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Kiminori Matsuyama and Philip Ushchev

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Centre for Economic Policy Research
187 boulevard Saint-Germain, 75007 Paris, France
2 Coldbath Square, London EC1R 5HL
Tel: +44 (0)20 7183 8801
www.cepr.org

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Kiminori Matsuyama - k-matsuyama@northwestern.edu
Northwestern University and CEPR

Philip Ushchev - ph.ushchev@gmail.com
ECARES, Université Libre De Bruxelles and CEPR

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with an Application to an Armington Model of Competitive Trade

Kiminori Matsuyama
Northwestern University

Philip Ushchev
ECARES, Université Libre de Bruxelles

September 19, 2026

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1. Introduction.

In recent years, a growing number of studies have explored some alternatives to the CES utility and production functions, which are ubiquitous across many fields of applied general equilibrium; see Matsuyama (2023, 2025) for some selective reviews. Virtually all these studies have focused, almost exclusively, on the implications of departing from CES on the markup and other pricing behaviors in imperfectly competitive settings. Of course, these are important issues to address, on which we have already written a series of papers and are planning to write many more. Yet, there is another important issue, which has escaped the attention in the literature. Departing from CES alters the measure of love-for-variety,[1] the consumer's desire to consume more variety of consumer goods and the productivity gains from using more variety of inputs. This changes the welfare implications of the variety of available goods even in the environment, where a variety change has no effect on the pricing of the goods. The goal of this paper is to fill this void in the literature.

Love-for-variety (LV) is a natural consequence of the quasi-concavity of the utility and production functions, that is, the convexity of the indifference and the isoquant curves. We will give the formal definition of LV in section 2, but heuristically, it can be expressed for a strictly quasi-concave, symmetric CRS production function, $X(\mathbf{x})$, as:

$$\mathcal{L}(V) \equiv \frac{\partial \ln W(V)}{\partial \ln V} > 0,$$

where V is the variety of inputs available and $W(V) \equiv \max_{\mathbf{x}} \left\{ X(\mathbf{x}) \middle| \int_0^V x_\omega d\omega \leq 1 \right\}$ is the maximized output per total units of inputs used.

Initiated by Dixit and Stiglitz (1977), popularized by Krugman (1980), Ethier (1982), and Romer (1987), the concept plays a central role across many fields of applied general equilibrium, as surveyed in Matsuyama (1995), most prominently in economic growth (Grossman and Helpman 1993, Gancia and Zilibotti 2005, and Acemoglu 2008),

[1]Different authors called this concept differently; e.g., "the desirability of variety" (Dixit and Stiglitz 1977), "love of variety" (Helpman and Krugman 1985, sec. 6.2), "taste for variety" (Benassy 1996), etc. In the context of input variety, Ethier (1982) called it "gains from an increased division of labor" and Romer (1987) "increasing returns due to specialization." We call it "love-for-variety," as in Parenti et al. (2017) and Thisse and Ushchev (2020), partly because it seems most common in the recent literature and partly because this term indicates that each consumer/producer *directly* enjoys more variety, which helps to set itself apart from alternatives in which more variety *indirectly* brings benefits to each consumer/producer.

international trade (Helpman and Krugman 1985), and economic geography (Fujita, Krugman, and Venables 1999). Even though commonly discussed in monopolistic competition settings, the concept is purely about preferences, and hence it is also useful in other contexts, independently of the market structure assumed, such as gains from trade in Armington-type competitive models, in which different countries produce different sets of goods, the migration decisions of the households across metropolitan areas, the traveler's choice between tourist destinations, and many others.

In spite of its importance, however, little is known about how love-for-variety depends on the underlying utility (production) functions and how it is related to the demand system derived from them. In a standard treatment, e.g., Matsuyama (1995, Sec.3A), $\mathcal{L}(V)$ is obtained under CES with gross substitutes.[2] It is equal to

$$\mathcal{L}(V) = \mathcal{L}^{CES} = \frac{1}{\sigma - 1},$$

where $\sigma > 1$ represents both the (constant) elasticity of substitution (ES) across different goods and the (constant) price elasticity (PE) of demand for each good. This expression has some appealing properties. First, LV is inversely related to both ES and PE. Second, knowing PE gives you everything you need to know about ES and LV. However, this expression also exhibits some unappealing properties. For example, LV is constant. Many find it implausible that love for consuming more variety of goods (or using more variety of inputs) would be independent of how much variety they have.[3] Many also find that the relation between PE, ES, and LV under CES is so hard-wired that it gives no flexibility. For this reason, many introduce what we call "the Benassy residual," which is meant to

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[2]CES is also assumed in most empirical assessments of welfare effects of new goods; see e.g., Feenstra (1994), Bils and Klenow (2001), and Broda and Weinstein (2006). A few exceptions include Feenstra and Weinstein (2017), which uses translog, and Errico and Lashkari (2025), which use HDIA demand systems.

[3]Indeed, some economists have expressed to us that, due to this unappealing feature of love-for-variety under CES, they prefer "the ideal-variety approach," e.g., Helpman and Krugman (1985, sec. 6.3). According to this approach, consumers are heterogeneous in taste, and each consumer buys just one type of the product closest to his/her ideal type. Thus, consumers have zero "love-for-variety." Yet adding more types of the product indirectly benefits each consumer, since it improves the chance of finding a type closer to his/her ideal type on average, and yet such indirect gains from product variety are diminishing, as the product space becomes congested. The ideal-variety approach is related to Lancaster (1966)'s "characteristic approach," where consumers derive utility from the characteristics or attributes embodied in goods, not from goods themselves. With the fixed dimension of characteristics, more variety of goods eventually runs into diminishing gains. Though these alternative approaches are popular in IO for studying the welfare effects of new goods, e.g., Hausman (1996), Nevo (2003), and Petrin (2002), just to name a few, they have not been used widely in applied general equilibrium due to their intractability.

capture direct externalities from the variety to TFP, or the term that is meant to capture the quality adjustment, to fill the gap between the love-for-variety inferred from, say, productivity growth, and the love-for-variety implied by CES.

But these features are not inherent limitations of the love-for-variety approach but an artifact of the CES assumption. In this paper, we first extend this approach for homothetic symmetric non-CES and investigate how LV changes when we depart from CES. How is LV related to ES and PE, which are distinct concepts outside of CES? How biased are our estimates of LV, and those of the Benassy residual or the quality adjustment term, if we incorrectly assume that the underlying demand system is CES when it is not? Under what conditions is LV diminishing in the variety of available goods? Does it help to depart from CES to introduce Marshall's 2nd law of demand (i.e., PE becomes higher at a higher price along the demand curve for each good)? And can we develop "the love-for-variety approach," which is flexible enough to allow for diminishing LV and, at the same time, tractable enough to be applicable to a wide range of applied general equilibrium models? These are some of the questions we address.

In Section 2, we first recall some general properties of homothetic symmetric functions with gross substitutes.[4] Then, we formally introduce the two measures, the substitutability measure, $\mathcal{S}(V)$, the elasticity of substitution evaluated at the symmetric points, and the love-for-variety measure, $\mathcal{L}(V)$, both of which can be expressed as functions of the available variety V only. Under CES, they are both constant and equal to $\mathcal{S}(V) = \sigma$ and $\mathcal{L}(V) = 1/(\sigma - 1)$. Once we depart from CES, they generally depend on V and

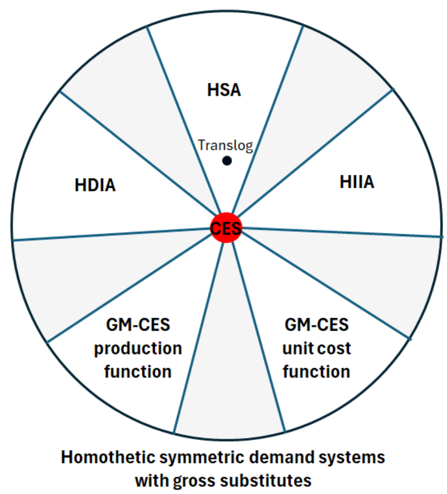
[4] It should be noted that homotheticity and symmetry are not so restrictive as they may seem, because one can nest homothetic symmetric functions into a nonhomothetic and/or asymmetric upper-tier function. In other words, homothetic symmetric non-CES can serve as building blocks to construct nonhomothetic asymmetric non-CES. Indeed, the homotheticity restriction is an advantage, as it makes our results applicable to a sector-level analysis in multi-sector models. (See, for example, Baqaee, Burnstein, Duprez, and Farhi 2024, which assume symmetry within what they call "types" and asymmetry across types.) Moreover, one of the messages of this paper is that homotheticity and symmetry are not strong enough — "anything goes" — so that one needs to look for tighter restrictions. Another message is that the standard measures of love-for-variety and gains from trade derived under CES require substantial modifications when we move away from CES, *even if we restrict ourselves to* homotheticity and symmetry. Obviously, these modifications would be necessary if we also allowed for nonhomotheticity and asymmetry. (Alternatively, Errico and Lashkari (2025) allow for a small asymmetric perturbation to symmetric HSA/HDIA/HIIA to offer a second-order local approximation of the price index.)

$$\mathcal{L}(V) \neq \frac{1}{\mathcal{S}(V) - 1},$$

and hence the RHS could no longer be used as the love-for-variety measure. [5] At the end of Section 2, we also discuss briefly generalized CES, proposed by Benassy (1996) in another attempt to break the tight link between ES and LV while preserving the CES demand structure.

Because homotheticity and symmetry are not strong enough to impose much restriction on the properties of the two functions, $\mathcal{S}(V)$ and $\mathcal{L}(V)$, we turn to five subclasses, which are pairwise-disjoint with the sole exception of CES, as shown in Figure 1. Table 1 displays the formulae for $\mathcal{S}(V)$ and $\mathcal{L}(V)$ under different classes.

Figure 1: Five Classes



In Section 3, we discuss the two classes of Geometric Mean of CES (GM-CES), obtained by taking the weighted geometric mean of CES unit cost functions and of CES production functions. Theorem 1 states that both $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are constant, and that their constant values, $\mathcal{S}^{GMCES}$and $\mathcal{L}^{GMCES}$, satisfy

$$\mathcal{L}^{GMCES} > \frac{1}{\mathcal{S}^{GMCES} - 1},$$

unless it is CES. Moreover, $\mathcal{L}^{GMCES}$ can be arbitrarily large, holding $\mathcal{S}^{GMCES}$ fixed. This suggests that the standard CES formula for LV underestimates the true value of LV and

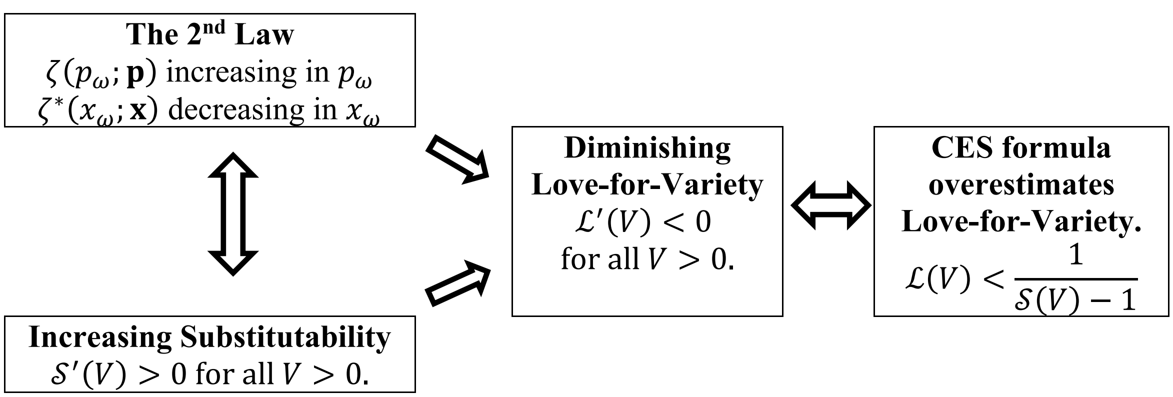
[5] Though many researchers have derived $\mathcal{S}(V)$ for some specific families of non-CES, such as translog, we are unaware of any previous attempt to derive the love-for-variety measure, $\mathcal{L}(V)$, for any non-CES. Instead, they take it for granted that $\mathcal{L} = 1/(\sigma - 1)$ under CES would be generalized to $\mathcal{L}(V) = 1/(\mathcal{S}(V) - 1)$ under non-CES, which is, unfortunately, not the case in general.

overestimates the Benassy residual or quality adjustment term under GM-CES, potentially by a wide margin.

In Section 4, we discuss the three remaining classes: HSA, HDIA, and HIIA, which stand for *Homothetic Single Aggregator*, *Homothetic Direct Implicit Additivity*, and *Homothetic Indirect Implicit Additivity*.[6] For each of these three classes, we show the following results, some of which are also illustrated in Figure 2. First, substitutability, $\mathcal{S}(V)$, is increasing in V, if and only if the 2nd law of demand holds (Theorem 2-i). Second, increasing (decreasing) substitutability, $\mathcal{S}'(V) > (<) 0$, implies diminishing (increasing) love-for-variety $\mathcal{L}'(V) < (>) 0$, but the converse is not true (Theorem 2-ii). Third, love-for-variety is constant, $\mathcal{L}'(V) = 0$, if and only if substitutability is constant, $\mathcal{S}'(V) = 0$, which occurs only under CES within these three classes (Theorem 2-iii).[7] Fourth, $\mathcal{L}'(V) < (>)\ 0$ is necessary and sufficient for

$$\mathcal{L}(V) < (>) \frac{1}{\mathcal{S}(V) - 1}.$$

Figure 2: The Relation between the Price Elasticity, Substitutability and Love-for Variety under the Three Classes of HSA/HDIA/HIIA.



[6]Matsuyama and Ushchev (2017) first introduced these three classes without symmetry and gross substitutability restrictions. Matsuyama (2023; 2025) discuss the relation between these three and other classes of non-CES demand systems in detail.

[7]These three results thus suggest that HSA/HDIA/HIIA offer a tractable way of capturing the intuition that gains from variety are diminishing, if different goods are more substitutable when more variety is available. As such, they can be valuable options to those who find the ideal-variety/Lancasterian approach more appealing than the love-for-variety approach under CES in predicting declining gains from more variety and yet find them intractable to be used in general equilibrium.

Thus, the standard CES formula for LV over(under)estimates the true value of LV and under(over)estimates the Benassy residual and the quality adjustment term in the case of diminishing (increasing) LV under the three classes (Theorem 3). We also show that the gap between $\mathcal{L}(V)$ and $1/[\mathcal{S}(V) - 1]$ goes asymptotically to zero, as V goes to infinity, under these three classes (Theorem 4).

It should be stressed that Theorems 1 through 4 are about love-for-variety, that is, the measure of the consumer's (producer's) desire for consuming (using) more variety of goods. As such, they hold regardless of what is assumed on the supply side of the variety change and hence are applicable to a wide range of models.

Nevertheless, in Section 5, we illustrate some implications of these results by applying them to a simple Armington model of competitive trade between two countries, which produce different sets of goods and differ in the country size, determined by the variety of goods produced. The competitive assumption means that each good is always supplied at its marginal cost, which is independent of the demand structure. Moreover, it ensures the efficiency of the equilibrium allocation, conditional on the variety of goods that becomes available in each country. Thus, all the implications on the gains from trade in this model come directly from the way the love-for-variety measure under non-CES differs from the one under CES. Table 2 displays the formulae for the gains from trade, $GT,$ under different classes of demand systems in this model.

After summarizing some broad implications of the model on the gains from trade (Theorem 5), we look into the five classes. Under the two classes of GM-CES, the gains from trade for a country satisfy

$$GT = \left(\frac{1}{\lambda}\right)^{\mathcal{L}^{GMCES}} \geq \left(\frac{1}{\lambda}\right)^{\frac{1}{\mathcal{S}^{GMCES}-1}} = \left(\frac{1}{\lambda}\right)^{\frac{1}{\epsilon^{GMCES}}},$$

where $0 < \lambda < 1$ is the country's domestic expenditure share, and $\epsilon^{GMCES} = \mathcal{S}^{GMCES} - 1 > 0$ is the trade elasticity, and the inequality is strict except the limit case of CES (Theorem 6). This formula preserves the key feature of the formula by Arkolakis, Costinot, and Rodríguez-Clare (2012), hereafter the ACR formula, in the sense that it is a decreasing power function of the domestic expenditure share[8]. However, it differs in the

[8]We define the gains from trade as the (gross) rate of increase in the country's welfare when the country moves from autarky to trade, given by the real consumption under trade divided by the real consumption

sense that the exponent is $\mathcal{L}^{GMCES}$, not $1/\epsilon^{GMCES}$. Moreover, $\mathcal{L}^{GMCES}$ can be arbitrarily large, even after controlling for $\epsilon^{GMCES} = \mathcal{S}^{GMCES} - 1 > 0$. Thus, the standard practice of estimating the gains from trade applying the ACR formula using the trade elasticity as a sufficient statistic underestimates the gains from trade under GM-CES, potentially by a wide margin, suggesting that the cases of GM-CES might offer a possible clue for "elusive gains from trade."[9]

Theorem 7 summarizes the gains from trade under HSA/HDIA/HIIA in this model. The ACR formula does not hold in that the gains from trade are no longer a power function of the domestic expenditure share, λ. In particular, even after controlling for λ, the gains from trade depend on the sizes of the two countries. For example, a proportional increase in the sizes of the two countries, which keeps their domestic expenditure shares unchanged, makes the gains from trade smaller under diminishing LV. This suggests that gains from trade are larger among smaller countries than among larger countries. Or, a smaller domestic expenditure of a country increases its gains from trade, but its implications depend on whether it is caused by a smaller size of this country or a larger size of its trading partner. If it is due to an increase in the size of its trading partner, the resulting increase in its gains from trade is bounded if these three classes of demand systems feature the choke price. In this sense, the ACR formula overestimates the gains from trade with a large country.

In Section 6, we offer some concluding remarks. Technical materials are in the Appendices.

2. General Homothetic Symmetric Functions.

under autarky. ACR defines the gains from trade as the (net) rate of decline in the country's welfare when the country moves from trade to autarky. If we follow their definition, the formula is $GT = 1 - \lambda^{\mathcal{L}^{GMCES}} > 1 - \lambda^{1/\epsilon^{GMCES}}$. We have chosen our definition, as it helps to keep the formulae simpler under many different classes of the functions that we explore. Of course, nothing of substance depends on this choice, as there is a one-to-one correspondence between the two definitions.

[9]This mechanism, which relies on the elasticity dispersion in GM-CES, is distinct from another mechanism, discussed in Matsuyama (2023; Section 6.2), using asymmetric HSA, obtained as a convex combination of Cobb-Douglas and CES, which exhibits the property that, as some goods become relatively scarce, the elasticity of substitution of such goods against the others approaches one from above. For countries which lack domestic supply of such goods, the welfare cost of autarky is infinity. The two mechanisms are complementary in suggesting possible clues for "elusive gains from trade" and can be combined, by using a two-tier demand system, where the lower-tier is GM-CES and the upper-tier is a convex combination of Cobb-Douglas and CES.

In what follows, we discuss general homothetic symmetric functions and the demand system derived from them in terms of the demand for inputs by producers with a monotone, strictly quasi-concave, symmetric CRS production function, $X = X(\mathbf{x})$. Here, $\mathbf{x} = \{x_\omega; \omega \in \overline{\Omega}\}$ is the input quantity vector, defined over $\overline{\Omega}$, a continuum of the set of all potential inputs. Symmetry means invariance under permutation; that is, $X(\mathbf{x}) = X(\mathbf{x}_T)$ for $\mathbf{x}_T = \{x_{T(\omega)}; \omega \in \overline{\Omega}\}$, where $T: \overline{\Omega} \to \overline{\Omega}$ is any Lebesgue-measure preserving one-to-one mapping. The set of all potential inputs, $\overline{\Omega}$, is divided into the set of available inputs, $\Omega \subset \overline{\Omega}$, and the set of unavailable inputs, $\overline{\Omega} \backslash \Omega$. That is, $x_\omega = 0$ for $\omega \in \overline{\Omega} \backslash \Omega$. We denote the Lebesgue measure of Ω by $V \equiv |\Omega|$. Our goal is to study how the effect of changing V on productivity is related to the demand system for inputs. To this end, it is necessary to assume that each input is inessential. That is, $x_\omega = 0$ for $\omega \in \overline{\Omega} \backslash \Omega$ does *not* imply $X(\mathbf{x}) = 0$, so that it is possible to produce a positive output, even when some potential inputs are unavailable.

2.1. Duality Theory: A Refresher.

We first recall some key results from the duality theory; see, e.g., Mas-Colell et al. (1995), and Jehle and Reny (2011). But in doing so, we restate them for the case of a continuum set of inputs. Let $\mathbf{p} = \{p_\omega; \omega \in \overline{\Omega}\}$ denote the input price vector, such that $p_\omega = \infty$ for $\omega \in \overline{\Omega} \backslash \Omega$ and $p_\omega < \infty$ for $\omega \in \Omega$. The inessentiality of inputs ensures that the unit cost function corresponding to this production function,

$$P(\mathbf{p}) \equiv \min_{\mathbf{x}} \left\{ \mathbf{px} \equiv \int_\Omega p_\omega x_\omega d\omega \,\middle|\, X(\mathbf{x}) \geq 1 \right\}, \tag{1}$$

is finite, and hence well-defined, even though $p_\omega = \infty$ for $\omega \in \overline{\Omega} \backslash \Omega$. Furthermore, it also satisfies monotonicity, strict quasi-concavity, linear homogeneity, and symmetry. The first-order condition of the minimization problem in eq.(1) yields the inverse demand curve for ω:

$$p_\omega = P(\mathbf{p}) \frac{\partial X(\mathbf{x})}{\partial x_\omega}; \quad \omega \in \Omega. \qquad (2)^{10}$$

[10]We note two technical issues. First, we discuss the partial derivatives like $\partial X(\mathbf{x})/\partial x_\omega$ and $\partial P(\mathbf{p})/\partial p_\omega$ heuristically in this paper, even though they are defined formally as Fréchet derivatives in the L^2-norm defined over a continuum set of available inputs, as in Parenti et al. (2017). Second, the first-order conditions in eq.(2) and eq.(4) assume the interior solution. This is ensured under CES or GM-CES introduced in Section 3. In general, however, it is important to address the possibility of the corner solution.

