\theta'(z) > c > 0$. Then, $\theta(z_0) = -\lim_{x\to\infty} \int_{z_0}^x \theta'(\xi) d\xi = -\lim_{x\to\infty} \int_{z_0}^x \xi\theta'(\xi) d\xi/\xi > \lim_{x\to\infty} \int_{z_0}^x c d\xi/\xi = \infty$, a contradiction.

$$\lim_{V\to\infty}\mathcal{L}(V) = -\lim_{z\to\bar{z}}\frac{\theta(z)}{z\theta'(z)} = -\lim_{z\to\bar{z}}\frac{\theta'(z)}{\theta'(z)+z\theta''(z)} = \lim_{z\to\bar{z}}\frac{1}{\zeta^{I}(z)-1} = \lim_{V\to\infty}\frac{1}{\mathcal{S}(V)-1}.$$

5. An Application to An Armington Model of Competitive Trade

Up to now, we showed how love-for-variety, --the utility (productivity) gains from expanding variety of consumer goods (inputs)--, depends on the underlying demand structure, when we go beyond the ubiquitous CES assumption. All the results so far are entirely about the implications of expanding variety on the demand side. These results are thus useful in any model, regardless of how one might model the supply side of expanding variety. For example, the range of available variety may change due to pure discovery. Or it may be due to trade liberalization. Or it may change due to various forms of innovation activities, which may be pursued by the public sector or by the private sectors. And the market structure of the private sectors could be perfectly competitive, monopolistic, oligopolistic or monopolistically competitive. The results are completely independent what one might assume on the supply side. For this reason, we deliberately chose not to take any stand on how variety might change on the supply side up to this point.

Nevertheless, it might be illuminating to show how the results so far might manifest themselves in an equilibrium model. In this section, we apply the results to a simple Armington model of competitive trade, where different countries produce different sets of goods, so that trade liberalization leads to an expanding variety for consumers. (Some other potential applications will be discussed in Section 6.) It should be pointed out that the goal of this exercise is not to develop a full-blown Armington model of trade without the CES assumption, which is realistic enough to bring to data. Such an analysis would require writing several papers. Instead, our goal here is modest; it aims to illustrate the implications of the results obtained so far, by highlighting some conceptual issues that are hidden under CES in a simple setting. For this reason, the model below abstracts from many features, which play prominent roles in the existing literature on the Armington models, such as the trade cost and asymmetry of demand for goods produced in different countries. Instead, asymmetry across the countries in this model lies in the variety of goods they produced.

5.1 A Simple Armington Model of Trade

Consider an Armington model of trade between the two countries, Home (H) and Foreign (F), which differ in labor supply, L and L^*, and in the set of goods they produce, Ω & Ω^*; $\Omega \cap \Omega^* = \emptyset$, with their masses given by $V \equiv |\Omega|$ & $V^* \equiv |\Omega^*|$. Otherwise, the two countries are identical. The representative consumers in both countries have the same homothetic symmetric preferences. Let w and w^* denote the wage rate at H and F. We normalize such that one unit of labor in each country can produce one unit of each good of that country. Thus, the price of any good produced at H is $p_\omega = w$ for all $\omega \in \Omega$ and the price of any good produced at F is $p_\omega^* = w^*$for all $\omega \in \Omega^*$. Because all the products produced in the same country are sold at the same price, they are consumed by the same amount in each country under the demand system that is symmetric across all goods.

So, let D and M^* denote H's demand and F's demand for each H good, so that the total demand for each H good is $D + M^*,$ and H's resource constraint is

$$V(D + M^*) = L.$$

Likewise, let M and D^* denote H's demand and F's demand for each F good, so that the total demand for $\omega \in \Omega^*$ is $M + D^*$, F's resource constraint is:

$$V^*(M + D^*) = L^*.$$

H's export value = F's import value, is wVM^*, while H's import value = F's export value, w^*V^*M, so that

$$wVM^* = w^*V^*M$$

is the balanced trade condition.[31]

Since everyone has the identical homothetic demand and faces the same paces, the relative demand for the H goods to the F goods is the same in both countries and inversely related to their relative price, as

$$\frac{D + M^*}{M + D^*} = \frac{D}{M} = \frac{M^*}{D^*} = g\left(\frac{w}{w^*}; V; V^*\right).$$

Here, $g(\cdot;\ ; V; V^*) > 0$ is the demand for an H good relative to an F good, which satisfies $g(w/w^*; V; V^*) \lesseqgtr 1 \Leftrightarrow w/w^* \gtreqless 1$, as shown in Appendix E.1. Thus, from the two

[31] From the two resource constraints, and the balanced trade condition, we can obtain both H's budget constraint, $wVD + w^*V^*M = wL$, and F's budget constraint, $wVM^* + w^*V^*D^* = w^*L^*$.

resource constraints, the condition that the relative supply (RS) is equal to the relative demand (RD) is given by

$$\frac{L/V}{L^*/V^*} = RS = RD = \frac{D + M^*}{M + D^*} = \frac{D}{M} = \frac{M^*}{D^*} = g\left(\frac{w}{w^*}; V; V^*\right),$$

In what follows, we consider the case where the two countries differ proportionally, so that the relative country size is the same regardless of whether it is measured in labor supply (the population size) or in product variety:

$$\frac{L}{L^*} = \frac{V}{V^*} \Leftrightarrow \frac{L}{V} = \frac{L^*}{V^*}.$$

Then,

$$\frac{D}{M} = \frac{M^*}{D^*} = g\left(\frac{w}{w^*}; V; V^*\right) = \frac{L/V}{L^*/V^*} = 1 \Rightarrow \frac{w}{w^*} = 1.$$

Hence we have factor (and good) price equalization. This simplifies the budget constraints of the two countries and the balanced trade condition as:

$$VD + V^*M = L; \quad VM^* + V^*D^* = L^*; \quad V^*M = VM^*,$$

which further implies leads to:

$$\frac{L}{L^*} = \frac{V}{V^*} = \frac{M}{M^*} = \frac{D}{D^*} \Leftrightarrow \frac{V}{L} = \frac{V^*}{L^*}; \frac{D}{L} = \frac{D^*}{L^*} = \frac{M}{L} = \frac{M^*}{L^*},$$

and the domestic expenditure shares at Home and Foreign are:

$$\lambda = \frac{VD}{L} = \frac{V}{V + V^*}; \quad \lambda^* = \frac{V^*D^*}{L^*} = \frac{V^*}{V + V^*}.$$

The H's trade/GDP ratio is $1 - \lambda = \lambda^*$, while The F's trade/GDP ratio is $1 - \lambda^* = \lambda$. Thus, in per capita term, the two countries become identical and the welfare effect of trade for Home is the same with an increase in the available variety from V to $V + V^* = V/\lambda$ and the welfare effect of trade for Foreign is the same with an increase in the available variety from V^* to $V + V^* = V^*/\lambda^*$. Thus, the gains from trade, defined as the rate of increase in the welfare when the two countries move from autarky to free trade, can be measured by:

$$\ln(GT) \equiv \ln\left[\frac{P(\mathbf{1}_{\Omega}^{-1})}{P(\mathbf{1}_{\Omega\cup\Omega^*}^{-1})}\right] = \int_{V}^{V+V^*} \mathcal{L}(v)\frac{dv}{v} = \int_{V}^{V/\lambda} \mathcal{L}(v)\frac{dv}{v} > 0;$$

$$\ln(GT^*) \equiv \ln\left[\frac{P(\mathbf{1}_{\Omega^*}^{-1})}{P(\mathbf{1}_{\Omega\cup\Omega^*}^{-1})}\right] = \int_{V^*}^{V+V^*} \mathcal{L}(v)\frac{dv}{v} = \int_{V^*}^{V^*/\lambda^*} \mathcal{L}(v)\frac{dv}{v} > 0.$$

Table 2: Home Gains from Trade (The domestic expenditure share is $\lambda = V/(V+V^*)$).

		Home Gains from Trade
General Formula		$\ln(GT) = \int_V^{V/\lambda} \mathcal{L}(v)\frac{dv}{v}$
CES		$\ln(GT) = \frac{1}{\sigma-1}\ln\left(\frac{1}{\lambda}\right)$
GM-CES unit cost function.		$\ln(GT) = \mathbb{E}_G\left[\frac{1}{\sigma-1}\right]\ln\left(\frac{1}{\lambda}\right) \geq \frac{1}{\mathbb{E}_G[\sigma]-1}\ln\left(\frac{1}{\lambda}\right)$
GM-CES production function.		$\ln(GT) = \mathbb{E}_G\left[\frac{1}{\sigma-1}\right]\ln\left(\frac{1}{\lambda}\right) \geq \frac{1}{\left[\mathbb{E}_G[1/\sigma]\right]^{-1}-1}\ln\left(\frac{1}{\lambda}\right)$
H.S.A.		$GT = \frac{s^{-1}(\lambda/V)}{s^{-1}(1/V)}\frac{\exp\left[\Phi\left(s^{-1}(\lambda/V)\right)\right]}{\exp\left[\Phi\left(s^{-1}(1/V)\right)\right]}.$
Special Cases of H.S.A.	Translog	$\ln(GT) = \frac{1-\lambda}{2\gamma V}$
	Generalized Translog	$\ln(GT) = \frac{(\gamma V)^{-1/\eta}}{\sigma-1}\frac{\eta}{1+\eta}\frac{1-(\lambda)^{1/\eta}}{1/\eta}$
	Constant Path-Through	$\ln(GT) = -\sum_{n=0}^{\infty}\frac{\rho}{1+(1-\rho)n}\frac{1-(\lambda)^{\frac{1-\rho}{\rho}(n+1)}}{\left[\sigma(\gamma V)^{\frac{1-\rho}{\rho}}\right]^{(n+1)}} + \ln\left[1+\frac{1-(\lambda)^{\frac{1-\rho}{\rho}}}{\sigma(\gamma V)^{\frac{1-\rho}{\rho}}-1}\right]^{\frac{\rho}{1-\rho}}$
HDIA		$GT = \frac{(\lambda/V)}{\phi^{-1}(\lambda/V)}\frac{\phi^{-1}(1/V)}{(1/V)}$
HIIA		$GT = \frac{\theta^{-1}(\lambda/V)}{\theta^{-1}(1/V)}$

Table 2 displays this general formula for the Home gains from trade, along with the formulae under the five classes. Note that, in general, even after controlling for the country's domestic expenditure share, gains from trade for the country depends on its product variety.

Before proceeding, let us point out that there exists even a simpler Armington model that generates the same gains from trade formulae shown in Table 2. That is, there is no production nor labor, only the endowments. Home is endowed with Q units of each $\omega \in \Omega$, and Foreign is endowed with Q units of each $\omega \in \Omega^*$, where Q is exogenous. Trade between the two countries are merely exchanges of their endowments. The equilibrium of this endowment-exchange model is isomorphic with the above model with $Q = L/V = L^*/V^*$.

5.2 The Effects of Country Sizes on Gains from Trade: General Implications.

It is straightforward to verify the following so that the proof is omitted.

