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DP19376

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Kiminori Matsuyama

INTERNATIONAL MACROECONOMICS AND FINANCE, INDUSTRIAL ORGANIZATION, INTERNATIONAL TRADE AND REGIONAL ECONOMICS, MONETARY ECONOMICS AND FLUCTUATIONS AND MACROECONOMICS AND GROWTH

CEPR

ISSN 0265-8003

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Discussion Paper DP19376
Published 19 August 2024
Submitted 16 August 2024

Centre for Economic Policy Research
33 Great Sutton Street, London EC1V 0DX, UK
Tel: +44 (0)20 7183 8801
www.cepr.org

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Abstract

This article reviews homothetic non-CES demand systems and their properties when applied to monopolistic competition, to offer the guidance to those looking for flexible and yet tractable ways of departing from CES. Under general homothetic symmetric non-CES, two measures, substitutability and love-for-variety, are introduced to identity the condition under which the equilibrium product variety is excessive or insufficient. Because homotheticity and symmetry alone impose little restriction to make further progress, we turn to the Homothetic Single Aggregator (H.S.A.) class. H.S.A. is more flexible than CES and translog, which are its special cases, and yet equally analytically tractable, because all cross-variety interactions are summarized by the single aggregator. Under H.S.A., substitutability is increasing in product variety iff Marshall's 2nd law holds, which is a sufficient condition for love-for-variety to be diminishing in product variety and for the equilibrium product variety to be excessive. H.S.A. remains tractable even under firm heterogeneity.

JEL Classification: N/A

Keywords: Monopolistic competition

Kiminori Matsuyama - k-matsuyama@northwestern.edu
Northwestern University and CEPR

Acknowledgements
This review originated from my graduate lectures at Northwestern University and University of Tokyo. I would like to thank the students for many questions. I also benefitted from the opportunity to test drive this review and receiving feedback at Federal Reserve Bank of Chicago. Actual writing was done during my visit to Becker Friedman Institute for Research in Economics at University of Chicago, whose hospitality is gratefully acknowledged. Much of the content is drawn from my joint work with Philip Ushchev, and this review benefitted greatly from his comments. All the shortcomings are mine.

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Kiminori Matsuyama
Northwestern University
k-matsuyama@northwestern.edu

Last Updated on 2024-08-16; 1:11 PM

prepared for *Annual Review of Economics,* 2025, vol. 17
https://doi.org/10.1146/annurev-economics-081624-082324

Abstract: This article reviews homothetic non-CES demand systems and their properties when applied to monopolistic competition, to offer the guidance to those looking for flexible and yet tractable ways of departing from CES. Under general homothetic symmetric non-CES, two measures, *substitutability* and *love-for-variety*, are introduced to identity the condition under which the equilibrium product variety is excessive or insufficient. Because homotheticity and symmetry alone impose little restriction to make further progress, we turn to the *Homothetic Single Aggregator* (H.S.A.) class. H.S.A. is more flexible than CES and translog, which are its special cases, and yet equally analytically tractable, because all cross-variety interactions are summarized by the single aggregator. Under H.S.A., substitutability is increasing in product variety iff Marshall's 2nd law holds, which is a sufficient condition for love-for-variety to be diminishing in product variety and for the equilibrium product variety to be excessive. H.S.A. remains tractable even under firm heterogeneity.

Keywords: Substitutability vs. Love-for-Variety; Equilibrium vs. Optimal; Homothetic Single Aggregator; 2nd and 3rd Laws of Demand; Firm Heterogeneity

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1. Introduction

We all know the CES demand system. It is ubiquitous in business cycle theory, economic growth, international trade, economic geography, among others. We love using CES, because it has many knife-edge properties, which help to make it tractable. At the same time, these knife-edge properties make CES too restrictive for many applications. Of course, many researchers have tried some non-CES demand systems, but they typically look for *alternatives,* such as linear-quadratic or translog, which come with their own drawbacks and limitations. What is needed is to *generalize* CES by relaxing some of its knife-edge properties in order to have more flexibility without losing too much tractability of CES.