Alternatively, from the duality theorem, we can instead start from a monotonic, strictly quasi-concave, linearly homogeneous, and symmetric unit cost function, $P(\mathbf{p})$, from which the CRS production function $X(\mathbf{x})$ is derived as follows:

$$X(\mathbf{x}) \equiv \min_{\mathbf{p}} \left\{ \mathbf{px} \equiv \int_{\Omega} p_\omega x_\omega d\omega \,\middle|\, P(\mathbf{p}) \geq 1 \right\}, \tag{3}$$

which satisfies monotonicity, strict quasi-concavity, and symmetry. The first-order condition of the minimization problem in eq.(3) yields the demand curve for ω:

$$x_\omega = \frac{\partial P(\mathbf{p})}{\partial p_\omega} X(\mathbf{x}); \quad \omega \in \Omega. \tag{4}$$

This is known as Shephard's lemma. The duality theorem thus allows us to use either $X(\mathbf{x})$ or $P(\mathbf{p})$ as the primitive of the CRS technology.

From either eq.(2) or eq.(4), Euler's theorem on linearly homogeneous functions implies that

$$\mathbf{px} \equiv \int_{\Omega} p_\omega x_\omega d\omega = P(\mathbf{p}) \left[\int_{\Omega} \frac{\partial X(\mathbf{x})}{\partial x_\omega} x_\omega d\omega \right] = \left[\int_{\Omega} p_\omega \frac{\partial P(\mathbf{p})}{\partial p_\omega} d\omega \right] X(\mathbf{x}) = P(\mathbf{p}) X(\mathbf{x}). \tag{5}$$

This identity means that the total cost of inputs is equal to the total value of output.

2.2. Budget Share, Price Elasticity of Demand and the 2nd Law of Demand.

From eq.(2), the budget share of ω can be written as a homogeneous function of degree 0 in $\mathbf{x}$ as:

$$s_\omega \equiv \frac{p_\omega x_\omega}{P(\mathbf{p}) X(\mathbf{x})} = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} \equiv s^*(x_\omega; \mathbf{x}) = s^*(1, \mathbf{x}/x_\omega); \;\; \omega \in \Omega. \tag{6}$$

From eq.(4), it can also be written as a homogeneous function of degree 0 in $\mathbf{p}$ as:

$$s_\omega \equiv \frac{p_\omega x_\omega}{P(\mathbf{p}) X(\mathbf{x})} = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} \equiv s(p_\omega; \mathbf{p}) = s(1, \mathbf{p}/p_\omega); \; \omega \in \Omega. \tag{7}$$

In what follows, we assume that inputs are gross substitutes; that is, the budget share of each input goes down as its price goes up (or as its quantity goes down) along its demand curve:[11]

That is, $x_\omega \geq 0$ may be binding, when p_ω exceeds "the choke price." Nevertheless, for expositional reasons, we postpone an explicit treatment of the choke price until Section 4, where it becomes relevant.

[11] Our definition of gross substitutability, a higher price of one good leads to lower expenditure on that good, differs from the standard definition, according to which a higher price of one good leads to higher expenditure on other goods. This standard definition would not work when each good is infinitesimal. Note

Definition: *Gross substitutability of inputs*

$$\frac{\partial \ln s(p_\omega; \mathbf{p})}{\partial \ln p_\omega} = \frac{\partial \ln s(1, \mathbf{p}/p_\omega)}{\partial \ln p_\omega} < 0 \Leftrightarrow \frac{\partial \ln s^*(x_\omega; \mathbf{x})}{\partial \ln x_\omega} = \frac{\partial \ln s^*(1, \mathbf{x}/x_\omega)}{\partial \ln x_\omega} > 0.$$

From eq.(6) and eq.(7), the price elasticity of demand for ω can be written both as a function of the prices and as a function of the quantities as follows:

$$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = 1 - \frac{\partial \ln s(p_\omega; \mathbf{p})}{\partial \ln p_\omega} \equiv \zeta(p_\omega; \mathbf{p}) = \zeta(1, \mathbf{p}/p_\omega) > 1; \tag{8}$$

$$\zeta_\omega \equiv -\left[\frac{\partial \ln p_\omega}{\partial \ln x_\omega}\right]^{-1} = \left[1 - \frac{\partial \ln s^*(x_\omega;\ \mathbf{x})}{\partial \ln x_\omega}\right]^{-1} \equiv \zeta^*(x_\omega; \mathbf{x}) = \zeta^*(1, \mathbf{x}/x_\omega)\ > 1. \tag{9}$$

Both functions are homogeneous of degree 0. Moreover, they show that gross substitutability is equivalent to assuming that the price elasticity is greater than one.

The literature on non-CES demand systems often focuses on the case where Marshall's 2nd law of demand (or the 2nd law for short) holds. That is, the price elasticity of demand for each input goes up as its price goes up (and hence its quantity goes down) along its demand curve.

Definition: *The 2nd law* holds if and only if

$$\frac{\partial \zeta(p_\omega; \mathbf{p})}{\partial p_\omega} = \frac{\partial \zeta(1, \mathbf{p}/p_\omega)}{\partial p_\omega} > 0 \Leftrightarrow \frac{\partial \zeta^*(x_\omega; \mathbf{x})}{\partial x_\omega} = \frac{\partial \zeta^*(1, \mathbf{x}/x_\omega)}{\partial x_\omega} < 0.$$

If the opposite inequality holds, we shall say that the anti-2nd law holds. CES is clearly a borderline case.

Note that eqs.(6)-(9) show that the budget share of $\omega \in \Omega$, s_ω, and its price elasticity of demand, ζ_ω, are both functions of $\mathbf{p}/p_\omega$ or $\mathbf{x}/x_\omega$. Even though symmetry implies that they are invariant under permutation, the budget share and the price elasticity still depend on the entire distribution of the prices (or the quantities) normalized by its own price (or its own quantity), which is infinite dimensional. For this reason, the cross-price interactions could be complicated under general homothetic symmetric demand systems.

2.3. Unit Quantity and Price Vectors.

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also that, under CES, $\sigma > 1$ ensures both inessentiality and gross substitutability, but, under non-CES, they need to be assumed separately.

To further characterize homothetic symmetric functions and demand systems, it is useful to define the unit quantity vector, $\mathbf{1}_\Omega \equiv \{(1_\Omega)_\omega; \omega \in \overline{\Omega}\}$ and the unit price vector, $\mathbf{1}_\Omega^{-1} \equiv \{(1_\Omega^{-1})_\omega; \omega \in \overline{\Omega}\}$, as follows:

$$(1_\Omega)_\omega \equiv \begin{cases} 1 & \text{for} \quad \omega \in \Omega \\ 0 & \text{for} \quad \omega \in \overline{\Omega}\backslash\Omega \end{cases}; \qquad (1_\Omega^{-1})_\omega \equiv \begin{cases} 1 & \text{for} \quad \omega \in \Omega \\ \infty & \text{for} \quad \omega \in \overline{\Omega}\backslash\Omega \end{cases},$$

which satisfies $\int_\Omega (1_\Omega)_\omega d\omega = \int_\Omega (1_\Omega^{-1})_\omega d\omega = |\Omega| \equiv V$. Then, symmetric quantity and price patterns among all the available inputs are expressed as:

$$\mathbf{x} = x\mathbf{1}_\Omega; \quad \mathbf{p} = p\mathbf{1}_\Omega^{-1},$$

where $x > 0$ and $p > 0$ are scalars.

2.4. Substitutability Measure Across Different Goods.

Since $\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$ are both homogeneous of degree zero, the price elasticity at symmetric patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$ or $\mathbf{x} = x\mathbf{1}_\Omega$, is $\zeta(1; \mathbf{1}_\Omega^{-1}) = \zeta^*(1; \mathbf{1}_\Omega)$, which is independent of p or x, and can be written as $\mathcal{S}(V)$, a function of V only. Moreover, Appendix A shows that $\mathcal{S}(V) = \zeta(1; \mathbf{1}_\Omega^{-1}) = \zeta^*(1; \mathbf{1}_\Omega)$ is equal to the Allen-Uzawa elasticity of substitution between every pair of inputs at the symmetric patterns.[12] Thus, we use the following definition for the substitutability across different inputs in the presence of available variety of inputs V:

Definition: *The substitutability measure* is defined by

$$\mathcal{S}(V) \equiv \zeta(1; \mathbf{1}_\Omega^{-1}) = \zeta^*(1; \mathbf{1}_\Omega) > 1. \tag{10}$$

In general, $\mathcal{S}(V)$ can be nonmonotonic in V. We shall call the case of $\mathcal{S}'(V) > (<) 0$ for all $V > 0$, the case of ***increasing (decreasing) substitutability***.

2.5. Love-for-Variety Measure.

Love-for-variety is commonly measured as the productivity gains from a higher V under the symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$, while holding the total amounts of inputs, xV, constant. That is,

$$\mathcal{L}(V) \equiv \partial \ln W(V)/\partial \ln V > 0, \text{ where } W(V) \equiv \max_{\mathbf{x}} \left\{ X(\mathbf{x}) \middle| \int_0^V x_\omega d\omega \leq 1 \right\}.$$

[12]Since the set of available inputs is a continuum, there is no need to look into the Morishima elasticity of substitution.

Or, using the unit quantity vector, $\mathbf{1}_\Omega$,

$$\left.\frac{d\ln X(\mathbf{x})}{d\ln V}\right|_{\mathbf{x}=x\mathbf{1}_\Omega, xV=const.} = \left.\frac{d\ln xX(\mathbf{1}_\Omega)}{d\ln V}\right|_{xV=const.} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0.$$

It is positive due to the strict quasi-concavity of $X(\mathbf{x})$ and could depend solely on V. This definition is the same as the one proposed by Benassy (1996, eq.(2)) for what he called "taste for variety," even though he applied it only to generalized CES, which will be discussed in Section 2.7.

Alternatively, we could also measure love-for-variety as the rate of decline in the unit cost under symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$, while holding the price of each input, p, constant. That is,

$$-\left.\frac{d\ln P(\mathbf{p})}{d\ln V}\right|_{\mathbf{p}=p\mathbf{1}_\Omega^{-1},\ p=const.} = -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} > 0,$$

which is also positive due to the monotonicity of $P(\mathbf{p})$ and could depend solely on V.

These two measures of love-for-variety are indeed identical. To see this, inserting $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$ and $\mathbf{x} = x\mathbf{1}_\Omega$ into eq.(5) yields $pxV = pP(\mathbf{1}_\Omega^{-1})xX(\mathbf{1}_\Omega)$, so that

$$P(\mathbf{1}_\Omega^{-1})X(\mathbf{1}_\Omega) = V \Longrightarrow -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0.$$

Hence, we use the two definitions interchangeably as the love-for-variety measure.

Definition. *The love-for-variety measure* is defined by:

$$\mathcal{L}(V) \equiv -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0. \tag{11}$$

In general, $\mathcal{L}(V)$ can be nonmonotonic in V. We shall call the case of $\mathcal{L}'(V) < (>) 0$ for all $V > 0$, the case of ***diminishing (increasing) love-for-variety***.

2.6. Standard CES.

In the case of standard CES with gross substitutes,

$$X(\mathbf{x}) = Z\left[\int_\Omega x_\omega^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \Leftrightarrow P(\mathbf{p}) = \frac{1}{Z}\left[\int_\Omega p_\omega^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}},$$

where $\sigma > 1$ is the (constant) elasticity of substitution parameter and Z is the TFP parameter. (In the context of spatial economics, Z is often assumed to vary across locations and interpreted as the locational affinity.) It is easy to verify:

$$\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x}) = \sigma > 1;\ \mathcal{S}(V) = \sigma > 1;\ \mathcal{L}(V) = \frac{1}{\sigma - 1} > 0.$$

Thus, under CES, the price elasticity of demand for each input is everywhere constant and equal to σ. Obviously, this implies that our substitutability measure, which is equal to the price elasticity evaluated at the symmetric patterns, $\mathcal{S}(V)$, is also equal to σ, and independent of V. Moreover, the love-for-variety measure, $\mathcal{L}(V)$, is also independent of V, and depends solely on the single parameter, σ, with a one-to-one inverse relation between the two.

Perhaps for these reasons, many authors claimed without proof that $\mathcal{S}(V)$ is constant only under CES, and/or that $\mathcal{S}(V)$ is the inverse measure of love-for-variety, even under general homothetic symmetric demand systems. Neither claim is correct. Indeed, the relation between the price elasticity, $\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$, substitutability, $\mathcal{S}(V)$, and love-for-variety, $\mathcal{L}(V)$, can be quite complex in general. First, whether the 2nd law holds or not tells us little about the sign of the derivative of $\mathcal{S}(V)$. This should not be surprising because the former is about how $\zeta(p_\omega; p\mathbf{1}_\Omega^{-1})$ responds to a change in p_ω, while the latter is about how $\zeta(1; \mathbf{1}_\Omega^{-1})$ responds to a change in V through its effect on $\mathbf{1}_\Omega^{-1}$. Second, as will be shown in Section 3, there are any number of families of homothetic symmetric non-CES in which $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are both independent of V, and they move in the same direction within each family. Indeed, knowing $\mathcal{S}(V)$ tells us little about $\mathcal{L}(V)$ in general. Again, this should not be surprising. $\mathcal{S}(V)$ shows how $\zeta(1; \mathbf{1}_\Omega^{-1})$ depends on V through its effect on $\mathbf{1}_\Omega^{-1}$, and hence is determined by the curvature of the demand system, hence by that of the marginal utility (productivity) function. In contrast, $\mathcal{L}(V)$ shows how $P(\mathbf{1}_\Omega^{-1})$ responds to a change in V through its effect on $\mathbf{1}_\Omega^{-1}$, and hence is determined by how the curvature of the utility (production) function depends on V. In short, "almost anything goes" under general homothetic symmetric functions and the demand systems derived from them. Therefore, to better understand the relation between $\mathcal{S}(V)$ and $\mathcal{L}(V)$ under non-CES, we turn to some subclasses in Section 3 and in Section 4.

2.7. Generalized CES à la Benassy (1996): A Digression.

Before proceeding, however, let us digress to discuss a previous attempt to break the tight relation between $\mathcal{S}(V)$ and $\mathcal{L}(V)$ under CES, while preserving the CES demand systems. Benassy (1996) proposed to generalize CES as:

$$X(\mathbf{x}) = Z(V)\left[\int_{\Omega} x_{\omega}{}^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \Leftrightarrow P(\mathbf{p}) = \frac{1}{Z(V)}\left[\int_{\Omega} p_{\omega}{}^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}},$$

by making TFP a function of V as $Z(V)$, justified by some sort of direct externalities from V to TFP. Under such modified CES,

$$\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x}) = \sigma > 1;\ \mathcal{S}(V) = \sigma > 1; \mathcal{L}(V) = \frac{1}{\sigma - 1} + \frac{\partial \ln Z(V)}{\partial \ln V}.$$

This demand system is still CES because it features constant price elasticity and constant substitutability. Yet this extension allows the gap between the "revealed" love-for-variety, $\mathcal{L}(V)$, which could be obtained from productivity growth data, and the love-for-variety implied by CES demand, $1/(\sigma-1)$, to be "accounted for" by $\partial \ln Z(V)/\partial \ln V$, the term we shall call "the Benassy residual," in analogy with the Solow residual in the growth accounting.[13]

Even if one believes in the presence of such direct externalities from V to TFP, one should note that any estimate of the Benassy residual hinges on the CES assumption.[14] In any case, adding the Benassy residual to the standard CES does not address some of our questions; that is, how $\mathcal{L}(V)$ depends on V under homothetic symmetric non-CES, particularly, the question of when $\mathcal{L}(V)$ is diminishing in V. Moreover, $\mathcal{L}(V) \neq 1/(\mathcal{S}(V) - 1)$ in general, even if TFP is independent of V, so that one could interpret non-CES as a way of microfounding the gap or the Benassy residual.

For the remainder of the paper, we present $\mathcal{L}(V)$ without the Benassy residual by assuming that Z is independent of V, i.e., $\partial \ln Z(V)/\partial \ln V = 0$. This is solely for the expositional simplicity. Adding the Benassy residual to any homothetic symmetric non-CES explored below is straightforward. One would just need to append $\partial \ln Z(V)/\partial \ln V$ to the expression for $\mathcal{L}(V)$; it would not affect the expression for $\mathcal{S}(V)$.

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[13]In addition, Benassy (1996) assumed that $\partial \ln Z(V)/\partial \ln V = \nu - 1/(\sigma - 1)$, so that $\mathcal{L}(V) = \nu$, which can be chosen independently of $\mathcal{S}(V) = \sigma$. If we assume instead $\partial \ln Z(V)/\partial \ln V$ is another parameter independent of $\mathcal{S}(V) = \sigma$, $\mathcal{L}(V)$ is still inversely related to $\mathcal{S}(V) = \sigma$.

[14]The same statement applies even if the residual is interpreted as the quality change.

3. The Two Classes of Geometric Means of CES (GM-CES).

We now look at two classes of homothetic symmetric functions, obtained by taking weighted geometric means of symmetric CES, thus containing symmetric CES as a special case. In what follows, we shall call these two classes "GM-CES" for short. As shown, $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are both independent of V under GM-CES, just like CES. Unlike CES, however, $\mathcal{L}(V)$ is no longer tightly linked to $\mathcal{S}(V)$, which only affects the lower bound as $\mathcal{L}(V) \geq 1/(\mathcal{S}(V) - 1)$, where the equality holds if and only if it is CES.

3.1. GM-CES Unit Cost Function.

The first class of GM-CES is defined by the set of the unit cost functions, $P(\mathbf{p})$, which can be constructed by taking the weighted geometric mean of symmetric CES unit cost functions:

$$\ln P(\mathbf{p}) \equiv \int_1^\infty \ln P(\mathbf{p}; \sigma)\, dG(\sigma) \equiv \mathbb{E}_G[\ln P(\mathbf{p}; \sigma)],$$

where

$$P(\mathbf{p}; \sigma) \equiv \left[\int_\Omega {p_\omega}^{1-\sigma}\, d\omega\right]^{1/(1-\sigma)}$$

is the unit cost function of the symmetric CES with the constant elasticity of substitution, $\sigma > 1$, and $G(\cdot)$ is a cdf of $\sigma \in (1, \infty)$, satisfying $G(1) = 0$ and $G(\infty) = 1$, with $\mathbb{E}_G[\cdot]$ denoting the expectation operator with respect to $G(\cdot)$. Being the weighted geometric mean of linearly homogeneous, symmetric and strictly quasi-concave functions, $P(\mathbf{p})$ is also linearly homogeneous, symmetric, and strictly quasi-concave. One possible interpretation is that there are many final goods, all of which generate CES demand for inputs but with different σ, and these final goods are aggregated by the Cobb-Douglas with $G(\sigma)$ being the cumulative share of the final goods whose demand for inputs have its constant elasticity of substitution less than or equal to σ. Indeed, we could even allow each final good to be produced with different GM-CES unit cost functions, and these final goods are aggregated by Cobb-Douglas. This is because taking a weighted geometric mean of any number of weighted geometric means of a class of functions generates a weighted geometric mean of that class of functions.[15]

3.2. GM-CES Production Function.

The second class of GM-CES is defined by the set of the production functions, $X(\mathbf{x})$, which can be constructed by taking the weighted geometric mean of symmetric CES production functions:

$$\ln X(\mathbf{x}) \equiv \int_1^\infty \ln X(\mathbf{x};\sigma)\, dG(\sigma) \equiv \mathbb{E}_G[\ln X(\mathbf{x};\sigma)]$$

where

$$X(\mathbf{x};\sigma) \equiv \left[\int_\Omega {x_\omega}^{1-1/\sigma}\, d\omega\right]^{\sigma/(\sigma-1)}$$

is the symmetric CES production function with the constant elasticity of substitution, $\sigma > 1$. Again, $G(\cdot)$ is a cdf of $\sigma \in (1,\infty)$, satisfying $G(1) = 0$ and $G(\infty) = 1$, with $\mathbb{E}_G[\cdot]$ denoting the expectation operator with respect to $G(\cdot)$. Again, being the weighted geometric means of linearly homogeneous, symmetric, and strictly quasi-concave functions, $X(\mathbf{x})$ is also linearly homogeneous, symmetric and strictly quasi-concave.

Appendix B proves the next theorem.

Theorem 1 (GM-CES): Under the two classes of GM-CES demand systems,

1-i): $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are both independent of V, whose values are given by

$$\mathcal{S}(V) = \mathbb{E}_G[\sigma] > 1;\;\; \mathcal{L}(V) = \mathbb{E}_G\left[\frac{1}{\sigma-1}\right] > 0.$$

for the GM-CES unit cost function, and by

$$\mathcal{S}(V) = \frac{1}{\mathbb{E}_G[1/\sigma]} > 1;\;\; \mathcal{L}(V) = \mathbb{E}_G\left[\frac{1}{\sigma-1}\right] > 0.$$

for the GM-CES production function.

[15] GM-CES should not be confused with nested-CES. GM-CES is *not* nested-CES, because the domain of $P(\mathbf{p};\sigma)$, Ω, is common across all $\sigma \in (1,\infty)$ in the definition of GM-CES, meaning that each input is used by many CES technologies with different σ's. In nested CES, the domain of $P(\mathbf{p};\sigma)$ depends on σ, as $\Omega(\sigma)$, such that $\Omega(\sigma) \cap \Omega(\sigma') = \emptyset$ if $\sigma \neq \sigma'$, meaning that the set of all inputs is partitioned into nests and that each input appears in the CES technologies with a particular σ. GM-CES is related to mixed CES, used in Adão, Costinot, and Donaldson (2017), in that the demand systems under the GM-CES unit cost function are special cases of mixed CES demand systems. We use GM-CES, partly because $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are both independent of V under GM-CES, which helps to point out some key issues in a crystallized manner, and partly because we are unable to ensure the integrability of mixed-CES in general. That is, we do not know of the existence of the utility (production) functions that generate mixed CES demand systems.

1-ii): $\mathcal{L}(V)$ can be arbitrarily large, without any upper bound, while its lower bound is given by:

$$\mathcal{L}(V) \geq \frac{1}{\mathcal{S}(V) - 1} > 0,$$

where the equality holds if and only if $G(\cdot)$ is degenerate, i.e., only under CES.