Theorem 5 (Gains from Trade):

5-i). Gains from trade are larger for the smaller country than for the larger country.

$$GT \gtreqless GT^* \Leftrightarrow V \lesseqgtr V^* \Leftrightarrow L \lesseqgtr L^* \Leftrightarrow \lambda \lesseqgtr \lambda^*$$

5-ii). When the two countries become proportionately larger, the gains from trade for both countries are smaller under diminishing love-for-variety, $\mathcal{L}'(\cdot) < 0$, because

$$\left.\frac{\partial \ln(GT)}{\partial \ln V}\right|_{\lambda=const.} = \mathcal{L}(V/\lambda) - \mathcal{L}(V) < 0;$$

$$\left.\frac{\partial \ln(GT^*)}{\partial \ln V^*}\right|_{\lambda^*=const.} = \mathcal{L}(V^*/\lambda^*) - \mathcal{L}(V^*) < 0.$$

5-iii) For any given V, Home's gains from trade are increasing in V^*, thus decreasing in $\lambda = V/(V + V^*)$, with

$$\left.\frac{\partial \ln(GT)}{\partial \ln V^*}\right|_{V=const.} = (1 - \lambda)\mathcal{L}(V/\lambda) > 0,$$

and its range is given by:

$$0 < \ln(GT) < \int_V^\infty \mathcal{L}(v)\frac{dv}{v}.$$

The upper bound is infinite if $\mathcal{L}(\infty) > 0$; it may be finite if $\mathcal{L}(\infty) = 0$. If finite, the upper bound is decreasing in V.

5-iv) For any given V^*, Home's gains from trade may be nonmonotone in V, hence in $\lambda = V/(V + V^*)$ in general. Under non-increasing love-for-variety, $\mathcal{L}'(\cdot) \leq 0$,

$$\left.\frac{\partial \ln(GT)}{\partial \ln V}\right|_{V^*=const.} = \lambda\mathcal{L}(V/\lambda) - \mathcal{L}(V) < 0,$$

so that Home's gains from trade is decreasing in V, and so in $\lambda = V/(V + V^*)$, with the range

$$0 < \ln(GT) < \int_0^{V^*} \mathcal{L}(v)\frac{dv}{v}.$$

The upper bound is finite if $\mathcal{L}(0) < \infty$; it may be infinite if $\mathcal{L}(0) = \infty$. If finite, the upper bound is increasing in V^*.

We now discuss the Home Gains from Trade under specific classes of demand systems. The formulae for gains from trade are also displayed in Table 2.

5.3. Gains from Trade under CES and GM-CES

Under CES, the substitutability and love-for-variety are both independent of V, whose constant values may be denoted as $\mathcal{S}^{CES} \equiv \sigma > 1$ and $\mathcal{L}^{CES} \equiv 1/(\sigma - 1) > 0$, respectively. By plugging the constant value of $\mathcal{L}^{CES}$ into the general GT formula,

$$\ln(GT) = \mathcal{L}^{CES} \ln\left(\frac{1}{\lambda}\right) = \frac{1}{\mathcal{S}^{CES} - 1} \ln\left(\frac{1}{\lambda}\right) = \frac{1}{\epsilon^{CES}} \ln\left(\frac{1}{\lambda}\right),$$

where $\epsilon^{CES} = \mathcal{S}^{CES} - 1$ is the trade elasticity under CES. This confirms the well-known ACR formula by Arkolaki, Costinot, and Rodríguez-Clare (2012).

As stated in Theorem 1, the substitutability and love-for-variety are both independent of V also under the two classes of GM-CES. By denoting their constant values by $\mathcal{S}^{GMCES}$ and $\mathcal{L}^{GMCES}$, gains from trade under GM-CES can be expressed as:

$$\ln(GT) = \mathcal{L}^{GMCES} \ln\left(\frac{1}{\lambda}\right) \geq \frac{1}{\mathcal{S}^{GMCES} - 1} \ln\left(\frac{1}{\lambda}\right) = \frac{1}{\epsilon^{GMCES}} \ln\left(\frac{1}{\lambda}\right)$$

where the inequality is strict whenever $G(\sigma)$ is non-degenerate. Moreover, Appendix E.1. shows that $\epsilon^{CES} = \mathcal{S}^{CES} - 1 > 0$ is the trade elasticity under GM-CES.

Thus, the ACR formula continues to hold under GM-CES in that

$$\ln(GT) = const. \times \ln\left(\frac{1}{\lambda}\right),$$

so that gains from trade is monotonically decreasing and goes to infinity, as $\lambda \to 0$, which occurs in this model, as $V/V^* = L/L^* \to 0$. Moreover, once the domestic expenditure share λ is controlled for, the absolute country sizes do not matter. Only the relative country size, which determines λ, matters; if the two countries are proportionately larger, to keep $V/V^* = L/L^*$ unchanged, gains from trade would not be affected. As should be clear from the derivation, the key feature of GM-CES that preserves the ACR formula is that $\mathcal{L}(V)$ is independent of V.

It is worth pointing out, however, that the ACR formula can be extended to GM-CES, only when "$const.$" in the above formula is interpreted as $\mathcal{L}^{GMCES}$. It would be incorrect to interpret "$const.$", as $1/(\mathcal{S}^{GMCES}-1)=1/\epsilon^{GMCES}$, which would give only the lower bound for the gains from trade as,

$$\ln(GT)\geq\frac{1}{\mathcal{S}^{GMCES}-1}\ln\left(\frac{1}{\lambda}\right)=\frac{1}{\epsilon^{GMCES}}\ln\left(\frac{1}{\lambda}\right)$$

As stated in Theorem 1, love-for-variety under GM-CES is no longer tightly linked to the substitutability, which only determines the lower bound. Indeed, holding $\mathcal{S}^{GMCES}$ fixed, $\mathcal{L}^{GMCES}$, and hence gains from trade, could be made arbitrarily large by changing the cdf, $G(\cdot)$. Moreover, $\epsilon^{GMCES}=\mathcal{S}^{GMCES}-1$ remains unchanged before and after trade liberalization. If one interprets this incorrectly as the evidence for CES and use the love-for-variety formula under CES, one underestimates the true gains from trade under GM-CES, potentially by a wide margin. [32]

5.4. Gains from Trade under H.S.A., HDIA, and HIIA.

We now turn to the three class of H.S.A., HDIA, and HIIA. As stated in Theorem 2 and illustrated in Figure 2, the love-for-variety measure under these three classes are no longer constant with the sole exception of CES. In particular, it is diminishing if the 2nd law holds globally. As a result, the ACR formula no longer holds.

The exact formulae for the Home gains from trade are derived in Appendices E.2.- E.4., also shown in Table 2. Needless to say, these formulae satisfy all the properties already shown to hold for general homothetic symmetric demand systems in Theorem 5. Moreover, we also show in Appendices E.2-E.4 that, under the three classes,

Theorem 6 (Gains from Trade): Under H.S.A., HDIA, and HIIA,

6-i) For any given V, Home's gains from trade are increasing in V^*, thus decreasing in $\lambda=V/(V+V^*)$, with the range,

$$0<\ln(GT)<\int_V^{\infty}\mathcal{L}(v)\frac{dv}{v},$$

whose upper bound is finite and decreasing in V, if and only if the choke price exists.

6-ii) For any given V^*, Home's gains from trade is decreasing in V, and so in $\lambda=V/(V+V^*)$ under non-increasing love-for-variety, $\mathcal{L}'(\cdot)\leq 0$, with the range

[32] Since GM-CES generates the demand systems that are special cases of mixed-CES demand systems used in Adao, Costinot and Donaldson (2017), this result also offers some insights for why they discovered that the gains from trade under mixed-CES dominates the gains from trade under CES.

$0 < \ln(GT) < \infty.$

Theorem 6-i) states that, under H.S.A., HDIA, and HIIA demand systems with the choke price, the Home's gains from trade with another country, no matter how big that country is, is limited. This is in stark contrast to the case of CES or GM-CES, under which Home's gains from trade with another country, becomes arbitrarily large, as its trade/GDP approaches to zero due to the large size of its trading partner. This suggests that the CES or GM-CES assumption may grossly overestimate the gains from trade with a large country. On the other hand, Theorem 6-ii) states that Home's gains from trade becomes arbitrarily large as its trade/GDP ratio approaches to zero, due to its small size.

6. Concluding Remarks

In this paper, we studied how love-for-variety is determined by the underlying demand structure outside of CES. Under general symmetric homothetic demand systems, both substitutability across different goods and love-for-variety are expressed as functions of the variety of available goods V only, as $\mathcal{S}(V)$ and $\mathcal{L}(V)$. Since the homotheticity alone imposes little restriction on the properties of these two functions, we turn to five classes of homothetic demand systems, which are pairwise disjoint with the sole exception of CES. Under the two classes of GM-CES, both $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are constant, and the standard CES formula for LV would underestimate the true LV under GM-CES. Under the three classes of H.S.A., HDIA, and HIIA, the 2nd law of demand is equivalent to increasing substitutability, both of which implies diminishing LV, which is the sufficient and necessary condition for the standard CES formula for LV would overestimate the true LV under these three classes. We also illustrated some implications of these results by applying them to a simple Armington model of trade and showed how biased our estimates of the gains from trade can be if we assume that the demand system is CES when it is not.

As pointed out earlier, the results in Theorems 1 through 4 are all about the properties of non-CES demand systems, hence independent of what one might assume on the supply side of the variety change. As such, these results are applicable to a wide range of models, not just to the Armington model of trade in this paper. One natural choice would be applications to monopolistic competition models with free entry, in which firms enter with their own products. Matsuyama and Ushchev (2020a) already explored some of the implications of H.S.A., HDIA, and HIIA in a Dixit-Stiglitz type monopolistic competition.[33] Another possible application is the Romer-type R&D based endogenous growth model with expanding variety, which predicts too little R&D in equilibrium. We conjecture that replacing CES by GM-CES makes equilibrium R&D move further below the optimal R&D, while replacing CES by H.S.A., HDIA, and HIIA with the 2nd law of demand could make R&D too much in equilibrium.

Moreover, because we have imposed homotheticity on demand systems, our results are readily applicable to multi-sector or multiple market segment settings. For example, one could embed any of the five classes we studied, i.e., two classes of GM-CES, H.S.A., HDIA, and HIIA, into an upper-tier demand system, which can be asymmetric and/or nonhomothetic. For this reason, we believe that potential applications are limitless. We are planning to explore some in our future work. At the same time, we hope the reader will find our results useful for their applications.

[33] However, Matsuyama and Ushchev (2020a) was written as the time when our knowledge of these three classes was more limited. Moreover, it does not consider GM-CES. So, we plan to revise that paper by greatly expanding it, by building on the results in this paper.

References

Acemoglu, D. (2008), Introduction to Modern Economic Growth, Princeton University Press.

Adao, R., A. Costinot, D. Donaldson, (2017), "Nonparametric Counterfactual Predictions in Neoclassical Models of International Trade" *American Economic Review*, 107 (3), 633-689.

Arkolakis, Costinot, and Rodríguez-Clare (2012), "New trade models, same old gains?" *American Economic Review*, 102 (1), 94-130.

Baqaee, D., E. Farhi, and K. Sangani (2024), "The Darwinian returns to scale," *Review of Economic Studies,* 91(3): 1373-1405.

Benassy, J. P. (1996). Taste for variety and optimum production patterns in monopolistic competition. *Economics Letters* 52: 41-47.

Bils, M., and P.J. Klenow (2001), "The Acceleration in variety growth," *American Economic Review* 91,2, 274-280.

Broda, C., and D.E. Weinstein (2006), "Globalization and the gains from variety," *Quarterly Journal of Economics*, 541-585.

Dixit, A. and JE. Stiglitz (1977), "Monopolistic competition and optimum product diversity," *American Economic Review* 67: 297-308.

Ethier, WJ. (1982): National and international returns to scale in the modern theory of international trade," *American Economic Review*, 72, 389-405.

Feenstra RC (1994), "New product varieties and the measurement of international prices," *American Economic Review*, 84, 157-177.

Feenstra, RC. (2003), "A Homothetic utility function for monopolistic competition models, without constant price elasticity," *Economics Letters* 78: 79-86.