Matsuyama (2023), "Non-CES Aggregators: A Guided Tour," reviewed several classes of non-CES demand systems and offered some guidance to those looking for flexible and yet tractable ways of departing from CES. Due to the space limitation, however, it focused on non-CES that are suited for applying to intersectoral demand systems, with special emphasis on nonhomotheticity and gross complementarities across sectors and the essentiality of goods and factors.

This review instead focuses on applications of *homothetic* non-CES to demand systems for differentiated products within a monopolistic competitive (MC) industry with free entry and endogenous product variety. This necessitates some additional restrictions on the class of demand systems studied. They are:

Endogenous range of inessential products: To allow for firms to enter or exit with their own products, demand systems need to be well-defined even when some products are unavailable or not yet invented.

Continuum of products: This ensures that firms cannot affect the industry-level variables, one of the defining features of MC that distinguishes it from oligopoly. Moreover, this helps tractability by making product variety a continuous variable.

***Gross substitutability across products*:** That is, the price elasticity of demand for each product is greater than one, or equivalently, the market share of each product is decreasing in its own price. This ensures that monopolistic competitive firms face a positive marginal revenue curve.

I will further restrict to:

Symmetric Demand Systems: This helps to highlight the supply-side heterogeneity across firms, such as productivity difference *a la* Melitz (2003), price setting *a la* Calvo (1983), and technology diffusion causing some but not all MC firms to face competitive fringes *a la* Judd (1985).

The restriction of homotheticity and symmetry is imposed mostly due to the page limitation.[1] Nevertheless, the reader should also note that homothetic and symmetric demand systems are not so restrictive as they may seem, because one can nest them into a nonhomothetic and/or asymmetric upper-tier demand system. In other words, homothetic symmetric non-CES can serve as building blocks to construct such nonhomothetic and/or asymmetric non-CES.

Here's the road map of this review. Section 2 offers a quick refresher on CES and its application to what I call the Dixit & Stiglitz (1977) environment, where MC firms are symmetric not only on the demand side but also on the supply side. Section 3 discusses general homothetic symmetric demand systems. Among others, this section introduces two measures, *substitutability* $\sigma(V)$ and *love-for-variety* $\mathcal{L}(V)$, both as functions of product variety V. These two measures help to characterize the demand system. Section 4 applies these general demand systems to the Dixit-Stiglitz environment. It characterizes the unique symmetric equilibrium, *under the assumption that it exists,* and conducts comparative statics, whose results depend on $\sigma(V)$. On the other hand, the optimal allocation depends on $\mathcal{L}(V)$. By comparing the equilibrium and the optimum, this section identifies the sufficient and necessary condition under which the equilibrium product variety is excessive, optimal, or insufficient. Yet, it is not possible to make further progress under general homothetic symmetric demand systems, because homotheticity and symmetry alone imposes little restriction on the relation between $\sigma(V)$ and $\mathcal{L}(V)$, so that "almost anything goes."

Section 5 thus turns to the subclass of homothetic symmetric demand systems with gross substitutes called *homothetic single aggregator* or H.S.A. This class of demand systems, which contains CES and translog (Feenstra 2003) as special cases, is characterized by the presence of a single price aggregator, a sufficient statistic, which captures everything one needs to know to understand the cross-variety interactions. Due to such a significant reduction in the

[1] There is another reason for imposing homotheticity. Most earlier studies of monopolistic competition under non-CES make use of nonhomothetic symmetric demand systems. For example, Dixit & Stiglitz (1977, Section II), Behrens & Murata (2007), Zhelobodko, et.al. (2012) and Mrázová & Neary (2017) use the directly explicitly additive (DEA) class of nonhomothetic symmetric demand systems. The indirectly explicitly additive (IEA) class used by Bertoletti & Etro (2017) and the linear-quadratic demand system used by Ottaviano et. al (2002) and Melitz & Ottaviano (2008) are also nonhomothetic and symmetric. This literature has been reviewed by Parenti et. al (2017) and Thisse & Ushchev (2018). See also Melitz (2018), which reviewed the work using the DEA class.