First, Theorem 1-i), also shown in the top half of Table 1, states that love-for-variety of both GM-CES unit cost and production functions is the arithmetic mean of love-for-variety of the component CES, while the substitutability of GM-CES unit cost (production) function is the arithmetic (harmonic) mean of the substitutability of the component CES. Since the arithmetic mean of σ and the harmonic mean of σ are equal to each other only when $G(\cdot)$ is degenerate, this implies that the two classes of GM-CES do not overlap with the sole exception of CES, as illustrated in Figure 1. Theorem 1-i) also means that it would be false to claim that $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are independent of V if and only if CES. Second, Theorem 1-ii) implies that, if one mistakenly interprets the constancy of $\mathcal{S}(V)$ as evidence for CES, one underestimates love-for-variety under GM-CES. Or if one obtains an estimate of $\mathcal{L}(V)$ directly from productivity growth data, one overestimates the Benassy residual and/or the quality adjustment term by attributing the gap between the revealed $\mathcal{L}(V)$ and the CES-implied love-for-variety, $[\mathcal{S}(V) - 1]^{-1}$, to them. Indeed, extending CES to GM-CES alone could potentially account for the gap. Third, because $\mathcal{S}(V)$ imposes only the lower bound on $\mathcal{L}(V)$, one could easily come up with any number of families of the cdf's, $G(\cdot)$, such that $\mathcal{L}(V)$ is positively related to $\mathcal{S}(V)$ within each family, which offers a counter-example to the claim that $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are always inversely related to each other.

4. The Three Classes of HSA/HDIA/HIIA.

Under CES and the two classes of GM-CES, $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are both independent of V. We now look for classes of homothetic symmetric functions and the demand systems derived from them, in which both $\mathcal{S}(V)$ and $\mathcal{L}(V)$ respond to a change in V, with the special attention to the cases of increasing substitutability, $\mathcal{S}'(V) > 0$, and diminishing love-for-variety, $\mathcal{L}'(V) < 0$. Intuitively, one might think that, if adding more variety makes different inputs more substitutable, the price elasticity of demand for each input would become larger, and love-for-variety would become smaller. As pointed out earlier, however, homotheticity and symmetry alone are not restrictive enough to capture this intuition: one needs to find some additional restrictions.

To this end, we look at three classes of homothetic symmetric functions, each named by its defining property. They are Homothetic Single Aggregator (HSA), Homothetic Direct Implicit Additivity (HDIA), and Homothetic Indirect Implicit Additivity (HIIA). In Appendices C.1 through C.3, we offer formal definitions, discuss some key properties, and derive the substitutability measure, $\mathcal{S}(V)$, and the love-for variety measure, $\mathcal{L}(V)$, for each of these three classes.

These three classes are useful for two reasons. First, these three classes and the two classes of GM-CES are pairwise-disjoint with the sole exception of CES, as seen in Figure 1. Thus, these classes offer alternative ways of departing from CES, while keeping CES as a special case. Second, each of the three classes generates the demand system with the property that the price elasticity of demand for ω can be expressed as

$$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = 1 - \frac{\partial \ln s(p_\omega; \mathbf{p})}{\partial \ln p_\omega} \equiv \zeta(p_\omega; \mathbf{p}) = \zeta\left(\frac{p_\omega}{\mathcal{A}(\mathbf{p})}\right) > 1,$$

where $\zeta: (0, \bar{z}) \to (1, \infty)$ is a function of a single variable, satisfying $\lim_{z \to \bar{z}} \zeta(z) = \infty$, if $\bar{z} < \infty$, and $\mathcal{A}(\mathbf{p})$ is linearly homogeneous in $\mathbf{p}$, whose value serves as a sufficient statistic that captures the interdependence of price elasticities across different inputs.[16] Thus, in these three classes, the price elasticity of demand for ω is a function of $p_\omega/\mathcal{A}(\mathbf{p})$, and hence responds to an increase in its own price, p_ω, and to a decline in $\mathcal{A}(\mathbf{p})$ in the same way, and hence also to an increase in V in the symmetric price patterns. Moreover, if $\bar{z} < \infty$, the choke price exists and can be expressed simply as $\bar{z}\mathcal{A}(\mathbf{p}) < \infty$.

These features impose the functional relation between $\mathcal{S}(V)$ and $\mathcal{L}(V)$ in these three classes, as summarized in Table 1. This functional relation enables us to establish the following theorems for these three classes. Due to some subtle differences across these three classes, it is more efficient to prove these theorems for each class separately, as we have done in Appendices C.1, C.2, and C.3.

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[16] Recall that, in general, the price elasticity of each input depends on $\mathbf{p}/p_\omega$ or $\mathbf{x}/x_\omega$, the entire distribution of the prices (or quantities) normalized by its own price (or quantity) as shown in eq.(8) and eq.(9).

Theorem 2: Under the three classes of HSA/HDIA/HIIA, 2-i) $\mathcal{S}'(V) > 0$ if and only if the 2nd law holds. 2-ii) $\mathcal{S}'(V) \gtreqless 0$ for all $V \in (V_0, \infty) \Rightarrow \mathcal{L}'(V) \lesseqgtr 0$ for all $V \in (V_0, \infty)$. 2-iii) $\mathcal{L}'(V) = 0$ for all $V \in (V_0, \infty) \Leftrightarrow \mathcal{S}'(V) = 0$ for all $V \in (V_0, \infty)$. In particular, $\mathcal{L}'(V) = 0$ for all $V > 0 \Leftrightarrow \mathcal{S}'(V) = 0$ for all $V > 0 \Leftrightarrow$ CES.
Theorem 3: Under the three classes of HSA/HDIA/HIIA, $$\mathcal{L}'(V) \lesseqgtr 0 \Leftrightarrow \mathcal{L}(V) \lesseqgtr \frac{1}{\mathcal{S}(V) - 1}.$$
Theorem 4: Under the three classes of HSA/HDIA/HIIA, $$\lim_{V\to\infty} \mathcal{L}(V) = \lim_{V\to\infty} \frac{1}{\mathcal{S}(V) - 1}.$$ In particular, $\lim_{V\to\infty} \mathcal{S}(V) = \infty \Leftrightarrow \lim_{V\to\infty} \mathcal{L}(V) = 0$.

Theorem 2-i) states that the 2nd law is equivalent to increasing substitutability. Theorem 2-ii), for $V_0 = 0$, states that, if the 2nd law (or equivalently, increasing substitutability) holds everywhere, love-for-variety is diminishing everywhere and that, if the anti-2nd law (or equivalently, diminishing substitutability) holds everywhere, love-for-variety is increasing everywhere. The converse is not true. Even if love-for-variety is diminishing everywhere, substitutability may not be increasing everywhere, and even if love-for-variety is diminishing everywhere, substitutability may not be increasing everywhere. However, Theorem 2-iii) states that constant love-for-variety is equivalent to constant substitutability, which occurs only under CES under the three classes.[17]

Theorem 3 states that, if one infers love-for-variety from the observed price elasticity under the (false) CES assumption, one over(under)estimates love-for-variety in the case of diminishing (increasing) love-for-variety. Or, if one obtains an estimate of $\mathcal{L}(V)$ directly from productivity growth data, one under(over)estimates the Benassy residual and/or the quality adjustment term by attributing all the gap between the revealed $\mathcal{L}(V)$ and the CES-implied love-for-variety, $[\mathcal{S}(V) - 1]^{-1}$ to it, when love-for-variety is diminishing (increasing). Figure 2 illustrates schematically the results of Theorem 2 and

[17]This also implies that the two classes of GM-CES and these three classes do not overlap with the sole exception of CES, as shown in Figure 1.

Theorem 3.[18] Theorem 4 states that this gap asymptotically disappears, as V goes to infinity.

5. An Application: An Armington Model of Competitive Trade.

Up to now, we studied how love-for-variety — the consumer's (producer's) preferences for more variety of consumer goods (inputs) — is related to the demand system, when we go beyond the ubiquitous CES assumption. All the results so far are entirely about the demand side of variety and thus useful in any model, regardless of how one might model a change in the variety of available goods.[19] For example, it may change, as the consumers and the producers move from one location to another. Or it may change due to the arrival of new goods, which may occur due to pure discovery.[20] Or it may change due to trade liberalization. Or it may change due to various forms of innovation activities, which may be conducted by the public sector or by the private sector. For the cases where such innovative activities are conducted by the private sectors, their market structures could be perfectly competitive, monopolistic, oligopolistic or monopolistically competitive. The results obtained above are completely independent of what one might assume on the supply side.

Nevertheless, it is illuminating to show how these results might manifest themselves in an applied general equilibrium model. In this section, we apply the results to a simple Armington model of competitive trade, where different countries produce different sets of goods, so that trade liberalization leads to more variety of goods for consumers. (Some other potential applications will be discussed in Section 6.) It should be noted that the goal of this exercise is not to develop a full-blown Armington model of trade without the CES assumption, which is realistic enough to bring to data. Such an analysis would require writing a series of papers. Instead, our aim here is to illustrate in a

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[18] When these homothetic symmetric non-CES demand systems are applied to the Dixit-Stiglitz (1977) model of monopolistic competition, it turns out that $\mathcal{L}(V) < [\mathcal{S}(V) - 1]^{-1}$ for all V is the condition under which the equilibrium entry is excessive; see Matsuyama (2025, Proposition 1). Thus, Theorems 2 and 3 jointly imply that the 2nd law of demand is sufficient for excessive equilibrium entry under HSA/HDIA/HIIA, the result first shown in Matsuyama and Ushchev (2020a; Theorems 1, 2, and 3).

[19] The careful reader might have noticed that we deliberately avoided using certain words like "the equilibrium," "firms," "market," "innovation," "markup rate" or "entry," when describing the model from Section 2 to Section 4.

[20] Indeed, what is "discovered" does not have to be new to everyone. Just think of the young generation who discovers the classic movies, or the Americans who discover sushi, etc.

simple setting the implications of the results obtained so far, by highlighting some conceptual issues that are hidden under CES. For this reason, the model below abstracts from many features, which play prominent roles in the existing literature on the Armington models, such as the trade cost and asymmetry of demand for goods produced in different countries. Instead, asymmetry across the countries in this model lies in the variety of goods they produce. The assumption of perfect competition also helps to isolate the implications of departing from CES on gains from trade only through its effects on love-for-variety, because it shuts down any possible effects on the markup that could show up under alternative market structures and ensures the efficiency of the equilibrium, conditional on the variety of goods that becomes available in each country.

5.1. A Simple Armington Model of Competitive Trade.

Consider an Armington model of trade between two countries, Home (H) and Foreign (F), which differ in labor supply, L and L^*, and in the set of goods they produce, Ω and Ω^*; $\Omega \cap \Omega^* = \emptyset$, with their masses given by $V \equiv |\Omega|$ and $V^* \equiv |\Omega^*|$. Otherwise, the two countries are identical. The representative consumers in both countries have the same homothetic preferences that are symmetric across all goods. Let w and w^* denote the wage rate at H and F. We normalize the units of labor and goods such that one unit of labor in each country can produce one unit of each good of that country. Thus, the price of any good produced at H is $p_\omega = w$ for all $\omega \in \Omega$ and the price of any good produced at F is $p_\omega^* = w^*$for all $\omega \in \Omega^*$. Because all the goods produced in the same country are sold at the same price and because the demand system is symmetric, they are consumed in the same amount in each country.

So, let D and M^* denote H's demand and F's demand for each H-good, so that the total demand for each H-good is $D + M^*,$ and H's resource constraint is

$$V(D + M^*) = L.$$

Likewise, let M and D^* denote H's demand and F's demand for each F-good, so that the total demand for $\omega \in \Omega^*$ is $M + D^*$, F's resource constraint is:

$$V^*(M + D^*) = L^*.$$

H's export value = F's import value, is wVM^*, while H's import value = F's export value, w^*V^*M, so that

$$wVM^* = w^*V^*M$$

is the balanced trade condition.[21]

Since everyone has the identical homothetic demand and faces the same prices, the relative demand for the H-goods to the F-goods is the same in both countries and inversely related to their relative price, as

$$\frac{D+M^*}{M+D^*} = \frac{D}{M} = \frac{M^*}{D^*} = g\left(\frac{w}{w^*}; V; V^*\right).$$

Here, $g(\cdot;\ ; V; V^*) > 0$ is the demand for an H-good relative to an F-good, which satisfies $g(w/w^*; V; V^*) \lesseqgtr 1 \Leftrightarrow w/w^* \gtreqless 1$, as shown in Appendix E.1. Thus, from the two resource constraints, the condition that the relative supply (RS) is equal to the relative demand (RD) is given by

$$\frac{L/V}{L^*/V^*} = RS = RD = \frac{D+M^*}{M+D^*} = \frac{D}{M} = \frac{M^*}{D^*} = g\left(\frac{w}{w^*}; V; V^*\right),$$

In what follows, we consider the case where the two countries differ only proportionally, so that the relative country size is the same regardless of whether it is measured in labor supply (the population size) or in variety of goods produced:

$$\frac{L}{L^*} = \frac{V}{V^*} \Leftrightarrow \frac{L}{V} = \frac{L^*}{V^*}.$$

Then,

$$\frac{D}{M} = \frac{M^*}{D^*} = g\left(\frac{w}{w^*}; V; V^*\right) = \frac{L/V}{L^*/V^*} = 1 \Rightarrow \frac{w}{w^*} = 1.$$

Hence we have factor (and good) price equalization. This simplifies the budget constraints of the two countries and the balanced trade condition as:

$$VD + V^*M = L; \quad VM^* + V^*D^* = L^*; \quad V^*M = VM^*,$$

which further leads to:

$$\frac{L}{L^*} = \frac{V}{V^*} = \frac{M}{M^*} = \frac{D}{D^*} \Leftrightarrow \frac{V}{L} = \frac{V^*}{L^*}; \frac{D}{L} = \frac{D^*}{L^*} = \frac{M}{L} = \frac{M^*}{L^*},$$

and the domestic expenditure shares at Home and Foreign are:

$$\lambda = \frac{VD}{L} = \frac{V}{V+V^*}; \qquad \lambda^* = \frac{V^*D^*}{L^*} = \frac{V^*}{V+V^*}.$$

The H's trade/GDP ratio is $1 - \lambda = \lambda^*$, while the F's trade/GDP ratio is $1 - \lambda^* = \lambda$.

[21] From the two resource constraints, and the balanced trade condition, we can obtain both H's budget constraint, $wVD + w^*V^*M = wL$, and F's budget constraint, $wVM^* + w^*V^*D^* = w^*L^*$.

Thus, in per capita terms, the two countries become identical and the welfare effect of trade for Home is the same as that of an increase in the available variety from V to $V + V^* = V/\lambda$ and the welfare effect of trade for Foreign is the same as that of an increase in the available variety from V^* to $V + V^* = V^*/\lambda^*$. Thus, the gains from trade, defined as the (gross) rate of increase in welfare when the two countries move from autarky to free trade, can be measured by:

$$\ln(GT) \equiv \ln\left[\frac{P(\mathbf{1}_{\Omega}^{-1})}{P(\mathbf{1}_{\Omega\cup\Omega^*}^{-1})}\right] = \int_{V}^{V+V^*} \mathcal{L}(v)\frac{dv}{v} = \int_{V}^{V/\lambda} \mathcal{L}(v)\frac{dv}{v} > 0;$$

$$\ln(GT^*) \equiv \ln\left[\frac{P\left(\mathbf{1}_{\Omega^*}^{-1}\right)}{P(\mathbf{1}_{\Omega\cup\Omega^*}^{-1})}\right] = \int_{V^*}^{V+V^*} \mathcal{L}(v)\frac{dv}{v} = \int_{V^*}^{V^*/\lambda^*} \mathcal{L}(v)\frac{dv}{v} > 0.$$

Table 2 displays this general formula for the Home gains from trade, along with the formulae under the five classes. Note that, in general, even after controlling for the country's domestic expenditure share, gains from trade for the country depends on the variety of goods produced in each country.

Before proceeding, let us point out that there exists even a simpler Armington model that generates the same gains from trade formulae shown in Table 2. That is, there is no production nor labor, only the endowments. Home is endowed with Q units of each $\omega \in \Omega$, and Foreign is endowed with Q units of each $\omega \in \Omega^*$, where Q is exogenous. Trade between the two countries is merely an exchange of their endowments. The equilibrium of this endowment-exchange model is isomorphic to the above model with $Q = L/V = L^*/V^*$.

5.2. The Effects of Country Sizes on Gains from Trade: Broad Implications.

Theorem 5 summarizes broad implications of how country sizes matter on gains from trade in this model. The proof is straightforward and hence is omitted.

Theorem 5 (Gains from Trade):

5-i). Gains from trade are larger for the smaller country than for the larger country.

$$GT \gtreqless GT^* \Leftrightarrow V \lesseqgtr V^* \Leftrightarrow L \lesseqgtr L^* \Leftrightarrow \lambda \lesseqgtr \lambda^*$$

5-ii). When the two countries become proportionately larger, the gains from trade for both countries are smaller under diminishing love-for-variety, $\mathcal{L}'(\cdot) < 0$, because

$$\left.\frac{\partial \ln(GT)}{\partial \ln V}\right|_{\lambda=const.} = \mathcal{L}(V/\lambda) - \mathcal{L}(V) < 0;$$

$$\left.\frac{\partial \ln(GT^*)}{\partial \ln V^*}\right|_{\lambda^*=const.} = \mathcal{L}(V^*/\lambda^*) - \mathcal{L}(V^*) < 0.$$

5-iii) For any given V, Home's gains from trade are increasing in V^*, thus decreasing in $\lambda = V/(V + V^*)$, with

$$\left.\frac{\partial \ln(GT)}{\partial \ln V^*}\right|_{V=const.} = (1 - \lambda)\mathcal{L}(V/\lambda) > 0,$$

and its range is given by:

$$0 < \ln(GT) < \int_V^\infty \mathcal{L}(v)\frac{dv}{v}.$$

The upper bound is infinite if $\mathcal{L}(\infty) > 0$; it may be finite if $\mathcal{L}(\infty) = 0$. If finite, the upper bound is decreasing in V.

5-iv) For any given V^*, Home's gains from trade may be nonmonotone in V, hence in $\lambda = V/(V + V^*)$ in general. Under non-increasing love-for-variety, $\mathcal{L}'(\cdot) \leq 0$,

$$\left.\frac{\partial \ln(GT)}{\partial \ln V}\right|_{V^*=const.} = \lambda\mathcal{L}(V/\lambda) - \mathcal{L}(V) < 0,$$

so that Home's gains from trade are decreasing in V, and so in $\lambda = V/(V + V^*)$, with the range

$$0 < \ln(GT) < \int_0^{V^*} \mathcal{L}(v)\frac{dv}{v}.$$

The upper bound is finite if $\mathcal{L}(0) < \infty$; it may be infinite if $\mathcal{L}(0) = \infty$. If finite, the upper bound is increasing in V^*.

These broad implications are quite intuitive, and so we skip the discussions. Instead, we move on to discuss specific implications on Home Gains from Trade obtained under the two classes of GM-CES and the three classes of HSA/HDIA/HIIA. The formulae for gains from trade under different non-CES are also displayed in Table 2.

5.3. Gains from Trade under CES and GM-CES.

Under CES, the substitutability and love-for-variety are both independent of V, whose constant values may be denoted as $\mathcal{S}^{CES} \equiv \sigma > 1$ and $\mathcal{L}^{CES} \equiv 1/(\sigma - 1) > 0$, respectively. By plugging the constant value of $\mathcal{L}^{CES}$ into the general GT formula,

$$\ln(GT) = \mathcal{L}^{CES} \ln\left(\frac{1}{\lambda}\right) = \frac{1}{\mathcal{S}^{CES} - 1} \ln\left(\frac{1}{\lambda}\right) = \frac{1}{\epsilon^{CES}} \ln\left(\frac{1}{\lambda}\right),$$

where $\epsilon^{CES} = \mathcal{S}^{CES} - 1$ is the trade elasticity under CES. This confirms the well-known ACR formula by Arkolakis, Costinot, and Rodríguez-Clare (2012).

As stated in Theorem 1, the substitutability and love-for-variety are both independent of V also under the two classes of GM-CES. By denoting their constant values by $\mathcal{S}^{GMCES}$ and $\mathcal{L}^{GMCES}$, we have:

Theorem 6 (Gains from Trade under GM-CES):

$$\ln(GT) = \mathcal{L}^{GMCES} \ln\left(\frac{1}{\lambda}\right) \geq \frac{1}{\mathcal{S}^{GMCES} - 1} \ln\left(\frac{1}{\lambda}\right) = \frac{1}{\epsilon^{GMCES}} \ln\left(\frac{1}{\lambda}\right)$$

where the inequality is strict whenever $G(\sigma)$ is non-degenerate.

The first equality follows from $\mathcal{L}(V) \equiv \mathcal{L}^{GMCES}$. The inequality from Theorem 1-ii). The second equality follows from $\epsilon^{GMCES} = \mathcal{S}^{GMCES} - 1 > 0$, as proved in Appendix E.1.

Thus, the ACR formula continues to hold under GM-CES in that

$$\ln(GT) = const. \times \ln\left(\frac{1}{\lambda}\right),$$

so that gains from trade are monotonically decreasing in λ and goes to infinity, as $\lambda \to 0$, which occurs in this model, as $V/V^* = L/L^* \to 0$. Moreover, once the domestic expenditure share λ is controlled for, the absolute country sizes do not matter. Only the relative country size, which determines λ, matters; if the two countries are made proportionately larger, to keep $V/V^* = L/L^*$ unchanged, gains from trade would not be affected. As should be clear from the derivation, the key feature of GM-CES that preserves the ACR formula is that $\mathcal{L}(V)$ is independent of V.

It is worth pointing out, however, that the ACR formula can be extended to GM-CES, only when "$const.$" in the above formula is interpreted as $\mathcal{L}^{GMCES}$. It would be incorrect to interpret "$const.$" as $1/(\mathcal{S}^{GMCES} - 1) = 1/\epsilon^{GMCES}$. As stated in Theorem 1, love-for-variety under GM-CES is no longer tightly linked to the substitutability, which only determines the lower bound. Indeed, holding $\mathcal{S}^{GMCES}$ fixed, $\mathcal{L}^{GMCES}$, and hence gains from trade, could be made arbitrarily large by changing the cdf, $G(\cdot)$. Moreover,

$\epsilon^{GMCES} = \mathcal{S}^{GMCES} - 1$ remains unchanged before and after trade liberalization. If one interprets this incorrectly as evidence for CES and uses the CES formula for the gains from trade, one underestimates the true gains from trade under GM-CES, potentially by a wide margin.[22]

5.4. Gains from Trade under the Three Classes of HSA/HDIA/HIIA.

We now turn to the three classes of HSA/HDIA/HIIA. As stated in Theorem 2 and illustrated in Figure 2, the love-for-variety measure under these three classes is no longer constant with the sole exception of CES. In particular, it is diminishing if the 2nd law holds globally. As a result, the ACR formula no longer holds.