Feenstra, RC., and D.E. Weinstein (2017), "Globalization, markups, and US welfare," *Journal of Political Economy,* 125(4), 1040-1074.

Fujita, M., P. Krugman, and A.J. Venables (1999), The Spatial Economy, MIT Press.

Fujiwara, I., and K. Matsuyama (2022), "Competition and the Phillips curve," Keio and Northwestern.

Gancia, G., and F. Zilibotti (2005), "Horizontal innovation in the theory of growth and development," Chapter 3 in P. Aghion and S. Durlauf, eds., Handbook of Economic Growth, Vol. 1A, Elsevier.

Grossman, GM and E. Helpman (1993) Innovation and growth in the global economy, MIT press.

Grossman, GM., E. Helpman, and H. Lhuillier (2023), "Supply chain resilience: should policy promote international diversification or reshoring," *Journal of Political Economy*, f131(12): 3462-3496.

Helpman, E., and P. Krugman (1985), Market Structure and Foreign Trade, MIT Press.

Hurwicz, L. and H. Uzawa (1971). "On the integrability of demand functions". In Chipman et al. Preferences, utility, and demand: A Minnesota symposium. New York: Harcourt, Brace, Jovanovich. Ch.6, 114–148.

Jehle, GA. and P. Reny (2011). Advanced Microeconomic Theory, 3rd edition. Pearson Education India.

Kimball, M. (1995), "The Quantitative analytics of the basic neomonetarist model," *Journal of Money, Credit and Banking* 27: 1241-77.

Krugman, P., (1980), "Scale economies, product differentiation, and the pattern of trade," *American Economic Review,* 950-959.
Mas-Colell, A., MD Whinston, JR Green (1995). Microeconomic Theory. New York: Oxford University Press.
Matsuyama, K. (1995), "Complementarities and cumulative processes in models of monopolistic competition," *Journal of Economic Literature* 33: 701-729.
Matsuyama, K., (2023), "Non-CES aggregators: A guided tour," *Annual Review of Economics*, vol.15, 235-265.
Matsuyama, K., (2025), "Homothetic Non-CES demand systems with applications to monopolistic competition," *Annual Review of Economics*, vol.17, forthcoming.
Matsuyama, K., and P. Ushchev (2017), "Beyond CES: three alternative classes of flexible homothetic demand systems," CEPR DP #12210.
Matsuyama, K., and P. Ushchev (2020a), "When does procompetitive entry imply excessive entry?" CEPR DP #14991.
Matsuyama, K., and P. Ushchev (2020b), "Constant Pass-Through," CEPR DP #15475.
Matsuyama, K., and P. Ushchev (2022a), "Destabilizing effects of market size in the dynamics of innovation," *Journal of Economic Theory*, 200, 105415.
Matsuyama, K., and P. Ushchev (2022b), "Selection and sorting of heterogeneous firms through competitive pressures," CEPR-DP17092.
Parenti, M., P. Ushchev, and J.-F. Thisse (2017), "Toward a theory of monopolistic competition," *Journal of Economic Theory,* 167, 86-115.
Ren K., and D.R. Zhang (2025), "Price markups or wage markdowns?" Northwestern University.
Romer, P. M. (1987), "Growth based on increasing returns due to specialization," *American Economic Review*, 77, 56-62.
Samuelson, PA., (1950), "The Problem of integrability in utility function," *Economica* 17 (68):355-385.
Thisse, J.-F., and P. Ushchev (2020), "Monopolistic competition without apology," in L.C.Corchon and M.A. Marini, eds., Handbook of Game Theory and Industrial Organization, Edward Elgar Publication.

Appendix A: Allen-Uzawa elasticity of substation at the symmetric patterns under general symmetric homothetic demand systems.

The Allen-Uzawa elasticity of substitution between two inputs, $\omega, \omega' \in \Omega$, are given by:

$$AES(p_\omega, p_{\omega'}, \mathbf{p}) = \frac{P(\mathbf{p})P_{\omega\omega'}(p_\omega, p_{\omega'}, \mathbf{p})}{x(p_\omega, \mathbf{p})x(p_{\omega'}, \mathbf{p})},$$

where $x(p_\omega, \mathbf{p})$ is the demand for ω per unit of output, while the functions $P_{\omega\omega'}(p_\omega, p_{\omega'}, \mathbf{p})$ are the ``second cross-derivatives'' of $P(\mathbf{p})$. The second-order Taylor approximation of $P(\mathbf{p})$ is

$$P(\mathbf{p} + \alpha\mathbf{h}) = P(\mathbf{p}) + \alpha \int_\Omega x(p_\omega, \mathbf{p})h_\omega d\omega + \frac{\alpha^2}{2}\int_\Omega \frac{\partial x(p_\omega, \mathbf{p})}{\partial p_\omega} h_\omega^2 d\omega$$

$$+ \frac{\alpha^2}{2}\int_\Omega \int_\Omega P_{\omega\omega'}(p_\omega, p_{\omega'}, \mathbf{p})h_\omega h_{\omega'}\, d\omega d\omega' + o(\alpha^2),$$

where $\mathbf{h}$ is a function over Ω, and α is a scalar. The linear homogeneity of $P(\mathbf{p})$ implies the following identity:

$$\int_\Omega \frac{\partial x(p_\omega, \mathbf{p})}{\partial p_\omega} p_\omega^2 d\omega + \int_\Omega \int_\Omega P_{\omega\omega'}(p_\omega, p_{\omega'}, \mathbf{p})p_\omega p_{\omega'}\, d\omega d\omega' = 0.$$

By setting $(p_\omega, \mathbf{p}) = (1, \mathbf{1}_\Omega^{-1})$ and $(p_\omega, p_{\omega'}, \mathbf{p}) = (1,1, \mathbf{1}_\Omega^{-1})$ in the identity, we obtain:

$$\left[\frac{\partial x(p_\omega, \mathbf{1}_\Omega^{-1})}{\partial p_\omega}\right]\Bigg|_{p_\omega=1} \underbrace{\left[\int_\Omega d\omega\right]}_{=V} + P_{\omega\omega'}(1,1, \mathbf{1}_\Omega^{-1}) \left[\underbrace{\int_\Omega \int_\Omega d\omega d\omega'}_{=V^2}\right] = 0.$$

Using the definition of $\mathcal{S}(V)$,

$$\mathcal{S}(V) \equiv \zeta(1; \mathbf{1}_\Omega^{-1}) = -\left[\frac{\partial \ln x(p_\omega, p\mathbf{1}_\Omega^{-1})}{\partial \ln p_\omega}\right]\Bigg|_{p_\omega=p}$$

$$\Rightarrow \quad \frac{\partial x(p_\omega, \mathbf{1}_\Omega^{-1})}{\partial p_\omega}\Bigg|_{p_\omega=1} = -\mathcal{S}(V)x(1, \mathbf{1}_\Omega^{-1}),$$

the above identity can be further rewritten as:

$$P_{\omega\omega'}(1,1, \mathbf{1}_\Omega^{-1}) = \frac{\mathcal{S}(V)}{V} x(1, \mathbf{1}_\Omega^{-1}).$$

Moreover, by setting $\mathbf{p} = P(\mathbf{1}_\Omega^{-1})$ in $P(\mathbf{p}) = \int_\Omega x(p_\omega, \mathbf{p})p_\omega d\omega$,

$$P(\mathbf{1}_\Omega^{-1}) = Vx(1, \mathbf{1}_\Omega^{-1}).$$

Thus, the Allen-Uzawa elasticity of substitution evaluated at a symmetric outcome:

$$AES_{\omega\omega'}(1,1,\mathbf{1}_\Omega^{-1}) = \frac{P(\mathbf{1}_\Omega^{-1})P_{\omega\omega'}(1,1,\mathbf{1}_\Omega^{-1})}{[x(1,\mathbf{1}_\Omega^{-1})]^2} = \frac{Vx(1,\mathbf{1}_\Omega^{-1})\frac{\mathcal{S}(V)}{V}x(1,\mathbf{1}_\Omega^{-1})}{[x(1,\mathbf{1}_\Omega^{-1})]^2} = \mathcal{S}(V).$$

Appendix B: Geometric Means of CES

This appendix characterizes the homothetic symmetric demand systems generated by the weighted geometric means of symmetric CES unit cost functions and of symmetric CES production functions, which we call the GM-CES unit cost function (Section 3.1) and the GM-CES production function (Section 3.2).

First, GM-CES unit cost function, $P(\mathbf{p})$, is defined by

$$\ln P(\mathbf{p}) \equiv \int_1^\infty \ln P(\mathbf{p};\sigma)\, dG(\sigma) \equiv \mathbb{E}_G[\ln P(\mathbf{p};\sigma)]$$

where $P(\mathbf{p};\sigma) \equiv \left[\int_\Omega p_\omega{}^{1-\sigma}\, d\omega\right]^{1/(1-\sigma)}$ and $G(\cdot)$ is the c.d.f. of $\sigma \in (1,\infty)$, with $\mathbb{E}_G[\cdot]$ denoting the expectation operator with respect to $G(\cdot)$. The demand for ω is

$$x_\omega = X(\mathbf{x})\frac{\partial P(\mathbf{p})}{\partial p_\omega} = P(\mathbf{p})X(\mathbf{x})\frac{\partial \ln P(\mathbf{p})}{\partial p_\omega} = P(\mathbf{p})X(\mathbf{x})\mathbb{E}_G\left[\frac{\partial \ln P(\mathbf{p};\sigma)}{\partial p_\omega}\right].$$

Thus, the budget share of ω is

$$s(p_\omega;\mathbf{p}) \equiv \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \mathbb{E}_G\left[\frac{\partial \ln P(\mathbf{p};\sigma)}{\partial \ln p_\omega}\right] = \mathbb{E}_G\left[\left(\frac{p_\omega}{P(\mathbf{p};\sigma)}\right)^{1-\sigma}\right]$$

The price elasticity of demand is thus

$$\zeta(p_\omega;\mathbf{p}) \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = 1 - \frac{\partial \ln s(p_\omega;\ \mathbf{p})}{\partial \ln p_\omega} = \frac{\mathbb{E}_G[\sigma p_\omega{}^{-\sigma}/[P(\mathbf{p};\sigma)]^{1-\sigma}]}{\mathbb{E}_G[p_\omega{}^{-\sigma}/[P(\mathbf{p};\sigma)]^{1-\sigma}]} > 1.$$

By evaluating this at the symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$,

$$\mathcal{S}(V) = \zeta(p;p\mathbf{1}_\Omega^{-1}) = \frac{\mathbb{E}_G[\sigma p^{-\sigma}/[P(p\mathbf{1}_\Omega^{-1};\sigma)]^{1-\sigma}]}{\mathbb{E}_G[p^{-\sigma}/[P(p\mathbf{1}_\Omega^{-1};\sigma)]^{1-\sigma}]} = \frac{\mathbb{E}_G[\sigma p^{-\sigma}/Vp^{1-\sigma}]}{\mathbb{E}_G[p^{-\sigma}/Vp^{1-\sigma}]} = \mathbb{E}_G[\sigma] > 1.$$

For the love-for-variety,

$$\mathcal{L}(V) \equiv -\frac{d \ln P(\mathbf{1}_\Omega^{-1})}{d \ln V} = \mathbb{E}_G\left[-\frac{d \ln P(\mathbf{1}_\Omega^{-1};\sigma)}{d \ln V}\right] = \mathbb{E}_G\left[\frac{1}{\sigma-1}\right].$$