dimensionality, H.S.A. is highly tractable yet more flexible than CES and translog. Moreover, it imposes much tighter relation between $\sigma(V)$ and $\mathcal{L}(V)$. Under H.S.A., Marshall's 2nd law of demand (i.e., the price elasticity of demand is increasing in its own price) is equivalent to increasing substitutability $\sigma'(V) > 0$, both of which are sufficient for diminishing love-for-variety, $\mathcal{L}'(V) < 0$. Armed with these results, Section 6 applies H.S.A. to the Dixit-Stiglitz environment. Under H.S.A., it is straightforward to show that the equilibrium is unique and symmetric. Moreover, the equilibrium product variety is excessive under diminishing love-for-variety. Therefore, Marshall's 2nd law, and equivalently, increasing substitutability are sufficient for excessive product variety.[2]

Then, Section 7 applies H.S.A. to the Melitz (2003) environment, where ex-ante symmetric firms learn their marginal costs after entry, drawn from a common distribution, and become ex-post heterogenous. Again, under H.S.A., it is straightforward to show that the equilibrium exists uniquely, and to conduct comparative statics. Section 8 discusses how H.S.A. can accommodate other types of firm heterogeneity. Appendix 1 explains why H.S.A. is more tractable than HDIA and HIIA. Appendix 2 lists some parametric families of H.S.A. for the quantitatively oriented reader who may want to use them for calibration and estimation.

Before proceeding, some caveats should be mentioned. First, this is a review of non-CES and of their key properties that play crucial roles when applied to monopolistic competition. My goal is to offer the guidance to those who are looking for tractable and yet flexible ways of departing from CES in their applications. And I hope that the readers will find useful building blocks for constructing their own models. But it is not intended to be a review of applications of monopolistic competition under non-CES to some topics in economics, whether they are in international trade, economic geography, economic growth, or Keynesian macro. Such a review needs a separate treatment, at least one in each topic, some of which I hope to write in the future. Second, because the materials reviewed here are theoretical in nature, I try not to sacrifice the

[2] Matsuyama & Ushchev (2020a, 2023) showed that many results in Sections 5 and 6 hold also in two other classes of homothetic symmetric demand systems: symmetric Homothetic Direct Implicit Additivity (HDIA) with gross substitutes, an extension of the Kimball (1995) aggregator with an endogenous product range, and symmetric Homothetic Indirect Implicit Additivity (HIIA) with gross substitutes. The three classes, H.S.A., HDIA, and HIIA, originally developed by Matsuyama & Ushchev (2017) without symmetry and gross substitutes restriction, all share CES as a special case, but are otherwise pairwise disjoint. HDIA and HIIA are less tractable than H.S.A. Some additional restrictions are needed just to ensure the uniqueness and the symmetry of the equilibrium in the Dixit-Stiglitz environment. Moreover, these two are not analytically tractable with firm heterogeneity. This is because the cross-variety interactions are captured by two aggregators under HDIA and HIIA, unlike one aggregator under H.S.A. For these reasons, I focus on H.S.A. from Section 5.

logical rigor, and yet try to keep the discussion as non-technical as possible. I offer the intuition and explain the logics behind the main results but skip many derivations. Furthermore, some regularity conditions, such as continuity and differentiability, are often not explicitly stated. Moreover, the space limitation prevents me from discussing any empirical evidence that motivates some assumptions. This review should thus be treated as a reading guide for the references cited, not as a substitute for reading them. Finally, I have encountered, repeatedly throughout years, several false claims about non-CES demand systems. Often taken for granted, these false claims are found not only in published and discussion papers but also heard in seminars, both by the speakers and by those in the audience. I have also seen them in the referee reports, both as an editor and as a submitting author. In this review, I explicitly discuss several fallacies and explain why they are wrong, but without citing any references. They are so widespread that I have no idea who should be given "credit" for starting each of them. Indeed, many of them are a kind of logical pitfalls, to which anyone could fall into. (I confess that I used to believe Fallacies #3 and #4 discussed in Section 3 myself.) By flagging these fallacies without finger-pointing, I am hoping to prevent misinformation from spreading, particularly, to the new generations of researchers.