The exact formulae for the Home gains from trade are derived in Appendices E.2–E.4. They are also shown in Table 2. Needless to say, these formulae satisfy all the properties already shown to hold for general homothetic symmetric demand systems in Theorem 5. Moreover, Appendices E.2–E.4 show that, under the three classes,

Theorem 7 (Gains from Trade under the Three Classes of HSA/HDIA/HIIA):

7-i) For any given V, Home's gains from trade are increasing in V^*, thus decreasing in $\lambda = V/(V + V^*)$, with the range,

$$0 < \ln(GT) < \int_V^\infty \mathcal{L}(v) \frac{dv}{v},$$

whose upper bound is finite and decreasing in V, if and only if the choke price exists.

7-ii) For any given V^*, Home's gains from trade are decreasing in V, and so in $\lambda = V/(V + V^*)$ under non-increasing love-for-variety, $\mathcal{L}'(\cdot) \leq 0$, with the range $0 < \ln(GT) < \infty$.

Theorem 7-i) states that, under HSA, HDIA, and HIIA demand systems with the choke price, the Home's gains from trade with another country, no matter how big that country is, is limited. This is in stark contrast to the case of CES or GM-CES, under which Home's gains from trade with Foreign become arbitrarily large, as $\lambda \to 0$ due to $V^* \to \infty$. This suggests that the CES or GM-CES assumption may grossly overestimate the

[22] Since GM-CES generates the demand systems that are special cases of mixed-CES demand systems used in Adão, Costinot and Donaldson (2017), this result also offers some insights for why they discovered that the gains from trade under mixed-CES dominate the gains from trade under CES.

gains from trade with a large country. On the other hand, Theorem 7-ii) states that Home's gains from trade becomes arbitrarily large as $\lambda \to 0$ due to $V \to 0$.

6. Concluding Remarks.

Love-for-variety, that is, the consumer's desire to consume more variety of consumer goods, is a natural consequence of the convexity of preferences. Yet the measure of love-for-variety in the existing literature has been derived only under CES. In this paper, we investigated how love-for-variety depends on the underlying utility and production functions outside of CES and how it is related to the demand systems derived from them. To this end, we first defined, for general homothetic symmetric functions, the two measures, the substitutability $\mathcal{S}(V)$ and love-for-variety $\mathcal{L}(V)$, both of which are functions of the variety of available goods V only. Since homotheticity and symmetry alone impose little restriction on the properties of these two functions, we turn to five classes of homothetic symmetric functions, which are pairwise-disjoint with the sole exception of CES. Under the two classes of GM-CES, both $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are constant, and the standard CES formula for LV would underestimate the true LV under GM-CES. Under the three classes of HSA/HDIA/HIIA, the 2nd law of demand and increasing substitutability are equivalent, and either of them implies diminishing LV, which is the necessary and sufficient condition for the standard CES formula to overestimate the true LV under these three classes.

We then illustrated some implications of these results by applying them to a simple Armington model of competitive trade and showed how biased our estimates of the gains from trade can be if we assume that the demand system is CES when it is not. Under the two classes of GM-CES, the ACR formula for the gains from trade holds, if modified with the (constant) love-for-variety, which means that the standard ACR formula with the (constant) trade elasticity underestimates the true gains from trade, possibly by a wide margin, thus suggesting a possible clue for "elusive gains from trade." Under the three classes of HSA/HDIA/HIIA with the 2nd law of demand, and hence, with diminishing love-for-variety, the ACR formula no longer holds. Even after controlling for the domestic expenditure share, the gains from trade are smaller among larger countries than among smaller countries. In the presence of the choke price, the gains from trade for a country is increasing as its trading partner gets larger, but it is bounded, unlike under CES or under GM-CES, which suggests that the ACR formula would overestimate the gains from trade with a large country. Note that, in this environment, departing from CES has no effect on the pricing of the goods, as they are always supplied at their marginal cost, which remains constant. Therefore, all the implications on the gains from trade come directly from the way the love-for-variety measure under non-CES differs from the one under CES. Though this Armington model is highly stylized, we conjecture that its basic messages, particularly, the standard CES formula for gains from trade under(over)estimates the gains from trade under GM-CES (under HSA/HDIA/HIIA with diminishing love-for-variety), would carry over to more general settings, which allow for other types of asymmetries and positive trade costs.

As pointed out earlier, the results in Theorems 1 through 4 are all about the properties of non-CES preferences and the demand side of the variety change, hence independent of what one might assume on the supply side of the variety change. As such, these results are applicable to a wide range of models, not just to the Armington model of trade in this paper. One natural choice would be applications to imperfect competition models with the introduction of new products. Indeed, we have a companion paper, Matsuyama and Ushchev (2020a), in which we explore some industrial policy implications of HSA/HDIA/HIIA in a Dixit-Stiglitz (1977)-type monopolistic competition.[23] Another possible application is the Romer-type R&D based endogenous growth model with expanding variety, which predicts too little R&D in equilibrium. We conjecture that replacing CES by GM-CES makes equilibrium R&D move further below the optimal R&D, while replacing CES by HSA/HDIA/HIIA with the 2nd law of demand could make equilibrium R&D excessive.

Moreover, because we have imposed homotheticity on the utility functions and the linear homogeneity of production functions, our results are readily applicable to multi-sector or multiple market segment settings. For example, one could embed any of the five classes we studied, i.e., two classes of GM-CES, and the three classes of

[23]However, Matsuyama and Ushchev (2020a) was written at a time when our knowledge of these three classes was more limited. Moreover, it did not make use of $\mathcal{L}(V)$, and consequently, the proofs of their main theorems are cumbersome and less transparent. Furthermore, it does not consider GM-CES. We are hence planning to rewrite that paper by greatly expanding it, by building on the results in this paper.

HSA/HDIA/HIIA, into an upper-tier demand system, which can be asymmetric and/or nonhomothetic. For this reason, we believe that potential applications are limitless. We are planning to explore some in our future work. At the same time, we hope the reader will find our results useful for their applications.

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Table 1: Substitutability and Love-for-Variety under CES and the Five Classes.

<table>
<tr><td colspan="2"></td><td>Substitutability: $\mathcal{S}(V)$</td><td>Love-for-Variety: $\mathcal{L}(V)$</td></tr>
<tr><td colspan="2">General Formula</td><td>$\zeta(1;\mathbf{1}_\Omega^{-1})$</td><td>$-\dfrac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V}$</td></tr>
<tr><td colspan="2">CES</td><td>σ</td><td>$\dfrac{1}{\sigma-1}$</td></tr>
<tr><td colspan="2">GM-CES
unit cost function</td><td>$\mathbb{E}_G[\sigma]$</td><td>$\mathbb{E}_G\left[\dfrac{1}{\sigma-1}\right]$</td></tr>
<tr><td colspan="2">GM-CES
production function</td><td>$\dfrac{1}{\mathbb{E}_G[1/\sigma]}$</td><td>$\mathbb{E}_G\left[\dfrac{1}{\sigma-1}\right]$</td></tr>
<tr><td colspan="2">HSA</td><td>$\zeta^S\left(s^{-1}\left(\dfrac{1}{V}\right)\right)$</td><td>$\Phi\left(s^{-1}\left(\dfrac{1}{V}\right)\right)=\dfrac{1}{\mathcal{E}_H\left(s^{-1}(1/V)\right)}$</td></tr>
<tr><td colspan="4">$\zeta^S(z)\equiv-\frac{zH''(z)}{H'(z)}>1;\ \frac{1}{\Phi(z)}=\mathcal{E}_H(z)\equiv-\frac{zH'(z)}{H(z)}>0;\ H(z)>0;\ H'(z)<0<H''(z).$</td></tr>
<tr><td rowspan="2">Special Case of HSA</td><td>Translog</td><td>$1+\gamma V$</td><td>$\dfrac{1}{2\gamma V}$</td></tr>
<tr><td>Generalized Translog</td><td>$1+(\sigma-1)(\gamma V)^{1/\eta}$</td><td>$\dfrac{1}{\sigma-1}\left(\dfrac{\eta}{1+\eta}\right)(\gamma V)^{-1/\eta}$</td></tr>
<tr><td colspan="2">HDIA</td><td>$\zeta^D\left(\phi^{-1}\left(\dfrac{1}{V}\right)\right)$</td><td>$\dfrac{1}{\mathcal{E}_\phi(\phi^{-1}(1/V))}-1$</td></tr>
<tr><td colspan="4">$\zeta^D(y)\equiv-\dfrac{\phi'(y)}{y\phi''(y)}>1;0<\mathcal{E}_\phi(y)\equiv\dfrac{y\phi'(y)}{\phi(y)}<1;\ \phi(y)>0;\phi'(y)>0>\phi''(y).$</td></tr>
<tr><td colspan="2">HIIA</td><td>$\zeta^I\left(\theta^{-1}\left(\dfrac{1}{V}\right)\right)$</td><td>$\dfrac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/V)\right)}$</td></tr>
<tr><td colspan="4">$\zeta^I(z)\equiv-\frac{z\theta''(z)}{\theta'(z)}>1;\ \mathcal{E}_\theta(z)\equiv-\frac{z\theta'(z)}{\theta(z)}>0;\ \theta(z)>0;\ \theta'(z)<0<\theta''(z).$</td></tr>
</table>

Table 2: Home Gains from Trade (The domestic expenditure share is $\lambda = V/(V+V^*)$.

		Home Gains from Trade
General Formula		$\ln(GT) = \int_V^{V/\lambda} \mathcal{L}(v)\frac{dv}{v}$
CES		$\ln(GT) = \frac{1}{\sigma-1}\ln\left(\frac{1}{\lambda}\right)$
GM-CES unit cost function		$\ln(GT) = \mathbb{E}_G\left[\frac{1}{\sigma-1}\right]\ln\left(\frac{1}{\lambda}\right) \geq \frac{1}{\mathbb{E}_G[\sigma]-1}\ln\left(\frac{1}{\lambda}\right)$
GM-CES production function		$\ln(GT) = \mathbb{E}_G\left[\frac{1}{\sigma-1}\right]\ln\left(\frac{1}{\lambda}\right) \geq \frac{1}{\left[\mathbb{E}_G[1/\sigma]\right]^{-1}-1}\ln\left(\frac{1}{\lambda}\right)$
HSA		$GT = \frac{s^{-1}(\lambda/V)}{s^{-1}(1/V)}\frac{\exp\left[\Phi\left(s^{-1}(\lambda/V)\right)\right]}{\exp\left[\Phi\left(s^{-1}(1/V)\right)\right]}$
Special Cases of HSA	Translog	$\ln(GT) = \frac{1-\lambda}{2\gamma V}$
	Generalized Translog	$\ln(GT) = \frac{(\gamma V)^{-1/\eta}}{\sigma-1}\frac{1}{1+1/\eta}\frac{1-\lambda^{1/\eta}}{1/\eta}$
HDIA		$GT = \frac{(\lambda/V)}{\phi^{-1}(\lambda/V)}\frac{\phi^{-1}(1/V)}{(1/V)}$
HIIA		$GT = \frac{\theta^{-1}(\lambda/V)}{\theta^{-1}(1/V)}$

Appendices:

Appendix A. Allen-Uzawa Elasticity of Substitution at the symmetric patterns under general homothetic symmetric demand systems.

Appendix B. The Two Classes of Geometric Means of CES.

Appendix C. The Three Classes of HSA/HDIA/HIIA.

Appendix C.1. The Homothetic Single Aggregator (HSA) class.

Appendix C.2. The Homothetic Direct Implicit Additivity (HDIA) class.

Appendix C.3. The Homothetic Indirect Implicit Additivity (HIIA) class.

Appendix C.4. An alternative (and equivalent) specification of the HSA class.

Appendix D. Proofs of Propositions S-1, D-1, I-1, and S*-1.

Appendix E. Technical Proofs for Section 5.

Appendix E.1. Relative Demand and Trade Elasticity.

Appendix E.2. Gains from Trade under HSA.

Appendix E.3. Gains from Trade under HDIA.

Appendix E.4. Gains from Trade under HIIA.

Appendix A. Allen-Uzawa elasticity of substitution at the symmetric patterns under general homothetic symmetric demand systems.

The Allen-Uzawa elasticity of substitution between two inputs, $\omega, \omega' \in \Omega$, is given by:

$$AES(p_\omega, p_{\omega'}, \mathbf{p}) = \frac{P(\mathbf{p})P_{\omega\omega'}(p_\omega, p_{\omega'}, \mathbf{p})}{x(p_\omega, \mathbf{p})x(p_{\omega'}, \mathbf{p})},$$

where $x(p_\omega, \mathbf{p})$ is the demand for ω per unit of output, while the functions $P_{\omega\omega'}(p_\omega, p_{\omega'}, \mathbf{p})$ are the "second cross-derivatives" of $P(\mathbf{p})$. The second-order Taylor approximation of $P(\mathbf{p})$ is

$$P(\mathbf{p} + \alpha\mathbf{h}) = P(\mathbf{p}) + \alpha \int_\Omega x(p_\omega, \mathbf{p})h_\omega d\omega + \frac{\alpha^2}{2}\int_\Omega \frac{\partial x(p_\omega, \mathbf{p})}{\partial p_\omega} h_\omega^2 d\omega$$
$$+ \frac{\alpha^2}{2}\int_\Omega \int_\Omega P_{\omega\omega'}(p_\omega, p_{\omega'}, \mathbf{p})h_\omega h_{\omega'}\, d\omega d\omega' + o(\alpha^2),$$

where $\mathbf{h}$ is a function over Ω, and α is a scalar. The linear homogeneity of $P(\mathbf{p})$ implies the following identity:

$$\int_\Omega \frac{\partial x(p_\omega, \mathbf{p})}{\partial p_\omega} p_\omega^2 d\omega + \int_\Omega \int_\Omega P_{\omega\omega'}(p_\omega, p_{\omega'}, \mathbf{p})p_\omega p_{\omega'}\, d\omega d\omega' = 0.$$

By setting $(p_\omega, \mathbf{p}) = (1, \mathbf{1}_\Omega^{-1})$ and $(p_\omega, p_{\omega'}, \mathbf{p}) = (1,1, \mathbf{1}_\Omega^{-1})$ in the identity, we obtain:

$$\left[\frac{\partial x(p_\omega, \mathbf{1}_\Omega^{-1})}{\partial p_\omega}\right]\Bigg|_{p_\omega=1} \underbrace{\left[\int_\Omega d\omega\right]}_{=V} + P_{\omega\omega'}(1,1, \mathbf{1}_\Omega^{-1})\left[\underbrace{\int_\Omega \int_\Omega d\omega d\omega'}_{=V^2}\right] = 0.$$

Using the definition of $\mathcal{S}(V)$,

$$\mathcal{S}(V) \equiv \zeta(1; \mathbf{1}_\Omega^{-1}) = -\left[\frac{\partial \ln x(p_\omega, p\mathbf{1}_\Omega^{-1})}{\partial \ln p_\omega}\right]\Bigg|_{p_\omega=p}$$

$$\Rightarrow \quad \frac{\partial x(p_\omega, \mathbf{1}_\Omega^{-1})}{\partial p_\omega}\Bigg|_{p_\omega=1} = -\mathcal{S}(V)x(1, \mathbf{1}_\Omega^{-1}),$$

the above identity can be further rewritten as:

$$P_{\omega\omega'}(1,1, \mathbf{1}_\Omega^{-1}) = \frac{\mathcal{S}(V)}{V}x(1, \mathbf{1}_\Omega^{-1}).$$

Moreover, by setting $\mathbf{p} = P(\mathbf{1}_\Omega^{-1})$ in $P(\mathbf{p}) = \int_\Omega x(p_\omega, \mathbf{p})p_\omega d\omega$,

$$P(\mathbf{1}_\Omega^{-1}) = Vx(1, \mathbf{1}_\Omega^{-1}).$$

Thus, the Allen-Uzawa elasticity of substitution evaluated at a symmetric outcome:

$$AES_{\omega\omega'}(1,1,\mathbf{1}_\Omega^{-1}) = \frac{P(\mathbf{1}_\Omega^{-1})P_{\omega\omega'}(1,1,\mathbf{1}_\Omega^{-1})}{[x(1,\mathbf{1}_\Omega^{-1})]^2} = \frac{Vx(1,\mathbf{1}_\Omega^{-1})\frac{\mathcal{S}(V)}{V}x(1,\mathbf{1}_\Omega^{-1})}{[x(1,\mathbf{1}_\Omega^{-1})]^2} = \mathcal{S}(V).$$

Appendix B. The Two Classes of Geometric Means of CES.

This appendix characterizes the homothetic symmetric demand systems generated by the weighted geometric means of symmetric CES unit cost functions and of symmetric CES production functions, which we call the GM-CES unit cost function (Section 3.1) and the GM-CES production function (Section 3.2).

First, GM-CES unit cost function, $P(\mathbf{p})$, is defined by

$$\ln P(\mathbf{p}) \equiv \int_1^\infty \ln P(\mathbf{p};\sigma)\, dG(\sigma) \equiv \mathbb{E}_G[\ln P(\mathbf{p};\sigma)]$$

where $P(\mathbf{p};\sigma) \equiv \left[\int_\Omega p_\omega{}^{1-\sigma}\, d\omega\right]^{1/(1-\sigma)}$ and $G(\cdot)$ is the c.d.f. of $\sigma \in (1,\infty)$, with $\mathbb{E}_G[\cdot]$ denoting the expectation operator with respect to $G(\cdot)$. The demand for ω is

$$x_\omega = X(\mathbf{x})\frac{\partial P(\mathbf{p})}{\partial p_\omega} = P(\mathbf{p})X(\mathbf{x})\frac{\partial \ln P(\mathbf{p})}{\partial p_\omega} = P(\mathbf{p})X(\mathbf{x})\mathbb{E}_G\left[\frac{\partial \ln P(\mathbf{p};\sigma)}{\partial p_\omega}\right].$$

Thus, the budget share of ω is

$$s(p_\omega;\mathbf{p}) \equiv \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \mathbb{E}_G\left[\frac{\partial \ln P(\mathbf{p};\sigma)}{\partial \ln p_\omega}\right] = \mathbb{E}_G\left[\left(\frac{p_\omega}{P(\mathbf{p};\sigma)}\right)^{1-\sigma}\right]$$

The price elasticity of demand is thus

$$\zeta(p_\omega;\mathbf{p}) \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = 1 - \frac{\partial \ln s(p_\omega;\ \mathbf{p})}{\partial \ln p_\omega} = \frac{\mathbb{E}_G[\sigma p_\omega{}^{-\sigma}/[P(\mathbf{p};\sigma)]^{1-\sigma}]}{\mathbb{E}_G[p_\omega{}^{-\sigma}/[P(\mathbf{p};\sigma)]^{1-\sigma}]} > 1.$$

By evaluating this at the symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$,

$$\mathcal{S}(V) = \zeta(p;p\mathbf{1}_\Omega^{-1}) = \frac{\mathbb{E}_G[\sigma p^{-\sigma}/[P(p\mathbf{1}_\Omega^{-1};\sigma)]^{1-\sigma}]}{\mathbb{E}_G[p^{-\sigma}/[P(p\mathbf{1}_\Omega^{-1};\sigma)]^{1-\sigma}]} = \frac{\mathbb{E}_G[\sigma p^{-\sigma}/Vp^{1-\sigma}]}{\mathbb{E}_G[p^{-\sigma}/Vp^{1-\sigma}]} = \mathbb{E}_G[\sigma] > 1.$$

For the love-for-variety,

$$\mathcal{L}(V) \equiv -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} = \mathbb{E}_G\left[-\frac{d\ln P(\mathbf{1}_\Omega^{-1};\sigma)}{d\ln V}\right] = \mathbb{E}_G\left[\frac{1}{\sigma-1}\right].$$

The lower bound for $\mathcal{L}(V)$ is obtained from Jensen's inequality,

$$\mathcal{L}(V) = \mathbb{E}_G\left[\frac{1}{\sigma-1}\right] \geq \frac{1}{\mathbb{E}_G[\sigma]-1} = \frac{1}{\mathcal{S}(V)-1},$$

where the equality holds if and only if $G(\cdot)$ is degenerate. To see that $\mathcal{L}(V)$ is unbounded from above, fix σ_0 and ε, such that $0 < \varepsilon < \min\{1 - 1/\sigma_0\,, 1/\sigma_0\} \le 1/2$, and let G_ε be a two-point distribution, assigning the mass equal to $\frac{\varepsilon\,(1-\varepsilon)}{1-2\varepsilon}\left(\sigma_0 - \frac{1}{1-\varepsilon}\right) > 0$ to $\sigma = \frac{1}{\varepsilon}$, and the mass equal to $\frac{\varepsilon\,(1-\varepsilon)}{1-2\varepsilon}\left(\frac{1}{\varepsilon} - \sigma_0\right) > 0$ to $\sigma = \frac{1}{1-\varepsilon}$. Then,

$$\mathcal{S}(V) = \mathbb{E}_{G_\varepsilon}(\sigma) = \frac{(1-\varepsilon)}{1-2\varepsilon}\left(\sigma_0 - \frac{1}{1-\varepsilon}\right) + \frac{\varepsilon}{1-2\varepsilon}\left(\frac{1}{\varepsilon} - \sigma_0\right) = \sigma_0,$$

$$\mathcal{L}(V) = \mathbb{E}_{G_\varepsilon}\left(\frac{1}{\sigma-1}\right) = \frac{\varepsilon}{1-\varepsilon}\frac{\varepsilon\,(1-\varepsilon)}{1-2\varepsilon}\left(\sigma_0 - \frac{1}{1-\varepsilon}\right) + \frac{1-\varepsilon}{\varepsilon}\frac{\varepsilon\,(1-\varepsilon)}{1-2\varepsilon}\left(\frac{1}{\varepsilon} - \sigma_0\right)$$

$$= \frac{\varepsilon^2}{1-2\varepsilon}\left(\sigma_0 - \frac{1}{1-\varepsilon}\right) + \frac{(1-\varepsilon)^2}{1-2\varepsilon}\left(\frac{1}{\varepsilon} - \sigma_0\right).$$

Clearly, $\mathcal{L}(V)$ is not bounded from above as $\varepsilon \searrow 0$. This completes the proof of Theorem 1 for the GM-CES unit cost function.