The lower bound for $\mathcal{L}(V)$ is obtained from Jensen's inequality,

$$\mathcal{L}(V) = \mathbb{E}_G\left[\frac{1}{\sigma-1}\right] \geq \frac{1}{\mathbb{E}_G[\sigma]-1} = \frac{1}{\mathcal{S}(V)-1},$$

where the equality holds if and only if $G(\cdot)$ is degenerate. To see that $\mathcal{L}(V)$ is unbounded from above, fix σ_0 and ε, such that $0 < \varepsilon < \min\{1 - 1/\sigma_0, 1/\sigma_0\} < 1/2$, and let G_ε be a two-point distribution, assigning the mass equal to $\frac{\varepsilon(1-\varepsilon)}{1-2\varepsilon}\left(\sigma_0 - \frac{1}{1-\varepsilon}\right) > 0$ to $\sigma = \frac{1}{\varepsilon}$, and the mass equal to $\frac{\varepsilon(1-\varepsilon)}{1-2\varepsilon}\left(\frac{1}{\varepsilon} - \sigma_0\right) > 0$ to $\sigma = \frac{1}{1-\varepsilon}$. Then,

$$\mathcal{S}(V) = \mathbb{E}_{G_\varepsilon}(\sigma) = \frac{(1-\varepsilon)}{1-2\varepsilon}\left(\sigma_0 - \frac{1}{1-\varepsilon}\right) + \frac{\varepsilon}{1-2\varepsilon}\left(\frac{1}{\varepsilon} - \sigma_0\right) = \sigma_0,$$

$$\mathcal{L}(V) = \mathbb{E}_{G_\varepsilon}\left(\frac{1}{\sigma-1}\right) = \frac{\varepsilon}{1-\varepsilon}\frac{\varepsilon(1-\varepsilon)}{1-2\varepsilon}\left(\sigma_0 - \frac{1}{1-\varepsilon}\right) + \frac{1-\varepsilon}{\varepsilon}\frac{\varepsilon(1-\varepsilon)}{1-2\varepsilon}\left(\frac{1}{\varepsilon} - \sigma_0\right)$$

$$= \frac{\varepsilon^2}{1-2\varepsilon}\left(\sigma_0 - \frac{1}{1-\varepsilon}\right) + \frac{(1-\varepsilon)^2}{1-2\varepsilon}\left(\frac{1}{\varepsilon} - \sigma_0\right).$$

Clearly, $\mathcal{L}(V)$ is not bounded from above as $\varepsilon \searrow 0$. This completes the proof of Theorem 1 for the GM-CES unit cost function.

Next, GM-CES production function, $X(\mathbf{x})$, defined by:

$$\ln X(\mathbf{x}) \equiv \int_1^\infty \ln X(\mathbf{x};\sigma)\, dG(\sigma) \equiv \mathbb{E}_G[\ln X(\mathbf{x};\sigma)]$$

where $X(\mathbf{x};\sigma) \equiv \left[\int_\Omega {x_\omega}^{1-1/\sigma}\, d\omega\right]^{\sigma/(\sigma-1)}$. The inverse demand for ω is

$$p_\omega = P(\mathbf{p})\frac{\partial X(\mathbf{x})}{\partial x_\omega} = P(\mathbf{p})X(\mathbf{x})\frac{\partial \ln X(\mathbf{x})}{\partial x_\omega} = P(\mathbf{p})X(\mathbf{x})\mathbb{E}_G\left[\frac{\partial \ln X(\mathbf{x};\sigma)}{\partial x_\omega}\right].$$

Thus, the budget share of ω is

$$s^*(x_\omega;\mathbf{x}) \equiv \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \mathbb{E}_G\left[\frac{\partial \ln X(\mathbf{x};\sigma)}{\partial \ln x_\omega}\right] = \mathbb{E}_G\left[\left(\frac{x_\omega}{X(\mathbf{x};\sigma)}\right)^{1-1/\sigma}\right].$$

Thus, the price elasticity of demand, $\zeta^*(x_\omega;\mathbf{x})$, satisfies

$$\zeta^*(x_\omega;\mathbf{x}) \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \left[1 - \frac{\partial \ln s^*(x_\omega;\ \mathbf{x})}{\partial \ln x_\omega}\right]^{-1} = \frac{\mathbb{E}_G\left[{x_\omega}^{-1/\sigma}/[X(\mathbf{x};\sigma)]^{1-1/\sigma}\right]}{\mathbb{E}_G\left[{x_\omega}^{-1/\sigma}/\sigma[X(\mathbf{x};\sigma)]^{1-1/\sigma}\right]} > 1.$$

By evaluating this at the symmetric quantity patterns $\mathbf{x} = x\mathbf{1}_\Omega$,

$$\frac{1}{\mathcal{S}(V)} \equiv \frac{1}{\zeta^*(x;x\mathbf{1}_\Omega)} \equiv \frac{\mathbb{E}_G\left[x^{-\frac{1}{\sigma}}/\sigma[X(x\mathbf{1}_\Omega;\sigma)]^{1-\frac{1}{\sigma}}\right]}{\mathbb{E}_G\left[x^{-\frac{1}{\sigma}}/[X(x\mathbf{1}_\Omega;\sigma)]^{1-\frac{1}{\sigma}}\right]} = \frac{\mathbb{E}_G\left[x^{-\frac{1}{\sigma}}/\sigma V x^{1-\frac{1}{\sigma}}\right]}{\mathbb{E}_G\left[x^{-\frac{1}{\sigma}}/V x^{1-\frac{1}{\sigma}}\right]} = \mathbb{E}_G\left[\frac{1}{\sigma}\right] < 1.$$

For the love-for-variety,

$$\mathcal{L}(V) \equiv \frac{d \ln X(\mathbf{1}_\Omega)}{d \ln V} - 1 = \mathbb{E}_G\left[\frac{d \ln X(\mathbf{1}_\Omega;\sigma)}{d \ln V} - 1\right] = \mathbb{E}_G\left[\frac{1}{\sigma-1}\right].$$

The lower bound for $\mathcal{L}(V)$ is obtained from Jensen's inequality,

$$\mathcal{L}(V) = \mathbb{E}_G\left[\frac{1}{\sigma - 1}\right] = \mathbb{E}_G\left[\frac{1/\sigma}{1 - 1/\sigma}\right] \geq \frac{\mathbb{E}_G[1/\sigma]}{1 - \mathbb{E}_G[1/\sigma]} = \frac{1/\mathcal{S}(V)}{1 - 1/\mathcal{S}(V)} = \frac{1}{\mathcal{S}(V) - 1},$$

where the equality holds if and only if $G(\cdot)$ is degenerate.

To see that $\mathcal{L}(V)$ is unbounded from above, consider the Pareto distribution of σ:

$$G(\sigma) = 1 - \left(\frac{\sigma_{min}}{\sigma}\right)^{\alpha}, \qquad \sigma \geq \sigma_{min} \equiv \frac{\alpha\sigma_0}{\alpha + 1} > 1,$$

where $\sigma_0 > 1$ and $\alpha > 1/(\sigma_0 - 1)$. The distribution and density of $x = 1/\sigma$ are given by

$$F(x) = (\sigma_{min}x)^{\alpha};\ f(x) = \alpha(\sigma_{min})^{\alpha}x^{\alpha-1}, \qquad x \in \left(0, \frac{1}{\sigma_{min}}\right).$$

Thus,

$$\frac{1}{\mathcal{S}(V)} = \mathbb{E}_G\left(\frac{1}{\sigma}\right) = \mathbb{E}_F(x) = \alpha(\sigma_{min})^{\alpha}\int_0^{1/\sigma_{min}} x^{\alpha}\,dx = \frac{1}{\sigma_{min}}\frac{\alpha}{\alpha + 1} = \frac{1}{\sigma_0} < 1;$$

$$\mathcal{L}(V) = \mathbb{E}_G\left(\frac{1}{\sigma - 1}\right) = \mathbb{E}_F\left(\frac{x}{1 - x}\right) = \mathbb{E}_F\left(\sum_{k=1}^{\infty} x^k\right) = \sum_{k=1}^{\infty}\mathbb{E}_F(x^k)$$

$$= \sum_{k=1}^{\infty}\alpha(\sigma_{min})^{\alpha}\int_0^{1/\sigma_{min}} x^{\alpha+k-1}\,dx$$

$$= \sum_{k=1}^{\infty}\frac{\alpha}{\alpha + k}\left(\frac{1}{\sigma_{min}}\right)^k = \sum_{k=1}^{\infty}\frac{\alpha}{\alpha + k}\left(\frac{\alpha + 1}{\alpha}\right)^k\left(\frac{1}{\sigma_0}\right)^k = \sum_{k=1}^{\infty}\frac{(1 + 1/\alpha)^k}{1 + k/\alpha}\left(\frac{1}{\sigma_0}\right)^k.$$

Hence,

$$\lim_{\alpha\to 1/(\sigma_0-1)}\mathcal{L}(V) = \lim_{\alpha\to 1/(\sigma_0-1)}\sum_{k=1}^{\infty}\frac{(1 + 1/\alpha)^k}{1 + k/\alpha}\left(\frac{1}{\sigma_0}\right)^k = \sum_{k=1}^{\infty}\frac{1}{1 + (\sigma_0 - 1)k}$$

$$> \int_1^{\infty}\frac{dz}{1 + (\sigma_0 - 1)z} = \left.\frac{\ln[1 + (\sigma_0 - 1)z]}{(\sigma_0 - 1)}\right]_1^{\infty} = \infty.$$

This completes the proof of Theorem 1 for the GM-CES production function.

Appendix C. An Alternative (and Equivalent) specification of the HSA class.

There exists an alternative (but equivalent) definition of H.S.A.. That is, a homothetic symmetric demand system for inputs with gross substitutes belongs to H.S.A. (*Homothetic Single Aggregator*) if there exists a function of a single variable, $s^*: \mathbb{R}_{++} \to$

$\mathbb{R}_+$ which is C^2 with $0 < \mathcal{E}_{s^*}(y) \equiv ys^{*\prime}(y)/s^*(y) < 1, s^*(0) = 0$ and $s^*(\infty) = \infty$, such that the budget share of $\omega \in \Omega$ can be written as:

$$s_\omega = \frac{\partial \ln X(\boldsymbol{x})}{\partial \ln x_\omega} = s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right), \tag{32}$$

where $A^*(\mathbf{x})$ is defined implicitly by the adding-up constraint:

$$\int_\Omega s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega \equiv 1 \tag{33}$$

By construction, $A^*(\mathbf{x})$ is linear homogenous in $\mathbf{x}$ for any fixed Ω and the budget shares of all inputs are added to up to one.

Note that the budget share of each input is increasing in its *normalized quantity*, $y_\omega \equiv x_\omega/A^*(\mathbf{x})$, which is defined as its own quantity x_ω divided by the *common quantity aggregator* $A^*(\mathbf{x})$.