Next, GM-CES production function, $X(\mathbf{x})$, defined by:

$$\ln X(\mathbf{x}) \equiv \int_1^\infty \ln X(\mathbf{x}; \sigma)\, dG(\sigma) \equiv \mathbb{E}_G[\ln X(\mathbf{x}; \sigma)]$$

where $X(\mathbf{x}; \sigma) \equiv \left[\int_\Omega {x_\omega}^{1-1/\sigma}\, d\omega\right]^{\sigma/(\sigma-1)}$. The inverse demand for ω is

$$p_\omega = P(\mathbf{p})\frac{\partial X(\mathbf{x})}{\partial x_\omega} = P(\mathbf{p})X(\mathbf{x})\frac{\partial \ln X(\mathbf{x})}{\partial x_\omega} = P(\mathbf{p})X(\mathbf{x})\mathbb{E}_G\left[\frac{\partial \ln X(\mathbf{x}; \sigma)}{\partial x_\omega}\right].$$

Thus, the budget share of ω is

$$s^*(x_\omega; \mathbf{x}) \equiv \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \mathbb{E}_G\left[\frac{\partial \ln X(\mathbf{x}; \sigma)}{\partial \ln x_\omega}\right] = \mathbb{E}_G\left[\left(\frac{x_\omega}{X(\mathbf{x}; \sigma)}\right)^{1-1/\sigma}\right].$$

Thus, the price elasticity of demand, $\zeta^*(x_\omega; \mathbf{x})$, satisfies

$$\zeta^*(x_\omega; \mathbf{x}) \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \left[1 - \frac{\partial \ln s^*(x_\omega;\ \mathbf{x})}{\partial \ln x_\omega}\right]^{-1} = \frac{\mathbb{E}_G\left[{x_\omega}^{-1/\sigma}/[X(\mathbf{x}; \sigma)]^{1-1/\sigma}\right]}{\mathbb{E}_G\left[{x_\omega}^{-1/\sigma}/\sigma[X(\mathbf{x}; \sigma)]^{1-1/\sigma}\right]} > 1.$$

By evaluating this at the symmetric quantity patterns $\mathbf{x} = x\mathbf{1}_\Omega$,

$$\frac{1}{\mathcal{S}(V)} \equiv \frac{1}{\zeta^*(x; x\mathbf{1}_\Omega)} \equiv \frac{\mathbb{E}_G\left[x^{-\frac{1}{\sigma}}/\sigma[X(x\mathbf{1}_\Omega; \sigma)]^{1-\frac{1}{\sigma}}\right]}{\mathbb{E}_G\left[x^{-\frac{1}{\sigma}}/[X(x\mathbf{1}_\Omega; \sigma)]^{1-\frac{1}{\sigma}}\right]} = \frac{\mathbb{E}_G\left[x^{-\frac{1}{\sigma}}/\sigma V x^{1-\frac{1}{\sigma}}\right]}{\mathbb{E}_G\left[x^{-\frac{1}{\sigma}}/V x^{1-\frac{1}{\sigma}}\right]} = \mathbb{E}_G\left[\frac{1}{\sigma}\right] < 1.$$

For the love-for-variety,

$$\mathcal{L}(V) \equiv \frac{d \ln X(\mathbf{1}_\Omega)}{d \ln V} - 1 = \mathbb{E}_G\left[\frac{d \ln X(\mathbf{1}_\Omega; \sigma)}{d \ln V} - 1\right] = \mathbb{E}_G\left[\frac{1}{\sigma-1}\right].$$

The lower bound for $\mathcal{L}(V)$ is obtained from Jensen's inequality,

$$\mathcal{L}(V) = \mathbb{E}_G\left[\frac{1}{\sigma - 1}\right] = \mathbb{E}_G\left[\frac{1/\sigma}{1 - 1/\sigma}\right] \geq \frac{\mathbb{E}_G[1/\sigma]}{1 - \mathbb{E}_G[1/\sigma]} = \frac{1/\mathcal{S}(V)}{1 - 1/\mathcal{S}(V)} = \frac{1}{\mathcal{S}(V) - 1},$$

where the equality holds if and only if $G(\cdot)$ is degenerate.

To see that $\mathcal{L}(V)$ is unbounded from above, consider the Pareto distribution of σ:

$$G(\sigma) = 1 - \left(\frac{\sigma_{min}}{\sigma}\right)^{\alpha}, \qquad \sigma \geq \sigma_{min} \equiv \frac{\alpha\sigma_0}{\alpha + 1} > 1,$$

where $\sigma_0 > 1$ and $\alpha > 1/(\sigma_0 - 1)$. The distribution and density of $x = 1/\sigma$ are given by

$$F(x) = (\sigma_{min}x)^{\alpha};\ f(x) = \alpha(\sigma_{min})^{\alpha}x^{\alpha-1}, \qquad x \in \left(0, \frac{1}{\sigma_{min}}\right).$$

Thus,

$$\frac{1}{\mathcal{S}(V)} = \mathbb{E}_G\left(\frac{1}{\sigma}\right) = \mathbb{E}_F(x) = \alpha(\sigma_{min})^{\alpha}\int_0^{1/\sigma_{min}} x^{\alpha}\,dx = \frac{1}{\sigma_{min}}\frac{\alpha}{\alpha + 1} = \frac{1}{\sigma_0} < 1;$$

$$\mathcal{L}(V) = \mathbb{E}_G\left(\frac{1}{\sigma - 1}\right) = \mathbb{E}_F\left(\frac{x}{1 - x}\right) = \mathbb{E}_F\left(\sum_{k=1}^{\infty} x^k\right) = \sum_{k=1}^{\infty}\mathbb{E}_F(x^k)$$

$$= \sum_{k=1}^{\infty}\alpha(\sigma_{min})^{\alpha}\int_0^{1/\sigma_{min}} x^{\alpha+k-1}\,dx$$

$$= \sum_{k=1}^{\infty}\frac{\alpha}{\alpha + k}\left(\frac{1}{\sigma_{min}}\right)^k = \sum_{k=1}^{\infty}\frac{\alpha}{\alpha + k}\left(\frac{\alpha + 1}{\alpha}\right)^k\left(\frac{1}{\sigma_0}\right)^k = \sum_{k=1}^{\infty}\frac{(1 + 1/\alpha)^k}{1 + k/\alpha}\left(\frac{1}{\sigma_0}\right)^k.$$

Hence,

$$\lim_{\alpha\to 1/(\sigma_0-1)}\mathcal{L}(V) = \lim_{\alpha\to 1/(\sigma_0-1)}\sum_{k=1}^{\infty}\frac{(1 + 1/\alpha)^k}{1 + k/\alpha}\left(\frac{1}{\sigma_0}\right)^k = \sum_{k=1}^{\infty}\frac{1}{1 + (\sigma_0 - 1)k}$$

$$> \int_1^{\infty}\frac{dz}{1 + (\sigma_0 - 1)z} = \frac{\ln[1 + (\sigma_0 - 1)z]}{(\sigma_0 - 1)}\Bigg]_1^{\infty} = \infty.$$

This completes the proof of Theorem 1 for the GM-CES production function.

Appendix C. The Three Classes of HSA/HDIA/HIIA.

This Appendix is divided into four subsections. The first three, Appendix C.1 on HSA, Appendix C.2. on HDIA, and Appendix C.3. on HIIA, are written in a way that they can be read independently, so that each of them can serve as the reference for the reader interested in using any of these three classes. Appendix C.4. discusses an alternative definition of HSA and proves that the two definitions are equivalent.

Appendix C.1. The Homothetic Single Aggregator (HSA) class.

Definition: A homothetic symmetric demand system for inputs with gross substitutes belongs to HSA (*Homothetic Single Aggregator*) if there exists a function of a single variable, $s: \mathbb{R}_{++} \rightarrow \mathbb{R}_{+}$, which is C^2 and strictly decreasing as long as $s(z) > 0$, with $\lim_{z\to 0} s(z) = \infty$ and $\lim_{z\to \bar{z}} s(z) = 0$, where $\bar{z} \equiv \inf\{z > 0 | s(z) = 0\}$, such that the budget share of $\omega \in \Omega$ can be written as:

$$s_\omega = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(\mathbf{p})}\right), \tag{12}$$

where $A(\mathbf{p})$ is defined implicitly by the adding-up constraint,

$$\int_\Omega s\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega \equiv 1. \tag{13}$$

By construction, $A(\mathbf{p})$ is linearly homogeneous in $\mathbf{p}$ for any fixed Ω and the budget shares of all inputs are added up to one.[24]

Before proceeding, it should be pointed out that there exists an alternative (but equivalent) definition of HSA For the sake of completeness, we discuss this alternative in Appendix C.4.

Some Special Cases: CES with gross substitutes is a special case where $s(z) = \gamma z^{1-\sigma}$ ($\sigma > 1$). Translog unit cost function is also a special case, where $s(z) = -\gamma \ln(z/\bar{z})$, for $\bar{z} < \infty$. Generalized Translog is defined by $s(z) = \gamma \left(1 - \frac{\sigma-1}{\eta} \ln(z)\right)^\eta =$

[24]For $s: \mathbb{R}_{++} \rightarrow \mathbb{R}_{+}$, satisfying the above conditions, a class of the budget share functions, $s_\gamma(z) \equiv \gamma s(z)$ for $\gamma > 0$, generate the same demand system with the same common price aggregator. This can be seen by reindexing the inputs, as $\omega' = \gamma\omega$, so that $\int_\Omega s_\gamma(p_\omega/A(\mathbf{p}))d\omega = \int_\Omega s(p_{\omega'}/A(\mathbf{p}))d\omega' = 1$. In this sense, $s_\gamma(z) \equiv \gamma s(z)$ for $\gamma > 0$ are all equivalent. Note also that a class of the budget share functions, $s_\lambda(z) \equiv s(\lambda z)$ for $\lambda > 0$, generate the same demand system, with $A_\lambda(\mathbf{p}) = \lambda A(\mathbf{p})$, because $s_\lambda(p_\omega/A_\lambda(\mathbf{p})) = s(\lambda p_\omega/A_\lambda(\mathbf{p})) = s(p_\omega/A(\mathbf{p}))$. In this sense, $s_\lambda(z) \equiv s(\lambda z)$ for $\lambda > 0$ are all equivalent.

$\gamma\left(-\frac{\sigma-1}{\eta}\ln(z/\bar{z})\right)^{\eta}$, $(\eta > 0)$, for $z < \bar{z} \equiv e^{\frac{\eta}{\sigma-1}}$, which is equivalent to translog for $\eta = 1$ and contains CES as the limit case, as $\eta \to \infty$.

Various families of HSA have been recently applied to a variety of monopolistic competition models.[25]

Key Properties:

Eqs. (12)-(13) state that the budget share of $\omega \in \Omega$ is decreasing in its *normalized price,* $z_\omega \equiv p_\omega/A(\mathbf{p})$, which is defined as its own price, p_ω, divided by the *common price aggregator,* $A(\mathbf{p})$. Note that the budget share function, $s(\cdot)$, is the primitive of HSA, while $A(\mathbf{p})$ is not, because it needs to be derived from $s(\cdot)$, using eq.(13). The monotonicity of $s(\cdot)$, combined with the assumptions, $\lim_{z\to 0} s(z) = \infty$ and $\lim_{z\to\bar{z}} s(z) = 0$, ensures that $A(\mathbf{p})$ is defined uniquely by eq.(13) for any $V \equiv |\Omega| > 0$. $A(\mathbf{p})$ is independent of ω, and thus captures "the average price" against which the prices of *all* inputs in Ω are measured. In other words, one could keep track of all the cross-price effects in the demand system by looking at a single aggregator, $A(\mathbf{p})$, which is the key feature of HSA[26] Note also that we allow for the possibility of $\bar{z} < \infty$, so that the budget share of ω is zero for $p_\omega \geq \bar{z}A(\mathbf{p})$. That is, $\bar{z}A(\mathbf{p})$ is the choke price. If $\bar{z} = \infty$, there is no choke price and the budget share for each input remains strictly positive for any positive price vector.

The price elasticity of $\omega \in \Omega$ can be written as a function of $z_\omega \equiv p_\omega/A(\mathbf{p}) < \bar{z}$ as

$$\zeta_\omega = \zeta(p_\omega; \mathbf{p}) = 1 - \frac{z_\omega s'(z_\omega)}{s(z_\omega)} \equiv \zeta^S(z_\omega) = \zeta^S\left(\frac{p_\omega}{A(\mathbf{p})}\right) > 1,$$

---

[25]See, e.g., Baqaee, Farhi, and Sangani (2024), Errico and Lashkari (2025), Fujiwara and Matsuyama (2026), Grossman, Helpman, and Lhuillier (2023), Matsuyama and Ushchev (2020a, 2020b, 2022a, 2022b), Ren and Zhang (2025). A large literature on monopolistic competition models under translog demand systems, which follows Feenstra (2003), may also be added to this list, because a symmetric translog unit cost function is a special case of symmetric HSA with gross substitutes.

[26]In contrast, that $s(\cdot)$ is independent of ω is not a defining feature of HSA, but due to the assumption that the underlying production function is symmetric. Generally, the HSA class of the production functions is defined by the property that the budget share of ω is given by $s_\omega(p_\omega/A(\mathbf{p}))$, where $A(\mathbf{p})$ is the unique solution to $\int_\Omega s_\omega(p_\omega/A(\mathbf{p}))d\omega = 1$. Note that $s_\omega(\cdot)$ may depend on ω but $A(\cdot)$ may not.

where $\zeta^S: (0, \bar{z}) \to (1, \infty)$ is C^1 for $z \in (0, \bar{z})$, and $\lim_{z \to \bar{z}} \zeta^S(z) = \infty$ if $\bar{z} < \infty$.[27] It turns out to be convenient to introduce another function, $H: (0, \bar{z}) \to \mathbb{R}_+$,

$$H(z) \equiv \int_z^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi > 0,$$

which satisfies $H'(z) < 0$ and $H''(z) > 0$, so that

$$\zeta^S(z) \equiv 1 - \frac{zs'(z)}{s(z)} \equiv -\frac{zH''(z)}{H'(z)} > 1. \tag{14}$$

In general, $\zeta^S(\cdot)$ can be nonmonotonic. Under CES, it is constant, $\zeta^{S\prime}(\cdot) = 0$. The 2nd law, $\partial\zeta(p_\omega; \mathbf{p})/\partial p_\omega > 0$, holds if and only if $\zeta^{S\prime}(\cdot) > 0$. The 2nd law holds for Generalized Translog with $\zeta^S(z_\omega) = 1 - \eta[\ln(z_\omega/\bar{z})]^{-1}$.

After deriving $A(\mathbf{p})$ from $s(\cdot)$, the unit cost function, $P(\mathbf{p})$, can be obtained by integrating eq.(12), which yields

$$\ln\left[\frac{A(\mathbf{p})}{cP(\mathbf{p})}\right] = \int_\Omega \left[\int_{p_\omega/A(\mathbf{p})}^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi\right] d\omega \equiv \int_\Omega H\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega \tag{15}$$

$$\equiv \int_\Omega s\left(\frac{p_\omega}{A(\mathbf{p})}\right) \Phi\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega$$

where

$$\Phi(z) \equiv \frac{1}{s(z)} \int_z^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi \equiv -\frac{H(z)}{zH'(z)} > 0$$

and c is a positive constant, which is proportional to TFP.[28] The unit cost function, $P(\mathbf{p})$, satisfies linear homogeneity, monotonicity, and strict quasi-concavity, and so does the corresponding production function, $X(\mathbf{x})$. This follows from Matsuyama and Ushchev (2017; Proposition 1-i)) and guarantees the integrability (in the sense of Samuelson 1950

[27]Conversely, from any C^1 function $\zeta^S: (0, \bar{z}) \to (1 + \varepsilon, \infty)$, satisfying $\lim_{z \to \bar{z}} \zeta^S(z) = \infty$ if $\bar{z} < \infty$, one could reverse-engineer as $s(z) = \gamma \exp\left[\int_{z_0}^{z} [1 - \zeta^S(\xi)]\, d\xi/\xi\right] > 0;\ z_0, z \in (0, \bar{z})$, where $\gamma = s(z_0)$ is a positive constant. One could thus use $\zeta^S(\cdot)$ instead of $s(\cdot)$, as a primitive of symmetric HSA with gross substitutes.
[28]The constant term in eq.(15), which appears by integrating eq.(12), cannot be pinned down. First, $A(\mathbf{p})$, the "average input price", depends on the unit of measurement of inputs, but not on the unit of measurement of the final good. In contrast, $P(\mathbf{p})$ is the cost of producing one unit of the final good, when the input prices are given by $\mathbf{p}$. Hence, it depends not only on the unit of measurement of inputs but also on that of the final good. Second, a change in TFP, though it affects $P(\mathbf{p})$, leaves the budget share of each input, and hence $A(\mathbf{p})$, unaffected.

and Hurwicz and Uzawa 1971) of HSA demand systems. It is important to note that, with the sole exception of CES, $A(\mathbf{p})/P(\mathbf{p})$ is not constant and depends on $\mathbf{p}$.[29] This can be verified by differentiating eq.(13) to yield

$$\frac{\partial \ln A(\mathbf{p})}{\partial \ln p_\omega} = \frac{z_\omega s'(z_\omega)}{\int_\Omega s'(z_{\omega'}) z_{\omega'} d\omega'} = \frac{[\zeta^S(z_\omega) - 1] s(z_\omega)}{\int_\Omega [\zeta^S(z_{\omega'}) - 1] s(z_{\omega'}) d\omega'},$$

which differs from

$$\frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s(z_\omega),$$

unless $\zeta^S(z)$ is independent of z, or equivalently, $s(z) = \gamma z^{1-\sigma}$ with $\zeta^S(z) = \sigma > 1$. This should not come as a surprise. After all, $A(\mathbf{p})$ is the "average input price", which captures the *cross-product effects* in the demand system, while $P(\mathbf{p})$ is the inverse measure of TFP, which captures the *productivity effects* of price changes. There is no reason to think *a priori* that they should move together.

Substitutability and Love-for-variety Measures under HSA:

For symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$, eq.(13) is simplified to

$$s\left(\frac{1}{A(\mathbf{1}_\Omega^{-1})}\right) V = 1 \Rightarrow \frac{1}{A(\mathbf{1}_\Omega^{-1})} = s^{-1}\left(\frac{1}{V}\right)$$

Hence, from eq.(14), the substitutability measure is given by:

$$\mathcal{S}(V) \equiv \zeta(1; \mathbf{1}_\Omega^{-1}) = \zeta^S\left(s^{-1}\left(\frac{1}{V}\right)\right) = -\frac{zH''(z)}{H'(z)}\bigg|_{z=s^{-1}(1/V)} > 1. \tag{16}$$

For the love-for-variety measure, from eq. (15),

$$-\ln[P(\mathbf{1}_\Omega^{-1})] = \ln\left[cs^{-1}\left(\frac{1}{V}\right)\right] + \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) \Rightarrow$$

$$-\frac{d \ln P(\mathbf{1}_\Omega^{-1})}{d \ln V} = -\left[\frac{d[\ln z + \Phi(z)]}{d \ln z} \Big/ \frac{d \ln s(z)}{d \ln z}\right]\bigg|_{z=s^{-1}(1/V)} = \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right)$$

so that

[29] This holds more generally, that is, for asymmetric HSA, as well as HSA with gross complements, as shown in Matsuyama and Ushchev (2017; Proposition 1-iii).

$$\mathcal{L}(V) \equiv -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} = \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) = -\frac{H(z)}{zH'(z)}\bigg|_{z=s^{-1}(1/V)}. \quad (17)[30]$$

The expressions for $\mathcal{S}(V)$ and $\mathcal{L}(V)$ under HSA in eq.(16) and eq. (17) are also displayed in Table 1.[31] Since $s^{-1}(1/V)$ is monotonically increasing in V,

$$\zeta^{S\prime}(\cdot) \gtreqless 0 \Leftrightarrow \mathcal{S}'(\cdot) \gtreqless 0; \qquad \Phi'(\cdot) \gtreqless 0 \Leftrightarrow \mathcal{L}'(\cdot) \gtreqless 0.$$

Proposition S-1 shows the relation between $\zeta^S(z)$ and $\Phi(z)$.

Proposition S-1:

$$\frac{z\Phi'(z)}{\Phi(z)} = \zeta^S(z) - 1 - \frac{1}{\Phi(z)} = \zeta^S(z) - \int_z^{\bar{z}} \zeta^S(\xi)w(\xi;z)d\xi.$$

where $w^S(\xi;z) \equiv -H'(\xi)/H(z)$, which satisfies $\int_z^{\bar{z}} w^S(\xi;z)d\xi = 1$. Hence,

$$\zeta^{S\prime}(z) \gtreqless 0, \forall z \in (z_0,\bar{z}) \implies \Phi'(z) \lesseqgtr 0, \forall z \in (z_0,\bar{z}).$$

The opposite is not true in general. However,

$$\zeta^{S\prime}(z) = 0, \forall z \in (z_0,\bar{z}) \Leftrightarrow \Phi'(z) = 0, \forall z \in (z_0,\bar{z}).$$

The proof of Proposition S-1 is in Appendix D.

By combining Proposition S-1, eq.(16) and eq.(17),

Proposition S-2: For $s(z_0)V_0 = 1$,

$$\zeta^{S\prime}(z) \gtreqless 0, \forall z \in (z_0,\bar{z}) \Leftrightarrow \mathcal{S}'(V) \gtreqless 0, \forall V \in (V_0,\infty);$$

$$\Phi'(z) \lesseqgtr 0, \forall z \in (z_0,\bar{z}) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0,\infty).$$

Moreover,

$$\mathcal{S}'(V) \gtreqless 0, \forall V \in (V_0,\infty) \Rightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0,\infty).$$

The opposite is not true in general. However,

$$\mathcal{S}'(V) = 0, \forall V \in (V_0,\infty) \Leftrightarrow \mathcal{L}'(V) = 0, \forall V \in (V_0,\infty).$$

Proposition S-3:

$$\mathcal{L}'(V) \lesseqgtr 0 \Leftrightarrow \frac{z\Phi'(z)}{\Phi(z)} = \zeta^S(z) - 1 - \frac{1}{\Phi(z)} \lesseqgtr 0 \Leftrightarrow \mathcal{L}(V) \lesseqgtr \frac{1}{\mathcal{S}(V)-1}.$$

The HSA portions of Theorems 2 and 3 follow from Propositions S-2 and S-3, respectively. For the HSA portion of Theorem 4, note

[30]Moreover, by evaluating eq.(15) at the symmetric price patterns, we can show $\mathcal{L}(V) = \ln[A(\mathbf{1}_\Omega^{-1})/cP(\mathbf{1}_\Omega^{-1})]$.

[31]Table 1 also displays $\mathcal{S}(V)$ and $\mathcal{L}(V)$ under Translog and Generalized Translog, special cases of HSA

$$\lim_{V\to\infty} \mathcal{L}(V) = \lim_{V\to\infty} \Phi\left(s^{-1}(1/V)\right) = \lim_{z\to\bar{z}} \Phi(z) = \lim_{z\to\bar{z}} \frac{1}{s(z)} \int_{z}^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi.$$

Hence, by applying L'Hôpital's rule,

$$\lim_{V\to\infty} \mathcal{L}(V) = \lim_{z\to\bar{z}} \left(-\frac{s(z)}{zs'(z)}\right) = \lim_{z\to\bar{z}} \frac{1}{\zeta^{s}(z)-1} = \lim_{V\to\infty} \frac{1}{\mathcal{S}(V)-1}.$$

Appendix C.2. The Homothetic Direct Implicit Additivity (HDIA) class

Definition: A homothetic symmetric demand system for inputs with gross substitutes belongs to HDIA (*Homothetic Direct Implicit Additivity*) if it is generated by the cost minimization of the competitive industry whose CRS production function, $X = X(\mathbf{x}) \equiv Z\hat{X}(\mathbf{x})$ can be defined implicitly by:

$$\int_{\Omega} \phi\left(\frac{Zx_{\omega}}{X(\mathbf{x})}\right) d\omega = \int_{\Omega} \phi\left(\frac{x_{\omega}}{\hat{X}(\mathbf{x})}\right) d\omega \equiv 1. \tag{18}$$

Here $\phi(\cdot)\colon \mathbb{R}_{+} \to \mathbb{R}_{+}$ is C^3, with $\phi'(y) > 0 > \phi''(y)$, $-y\phi''(y)/\phi'(y) < 1$ for $\forall y \in (0,\infty)$ and $\phi(0) = 0$, $\phi(\infty) = \infty$, $\phi'(\infty) = 0$, and independent of $Z > 0$, the TFP parameter. Note that, unlike eq.(13), the adding-up constraint of the HSA, eq.(18) defines the production function $X(\mathbf{x})$ directly.[32] CES with gross substitutes is a special case where $\phi(y) = (y)^{1-1/\sigma}$ ($\sigma > 1$). Symmetric HDIA defined as above may be viewed as an extension of the Kimball (1995) aggregator in that the set of available inputs Ω, and its Lebesgue measure, $V \equiv |\Omega|$, can vary.

Key Properties:

From the cost minimization problem, eq.(1), subject to eq.(18), we obtain the inverse demand curve,

$$p_{\omega} = B(\mathbf{p})\phi'\left(\frac{Zx_{\omega}}{X(\mathbf{x})}\right) = B(\mathbf{p})\phi'\left(\frac{x_{\omega}}{\hat{X}(\mathbf{x})}\right), \tag{19}$$

and hence the demand curve,

[32] This means that, unlike HSA but similar to HIIA defined in Appendix C.3, we do not need to worry about the integrability of HDIA. Note also that $\hat{X}(\mathbf{x}) = X(\mathbf{x})/Z$ defined by eq.(18) is invariant of TFP, $Z > 0$, by construction. Thus, an increase in Z causes a proportionate increase in $X(\mathbf{x})$. This allows us to examine the effect of TFP without shifting $\phi(\cdot)$. Alternatively, we could have defined $X(\mathbf{x})$ by $\int_{\Omega} \phi(x_{\omega}/X(\mathbf{x}))d\omega = 1$, as in Matsuyama and Ushchev (2017). Though mathematically equivalent, this definition requires that $\phi(\cdot)$ no longer be independent of TFP, which would make it harder to show that $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are independent of TFP.

$$x_\omega = (\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\hat{X}(\mathbf{x}) = (\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\frac{X(\mathbf{x})}{Z},$$

where $B(\mathbf{p})$ is defined by:

$$\int_\Omega \phi\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right)d\omega \equiv 1.$$

This shows that the choke price is equal to $B(\mathbf{p})\phi'(0)$ if $\phi'(0) < \infty$, and that there is no choke price if $\phi'(0) = \infty$. The unit cost function is:

$$P(\mathbf{p}) = \frac{\hat{P}(\mathbf{p})}{Z} \equiv \frac{1}{Z}\int_\Omega p_\omega (\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)d\omega.$$

Clearly, both $B(\mathbf{p})$ and $\hat{P}(\mathbf{p})$ are linearly homogeneous in $\mathbf{p}$, and independent of $Z > 0$. Hence, an increase in TFP, Z, causes a proportional decline in the unit cost function, $P(\mathbf{p}) = \hat{P}(\mathbf{p})/Z$.

The budget share, $s_\omega = s(p_\omega; \mathbf{p}) = s^*(x_\omega; \mathbf{x})$, is:

$$\frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{p_\omega}{\hat{P}(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) = \phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right)\frac{x_\omega}{C^*(\mathbf{x})}. \tag{20}$$

where

$$C^*(\mathbf{x}) \equiv \int_\Omega \phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right)x_\omega d\omega$$

is linearly homogeneous in $\mathbf{x}$ and independent of $Z > 0$, and satisfies the identity

$$\frac{\hat{P}(\mathbf{p})}{B(\mathbf{p})} = \int_\Omega \frac{p_\omega}{B(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)d\omega = \int_\Omega \phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right)\frac{x_\omega}{\hat{X}(\mathbf{x})}d\omega = \frac{C^*(\mathbf{x})}{\hat{X}(\mathbf{x})}. \tag{21}$$

Eqs.(20)-(21) show that the budget share under HDIA is a function of the two relative prices, $p_\omega/\hat{P}(\mathbf{p})$ and $p_\omega/B(\mathbf{p})$, or a function of the two relative quantities, $x_\omega/\hat{X}(\mathbf{x})$ and $x_\omega/C^*(\mathbf{x})$, unless $\hat{P}(\mathbf{p})/B(\mathbf{p}) = C^*(\mathbf{x})/\hat{X}(\mathbf{x})$ is a positive constant, which occurs if and only if it is CES. Thus, HDIA and HSA do not overlap with the sole exception of CES.[33]

From the inverse demand curve, eq.(19), the price elasticity of demand can be written as a function of a single variable, $y_\omega \equiv x_\omega/\hat{X}(\mathbf{x})$ as:

$$\zeta_\omega = \zeta^*(x_\omega; \boldsymbol{x}) = -\frac{\phi'(y_\omega)}{y\phi''(y_\omega)} \equiv \zeta^D(y_\omega) = \zeta^D\left(\frac{x_\omega}{\hat{X}(\boldsymbol{x})}\right) > 1 \tag{22}$$

[33]This statement is a special case of Proposition 2-(ii) in Matsuyama and Ushchev (2017).

where $\zeta^D(y) > 1$ ensures gross substitutability. Using eq.(19), it can also be written as a function of $p_\omega/B(\mathbf{p}) = \phi'(y_\omega)$ as:

$$\zeta_\omega = \zeta(p_\omega; \mathbf{p}) = \zeta^D\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right) > 1.$$

Under CES, $\zeta^{D'}(\cdot) = 0$. The 2nd law, $\partial\zeta^*(x_\omega; \mathbf{x})/\partial x_\omega < 0$, holds if and only if $\zeta^{D'}(\cdot) < 0$.

Substitutability and Love-for-variety measures under HDIA:

For symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$, eq.(18) is simplified to

$$\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right)V = 1 \;\Rightarrow \frac{1}{\hat{X}(\mathbf{1}_\Omega)} = \phi^{-1}\left(\frac{1}{V}\right).$$

Hence, from eq.(22), the substitutability measure is given by:

$$\mathcal{S}(V) \equiv \zeta^*(1; \mathbf{1}_\Omega) = \;\zeta^D\left(\phi^{-1}\left(\frac{1}{V}\right)\right) = -\left.\frac{\phi'(y)}{y\phi''(y)}\right|_{y=\phi^{-1}(1/V)} > 1. \tag{23}$$

The love-for-variety measure under HDIA is given by:

$$\mathcal{L}(V) \equiv \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 = \frac{1}{\mathcal{E}_\phi\left(\phi^{-1}(1/V)\right)} - 1 \equiv \left.\frac{\phi(y)}{y\phi'(y)}\right|_{y=\phi^{-1}(1/V)} - 1 > 0 \tag{24}$$

where

$$0 < \mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)} < 1.$$ [34]

The expressions for $\mathcal{S}(V)$ and $\mathcal{L}(V)$ under HDIA in eq.(23) and eq.(24) are also displayed in Table 1. Since $\phi^{-1}(1/V)$ is monotonically decreasing in V,

$$\zeta^{D\prime}(\cdot) \lesseqqgtr 0 \Leftrightarrow \mathcal{S}'(\cdot) \gtreqqless 0; \quad \mathcal{E}'_\phi(\cdot) \gtreqqless 0 \Leftrightarrow \mathcal{L}'(\cdot) \lesseqqgtr 0,$$

Proposition D-1 shows the relation between $\zeta^D(y)$ and $\mathcal{E}_\phi(y)$.

> **Proposition D-1:**
>
> $$\frac{y\mathcal{E}'_\phi(y)}{\mathcal{E}_\phi(y)} = 1 - \frac{1}{\zeta^D(y)} - \mathcal{E}_\phi(y) = \int_0^y \left[\frac{1}{\zeta^D(\xi)}\right] w^D(\xi; y)d\xi - \frac{1}{\zeta^D(y)},$$
>
> where $w^D(\xi; y) \equiv \phi'(\xi)/\phi(y) > 0$, which satisfies $\int_0^y w^D(\xi; y)\,d\xi = 1$. Hence,

[34]Moreover, by evaluating eq.(21) at the symmetric price and quantity patterns, one can show that
$$\frac{\hat{P}(\mathbf{1}_\Omega^{-1})}{B(\mathbf{1}_\Omega^{-1})} = \frac{C^*(\mathbf{1}_\Omega)}{\hat{X}(\mathbf{1}_\Omega)} = \int_\Omega \mathcal{E}_\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right)\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right)d\omega = \mathcal{E}_\phi\left(\phi^{-1}\left(\frac{1}{V}\right)\right) \Rightarrow \mathcal{L}(V) = \frac{B(\mathbf{1}_\Omega^{-1})}{\hat{P}(\mathbf{1}_\Omega^{-1})} - 1 = \frac{\hat{X}(\mathbf{1}_\Omega)}{C^*(\mathbf{1}_\Omega)} - 1.$$

$$\zeta^{D'}(y) \lesseqgtr 0, \forall y \in (0, y_0) \Rightarrow \mathcal{E}'_\phi(y) \lesseqgtr 0, \forall y \in (0, y_0).$$

The opposite is not true in general. However,

$$\zeta^{D'}(y) = 0, \forall y \in (0, y_0) \Leftrightarrow \mathcal{E}'_\phi(y) = 0, \forall y \in (0, y_0).$$

The proof of Proposition D-1 is in Appendix D.

By combining Proposition D-1, eq.(23) and eq.(24),

Proposition D-2: For $\phi(y_0)V_0 = 1$,

$$\zeta^{D'}(y) \lesseqgtr 0\ \forall y \in (0, y_0) \Leftrightarrow \mathcal{S}'(V) \gtreqless 0, \forall V \in (V_0, \infty);$$

$$\mathcal{E}'_\phi(y) \lesseqgtr 0, \forall y \in (0, y_0) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

Moreover,

$$\mathcal{S}'(V) \gtreqless 0, \forall V \in (V_0, \infty) \Rightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

The opposite is not true in general. However,

$$\mathcal{S}'(V) = 0, \forall V \in (V_0, \infty) \Leftrightarrow \mathcal{L}'(V) = 0, \forall V \in (V_0, \infty).$$

Proposition D-3:

$$\mathcal{L}'(V) \lesseqgtr 0 \Leftrightarrow \frac{y\mathcal{E}'_\phi(y)}{\mathcal{E}_\phi(y)} = 1 - \frac{1}{\zeta^D(y)} - \mathcal{E}_\phi(y) \lesseqgtr 0 \Leftrightarrow \mathcal{L}(V) \lesseqgtr \frac{1}{\mathcal{S}(V) - 1}.$$

The HDIA portions of Theorems 2 and 3 follow from Propositions D-2 and D-3, respectively. For the HDIA portion of Theorem 4, note

$$\lim_{V\to\infty} \mathcal{L}(V) = \lim_{V\to\infty} \frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/V))} - 1 = \lim_{y\to 0} \frac{\phi(y)}{y\phi'(y)} - 1.$$

Hence, by applying L' Hôpital's rule,

$$\lim_{V\to\infty} \mathcal{L}(V) = \lim_{y\to 0} \frac{\phi'(y)}{\phi'(y) + y\phi''(y)} - 1 = \lim_{y\to 0} \frac{\zeta^D(y)}{\zeta^D(y) - 1} - 1 = \lim_{V\to\infty} \frac{1}{\mathcal{S}(V) - 1}.$$

Appendix C.3. The Homothetic Indirect Implicit Additivity (HIIA) class.

Definition: A homothetic symmetric demand system for inputs with gross substitutes belongs to HIIA (*Homothetic Indirect Implicit Additivity*) if it is generated by the cost minimization of the competitive industry whose unit cost function, $P = P(\mathbf{p}) = \hat{P}(\mathbf{p})/Z$, can be defined implicitly by:

$$\int_\Omega \theta\left(\frac{p_\omega}{ZP(\mathbf{p})}\right) d\omega = \int_\Omega \theta\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) d\omega = 1, \tag{25}$$

where $\theta(\cdot): \mathbb{R}_{++} \to \mathbb{R}_{+}$ is C^3, with $\theta'(z) < 0 < \theta''(z)$, and $-z\theta''(z)/\theta'(z) > 1$, for $\theta(z) > 0$ with $\lim_{z\to 0}\theta(z) = \infty$ and $\lim_{z\to\bar{z}}\theta(z) = 0$, where $\bar{z} \equiv \inf\{z > 0 | \theta(z) = \theta'(z) = 0\}$, and it is independent of $Z > 0$, the TFP parameter. If $\bar{z} < \infty$, the choke price is equal to $\hat{P}(\mathbf{p})\bar{z} = ZP(\mathbf{p})\bar{z}$, and it satisfies $\lim_{z\to\bar{z}}[-z\theta''(z)/\theta'(z)] = \infty$. If $\bar{z} = \infty$, the choke price does not exist and demand for each input always remains positive for any positive price vector. Note that, unlike eq.(13), the adding-up constraint of HSA, eq.(25) defines the unit cost function $P(\mathbf{p})$ directly.[35] CES with gross substitutes is a special case where $\theta(z) = (z)^{1-\sigma}$ ($\sigma > 1$).

Key Properties:

From the minimization problem, eq.(3), subject to eq.(25), we obtain the demand curve,

$$x_\omega = -B^*(\mathbf{x})\theta'\left(\frac{p_\omega}{ZP(\mathbf{p})}\right) = -B^*(\mathbf{x})\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) > 0 \tag{26}$$

for $z < \bar{z}$, and hence the inverse demand curve,

$$p_\omega = \hat{P}(\mathbf{p})(-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right) = ZP(\mathbf{p})(-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right),$$

where $B^*(\mathbf{x}) > 0$ is defined by

$$\int_\Omega \theta\left((-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\right)d\omega \equiv 1.$$

Thus, the choke price is $\hat{P}(\mathbf{p})\bar{z} = ZP(\mathbf{p})\bar{z}$, if $\bar{z} < \infty$. The production function is

$$X(\mathbf{x}) = Z\hat{X}(\mathbf{x}) \equiv Z\int_\Omega (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)x_\omega d\omega.$$

Clearly, both $B^*(\mathbf{x})$ and $\hat{X}(\mathbf{x})$ are linearly homogeneous in $\mathbf{x}$ and independent of $Z > 0$, by construction. Thus, an increase in TFP, Z, causes a proportional increase in $X(\mathbf{x}) = Z\hat{X}(\mathbf{x})$.

The budget share, $s_\omega = s(p_\omega; \boldsymbol{p}) = s^*(x_\omega; \mathbf{x})$, is

[35]This means that, unlike HSA but similar to HDIA defined in Appendix C.2, we do not need to worry about the integrability of HIIA. Note also that $\hat{P}(\mathbf{p}) = ZP(\mathbf{p})$ defined by eq.(25) is invariant of TFP, $Z > 0$, by construction. Thus, an increase in Z causes a proportionate decline in $P(\mathbf{p})$. This allows us to examine the effect of TFP without shifting $\theta(\cdot)$. Alternatively, we could define $P(\mathbf{p})$ by $\int_\Omega \theta(p_\omega/P(\mathbf{p}))d\omega = 1$, as in Matsuyama and Ushchev (2017). Though mathematically equivalent, this definition requires that $\theta(\cdot)$ no longer be independent of TFP, which would make it harder to show that $\mathcal{S}(V)$ and $\mathcal{L}(V)$ are independent of TFP.

$$\frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{p_\omega}{C(\mathbf{p})}\left[-\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)\right] = (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\frac{x_\omega}{\hat{X}(\mathbf{x})}, \tag{27}$$

where

$$C(\mathbf{p}) \equiv \int_\Omega p_\omega\left[-\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)\right]d\omega > 0$$

is linearly homogeneous in $\mathbf{p}$, and independent of $Z > 0$ and satisfies the identity,

$$\frac{C(\mathbf{p})}{\hat{P}(\mathbf{p})} = \int_\Omega \frac{p_\omega}{\hat{P}(\mathbf{p})}\left[-\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)\right]d\omega = \int_\Omega (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\frac{x_\omega}{B^*(\mathbf{x})}d\omega = \frac{\hat{X}(\mathbf{x})}{B^*(\mathbf{x})}. \tag{28}$$

Eqs.(27)-(28) show that the budget share under HIIA is a function of the two relative prices, $p_\omega/\hat{P}(\mathbf{p})$ and $p_\omega/C(\mathbf{p})$, or a function of the two relative quantities, $x_\omega/\hat{X}(\mathbf{x})$ and $x_\omega/B^*(\mathbf{x})$, unless $C(\mathbf{p})/\hat{P}(\mathbf{p}) = \hat{X}(\mathbf{x})/B^*(\mathbf{x})$ is a positive constant, which occurs if and only if it is CES. Thus, HIIA and HSA do not overlap with the sole exception of CES.[36] Furthermore, by comparing the expressions for the budget share under HDIA and the budget share under HIIA, one could see that HDIA and HIIA do not overlap with the sole exception of CES.[37]

From the demand curve, eq.(26), the price elasticity of demand can be written as a function of a single variable, $z_\omega \equiv p_\omega/\hat{P}(\mathbf{p})$, as:

$$\zeta_\omega = \zeta(p_\omega; \boldsymbol{p}) = -\frac{z_\omega\theta''(z_\omega)}{\theta'(z_\omega)} \equiv \zeta^I(z_\omega) = \zeta^I\left(\frac{p_\omega}{\hat{P}(\boldsymbol{p})}\right) > 1, \tag{29}$$

where $\zeta^I(z) > 1$ ensures gross substitutability. Using eq.(26), it can also be written as a function of $x_\omega/B^*(\mathbf{x}) = -\theta'(z_\omega)$ as:

$$\zeta_\omega \equiv \zeta^*(x_\omega;\ \mathbf{x}) = \zeta^I\left((-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\right) > 1.$$

Under CES, $\zeta^{I'}(\cdot) = 0$. The 2nd law, $\partial\zeta(p_\omega; \mathbf{p})/\partial p_\omega > 0$, holds if and only if $\zeta^{I'}(\cdot) > 0$.

Substitutability and Love-for-variety measures under HIIA:

For symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$, eq.(25) is simplified to

$$\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)V = 1 \Rightarrow \frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})} = \theta^{-1}\left(\frac{1}{V}\right).$$

[36] This statement is a special case of Proposition 3-(ii) in Matsuyama and Ushchev (2017).

[37] This statement is a special case of Proposition 4-(iii) in Matsuyama and Ushchev (2017).

Hence, from eq.(29), the substitutability measure under HIIA is given by:

$$\mathcal{S}(V) \equiv \zeta(1;\mathbf{1}_\Omega^{-1}) = \zeta^I\left(\theta^{-1}\left(\frac{1}{V}\right)\right) = -\frac{z\theta''(z)}{\theta'(z)}\bigg|_{z=\theta^{-1}(1/V)} > 1. \tag{30}$$

The love-for-variety measure under HIIA is given by:

$$\mathcal{L}(V) \equiv -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} = \frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/V)\right)} \equiv -\frac{\theta(z)}{z\theta'(z)}\bigg|_{z=\theta^{-1}(1/V)} > 0. \tag{31}$$

where

$$\mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} > 0.$$ [38]

The expressions for $\mathcal{S}(V)$ and $\mathcal{L}(V)$ under HIIA in eq.(30) and eq.(31) are also displayed in Table 1. Since $\theta^{-1}(1/V)$ is monotonically increasing in V,

$$\zeta^{I'}(\cdot) \gtreqless 0 \Leftrightarrow \mathcal{S}'(\cdot) \gtreqless 0; \quad \mathcal{E}_\theta'(\cdot) \lesseqgtr 0 \Leftrightarrow \mathcal{L}'(\cdot) \gtreqless 0.$$

Proposition I-1 shows the relation between $\zeta^I(z)$ and $\mathcal{E}_\theta(z)$.

Proposition I-1:

$$\frac{z\mathcal{E}_\theta'(z)}{\mathcal{E}_\theta(z)} = \mathcal{E}_\theta(z) + 1 - \zeta^I(z) = \int_z^{\overline{z}} \zeta^I(\xi)w^I(\xi;z)d\xi - \zeta^I(z).$$

where $w^I(\xi;z) \equiv -\theta'(\xi)/\theta(z) > 0$, which satisfies $\int_z^{\overline{z}} w^I(\xi;z)\,d\xi = 1$.

$$\zeta^{I'}(z) \gtreqless 0, \forall z \in (z_0,\overline{z}) \quad \Rightarrow \quad \mathcal{E}_\theta'(z) \gtreqless 0, \forall z \in (z_0,\overline{z}).$$

The opposite is not true in general. However,

$$\zeta^{I'}(z) = 0, \forall z \in (z_0,\overline{z}) \quad \Leftrightarrow \quad \mathcal{E}_\theta'(z) = 0, \forall z \in (z_0,\overline{z}).$$

The proof of Proposition I-1 is in Appendix D.