The price elasticity of $\omega \in \Omega$ can be written as a function of $y_\omega \equiv x_\omega/A^*(\mathbf{x})$ as:

$$\zeta_\omega = \zeta^*(x_\omega; \mathbf{x}) = \left[1 - \frac{y_\omega s^{*\prime}(y_\omega)}{s^*(y_\omega)}\right]^{-1} \equiv \zeta^{S*}(y_\omega) = \zeta^{S*}\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) > 1,$$

where $\zeta^{S*}: (0,\infty) \to (1,\infty)$ is C^1. Note that $0 < \mathcal{E}_{s^*}(y) \equiv ys^{*\prime}(y)/s^*(y) < 1$ ensures $\zeta^{S*}(y_\omega) > 1$, that is, *gross substitutability.*[34]

It turns out to be convenient to introduce another function, $H^*: \mathbb{R}_{++} \to \mathbb{R}_+$,

$$H^*(y) \equiv \int_0^y \frac{s^*(\xi^*)}{\xi^*} d\xi^*,$$

which satisfies $H^{*\prime}(y) > 0$ and $H^{*\prime\prime}(y) < 0$ and

$$\zeta^{S*}(y) \equiv \left[1 - \frac{ys^{*\prime}(y)}{s^*(y)}\right]^{-1} \equiv -\frac{H^{*\prime}(y)}{yH^{*\prime\prime}(y)} > 1. \tag{34}$$

In general, $\zeta^{S*}(\cdot)$ can be nonmonotonic. Under CES, given by $s^*(y) = \gamma^{1/\sigma}(y)^{1-1/\sigma}$, it is constant, $\zeta^{S*\prime}(y) = 0$. The 2nd law, $\partial\zeta(x_\omega; \mathbf{x})/\partial x_\omega < 0$, holds if and only if $\zeta^{S*\prime}(\cdot) < 0$. The choke price exists if and only if $\lim_{y\to 0} s^{*\prime}(y) = s^{*\prime}(0) < \infty$, which implies $\lim_{y\to 0} ys^{*\prime}(y)/s^*(y) = 1$ and hence $\lim_{y\to 0} \zeta^{S*}(y) = \infty$. Translog corresponds to $s^*(y)$,

[34] Conversely, from any continuously differentiable $\zeta^*: (0,\infty) \to (1,\infty)$, one could reverse-engineer as $s^*(y) = \gamma^* \exp\left[\int_{y_0}^{y}\left[1 - \frac{1}{\zeta^*(\xi^*)}\right]\frac{d\xi^*}{\xi^*}\right] > 0$, where $\gamma^* = s^*(y_0)$ is a positive constant. Thus, we could also use $\zeta^*(\cdot)$ instead of $s^*(\cdot)$ as a primitive of symmetric H.S.A. with gross substitutes.

defined implicitly by $s^* \exp(s^*/\gamma) \equiv \bar{z}y$, for $\bar{z} < \infty$. CoPaTh corresponds to $s^*(y) = \left[\frac{1}{\sigma} + \left(1 - \frac{1}{\sigma}\right) y^{-\frac{1-\rho}{\rho}}\right]^{-\frac{\rho}{1-\rho}} = \left[1 - \bar{z}^{-\frac{1-\rho}{\rho}} + (y\bar{z})^{-\frac{1-\rho}{\rho}}\right]^{-\frac{\rho}{1-\rho}}$ with $\bar{z} = s^{*\prime}(0) = \left(1 - \frac{1}{\sigma}\right)^{-\frac{\rho}{1-\rho}}$.

After deriving $A^*(\mathbf{x})$ from $s^*(\cdot)$, the production function, $X(\mathbf{x})$, can be obtained by integrating eq.(32), which yields

$$\ln\left[\frac{X(\mathbf{x})}{c^* A^*(\mathbf{x})}\right] = \int_{\Omega} \left[\int_0^{\frac{x_\omega}{A^*(\mathbf{x})}} \frac{s^*(\xi^*)}{\xi^*} d\xi^*\right] d\omega \equiv \int_{\Omega} H^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega \tag{35}$$

$$\equiv \int_{\Omega} s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) \Phi^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega$$

where c^* is a positive constant, which is proportional to TFP and

$$\Phi^*(y) \equiv \frac{1}{s^*(y)} \int_0^y \frac{s^*(\xi^*)}{\xi^*} d\xi^* \equiv \frac{H^*(y)}{yH^{*\prime}(y)} > 1,$$

where the inequality follows from $\mathcal{E}_{s^*}(y) \equiv ys^{*\prime}(y)/s^*(y) < 1$, which implies that $s^*(y)/y$ is decreasing in y, and hence $H^*(y)$ is concave.

Note that $X(\mathbf{x})/A^*(\mathbf{x})$ is constant, if and only if it is CES. To see this, differentiating eq.(33) yields,

$$\frac{\partial \ln A^*(\mathbf{x})}{\partial \ln x_\omega} = \frac{y_\omega s^{*\prime}(y_\omega)}{\int_\Omega s^{*\prime}(y_{\omega'}) y_{\omega'} d\omega'} = \frac{\left[1 - \frac{1}{\zeta^*(y_\omega)}\right] s^*(y_\omega)}{\int_\Omega \left[1 - \frac{1}{\zeta^*(y_{\omega'})}\right] s^*(y_{\omega'}) d\omega'},$$

which differs from

$$\frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} = s^*(y_\omega),$$

unless $\zeta^*(y_\omega)$ is constant.

For symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$, eq.(33) is simplified to

$$s^*\left(\frac{1}{A^*(\mathbf{1}_\Omega)}\right) V = 1 \Rightarrow \frac{1}{A^*(\mathbf{1}_\Omega)} = s^{*-1}\left(\frac{1}{V}\right).$$

Hence, from eq.(34), the substitutability measure is given by:

$$\mathcal{S}(V) \equiv \zeta^*(1; \mathbf{1}_\Omega) = \zeta^{S^*}\left(s^{*-1}\left(\frac{1}{V}\right)\right) = -\frac{H^{*\prime}(y)}{yH^{*\prime\prime}(y)}\bigg|_{y=s^{*-1}(1/V)} > 1 \tag{36}$$

For the love-for-variety measure, from eq.(35),

$$\ln X(\mathbf{1}_\Omega) = \ln c^* + \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right) - \ln s^{*-1}\left(\frac{1}{V}\right) \Rightarrow$$

$$\frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} = \left[\frac{d[\ln y - \Phi^*(y)]}{d\ln y} \Big/ \frac{d\ln s^*(y)}{d\ln y}\right]_{y=s^{*-1}(1/V)} = \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right)$$

so that

$$\mathcal{L}(V) \equiv \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 = \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right) - 1 = \left.\frac{H^*(y)}{yH^{*\prime}(y)}\right|_{y=s^{*-1}(1/V)} - 1 \qquad (37)^{35}$$

Since $s^{*-1}(1/V)$ is monotonically decreasing in V, eqs.(36)-(37) imply

$$\zeta^{S*\prime}(\cdot) \lesseqgtr 0 \Leftrightarrow \mathcal{S}'(\cdot) \gtreqless 0;\ \Phi^{*\prime}(\cdot) \gtreqless 0 \Leftrightarrow \mathcal{L}'(\cdot) \lesseqgtr 0,$$

Proposition S*-1 shows the relation between $\zeta^{S*}(y)$ and $\Phi^*(y)$.

Proposition S*-1

$$\frac{y\Phi^{*\prime}(y)}{\Phi^*(y)} = \frac{1}{\Phi^*(y)} - 1 + \frac{1}{\zeta^{S*}(y)} = \frac{1}{\zeta^{S*}(y)} - \int_0^y \left[\frac{1}{\zeta^{S*}(\xi^*)}\right] w^*(\xi^*; y)d\xi^*.$$

where $w^{S*}(\xi^*; y) \equiv H^{*\prime}(\xi^*)/H^*(y) > 0$, which satisfies $\int_0^y w^{S*}(\xi^*; y)d\xi^* = 1$. Hence,

$$\zeta^{S*\prime}(y) \lesseqgtr 0, \forall y \in (0, y_0) \Rightarrow \Phi^{*\prime}(y) \gtreqless 0, \forall y \in (0, y_0).$$

The opposite is not true in general. However,

$$\zeta^{S*\prime}(y) = 0, \forall y \in (0, y_0) \Leftrightarrow \Phi^{*\prime}(y) = 0, \forall y \in (0, y_0).$$

The proof of Proof of S*-1 is in Appendix D.

By combining Proposition S*-1, eq.(36) and eq.(37),

Proposition S*-2: For $s^*(y_0)V_0 = 1$,

$$\zeta^{S*\prime}(y) \lesseqgtr 0, \forall y \in (0, y_0) \Leftrightarrow \mathcal{S}'(V) \gtreqless 0, \forall V \in (V_0, \infty);$$

$$\Phi^{*\prime}(y) \gtreqless 0, \forall y \in (0, y_0) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

Moreover,

$$\mathcal{S}'(V) \gtreqless 0, \forall V \in (V_0, \infty) \Rightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

The opposite is not true in general. However,

$$\mathcal{S}'(V) = 0, \forall V \in (V_0, \infty) \Leftrightarrow \mathcal{L}'(V) = 0, \forall V \in (V_0, \infty).$$

[35] Moreover, by evaluating eq.(35) at the symmetric quantity patterns, $\mathcal{L}(V) = \ln[X(\mathbf{1}_\Omega)/c^*A^*(\mathbf{1}_\Omega)] - 1$.

Proposition S*-3:

$$\mathcal{L}'(V) \lesseqgtr 0 \Leftrightarrow \frac{y\Phi^{*\prime}(y)}{\Phi^*(y)} = \frac{1}{\Phi^*(y)} - 1 + \frac{1}{\zeta^{S*}(y)} \gtreqless 0 \Leftrightarrow \mathcal{L}(V) \lesseqgtr \frac{1}{\mathcal{S}(V) - 1}$$

As shown below, the two definitions of H.S.A. are isomorphic. Therefore, the H.S.A. portions of Theorems 2 and 3 also follow from Proposition S*-2 and S*-3, respectively. For the H.S.A. portion of Theorem 4, note

$$\lim_{V\to\infty} \mathcal{L}(V) = \lim_{V\to\infty} \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right) - 1 = \lim_{y\to 0} \frac{1}{s^*(y)} \int_0^y \frac{s^*(\xi^*)}{\xi^*} d\xi^* - 1.$$

Hence, by applying L'Hopital's rule,

$$\lim_{V\to\infty} \mathcal{L}(V) = \lim_{y\to 0} \frac{s^*(y)}{ys^{*\prime}(y)} - 1 = \lim_{y\to 0} \frac{\zeta^*(y)}{\zeta^*(y) - 1} - 1 = \lim_{V\to\infty} \frac{1}{\mathcal{S}(V) - 1}.$$

Indeed, these two definitions of H.S.A. are equivalent.[36] The isomorphism between the two is given by the one-to-one mapping between $s(z) \leftrightarrow s^*(y)$, defined by:

$$s^*(y) = s\left(\frac{s^*(y)}{y}\right); \qquad s(z) = s^*\left(\frac{s(z)}{z}\right).$$

Differentiating either of these two equalities yields the identity,

$$\zeta^{S*}(y) \equiv \left[1 - \frac{d\ln s^*(y)}{d\ln y}\right]^{-1} = \zeta^S(z) \equiv 1 - \frac{d\ln s(z)}{d\ln z} > 1,$$

which shows that $0 < \mathcal{E}_{s^*}(y) \equiv \frac{d\ln s^*(y)}{d\ln y} < 1$ is equivalent to $\mathcal{E}_s(z) \equiv \frac{d\ln s(z)}{d\ln z} < 0$.

Furthermore, the normalized quantity, $y_\omega \equiv x_\omega / A^*(\mathbf{x})$, and the normalized price, $z_\omega \equiv p_\omega / A(\mathbf{p})$, are negatively related as

$$z_\omega = \frac{s^*(y_\omega)}{y_\omega} \Leftrightarrow y_\omega = \frac{s(z_\omega)}{z_\omega},$$

$$\frac{dy_\omega}{y_\omega} = -\zeta(z_\omega)\frac{dz_\omega}{z_\omega} \Leftrightarrow \frac{dz_\omega}{z_\omega} = -\frac{1}{\zeta^*(y_\omega)}\frac{dy_\omega}{y_\omega}$$

and

$$\frac{z_\omega \zeta^{S\prime}(z_\omega)}{y_\omega \zeta^{S*\prime}(y_\omega)} = -\zeta^S(z_\omega) = -\zeta^{S*}(y_\omega) < 0.$$

In addition, if $\lim_{y\to 0} s^{*\prime}(y) < \infty$, then $\lim_{y\to 0} \zeta^{S*}(y) = \infty$ and

[36] This isomorphism has been shown for the broader class of H.S.A., which allows for asymmetry as well as gross complements; see Matsuyama and Ushchev (2017, sec. 3, Remark 3).

$$\lim_{y \to 0} \frac{s^*(y)}{y} = \lim_{y \to 0} s^{*\prime}(y) = \bar{z} \equiv \inf\{z > 0 | s(z) = 0\} < \infty$$

is the (normalized) choke price.