By combining Proposition I-1, eq.(30) and eq.(31),

Proposition I-2: For $\theta(z_0)V_0 = 1$,

$$\zeta^{I'}(z) \gtreqless 0, \forall z \in (z_0,\overline{z}) \Leftrightarrow \mathcal{S}'(V) \gtreqless 0, \forall V \in (V_0,\infty);$$

$$\mathcal{E}_\theta'(z) \gtreqless 0, \forall z \in (z_0,\overline{z}) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0,\infty).$$

Moreover,

[38] Moreover, by evaluating eq.(28) at the symmetric price and quantity patterns, one can show that

$$\frac{C(\mathbf{1}_\Omega^{-1})}{\hat{P}(\mathbf{1}_\Omega^{-1})} = \frac{\hat{X}(\mathbf{1}_\Omega)}{B^*(\mathbf{1}_\Omega)} = \int_\Omega \mathcal{E}_\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)d\omega = \mathcal{E}_\theta\left(\theta^{-1}\left(\frac{1}{V}\right)\right) \Rightarrow \mathcal{L}(V) = \frac{\hat{P}(\mathbf{1}_\Omega^{-1})}{C(\mathbf{1}_\Omega^{-1})} = \frac{B^*(\mathbf{1}_\Omega)}{\hat{X}(\mathbf{1}_\Omega)}.$$

$$\mathcal{S}'(V) \gtreqless 0, \forall V \in (V_0, \infty) \Longrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

The opposite is not true in general. However,

$$\mathcal{S}'(V) = 0, \forall V \in (V_0, \infty) \Leftrightarrow \mathcal{L}'(V) = 0, \forall V \in (V_0, \infty).$$

Proposition I-3:

$$\mathcal{L}'(V) \lesseqgtr 0 \Leftrightarrow \frac{z\mathcal{E}_\theta'(z)}{\mathcal{E}_\theta(z)} = \mathcal{E}_\theta(z) + 1 - \zeta^I(z) \gtreqless 0 \Leftrightarrow \mathcal{L}(V) \lesseqgtr \frac{1}{\mathcal{S}(V) - 1}.$$

The HIIA portions of Theorems 2 and 3 follow from Propositions I-2 and I-3, respectively. For the HIIA portion of Theorem 4, note that $\bar{z}\theta'(\bar{z}) = 0$ holds.[39] Thus, by applying L'Hôpital's rule,

$$\lim_{V\to\infty} \mathcal{L}(V) = -\lim_{z\to\bar{z}} \frac{\theta(z)}{z\theta'(z)} = -\lim_{z\to\bar{z}} \frac{\theta'(z)}{\theta'(z) + z\theta''(z)} = \lim_{z\to\bar{z}} \frac{1}{\zeta^I(z) - 1} = \lim_{V\to\infty} \frac{1}{\mathcal{S}(V) - 1}.$$

Appendix C.4. An Alternative (and Equivalent) specification of the HSA class.

There exists an alternative (but equivalent) definition of HSA That is, a homothetic symmetric demand system for inputs with gross substitutes belongs to HSA (*Homothetic Single Aggregator*) if there exists a function of a single variable, $s^*: \mathbb{R}_{++} \to \mathbb{R}_+$ which is C^2 with $0 < \mathcal{E}_{s^*}(y) \equiv ys^{*\prime}(y)/s^*(y) < 1, s^*(0) = 0$ and $s^*(\infty) = \infty$, such that the budget share of $\omega \in \Omega$ can be written as:

$$s_\omega = \frac{\partial\, ln\, X(\boldsymbol{x})}{\partial\, ln\, x_\omega} = s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right), \tag{32}$$

where $A^*(\mathbf{x})$ is defined implicitly by the adding-up constraint:

$$\int_\Omega s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega \equiv 1 \tag{33}$$

By construction, $A^*(\mathbf{x})$ is linearly homogeneous in $\mathbf{x}$ for any fixed Ω and the budget shares of all inputs are added up to one.

Note that the budget share of each input is increasing in its *normalized quantity*, $y_\omega \equiv x_\omega/A^*(\mathbf{x})$, which is defined as its own quantity x_ω divided by the *common quantity aggregator* $A^*(\mathbf{x})$.

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[39]For $\bar{z} < \infty$, this follows from $\theta'(\bar{z}) = 0$. For $\bar{z} = \infty$, suppose the contrary, so that there exists $z_0 > 0$ such that, for all $z > z_0$, $-z\theta'(z) > c > 0$. Then, $\theta(z_0) = -\lim_{x\to\infty} \int_{z_0}^{x} \theta'(\xi)d\xi = -\lim_{x\to\infty} \int_{z_0}^{x} \xi\theta'(\xi)d\xi/\xi > \lim_{x\to\infty} \int_{z_0}^{x} cd\xi/\xi = \infty$, a contradiction.

The price elasticity of $\omega \in \Omega$ can be written as a function of $y_\omega \equiv x_\omega / A^*(\mathbf{x})$ as:

$$\zeta_\omega = \zeta^*(x_\omega; \mathbf{x}) = \left[1 - \frac{y_\omega s^{*\prime}(y_\omega)}{s^*(y_\omega)}\right]^{-1} \equiv \zeta^{S*}(y_\omega) = \zeta^{S*}\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) > 1,$$

where $\zeta^{S*}: (0, \infty) \to (1, \infty)$ is C^1. Note that $0 < \mathcal{E}_{s^*}(y) \equiv y s^{*\prime}(y)/s^*(y) < 1$ ensures $\zeta^{S*}(y_\omega) > 1$, that is, *gross substitutability.*[40]

It turns out to be convenient to introduce another function, $H^*: \mathbb{R}_{++} \to \mathbb{R}_+$,

$$H^*(y) \equiv \int_0^y \frac{s^*(\xi^*)}{\xi^*} d\xi^*,$$

which satisfies $H^{*\prime}(y) > 0$ and $H^{*\prime\prime}(y) < 0$ and

$$\zeta^{S*}(y) \equiv \left[1 - \frac{y {s^*}'(y)}{s^*(y)}\right]^{-1} \equiv -\frac{{H^*}'(y)}{y {H^*}''(y)} > 1. \tag{34}$$

In general, $\zeta^{S*}(\cdot)$ can be nonmonotonic. Under CES, given by $s^*(y) = \gamma^{1/\sigma}(y)^{1-1/\sigma}$, it is constant, $\zeta^{S*\prime}(y) = 0$. The 2nd law, $\partial\zeta(x_\omega; \mathbf{x})/\partial x_\omega < 0$, holds if and only if $\zeta^{S*\prime}(\cdot) < 0$. The choke price exists if and only if $\lim_{y\to 0} s^{*\prime}(y) = s^{*\prime}(0) < \infty$, which implies $\lim_{y \to 0} y s^{*\prime}(y)/s^*(y) = 1$ and hence $\lim_{y\to 0} \zeta^{S*}(y) = \infty$. Translog corresponds to $s^*(y)$, defined implicitly by $s^* \exp(s^*/\gamma) \equiv \bar{z} y$, for $\bar{z} < \infty$.

After deriving $A^*(\mathbf{x})$ from $s^*(\cdot)$, the production function, $X(\mathbf{x})$, can be obtained by integrating eq.(32), which yields

$$\ln\left[\frac{X(\mathbf{x})}{c^* A^*(\mathbf{x})}\right] = \int_\Omega \left[\int_0^{\frac{x_\omega}{A^*(\mathbf{x})}} \frac{s^*(\xi^*)}{\xi^*} d\xi^*\right] d\omega \equiv \int_\Omega H^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega \tag{35}$$

$$\equiv \int_\Omega s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) \Phi^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega$$

where c^* is a positive constant, which is proportional to TFP and

$$\Phi^*(y) \equiv \frac{1}{s^*(y)} \int_0^y \frac{s^*(\xi^*)}{\xi^*} d\xi^* \equiv \frac{H^*(y)}{y {H^*}'(y)} > 1,$$

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[40] Conversely, from any continuously differentiable $\zeta^{S*}: (0, \infty) \to (1, \infty)$, one could reverse-engineer as $s^*(y) = \gamma^* \exp\left[\int_{y_0}^{y} \left[1 - \frac{1}{\zeta^{S*}(\xi^*)}\right] \frac{d\xi^*}{\xi^*}\right] > 0$, where $\gamma^* = s^*(y_0)$ is a positive constant. Thus, we could also use $\zeta^{S*}(\cdot)$ instead of $s^*(\cdot)$ as a primitive of symmetric HSA with gross substitutes.

where the inequality follows from $\mathcal{E}_{s^*}(y) \equiv ys^{*\prime}(y)/s^*(y) < 1$, which implies that $s^*(y)/y$ is decreasing in y, and hence $H^*(y)$ is concave.

Note that $X(\mathbf{x})/A^*(\mathbf{x})$ is constant, if and only if it is CES. To see this, differentiating eq.(33) yields,

$$\frac{\partial \ln A^*(\mathbf{x})}{\partial \ln x_\omega} = \frac{y_\omega {s^*}'(y_\omega)}{\int_\Omega s^{*\prime}(y_{\omega'}) y_{\omega'} d\omega'} = \frac{\left[1 - \frac{1}{\zeta^*(y_\omega)}\right] s^*(y_\omega)}{\int_\Omega \left[1 - \frac{1}{\zeta^*(y_{\omega'})}\right] s^*(y_{\omega'}) d\omega'},$$

which differs from

$$\frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} = s^*(y_\omega),$$

unless $\zeta^*(y_\omega)$ is constant.

For symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$, eq.(33) is simplified to

$$s^*\left(\frac{1}{A^*(\mathbf{1}_\Omega)}\right) V = 1 \Rightarrow \frac{1}{A^*(\mathbf{1}_\Omega)} = s^{*-1}\left(\frac{1}{V}\right).$$

Hence, from eq.(34), the substitutability measure is given by:

$$\mathcal{S}(V) \equiv \zeta^*(1; \mathbf{1}_\Omega) = \zeta^{S*}\left(s^{*-1}\left(\frac{1}{V}\right)\right) = -\left.\frac{{H^*}'(y)}{y{H^*}''(y)}\right|_{y=s^{*-1}(1/V)} > 1 \tag{36}$$

For the love-for-variety measure, from eq.(35),

$$\ln X(\mathbf{1}_\Omega) = \ln c^* + \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right) - \ln s^{*-1}\left(\frac{1}{V}\right) \Rightarrow$$

$$\frac{d \ln X(\mathbf{1}_\Omega)}{d \ln V} = \left[\frac{d[\ln y - \Phi^*(y)]}{d \ln y} \Big/ \frac{d \ln s^*(y)}{d \ln y}\right]_{\mathrm{y}=s^{*-1}(1/V)} = \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right)$$

so that

$$\mathcal{L}(V) \equiv \frac{d \ln X(\mathbf{1}_\Omega)}{d \ln V} - 1 = \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right) - 1 = \left.\frac{H^*(y)}{y{H^*}'(y)}\right|_{\mathrm{y}=s^{*-1}(1/V)} - 1 \tag{37}$$ [41]

Since $s^{*-1}(1/V)$ is monotonically decreasing in V, eqs.(36)-(37) imply

$$\zeta^{S*\prime}(\cdot) \lesseqgtr 0 \Leftrightarrow \mathcal{S}'(\cdot) \gtreqless 0; \; \Phi^{*\prime}(\cdot) \gtreqless 0 \Leftrightarrow \mathcal{L}'(\cdot) \lesseqgtr 0,$$

Proposition S*-1 shows the relation between $\zeta^{S*}(y)$ and $\Phi^*(y)$.

Proposition S*-1:

[41] Moreover, by evaluating eq.(35) at the symmetric quantity patterns, $\mathcal{L}(V) = \ln[X(\mathbf{1}_\Omega)/c^* A^*(\mathbf{1}_\Omega)] - 1$.

$$\frac{y\Phi^{*\prime}(y)}{\Phi^*(y)} = \frac{1}{\Phi^*(y)} - 1 + \frac{1}{\zeta^{S*}(y)} = \frac{1}{\zeta^{S*}(y)} - \int_0^y \left[\frac{1}{\zeta^{S*}(\xi^*)}\right] w^{S*}(\xi^*; y)d\xi^*.$$

where $w^{S*}(\xi^*; y) \equiv H^{*\prime}(\xi^*)/H^*(y) > 0$, which satisfies $\int_0^y w^{S*}(\xi^*; y)d\xi^* = 1$. Hence,

$$\zeta^{S*\prime}(y) \lesseqgtr 0, \forall y \in (0, y_0) \Longrightarrow \Phi^{*\prime}(y) \gtreqless 0, \forall y \in (0, y_0).$$

The opposite is not true in general. However,

$$\zeta^{S*\prime}(y) = 0, \forall y \in (0, y_0) \Leftrightarrow \Phi^{*\prime}(y) = 0, \forall y \in (0, y_0).$$

The proof of Proposition S*-1 is in Appendix D.

By combining Proposition S*-1, eq.(36) and eq.(37),

Proposition S*-2: For $s^*(y_0)V_0 = 1$,

$$\zeta^{S*\prime}(y) \lesseqgtr 0, \forall y \in (0, y_0) \Leftrightarrow \mathcal{S}'(V) \gtreqless 0, \forall V \in (V_0, \infty);$$

$$\Phi^{*\prime}(y) \gtreqless 0, \forall y \in (0, y_0) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

Moreover,

$$\mathcal{S}'(V) \gtreqless 0, \forall V \in (V_0, \infty) \Longrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (V_0, \infty).$$

The opposite is not true in general. However,

$$\mathcal{S}'(V) = 0, \forall V \in (V_0, \infty) \Leftrightarrow \mathcal{L}'(V) = 0, \forall V \in (V_0, \infty).$$

Proposition S*-3:

$$\mathcal{L}'(V) \lesseqgtr 0 \Leftrightarrow \frac{y\Phi^{*\prime}(y)}{\Phi^*(y)} = \frac{1}{\Phi^*(y)} - 1 + \frac{1}{\zeta^{S*}(y)} \gtreqless 0 \Leftrightarrow \mathcal{L}(V) \lesseqgtr \frac{1}{\mathcal{S}(V) - 1}$$

As shown below, the two definitions of HSA are isomorphic. Therefore, the HSA portions of Theorems 2 and 3 also follow from Propositions S*-2 and S*-3, respectively. For the HSA portion of Theorem 4, note

$$\lim_{V\to\infty} \mathcal{L}(V) = \lim_{V\to\infty} \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right) - 1 = \lim_{y\to 0} \frac{1}{s^*(y)} \int_0^y \frac{s^*(\xi^*)}{\xi^*} d\xi^* - 1.$$

Hence, by applying L'Hôpital's rule,

$$\lim_{V\to\infty} \mathcal{L}(V) = \lim_{y\to 0} \frac{s^*(y)}{ys^{*\prime}(y)} - 1 = \lim_{y\to 0} \frac{\zeta^*(y)}{\zeta^*(y) - 1} - 1 = \lim_{V\to\infty} \frac{1}{\mathcal{S}(V) - 1}.$$

Equivalence between these two definitions of HSA:[42] The isomorphism between the two is given by the one-to-one mapping between $s(z) \leftrightarrow s^*(y)$, defined by:

$$s^*(y) = s\left(\frac{s^*(y)}{y}\right); \qquad s(z) = s^*\left(\frac{s(z)}{z}\right).$$

Differentiating either of these two equalities yields the identity,

$$\zeta^{S*}(y) \equiv \left[1 - \frac{d\ln s^*(y)}{d\ln y}\right]^{-1} = \zeta^S(z) \equiv 1 - \frac{d\ln s(z)}{d\ln z} > 1,$$

which shows that $0 < \mathcal{E}_{s^*}(y) \equiv \frac{d\ln s^*(y)}{d\ln y} < 1$ is equivalent to $\mathcal{E}_s(z) \equiv \frac{d\ln s(z)}{d\ln z} < 0$.

Furthermore, the normalized quantity, $y_\omega \equiv x_\omega / A^*(\mathbf{x})$, and the normalized price, $z_\omega \equiv p_\omega / A(\mathbf{p})$, are negatively related as

$$z_\omega = \frac{s^*(y_\omega)}{y_\omega} \Leftrightarrow y_\omega = \frac{s(z_\omega)}{z_\omega},$$

$$\frac{dy_\omega}{y_\omega} = -\zeta(z_\omega)\frac{dz_\omega}{z_\omega} \Leftrightarrow \frac{dz_\omega}{z_\omega} = -\frac{1}{\zeta^*(y_\omega)}\frac{dy_\omega}{y_\omega}$$

and

$$\frac{z_\omega \zeta^{S\prime}(z_\omega)}{y_\omega \zeta^{S*\prime}(y_\omega)} = -\zeta^S(z_\omega) = -\zeta^{S*}(y_\omega) < 0.$$

In addition, if $\lim_{y\to 0} s^{*\prime}(y) < \infty$, then $\lim_{y\to 0} \zeta^{S*}(y) = \infty$ and

$$\lim_{y\to 0}\frac{s^*(y)}{y} = \lim_{y\to 0} s^{*\prime}(y) = \bar{z} \equiv \inf\{z > 0 | s(z) = 0\} < \infty$$

is the (normalized) choke price.

Moreover,

$$\frac{p_\omega x_\omega}{A(\mathbf{p})A^*(\mathbf{x})} = y_\omega z_\omega = s(z_\omega) = s^*(y_\omega) = \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})}$$

hence we have the identity,

$$c\exp\left[\int_\Omega s(z_\omega)\Phi(z_\omega)d\omega\right] = c\exp\left[\int_\Omega H(z_\omega)d\omega\right] = \frac{A(\mathbf{p})}{P(\mathbf{p})} = \frac{X(\mathbf{x})}{A^*(\mathbf{x})}$$

$$= c^*\exp\left[\int_\Omega s^*(y_\omega)\Phi^*(y_\omega)\,d\omega\right] = c^*\exp\left[\int_\Omega H^*(y_\omega)\,d\omega\right]$$

[42]This isomorphism has been shown for the broader class of HSA, which allows for asymmetry as well as gross complements; see Matsuyama and Ushchev (2017, sec. 3, Remark 3).

which is a positive constant if and only if it is CES. Furthermore, using

$$s(\xi) = s^*(\xi^*) = \xi\xi^* \to \frac{d\xi^*}{\xi^*} = \left[\frac{\xi s'(\xi)}{s(\xi)} - 1\right]\frac{d\xi}{\xi} \to s^*(\xi^*)\frac{d\xi^*}{\xi^*} = \left[s'(\xi) - \frac{s(\xi)}{\xi}\right]d\xi$$

$$\xi = z \leftrightarrow \xi^* = y;\ \xi = \overline{z} \leftrightarrow \xi^* = 0,$$

we have

$$\Phi^*(y) - \Phi(z) \equiv \frac{1}{s^*(y)}\int_0^y \frac{s^*(\xi^*)}{\xi^*}d\xi^* - \frac{1}{s(z)}\int_z^{\overline{z}} \frac{s(\xi)}{\xi}d\xi$$

$$= \frac{1}{s(z)}\int_{\overline{z}}^{z} \left[s'(\xi) - \frac{s(\xi)}{\xi}\right]d\xi - \frac{1}{s(z)}\int_z^{\overline{z}} \frac{s(\xi)}{\xi}d\xi = 1.$$

Since $w^S(\xi; z) = \frac{s(\xi)/\xi}{\int_z^{\overline{z}}[s(\xi')/\xi']d\xi'} \Leftrightarrow s(z)\Phi(z)w^S(\xi; z) = \frac{s(\xi)}{\xi}$ and $w^{S*}(\xi^*; y) = \frac{s^*(\xi^*)/\xi^*}{\int_0^y\left[s^*\left(\xi^{*\prime}\right)/\xi^{*\prime}\right]d\xi^{*\prime}} \Leftrightarrow s^*(y)\Phi^*(y)w^{S*}(\xi^*; y) = \frac{s^*(\xi^*)}{\xi^*}$, this implies

$$\frac{\xi w^S(\xi; z)}{\xi^* w^{S*}(\xi^*; y)} = \frac{\Phi^*(y)}{\Phi(z)} = 1 + \frac{1}{\Phi(z)} = \frac{\Phi^*(y)}{\Phi^*(y) - 1},$$

$$\ln\left(\frac{c}{c^*}\right) = \int_\Omega [s^*(y_\omega)\Phi^*(y_\omega) - s(z_\omega)\Phi(z_\omega)]d\omega = \int_\Omega [H^*(y_\omega) - H(z_\omega)]d\omega$$

$$= \int_\Omega s(z_\omega)d\omega = 1.$$

and

$$\mathcal{L}(V) = \Phi\left(s^{-1}(1/V)\right) = \Phi^*\left(s^{*-1}(1/V)\right) - 1.$$

Appendix D. Proofs of Propositions S-1, D-1, I-1, and S*-1.