Moreover,

$$\frac{p_\omega x_\omega}{A(\mathbf{p})A^*(\mathbf{x})} = y_\omega z_\omega = s(z_\omega) = s^*(y_\omega) = \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})}$$

hence we have the identity,

$$c \exp\left[\int_\Omega s(z_\omega)\Phi(z_\omega) d\omega\right] = c \exp\left[\int_\Omega H(z_\omega) d\omega\right] = \frac{A(\mathbf{p})}{P(\mathbf{p})} = \frac{X(\mathbf{x})}{A^*(\mathbf{x})}$$

$$= c^* \exp\left[\int_\Omega s^*(y_\omega)\Phi^*(y_\omega)\, d\omega\right] = c^* \exp\left[\int_\Omega H^*(y_\omega)\, d\omega\right]$$

which is a positive constant if and only if it is CES. Furthermore, using

$$s(\xi) = s^*(\xi^*) = \xi\xi^* \to \frac{d\xi^*}{\xi^*} = \left[\frac{\xi s'(\xi)}{s(\xi)} - 1\right]\frac{d\xi}{\xi} \to s^*(\xi^*)\frac{d\xi^*}{\xi^*} = \left[s'(\xi) - \frac{s(\xi)}{\xi}\right] d\xi$$

$$\xi = z \leftrightarrow \xi^* = y;\ \xi = \overline{z} \leftrightarrow \xi^* = 0,$$

we have

$$\Phi^*(y) - \Phi(z) \equiv \frac{1}{s^*(y)} \int_0^y \frac{s^*(\xi^*)}{\xi^*} d\xi^* - \frac{1}{s(z)} \int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi$$

$$= \frac{1}{s(z)} \int_{\bar{z}}^{z} \left[s'(\xi) - \frac{s(\xi)}{\xi}\right] d\xi - \frac{1}{s(z)} \int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi = 1.$$

Since $w^S(\xi; z) = \frac{s(\xi)/\xi}{\int_z^{\bar{z}} [s(\xi')/\xi'] d\xi'} \Leftrightarrow s(z)\Phi(z)w^S(\xi; z) = \frac{s(\xi)}{\xi}$ and $w^{S*}(\xi^*; y) = \frac{s^*(\xi^*)/\xi^*}{\int_0^y \left[s^*\left(\xi^{*\prime}\right)/\xi^{*\prime}\right] d\xi^{*\prime}} \Leftrightarrow s^*(y)\Phi^*(y)w^{S*}(\xi^*; y) = \frac{s^*(\xi^*)}{\xi^*}$, this implies

$$\frac{\xi w^S(\xi; z)}{\xi^* w^{S*}(\xi^*; y)} = \frac{\Phi^*(y)}{\Phi(z)} = 1 + \frac{1}{\Phi(z)} = \frac{\Phi^*(y)}{\Phi^*(y) - 1},$$

$$\ln\left(\frac{c}{c^*}\right) = \int_\Omega [s^*(y_\omega)\Phi^*(y_\omega) - s(z_\omega)\Phi(z_\omega)] d\omega = \int_\Omega [H^*(y_\omega) - H(z_\omega)] d\omega$$

$$= \int_\Omega s(z_\omega) d\omega = 1.$$

and

$$\mathcal{L}(V) = \Phi\big(s^{-1}(1/V)\big) = \Phi^*\big(s^{*-1}(1/V)\big) - 1.$$

Appendix D. Proofs of Propositions S-1, D-1, I-1, and S*-1.

Proof of Proposition S-1: Let $w^S(\xi; z) \equiv -\frac{H'(\xi)}{H(z)} > 0$, satisfying $\int_z^{\overline{z}} w^S(\xi; z)\, d\xi = 1$, because $H(\overline{z}) = 0$. Then, because $\overline{z}H'(\overline{z}) = -s(\overline{z}) = 0$,

$$\int_z^{\overline{z}} [\zeta^S(\xi) - 1] w^S(\xi; z) d\xi = \frac{\int_z^{\overline{z}} [\xi H''(\xi) + H'(\xi)] d\xi}{H(z)} = \frac{\int_z^{\overline{z}} d[\xi H'(\xi)]}{H(z)} = -\frac{zH'(z)}{H(z)}$$

$$= \frac{1}{\Phi(z)}.$$

Thus,

$$\frac{z\Phi'(z)}{\Phi(z)} = \frac{zH'(z)}{H(z)} - 1 - \frac{zH''(z)}{H'(z)} = \zeta^S(z) - 1 - \frac{1}{\Phi(z)} = \zeta^S(z) - \int_z^{\overline{z}} \zeta^S(\xi) w^S(\xi; z) d\xi,$$

from which

$$\zeta^{S\prime}(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \implies \Phi'(z) \lesseqgtr 0, \forall z \in (z_0, \overline{z}).$$

Furthermore, $\Phi'(z) = 0$ for $z \in (z_0, \overline{z})$ implies $\zeta^S(z) = 1 + 1/\Phi(z)$, which is hence constant and thus $\zeta^{S\prime}(z) = 0$ for $z \in (z_0, \overline{z})$. This completes the proof. ■

Proof of Proposition D-1. Let $w^D(\xi; y) \equiv \phi'(\xi)/\phi(y) > 0$, satisfying $\int_0^{y} w^D(\xi;\, y)\, d\xi = 1$. Then,

$$\int_0^{y} \left[1 - \frac{1}{\zeta^D(\xi)}\right] w^D(\xi; y) d\xi = \frac{\int_0^{y} [\xi \phi''(\xi) + \phi'(\xi)] d\xi}{\phi(y)} = \frac{\int_0^{y} d[\xi \phi'(\xi)]}{\phi(y)} = \frac{y\phi'(y)}{\phi(y)}$$

$$\equiv \mathcal{E}_\phi(y).$$

Thus,

$$\frac{y\mathcal{E}_\phi'(y)}{\mathcal{E}_\phi(y)} = 1 - \frac{1}{\zeta^D(y)} - \mathcal{E}_\phi(y) = \int_0^{y} \left[\frac{1}{\zeta^D(\xi)}\right] w^D(\xi; y) d\xi - \frac{1}{\zeta^D(y)},$$

from which

$$\zeta^{D\prime}(y) \lesseqgtr 0, \forall y \in (0, y_0) \implies \mathcal{E}_\phi'(y) \lesseqgtr 0, \forall y \in (0, y_0).$$

Furthermore, $\mathcal{E}'_{\phi}(y) = 0$ for $y \in (0, y_0)$ implies $\zeta^D(y) = 1/[1 - \mathcal{E}_{\phi}(y)]$, which is hence constant, and thus $\zeta^{D\prime}(y) = 0$ for $y \in (0, y_0)$. This completes the proof. ■

Proof of Proposition I-1: The proof is analogous to that of Proposition S-1. Let $w^I(\xi; z) \equiv -\theta'(\xi)/\theta(z) > 0$, satisfying $\int_z^{\overline{z}} w^I(\xi; z)\, d\xi = 1$, because $\theta(\overline{z}) = 0$. Then, because $\overline{z}\theta'(\overline{z}) = 0$,[37]

$$\int_z^{\overline{z}} [\zeta^I(\xi) - 1] w^I(\xi; z) d\xi = \frac{\int_z^{\overline{z}} [\theta'(\xi) + \xi\theta''(\xi)] d\xi}{\theta(z)} = \frac{\int_z^{\overline{z}} d[\xi\theta'(\xi)]}{\theta(z)} = -\frac{z\theta'(z)}{\theta(z)} \equiv \mathcal{E}_\theta(z).$$

Thus,

$$\frac{z\mathcal{E}'_\theta(z)}{\mathcal{E}_\theta(z)} = \mathcal{E}_\theta(z) + 1 - \zeta^I(z) = \int_z^{\overline{z}} \zeta^I(\xi) w^I(\xi; z) d\xi - \zeta^I(z),$$

from which

$$\zeta^{I\prime}(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \quad \Rightarrow \quad \mathcal{E}'_\theta(z) \gtreqless 0, \forall z \in (z_0, \overline{z}).$$

Furthermore, $\mathcal{E}'_\theta(z) = 0$ for $z \in (z_0, \overline{z})$ implies $\zeta^I(z) = 1 + \mathcal{E}_\theta(z)$, which is hence constant and thus $\zeta^{I\prime}(z) = 0$ for $z \in (z_0, \overline{z})$. This completes the proof. ■

Proof of Proposition S*-1: The proof is analogous to that of Proposition D-1. Let $w^{S*}(\xi^*; y) \equiv H^{*\prime}(\xi^*)/H^*(y) > 0$, satisfying $\int_0^y w^{S*}(\xi^*; y) d\xi^* = 1$. Then,

$$\int_0^y \left[1 - \frac{1}{\zeta^{S*}(\xi^*)}\right] w^{S*}(\xi^*; y) d\xi^* = \frac{\int_0^y [\xi^* H^{*\prime\prime}(\xi^*) + H^{*\prime}(\xi)] d\xi^*}{H^*(y)} = \frac{\int_0^y d[\xi^* H^{*\prime}(\xi^*)]}{H^*(y)} = \frac{yH^{*\prime}(y)}{H^*(y)} = \frac{1}{\Phi^*(y)}.$$

Thus,

$$\frac{y\Phi^{*\prime}(y)}{\Phi^*(y)} = \frac{yH^{*\prime}(y)}{H^*(y)} - 1 - \frac{yH^{*\prime\prime}(y)}{H^{*\prime}(y)} = \frac{1}{\Phi^*(y)} - 1 + \frac{1}{\zeta^{S*}(y)} = \frac{1}{\zeta^{S*}(y)} - \int_0^y \frac{w^{S*}(\xi^*; y)}{\zeta^{S*}(\xi^*)} d\xi^*,$$

from which

[37] See footnote 28.

$$\zeta^{S*\prime}(y) \lesseqgtr 0, \forall y \in (0, y_0) \Rightarrow \Phi^{*'}(y) \gtreqless 0, \forall y \in (0, y_0).$$

Furthermore, $\Phi^{*'}(y) = 0$, for $y \in (0, y_0)$ implies $\zeta^{S*}(y) = \Phi^*(y)/[\Phi^*(y) - 1]$, which is hence constant and thus $\zeta^{S*'}(y) = 0$ for $y \in (0, y_0)$. This completes the proof. ■

Appendix E: Technical Proofs for Section 5.

Appendix E.1. (Relative Demand and Trade Elasticity): We now show that the demand for an H good relative to an F good can be written as $g(w/w^*; V; V^*)$, which satisfies the property, $g(w/w^*; V; V^*) \lesseqgtr 1 \Leftrightarrow w/w^* \gtreqless 1$. From Shepherd's Lemma,

$$D = \frac{\partial P(\mathbf{p})}{\partial p_\omega}\bigg|_{\mathbf{p}=(w\mathbf{1}_\Omega^{-1};\, w^*\mathbf{1}_{\Omega^*}^{-1})} X(D\mathbf{1}_\Omega; M\mathbf{1}_{\Omega^*}) \quad for\ \omega \in \Omega$$

$$M = \frac{\partial P(\mathbf{p})}{\partial p_{\omega^*}}\bigg|_{\mathbf{p}=(w\mathbf{1}_\Omega^{-1};\, w^*\mathbf{1}_{\Omega^*}^{-1})} X(D\mathbf{1}_\Omega; M\mathbf{1}_{\Omega^*}) \quad for\ \omega^* \in \Omega^*$$

By taking the ratio, the relative demand is

$$\frac{D}{M} = \frac{\left.\frac{\partial P(\mathbf{p})}{\partial p_\omega}\right|_{\mathbf{p}=(w\mathbf{1}_\Omega^{-1};\, w^*\mathbf{1}_{\Omega^*}^{-1})}}{\left.\frac{\partial P(\mathbf{p})}{\partial p_{\omega^*}}\right|_{\mathbf{p}=(w\mathbf{1}_\Omega^{-1};\, w^*\mathbf{1}_{\Omega^*}^{-1})}} = g\left(\frac{w}{w^*}; V;\ V^*\right).$$

and the strictly quasi-concavity of $P(\mathbf{p})$ implies that this function is decreasing in w/w^*. Moreover, the symmetry implies that it is equal to 1 at $w/w^* = 1$, from which the result follows.

Next, by applying the above expression for GM-CES unit cost functions,

$$\frac{w}{w^*}\frac{D}{M} = \frac{w}{w^*} g\left(\frac{w}{w^*}; V;\ V^*\right) = \frac{\mathbb{E}_G\left[\frac{w^{1-\sigma}}{Vw^{1-\sigma} + V^* {w^*}^{1-\sigma}}\right]}{\mathbb{E}_G\left[\frac{{w^*}^{1-\sigma}}{Vw^{1-\sigma} + V^* {w^*}^{1-\sigma}}\right]}.$$

Log-differentiating the above expression by $t = w/w^*$ yields

$$1 + \frac{d\ln(D/M)}{d\ln(w/w^*)} = \frac{\mathbb{E}_G\left[\frac{1-\sigma}{(t)^{1-\sigma}}\left[\frac{(t)^{1-\sigma}}{V(t)^{1-\sigma} + V^*}\right]^2 V^*\right]}{\mathbb{E}_G\left[\frac{(t)^{1-\sigma}}{V(t)^{1-\sigma} + V^*}\right]} + \frac{\mathbb{E}_G\left[\frac{1-\sigma}{(t)^{\sigma-1}}\left[\frac{1}{V(t)^{1-\sigma} + V^*}\right]^2 V\right]}{\mathbb{E}_G\left[\frac{1}{V(t)^{1-\sigma} + V^*}\right]}.$$

By evaluating at $t = w/w^* = 1 = g(1; V;\ V^*)$, the trade elasticity in equilibrium is equal to

$$-\frac{d\ln(D/M)}{d\ln(w/w^*)}\bigg|_{t=w/w^*=1} - 1\ = \mathbb{E}_G[\sigma] - 1 = \mathcal{S}^{GMCMS} - 1.$$

Likewise, by applying the inverse relative demand for GM-CES production functions,

$$\frac{w}{w^*}\frac{D}{M} = g^{-1}\left(\frac{D}{M}; V;\ V^*\right)\frac{D}{M} = \frac{\mathbb{E}_G\left[\frac{D^{1-1/\sigma}}{VD^{1-1/\sigma} + V^*M^{1-1/\sigma}}\right]}{\mathbb{E}_G\left[\frac{M^{1-1/\sigma}}{VD^{1-1/\sigma} + V^*M^{1-1/\sigma}}\right]}$$

By setting $u = D/M$, log-differentiating the above expression yields

$$1 + \frac{d\ln(w/w^*)}{d\ln(D/M)}$$
$$= \frac{\mathbb{E}_G\left[\frac{1-1/\sigma}{u^{1-1/\sigma}}\left[\frac{u^{1-1/\sigma}}{Vu^{1-1/\sigma} + V^*}\right]^2 V^*\right]}{\mathbb{E}_G\left[\frac{u^{1-1/\sigma}}{Vu^{1-1/\sigma} + V^*}\right]} + \frac{\mathbb{E}_G\left[\frac{1-1/\sigma}{u^{1/\sigma-1}}\left[\frac{1}{Vu^{1-1/\sigma} + V^*}\right]^2 V\right]}{\mathbb{E}_G\left[\frac{1}{Vu^{1-1/\sigma} + V^*}\right]}.$$

By evaluating this expression at $t = w/w^* = 1$ and $u = D/M = 1$,

$$1 + \frac{d\ln t}{d\ln u}\bigg|_{t=w/w^*=1} = \frac{V^*}{V + V^*}\mathbb{E}_G\left[1 - \frac{1}{\sigma}\right] + \frac{V}{V + V^*}\mathbb{E}_G\left[1 - \frac{1}{\sigma}\right] = 1 - \mathbb{E}_G\left[\frac{1}{\sigma}\right].$$

Hence, the trade elasticity in equilibrium is

$$-\frac{d\ln(D/M)}{d\ln(w/w^*)}\bigg|_{t=w/w^*=1} - 1 = \frac{d\ln u}{d\ln t}\bigg|_{t=w/w^*=1} - 1\ = \frac{1}{\mathbb{E}_G[1/\sigma]} - 1 = \mathcal{S}^{GMCMS} - 1.$$

Appendix E.2. (Gains from Trade under H.S.A.)

By plugging in the expression of the love-for-variety under H.S.A. in Table 1 into the general formula for the gains from trade in Table 2,

$$\ln(GT) = \int_V^{V/\lambda} \Phi\left(s^{-1}\left(\frac{1}{v}\right)\right)\frac{dv}{v}.$$

Using the change of variables, $s^{-1}\left(\frac{1}{v}\right) = \xi \Leftrightarrow v = \frac{1}{s(\xi)}$, which implies $\frac{dv}{v} = -\frac{\xi s'(\xi)}{s(\xi)}\frac{d\xi}{\xi} = (\zeta^S(\xi) - 1)\frac{d\xi}{\xi}$, and the identity, $\Phi(\xi)(\zeta^S(\xi) - 1) = 1 + \xi\Phi'(\xi)$,

$$\ln(GT) = \int_V^{V/\lambda} \Phi\left(s^{-1}\left(\frac{1}{v}\right)\right)\frac{dv}{v} = \int_{s^{-1}(1/V)}^{s^{-1}(\lambda/V)} \Phi(\xi)(\zeta^S(\xi) - 1)\frac{d\xi}{\xi}$$

$$= \int_{s^{-1}(1/V)}^{s^{-1}(\lambda/V)} [1 + \xi\Phi'(\xi)]\frac{d\xi}{\xi}$$

$$= \int_{s^{-1}(1/V)}^{s^{-1}(\lambda/V)} \frac{d\xi}{\xi} + \int_{s^{-1}(1/V)}^{s^{-1}(\lambda/V)} \Phi'(\xi)d\xi = \ln\left[\frac{s^{-1}(\lambda/V)}{s^{-1}(1/V)}\right] + \Phi\left(s^{-1}\left(\frac{\lambda}{V}\right)\right) - \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right).$$

Hence, the expression for the Home gains from trade under H.S.A. is given by:

$$GT = \frac{s^{-1}(\lambda/V)}{s^{-1}(1/V)}\frac{\exp[\Phi(s^{-1}(\lambda/V))]}{\exp[\Phi(s^{-1}(1/V))]} = \frac{s^{-1}((1-\lambda)/V^*)}{s^{-1}((1-\lambda)/\lambda V^*)}\frac{\exp[\Phi(s^{-1}((1-\lambda)/V^*))]}{\exp[\Phi(s^{-1}((1-\lambda)/\lambda V^*))]},$$

as shown in Table 2. For translog, we have

$$s^{-1}\left(\frac{1}{V}\right) = \bar{z}\exp\left[-\frac{1}{\gamma V}\right];\ \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) = \frac{1}{2\gamma V} \rightarrow \ln(GT) = \frac{1-\lambda}{2\gamma V}$$

For generalized translog, we have

$$s^{-1}\left(\frac{1}{V}\right) = \bar{z}\exp\left[-\frac{\eta}{(\sigma-1)}(\gamma V)^{-1/\eta}\right];\ \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) = \frac{1}{\sigma-1}\frac{\eta}{1+\eta}(\gamma V)^{-1/\eta}$$

so that

$$\ln(GT) = \frac{1}{(\sigma-1)}\frac{\eta}{1+\eta}\left(\frac{1}{\gamma V}\right)^{1/\eta}\frac{1-(\lambda)^{1/\eta}}{1/\eta}$$

For CoPaTh, see Appendix F.

We now prove the H.S.A. portion of Theorem 6. First, for any given $V > 0, GT$ is increasing in V^* and decreasing in $\lambda = V/(V + V^*)$ with the range,

$$1 < GT < \frac{\overline{z}}{s^{-1}(1/V)}\frac{\exp[\Phi(\overline{z})]}{\exp[\Phi(s^{-1}(1/V))]}$$

Clearly, the upper bound is finite and decreasing in V, if and only if $\overline{z} < \infty$, i.e., if and only if the choke price exists. Next, for any given $V^* > 0$, GT is monotonically decreasing in V and in λ under non-increasing love-for-variety. Furthermore, by letting

$V \to 0$ and $\lambda = V/(V + V^*) \to 0$, the upper bound goes to infinity,[38] because $GT < \frac{s^{-1}(1/V^*)\exp\left[\Phi\left(s^{-1}(1/V^*)\right)\right]}{\lim_{z\to 0}[z\exp[\Phi(z)]]} = \infty$.

Appendix E.3. (Gains from Trade under HDIA)

By plugging in the expression of the love-for-variety under HDIA. in Table 1 into the general formula for the gains from trade in Table 2,

$$\ln(GT) = \int_V^{V/\lambda} \left[\frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/v))} - 1\right]\frac{dv}{v}.$$

Using the change of variables, $\phi^{-1}\left(\frac{1}{v}\right) = \xi \Leftrightarrow v = \frac{1}{\phi(\xi)}$, which implies $\frac{dv}{v} = -\frac{\xi\phi'(\xi)}{\phi(\xi)}\frac{d\xi}{\xi} = -\mathcal{E}_\phi(\xi)\frac{d\xi}{\xi}$,

$$\ln(GT) = \int_V^{V/\lambda} \left[\frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/v))} - 1\right]\frac{dv}{v} = \int_{\phi^{-1}(1/V)}^{\phi^{-1}(\lambda/V)} \left[\mathcal{E}_\phi(\xi) - 1\right]\frac{d\xi}{\xi}$$

$$= -\int_{\phi^{-1}(1/V)}^{\phi^{-1}(\lambda/V)} \frac{d\xi}{\xi} + \int_{\phi^{-1}(1/V)}^{\phi^{-1}(\lambda/V)} \left[\mathcal{E}_\phi(\xi)\right]\frac{d\xi}{\xi} = \ln\left[\frac{(\lambda/V)}{\phi^{-1}(\lambda/V)}\frac{\phi^{-1}(1/V)}{(1/V)}\right].$$

Hence, the expression for the Home gains from trade under HDIA is given by:

$$GT = \frac{(\lambda/V)}{\phi^{-1}(\lambda/V)}\frac{\phi^{-1}(1/V)}{(1/V)} = \frac{((1-\lambda)/V^*)}{\phi^{-1}((1-\lambda)/V^*)}\frac{\phi^{-1}((1-\lambda)/\lambda V^*)}{((1-\lambda)/\lambda V^*)},$$

as shown in Table 2.