Proof of Proposition S-1: Let $w^S(\xi; z) \equiv -\frac{H'(\xi)}{H(z)} > 0$, satisfying $\int_z^{\overline{z}} w^S(\xi; z)\, d\xi = 1$, because $H(\overline{z}) = 0$. Then, because $\overline{z}H'(\overline{z}) = -s(\overline{z}) = 0$,

$$\int_z^{\overline{z}} [\zeta^S(\xi) - 1]w^S(\xi; z)d\xi = \frac{\int_z^{\overline{z}}[\xi H''(\xi) + H'(\xi)]d\xi}{H(z)} = \frac{\int_z^{\overline{z}} d[\xi H'(\xi)]}{H(z)} = -\frac{zH'(z)}{H(z)}$$

$$= \frac{1}{\Phi(z)}.$$

Thus,

$$\frac{z\Phi'(z)}{\Phi(z)} = \frac{zH'(z)}{H(z)} - 1 - \frac{zH''(z)}{H'(z)} = \zeta^S(z) - 1 - \frac{1}{\Phi(z)} = \zeta^S(z) - \int_z^{\overline{z}} \zeta^S(\xi) w^S(\xi; z) d\xi,$$

from which

$$\zeta^{S\prime}(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \implies \Phi'(z) \lesseqgtr 0, \forall z \in (z_0, \overline{z}).$$

Furthermore, $\Phi'(z) = 0$ for $z \in (z_0, \overline{z})$ implies $\zeta^S(z) = 1 + 1/\Phi(z)$, which is hence constant and thus $\zeta^{S\prime}(z) = 0$ for $z \in (z_0, \overline{z})$. This completes the proof. ■

Proof of Proposition D-1: Let $w^D(\xi; y) \equiv \phi'(\xi)/\phi(y) > 0$, satisfying $\int_0^{y} w^D(\xi;\ y)\, d\xi = 1$. Then,

$$\int_0^{y} \left[1 - \frac{1}{\zeta^D(\xi)}\right] w^D(\xi; y) d\xi = \frac{\int_0^{y} [\xi\phi''(\xi) + \phi'(\xi)] d\xi}{\phi(y)} = \frac{\int_0^{y} d[\xi\phi'(\xi)]}{\phi(y)} = \frac{y\phi'(y)}{\phi(y)}$$
$$\equiv \mathcal{E}_\phi(y).$$

Thus,

$$\frac{y\mathcal{E}'_\phi(y)}{\mathcal{E}_\phi(y)} = 1 - \frac{1}{\zeta^D(y)} - \mathcal{E}_\phi(y) = \int_0^{y} \left[\frac{1}{\zeta^D(\xi)}\right] w^D(\xi; y) d\xi - \frac{1}{\zeta^D(y)},$$

from which

$$\zeta^{D\prime}(y) \lesseqgtr 0, \forall y \in (0, y_0) \implies \mathcal{E}'_\phi(y) \lesseqgtr 0, \forall y \in (0, y_0).$$

Furthermore, $\mathcal{E}'_\phi(\ y) = 0$ for $y \in (0, y_0)$ implies $\zeta^D(y) = 1/\big[1 - \mathcal{E}_\phi(y)\big]$, which is hence constant, and thus $\zeta^{D\prime}(y) = 0$ for $y \in (0, y_0)$. This completes the proof. ■

Proof of Proposition I-1: The proof is analogous to that of Proposition S-1. Let $w^I(\xi; z) \equiv -\theta'(\xi)/\theta(z) > 0$, satisfying $\int_z^{\overline{z}} w^I(\xi; z)\, d\xi = 1$, because $\theta(\overline{z}) = 0$. Then, because $\overline{z}\theta'(\overline{z}) = 0$,[43]

$$\int_z^{\overline{z}} [\zeta^I(\xi) - 1] w^I(\xi; z) d\xi = \frac{\int_z^{\overline{z}} [\theta'(\xi) + \xi\theta''(\xi)] d\xi}{\theta(z)} = \frac{\int_z^{\overline{z}} d[\xi\theta'(\xi)]}{\theta(z)} = -\frac{z\theta'(z)}{\theta(z)}$$
$$\equiv \mathcal{E}_\theta(z).$$

Thus,

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[43] See footnote 39.

$$\frac{z\mathcal{E}_\theta'(z)}{\mathcal{E}_\theta(z)} = \mathcal{E}_\theta(z) + 1 - \zeta^I(z) = \int_z^{\overline{z}} \zeta^I(\xi) w^I(\xi; z) d\xi - \zeta^I(z),$$

from which

$$\zeta^{I'}(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \quad \Longrightarrow \quad \mathcal{E}_\theta'(z) \gtreqless 0, \forall z \in (z_0, \overline{z}).$$

Furthermore, $\mathcal{E}_\theta'(z) = 0$ for $z \in (z_0, \overline{z})$ implies $\zeta^I(z) = 1 + \mathcal{E}_\theta(z)$, which is hence constant and thus $\zeta^{I'}(z) = 0$ for $z \in (z_0, \overline{z})$. This completes the proof. ■

Proof of Proposition S*-1: The proof is analogous to that of Proposition D-1. Let $w^{S*}(\xi^*; y) \equiv H^{*\prime}(\xi^*)/H^*(y) > 0$, satisfying $\int_0^y w^{S*}(\xi^*; y) d\xi^* = 1$. Then,

$$\int_0^y \left[1 - \frac{1}{\zeta^{S*}(\xi^*)}\right] w^{S*}(\xi^*; y) d\xi^* = \frac{\int_0^y [\xi^* H^{*\prime\prime}(\xi^*) + H^{*\prime}(\xi^*)] d\xi^*}{H^*(y)} = \frac{\int_0^y d[\xi^* H^{*\prime}(\xi^*)]}{H^*(y)}$$

$$= \frac{yH^{*\prime}(y)}{H^*(y)} = \frac{1}{\Phi^*(y)}.$$

Thus,

$$\frac{y\Phi^{*\prime}(y)}{\Phi^*(y)} = \frac{yH^{*\prime}(y)}{H^*(y)} - 1 - \frac{yH^{*''}(y)}{H^{*\prime}(y)} = \frac{1}{\Phi^*(y)} - 1 + \frac{1}{\zeta^{S*}(y)}$$

$$= \frac{1}{\zeta^{S*}(y)} - \int_0^y \frac{w^{S*}(\xi^*; y)}{\zeta^{S*}(\xi^*)} d\xi^*,$$

from which

$$\zeta^{S*\prime}(y) \lesseqgtr 0, \forall y \in (0, y_0) \Longrightarrow \Phi^{*'}(y) \gtreqless 0, \forall y \in (0, y_0).$$

Furthermore, $\Phi^{*'}(y) = 0$, for $y \in (0, y_0)$ implies $\zeta^{S*}(y) = \Phi^*(y)/[\Phi^*(y) - 1]$, which is hence constant and thus $\zeta^{S*'}(y) = 0$ for $y \in (0, y_0)$. This completes the proof. ■

Appendix E. Technical Proofs for Section 5.

Appendix E.1. Relative Demand and Trade Elasticity.

We now show that the demand for an H good relative to an F good can be written as $g(w/w^*; V; V^*)$, which satisfies the property, $g(w/w^*; V; V^*) \lesseqgtr 1 \Leftrightarrow w/w^* \gtreqless 1$. From Shephard's lemma,

$$D = \frac{\partial P(\mathbf{p})}{\partial p_\omega}\bigg|_{\mathbf{p}=(w\mathbf{1}_\Omega^{-1};\, w^*\mathbf{1}_{\Omega^*}^{-1})} X(D\mathbf{1}_\Omega; M\mathbf{1}_{\Omega^*}) \quad \text{for } \omega \in \Omega$$

$$M = \frac{\partial P(\mathbf{p})}{\partial p_{\omega^*}}\bigg|_{\mathbf{p}=(w\mathbf{1}_\Omega^{-1};\, w^*\mathbf{1}_{\Omega^*}^{-1})} X(D\mathbf{1}_\Omega; M\mathbf{1}_{\Omega^*}) \quad \text{for } \omega^* \in \Omega^*$$

By taking the ratio, the relative demand is

$$\frac{D}{M} = \frac{\left.\frac{\partial P(\mathbf{p})}{\partial p_\omega}\right|_{\mathbf{p}=(w\mathbf{1}_\Omega^{-1};\, w^*\mathbf{1}_{\Omega^*}^{-1})}}{\left.\frac{\partial P(\mathbf{p})}{\partial p_{\omega^*}}\right|_{\mathbf{p}=(w\mathbf{1}_\Omega^{-1};\, w^*\mathbf{1}_{\Omega^*}^{-1})}} = g\left(\frac{w}{w^*}; V;\; V^*\right).$$

and the strictly quasi-concavity of $P(\mathbf{p})$ implies that this function is decreasing in w/w^*. Moreover, the symmetry implies that it is equal to 1 at $w/w^* = 1$, from which the result follows.

Next, by applying the above expression for GM-CES unit cost functions,

$$\frac{w}{w^*}\frac{D}{M} = \frac{w}{w^*} g\left(\frac{w}{w^*}; V;\; V^*\right) = \frac{\mathbb{E}_G\left[\frac{w^{1-\sigma}}{Vw^{1-\sigma} + V^* w^{*1-\sigma}}\right]}{\mathbb{E}_G\left[\frac{w^{*1-\sigma}}{Vw^{1-\sigma} + V^* w^{*1-\sigma}}\right]}.$$

Log-differentiating the above expression by $t = w/w^*$ yields

$$1 + \frac{d\ln(D/M)}{d\ln(w/w^*)} = \frac{\mathbb{E}_G\left[\frac{1-\sigma}{(t)^{1-\sigma}}\left[\frac{(t)^{1-\sigma}}{V(t)^{1-\sigma} + V^*}\right]^2 V^*\right]}{\mathbb{E}_G\left[\frac{(t)^{1-\sigma}}{V(t)^{1-\sigma} + V^*}\right]} + \frac{\mathbb{E}_G\left[\frac{1-\sigma}{(t)^{\sigma-1}}\left[\frac{1}{V(t)^{1-\sigma} + V^*}\right]^2 V\right]}{\mathbb{E}_G\left[\frac{1}{V(t)^{1-\sigma} + V^*}\right]}.$$

By evaluating at $t = w/w^* = 1 = g(1; V;\; V^*)$, the trade elasticity in equilibrium is equal to

$$-\frac{d\ln(D/M)}{d\ln(w/w^*)}\bigg|_{t=w/w^*=1} - 1 = \mathbb{E}_G[\sigma] - 1 = \mathcal{S}^{GMCES} - 1.$$

Likewise, by applying the inverse relative demand for GM-CES production functions,

$$\frac{w}{w^*}\frac{D}{M} = g^{-1}\left(\frac{D}{M}; V;\; V^*\right)\frac{D}{M} = \frac{\mathbb{E}_G\left[\frac{D^{1-1/\sigma}}{VD^{1-1/\sigma} + V^* M^{1-1/\sigma}}\right]}{\mathbb{E}_G\left[\frac{M^{1-1/\sigma}}{VD^{1-1/\sigma} + V^* M^{1-1/\sigma}}\right]}$$

By setting $u = D/M$, log-differentiating the above expression yields

$$1 + \frac{d\ln(w/w^*)}{d\ln(D/M)}$$

$$= \frac{\mathbb{E}_G\left[\frac{1-1/\sigma}{u^{1-1/\sigma}}\left[\frac{u^{1-1/\sigma}}{Vu^{1-1/\sigma}+V^*}\right]^2 V^*\right]}{\mathbb{E}_G\left[\frac{u^{1-1/\sigma}}{Vu^{1-1/\sigma}+V^*}\right]} + \frac{\mathbb{E}_G\left[\frac{1-1/\sigma}{u^{1/\sigma-1}}\left[\frac{1}{Vu^{1-1/\sigma}+V^*}\right]^2 V\right]}{\mathbb{E}_G\left[\frac{1}{Vu^{1-1/\sigma}+V^*}\right]}.$$

By evaluating this expression at $t = w/w^* = 1$ and $u = D/M = 1$,

$$1 + \left.\frac{d\ln t}{d\ln u}\right|_{t=w/w^*=1} = \frac{V^*}{V+V^*}\mathbb{E}_G\left[1-\frac{1}{\sigma}\right] + \frac{V}{V+V^*}\mathbb{E}_G\left[1-\frac{1}{\sigma}\right] = 1 - \mathbb{E}_G\left[\frac{1}{\sigma}\right].$$

Hence, the trade elasticity in equilibrium is

$$-\left.\frac{d\ln(D/M)}{d\ln(w/w^*)}\right|_{t=w/w^*=1} - 1 = -\left.\frac{d\ln u}{d\ln t}\right|_{t=w/w^*=1} - 1 = \frac{1}{\mathbb{E}_G[1/\sigma]} - 1 = \mathcal{S}^{GMCES} - 1.$$

Appendix E.2. Gains from Trade under HSA.

By plugging in the expression of the love-for-variety under HSA in Table 1 into the general formula for the gains from trade in Table 2,

$$\ln(GT) = \int_V^{V/\lambda} \Phi\left(s^{-1}\left(\frac{1}{v}\right)\right)\frac{dv}{v}.$$

Using the change of variables, $s^{-1}\left(\frac{1}{v}\right) = \xi \Leftrightarrow v = \frac{1}{s(\xi)}$, which implies $\frac{dv}{v} = -\frac{\xi s'(\xi)}{s(\xi)}\frac{d\xi}{\xi} = (\zeta^S(\xi)-1)\frac{d\xi}{\xi}$, and the identity, $\Phi(\xi)(\zeta^S(\xi)-1) = 1 + \xi\Phi'(\xi)$,

$$\ln(GT) = \int_V^{V/\lambda} \Phi\left(s^{-1}\left(\frac{1}{v}\right)\right)\frac{dv}{v} = \int_{s^{-1}(1/V)}^{s^{-1}(\lambda/V)} \Phi(\xi)(\zeta^S(\xi)-1)\frac{d\xi}{\xi}$$

$$= \int_{s^{-1}(1/V)}^{s^{-1}(\lambda/V)} [1+\xi\Phi'(\xi)]\frac{d\xi}{\xi}$$

$$= \int_{s^{-1}(1/V)}^{s^{-1}(\lambda/V)} \frac{d\xi}{\xi} + \int_{s^{-1}(1/V)}^{s^{-1}(\lambda/V)} \Phi'(\xi)d\xi = \ln\left[\frac{s^{-1}(\lambda/V)}{s^{-1}(1/V)}\right] + \Phi\left(s^{-1}\left(\frac{\lambda}{V}\right)\right) - \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right).$$

Hence, the expression for the Home gains from trade under HSA is given by:

$$GT = \frac{s^{-1}(\lambda/V)}{s^{-1}(1/V)} \frac{\exp\left[\Phi\left(s^{-1}(\lambda/V)\right)\right]}{\exp\left[\Phi\left(s^{-1}(1/V)\right)\right]} = \frac{s^{-1}((1-\lambda)/V^*)}{s^{-1}((1-\lambda)/\lambda V^*)} \frac{\exp\left[\Phi\left(s^{-1}((1-\lambda)/V^*)\right)\right]}{\exp\left[\Phi\left(s^{-1}((1-\lambda)/\lambda V^*)\right)\right]},$$

as shown in Table 2. For translog, we have

$$s^{-1}\left(\frac{1}{V}\right) = \bar{z}\exp\left[-\frac{1}{\gamma V}\right];\ \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) = \frac{1}{2\gamma V} \rightarrow \ln(GT) = \frac{1-\lambda}{2\gamma V}$$

For generalized translog,

$$s^{-1}\left(\frac{1}{V}\right) = \bar{z}\exp\left[-\frac{\eta}{(\sigma-1)}(\gamma V)^{-1/\eta}\right];\ \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) = \frac{1}{\sigma-1}\frac{\eta}{1+\eta}(\gamma V)^{-1/\eta}$$

so that

$$\ln(GT) = \frac{1}{(\sigma-1)}\frac{\eta}{1+\eta}\left(\frac{1}{\gamma V}\right)^{1/\eta}\frac{1-(\lambda)^{1/\eta}}{1/\eta}.$$

We now prove the HSA portion of Theorem 7. First, for any given $V > 0, GT$ is increasing in V^* and decreasing in $\lambda = V/(V+V^*)$ with the range,

$$1 < GT < \frac{\bar{z}}{s^{-1}(1/V)}\frac{\exp[\Phi(\bar{z})]}{\exp\left[\Phi\left(s^{-1}(1/V)\right)\right]}$$

Clearly, the upper bound is finite and decreasing in V, if and only if $\bar{z} < \infty$, i.e., if and only if the choke price exists. Next, for any given $V^* > 0$, GT is monotonically decreasing in V and in λ under non-increasing love-for-variety. Furthermore, by letting $V \rightarrow 0$ and $\lambda = V/(V+V^*) \rightarrow 0$, the upper bound goes to infinity,[44] because $GT < \frac{s^{-1}(1/V^*)\exp\left[\Phi\left(s^{-1}(1/V^*)\right)\right]}{\lim_{z\rightarrow 0}[z\exp[\Phi(z)]]} = \infty$.

Appendix E.3. Gains from Trade under HDIA.

By plugging in the expression of the love-for-variety under HDIA. in Table 1 into the general formula for the gains from trade in Table 2,

[44] To see $\lim_{z\rightarrow 0}[z\exp[\Phi(z)]] = 0$, we show $\lim_{z\rightarrow 0}[\ln z + \Phi(z)] = -\infty$. From integration by parts, $\Phi(z) \equiv \frac{1}{s(z)}\int_z^{\bar{z}}\frac{s(\xi)}{\xi}d\xi = \frac{-s(z)\ln z - \int_z^{\bar{z}} s'(\xi)\ln\xi d\xi}{s(z)} = -\ln z - \frac{1}{s(z)}\int_z^{\bar{z}} s'(\xi)\ln\xi\, d\xi$. Hence, $\ln z + \Phi(z) = -\frac{1}{s(z)}\int_z^{\bar{z}} s'(\xi)\ln\xi\, d\xi = \frac{\int_z^{\bar{z}} s'(\xi)\ln\xi d\xi}{\int_z^{\bar{z}} s'(\xi)d\xi}$. Since $s(z) \rightarrow \infty$ as $z \rightarrow 0$, the numerator and denominator both go to $+/-\infty$ as $z \rightarrow 0$. Applying L'Hôpital's rule, $\lim_{z\rightarrow 0}[\ln z + \Phi(z)] = \lim_{z\rightarrow 0}\frac{\int_z^{\bar{z}} s'(\xi)\ln\xi d\xi}{\int_z^{\bar{z}} s'(\xi)d\xi} = \lim_{z\rightarrow 0}\frac{s'(z)\ln z}{s'(z)} = \lim_{z\rightarrow 0}(\ln z) = -\infty$.

$$\ln(GT) = \int_V^{V/\lambda}\left[\frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/v))} - 1\right]\frac{dv}{v}.$$

Using the change of variables, $\phi^{-1}\left(\frac{1}{v}\right) = \xi \Leftrightarrow v = \frac{1}{\phi(\xi)}$, which implies $\frac{dv}{v} = -\frac{\xi\phi'(\xi)}{\phi(\xi)}\frac{d\xi}{\xi} = -\mathcal{E}_\phi(\xi)\frac{d\xi}{\xi}$,

$$\ln(GT) = \int_V^{V/\lambda}\left[\frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/v))} - 1\right]\frac{dv}{v} = \int_{\phi^{-1}(1/V)}^{\phi^{-1}(\lambda/V)}\left[\mathcal{E}_\phi(\xi) - 1\right]\frac{d\xi}{\xi}$$

$$= -\int_{\phi^{-1}(1/V)}^{\phi^{-1}(\lambda/V)}\frac{d\xi}{\xi} + \int_{\phi^{-1}(1/V)}^{\phi^{-1}(\lambda/V)}\left[\mathcal{E}_\phi(\xi)\right]\frac{d\xi}{\xi} = \ln\left[\frac{(\lambda/V)}{\phi^{-1}(\lambda/V)}\frac{\phi^{-1}(1/V)}{(1/V)}\right].$$

Hence, the expression for the Home gains from trade under HDIA is given by:

$$GT = \frac{(\lambda/V)}{\phi^{-1}(\lambda/V)}\frac{\phi^{-1}(1/V)}{(1/V)} = \frac{((1-\lambda)/V^*)}{\phi^{-1}((1-\lambda)/V^*)}\frac{\phi^{-1}((1-\lambda)/\lambda V^*)}{((1-\lambda)/\lambda V^*)},$$

as shown in Table 2.

We now prove the HDIA portion of Theorem 7. First, for any given $V > 0, GT$ is increasing in V^* and decreasing in $\lambda = V/(V+V^*)$ with the range,

$$1 < GT < \lim_{\lambda\to 0}\left[\frac{(\lambda/V)}{\phi^{-1}(\lambda/V)}\right]\frac{\phi^{-1}(1/V)}{(1/V)} = \lim_{y\to 0}\left[\frac{\phi(y)}{y}\right]\frac{\phi^{-1}(1/V)}{(1/V)} = \phi'(0)\frac{\phi^{-1}(1/V)}{(1/V)}$$

The upper bound is thus finite if and only if $\phi'(0) < \infty$, that is, if and only if the choke price exists. Next, for any given $V^* > 0$, GT is monotonically decreasing in V and in λ under non-increasing love-for-variety. Furthermore, by letting $V \to 0$ and $\lambda = V/(V+V^*) \to 0$, the upper bound goes to infinity, because

$$GT = \frac{(1-\lambda)/V^*}{\phi^{-1}((1-\lambda)/V^*)}\frac{\phi^{-1}((1-\lambda)/\lambda V^*)}{((1-\lambda)/\lambda V^*)} < \frac{1/V^*}{\phi^{-1}(1/V^*)}\lim_{\lambda\to 0}\left[\frac{\phi^{-1}((1-\lambda)/\lambda V^*)}{((1-\lambda)/\lambda V^*)}\right]$$

$$= \frac{1/V^*}{\phi^{-1}(1/V^*)}\lim_{y\to\infty}\left[\frac{y}{\phi(y)}\right] = \frac{1/V^*}{\phi^{-1}(1/V^*)}\lim_{y\to\infty}\left[\frac{1}{\phi'(y)}\right] = \infty.$$

Appendix E.4. Gains from Trade under HIIA.

By plugging in the expression of the love-for-variety under HIIA in Table 1 into the general formula for the gains from trade in Table 2,

$$\ln(GT) = \int_V^{V/\lambda}\frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/v)\right)}\frac{dv}{v}.$$

Using the change of variables, $\theta^{-1}\left(\frac{1}{v}\right) = \xi \Leftrightarrow v = \frac{1}{\theta(\xi)}$, which implies $\frac{dv}{v} = -\frac{\xi\theta'(\xi)}{\theta(\xi)}\frac{d\xi}{\xi} = \mathcal{E}_\theta(\xi)\frac{d\xi}{\xi}$,

$$\ln(GT) = \int_V^{V/\lambda} \frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/v)\right)}\frac{dv}{v} = \int_{\theta^{-1}(1/V)}^{\theta^{-1}(\lambda/V)} \frac{d\xi}{\xi} = \ln\left(\frac{\theta^{-1}(\lambda/V)}{\theta^{-1}(1/V)}\right).$$

Hence, the expression for the Home gains from trade under HIIA is given by:

$$GT = \frac{\theta^{-1}(\lambda/V)}{\theta^{-1}(1/V)} = \frac{\theta^{-1}((1-\lambda)/V^*)}{\theta^{-1}((1-\lambda)/\lambda V^*)},$$

as shown in Table 2.

We now prove the HIIA portion of Theorem 7. First, for any given $V > 0, GT$ is increasing in V^* and decreasing in $\lambda = V/(V + V^*)$ with the range,

$$1 < GT < \frac{\lim_{\lambda\to 0} \theta^{-1}(\lambda/V)}{\theta^{-1}(1/V)} = \frac{\overline{z}}{\theta^{-1}(1/V)}.$$

Thus, the upper bound is finite and decreasing in V, if and only if $\overline{z} < \infty$, that is, if and only if the choke price exists. Next, for any given $V^* > 0$, GT is monotonically decreasing in V and in λ under non-increasing love-for-variety. Furthermore, by letting $V \to 0$ and $\lambda = V/(V + V^*) \to 0$,

$$1 < GT < \frac{\theta^{-1}(1/V^*)}{\theta^{-1}(\infty)} = \infty.$$