We now prove the HDIA portion of Theorem 6. First, for any given $V > 0, GT$ is increasing in V^* and decreasing in $\lambda = V/(V + V^*)$ with the range,

$$1 < GT < \lim_{\lambda\to 0}\left[\frac{(\lambda/V)}{\phi^{-1}(\lambda/V)}\right]\frac{\phi^{-1}(1/V)}{(1/V)} = \lim_{y\to 0}\left[\frac{\phi(y)}{y}\right]\frac{\phi^{-1}(1/V)}{(1/V)} = \phi'(0)\frac{\phi^{-1}(1/V)}{(1/V)}$$

---

[38] To see $\lim_{z\to 0}[z\exp[\Phi(z)]] = 0$, we show $\lim_{z\to 0}[\ln z + \Phi(z)] = -\infty$. From integration by parts, $\Phi(z) \equiv \frac{1}{s(z)}\int_z^{\bar{z}}\frac{s(\xi)}{\xi}d\xi = \frac{-s(z)\ln z - \int_z^{\bar{z}} s'(\xi)\ln\xi d\xi}{s(z)} = -\ln z - \frac{1}{s(z)}\int_z^{\bar{z}} s'(\xi)\ln\xi\, d\xi$. Hence, $\ln z + \Phi(z) = -\frac{1}{s(z)}\int_z^{\bar{z}} s'(\xi)\ln\xi\, d\xi = \frac{\int_z^{\bar{z}} s'(\xi)\ln\xi d\xi}{\int_z^{\bar{z}} s'(\xi)d\xi}$. Since $s(z) \to \infty$ as $z \to 0$, the numerator and denominator both go to $+/-\infty$ as $z \to 0$. Applying L'Hôpital's rule, $\lim_{z\to 0}[\ln z + \Phi(z)] = \lim_{z\to 0}\frac{\int_z^{\bar{z}} s'(\xi)\ln\xi d\xi}{\int_z^{\bar{z}} s'(\xi)d\xi} = \lim_{z\to 0}\frac{s'(z)\ln z}{s'(z)} = \lim_{z\to 0}(\ln z) = -\infty$.

The upper bound is thus finite if and only if $\phi'(0) < \infty$, that is, if and only if the choke price exists. Next, for any given $V^* > 0$, GT is monotonically decreasing in V and in λ under non-increasing love-for-variety. Furthermore, by letting $V \to 0$ and $\lambda = V/(V+V^*) \to 0$, the upper bound goes to infinity, because

$$GT = \frac{(1-\lambda)/V^*}{\phi^{-1}((1-\lambda)/V^*)} \frac{\phi^{-1}((1-\lambda)/\lambda V^*)}{((1-\lambda)/\lambda V^*)} < \frac{1/V^*}{\phi^{-1}(1/V^*)} \lim_{\lambda \to 0} \left[\frac{\phi^{-1}((1-\lambda)/\lambda V^*)}{((1-\lambda)/\lambda V^*)} \right]$$

$$= \frac{1/V^*}{\phi^{-1}(1/V^*)} \lim_{y \to \infty} \left[\frac{y}{\phi(y)} \right] = \frac{1/V^*}{\phi^{-1}(1/V^*)} \lim_{y \to \infty} \left[\frac{1}{\phi'(y)} \right] = \infty.$$

Appendix E.4 (Gains from Trade under HIIA)

By plugging in the expression of the love-for-variety under HIIA in Table 1 into the general formula for the gains from trade in Table 2,

$$\ln(GT) = \int_V^{V/\lambda} \frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/v)\right)} \frac{dv}{v}.$$

Using the change of variables, $\theta^{-1}\left(\frac{1}{v}\right) = \xi \Leftrightarrow v = \frac{1}{\theta(\xi)}$, which implies $\frac{dv}{v} = -\frac{\xi\theta'(\xi)}{\theta(\xi)} \frac{d\xi}{\xi} = \mathcal{E}_\theta(\xi) \frac{d\xi}{\xi}$,

$$\ln(GT) = \int_V^{V/\lambda} \frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/v)\right)} \frac{dv}{v} = \int_{\theta^{-1}(1/V)}^{\theta^{-1}(\lambda/V)} \frac{d\xi}{\xi} = \ln\left(\frac{\theta^{-1}(\lambda/V)}{\theta^{-1}(1/V)}\right).$$

Hence, the expression for the Home gains from trade under HIIA is given by:

$$GT = \frac{\theta^{-1}(\lambda/V)}{\theta^{-1}(1/V)} = \frac{\theta^{-1}((1-\lambda)/V^*)}{\theta^{-1}((1-\lambda)/\lambda V^*)},$$

as shown in Table 2.

We now prove the HIIA portion of Theorem 6. First, for any given $V > 0, GT$ is increasing in V^* and decreasing in $\lambda = V/(V+V^*)$ with the range,

$$1 < GT < \frac{\lim_{\lambda \to 0} \theta^{-1}(\lambda/V)}{\theta^{-1}(1/V)} = \frac{\overline{z}}{\theta^{-1}(1/V)}.$$

Thus, the upper bound is finite and decreasing in V, if and only if $\overline{z} < \infty$, that is, if and only if the choke price exists. Next, for any given $V^* > 0$, GT is monotonically decreasing in V and in λ under non-increasing love-for-variety. Furthermore, by letting $V \to 0$ and $\lambda = V/(V+V^*) \to 0$,

$$1 < GT < \frac{\theta^{-1}(1/V^*)}{\theta^{-1}(\infty)} = \infty.$$

Appendix F. (CoPaTh family of H.S.A.)

We derive the formulae for $\mathcal{S}(V) = \zeta^s\left(s^{-1}\left(\frac{1}{V}\right)\right)$, $\mathcal{L}(V) = \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right)$ and GT under the CoPaTh family of H.S.A. First, from

$$s(z) = \gamma\sigma^{\frac{\rho}{1-\rho}}\left[1 - \left(\frac{z}{\bar{z}}\right)^{\frac{1-\rho}{\rho}}\right]^{\frac{\rho}{1-\rho}} \; for\; z < \bar{z};\;\; 0 < \rho < 1,$$

we have

$$z = s^{-1}\left(\frac{1}{V}\right) = \bar{z}\left\{1 - \frac{1}{\sigma}\left[\frac{1}{\gamma V}\right]^{\frac{1-\rho}{\rho}}\right\}^{\frac{\rho}{1-\rho}} < \bar{z}.$$

By inserting this expression to $\zeta^S(z) = \left[1 - \left(\frac{z}{\bar{z}}\right)^{\frac{1-\rho}{\rho}}\right]^{-1} > 1$,

$$\mathcal{S}(V) = \zeta^s\left(s^{-1}\left(\frac{1}{V}\right)\right) = \sigma[\gamma V]^{\frac{1-\rho}{\rho}} > 1.$$

Second, note that:

$$\Phi(z) = \frac{1}{s(z)}\int_z^{\bar{z}} \frac{s(\xi)}{\xi}\mathrm{d}\xi = \left[1 - \left(\frac{z}{\bar{z}}\right)^{\frac{1}{\nu}}\right]^{-\nu}\int_z^{\bar{z}}\left[1 - \left(\frac{\xi}{\bar{z}}\right)^{\frac{1}{\nu}}\right]^{\nu}\frac{d\xi}{\xi}$$

where $\nu = \frac{\rho}{1-\rho}$. From Appendix A in Fujiwara and Matsuyama (2023), the integral can be represented in the terms of hypergeometric series, as follows.

$$\int_z^{\bar{z}}\left[1 - \left(\frac{\xi}{\bar{z}}\right)^{\frac{1}{\nu}}\right]^{\nu}\frac{d\xi}{\xi} = \frac{\nu}{1+\nu}\left[1 - \left(\frac{z}{\bar{z}}\right)^{\frac{1}{\nu}}\right]^{\nu+1}\sum_{n=0}^{\infty}\frac{(1)_n(1+\nu)_n}{(2+\nu)_n n!}\left[1 - \left(\frac{z}{\bar{z}}\right)^{\frac{1}{\nu}}\right]^n,$$

where $(q)_n$ is the Pocchammer's symbol defined, for any real q and any non-negative integer n, as follows:

$$(q)_n = \begin{cases} 1, & n = 0; \\ q(q+1)\dots(q+n-1), & n > 0. \end{cases}$$

In particular, $(1)_n = n!$, hence, the coefficient in the hypergeometric series can be simplified as:

$$\frac{(1)_n(1+\nu)_n}{(2+\nu)_n n!} = \frac{n!\,(1+\nu)_n}{(2+\nu)_n n!} = \frac{(1+\nu)_n}{(2+\nu)_n} = \frac{(1+\nu)(2+\nu)\dots(n+\nu)}{(2+\nu)(3+\nu)\dots(1+n+\nu)} = \frac{1+\nu}{1+\nu+n}$$

Hence, the above identity can be further simplified to:

$$\int_z^{\bar{z}} \left[1-\left(\frac{\xi}{\bar{z}}\right)^{\frac{1}{\nu}}\right]^{\nu} \frac{d\xi}{\xi} = \frac{\nu}{1+\nu}\left[1-\left(\frac{z}{\bar{z}}\right)^{\frac{1}{\nu}}\right]^{\nu+1} \sum\nolimits_{n=0}^{\infty} \frac{1+\nu}{1+\nu+n}\left[1-\left(\frac{z}{\bar{z}}\right)^{\frac{1}{\nu}}\right]^{n}$$

$$= \left[1-\left(\frac{z}{\bar{z}}\right)^{\frac{1}{\nu}}\right]^{\nu} \sum\nolimits_{n=0}^{\infty} \frac{\nu}{1+\nu+n}\left[1-\left(\frac{z}{\bar{z}}\right)^{\frac{1}{\nu}}\right]^{n+1}.$$

Plugging this back to the expression for $\Phi(z)$:

$$\Phi(z) = \left[1-\left(\frac{z}{\bar{z}}\right)^{\frac{1}{\nu}}\right]^{-\nu} \int_z^{\bar{z}} \left[1-\left(\frac{\xi}{\bar{z}}\right)^{\frac{1}{\nu}}\right]^{\nu} \frac{d\xi}{\xi} = \sum\nolimits_{n=0}^{\infty} \frac{\nu}{1+\nu+n}\left[1-\left(\frac{z}{\bar{z}}\right)^{\frac{1}{\nu}}\right]^{n+1}$$

Or, using that $\nu = \frac{\rho}{1-\rho}$ and $1-\left(\frac{z}{\bar{z}}\right)^{\frac{1}{\nu}} = \frac{1}{\zeta^s(z)}$,

$$\Phi(z) = \sum\nolimits_{n=0}^{\infty} \frac{\rho}{1+(1-\rho)n}\left[\frac{1}{\zeta^s(z)}\right]^{n+1},$$

from which

$$\Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) = \sum\nolimits_{n=0}^{\infty} \frac{\rho}{1+(1-\rho)n}\left[\frac{1}{\mathcal{S}(V)}\right]^{n+1} = \sum\nolimits_{n=0}^{\infty} \frac{\rho}{1+(1-\rho)n}\left[\frac{1}{\sigma(\gamma V)^{\frac{1-\rho}{\rho}}}\right]^{n+1}.$$

Using this expression to the gains from trade, we obtain

$$\ln GT = -\sum\nolimits_{n=0}^{\infty} \frac{\rho}{1+(1-\rho)n} \frac{1-(\lambda)^{\frac{1-\rho}{\rho}(n+1)}}{\sigma(\gamma V)^{\frac{1-\rho}{\rho}(n+1)}} + \ln\left[1+\frac{1-(\lambda)^{\frac{1-\rho}{\rho}}}{\sigma(\gamma V)^{\frac{1-\rho}{\rho}}-1}\right]^{\frac{\rho}{1-\rho}}.$$