2. Dixit Stiglitz under CES: A Quick Refresher

We discuss CES demand systems in terms of demand for differentiated intermediate inputs generated by a competitive industry that produces a single final good, using symmetric CES production function,

$$X = X(\mathbf{x}) = Z\left[\int_{\Omega} (x_\omega)^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}}.$$

Here $\mathbf{x} = \{x_\omega; \omega \in \Omega\}$ is an input quantity vector, where Ω is the set of input varieties, indexed by ω, that are available in equilibrium, whose mass is denoted by $V \equiv |\Omega|$. Under CES, the elasticity of substitution across varieties is a parameter, $\sigma > 1$, and $Z > 0$ is TFP.

2.1 CES Demand System

Facing $\mathbf{p} = \{p_\omega; \omega \in \Omega\}$, the input price vector, the competitive industry chooses $\mathbf{x}$ to minimize the production cost, which leads to the unit cost function,

$$P = P(\mathbf{p}) \equiv \min_{\mathbf{x}} \left\{ \mathbf{px} \equiv \int_{\Omega} p_\omega x_\omega d\omega \,\middle|\, X(\mathbf{x}) \geq 1 \right\} = \frac{1}{Z} \left[\int_{\Omega} (p_\omega)^{1-\sigma} d\omega \right]^{\frac{1}{1-\sigma}},$$

with demand for ω

$$x_\omega = \left(\frac{p_\omega}{ZP(\mathbf{p})} \right)^{-\sigma} \frac{X(\mathbf{x})}{Z} = \frac{(p_\omega)^{-\sigma}}{\left(ZP(\mathbf{p})\right)^{1-\sigma}} E,$$

and the budget share of ω

$$s_\omega \equiv \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{p_\omega x_\omega}{E} = \left(\frac{p_\omega}{ZP(\mathbf{p})} \right)^{1-\sigma} = \left(\frac{Z x_\omega}{X(\mathbf{x})} \right)^{1-\frac{1}{\sigma}}$$

where $E \equiv P(\mathbf{p})X(\mathbf{x}) = \mathbf{px}$**,** is the size of this industry, and hence market size for differentiated inputs, which we treat as given.

2.2 The Environment

We now apply CES to what I shall call **the Dixit-Stiglitz environment.** There exists a single primary factor of production, "labor," taken as numeraire. Each differentiated intermediate input, $\omega \in \Omega$, is produced from "labor" and sold exclusively by a single monopolistically competitive (MC) firm, also indexed by $\omega \in \Omega$. These MC firms are symmetric. Not only their products enter symmetrically in the demand system, but also share the same technology. Each firm needs to hire $F + \psi x_\omega$ units of "labor" to supply x_ω units of its own product. Here F is the fixed cost, a combination of **the entry/innovation cost,** required to develop its own product and to enter the market, and of **the overhead cost**, required to stay in the market; ψx_ω is the production cost, or "employment," where ψ is a constant marginal cost of production, and the inverse of productivity. Finally, there is free-entry to the market. Firms enter/exit until their gross profit is equalized to the fixed cost, $\Pi_\omega = F$. This ensures that there is no excess profit in equilibrium, and that the total "labor" demand of this sector is $L = \mathbf{px} = P(\mathbf{p})X(\mathbf{x}) = E$.[3]

2.3 Equilibrium:

[3]Notice that no assumption is made on how this sector interacts with the rest of the economy, except E is the aggregate spending on this sector, which leads to this sector's "labor" demand, $L = E$. Of course, one could assume that the representative household, endowed with L units of "labor", consumes only the final good produced in this sector, so that its budget constraint leads to $L = E$. However, the sector-level analysis in this review does not need to make such an assumption.

As the sole producer of its own product, each MC firm sets its price, p_ω, to maximize its gross profit,

$$\Pi_\omega = (p_\omega - \psi)x_\omega = \frac{(p_\omega - \psi)(p_\omega)^{-\sigma}}{\big(ZP(\mathbf{p})\big)^{1-\sigma}}E,$$

holding the industry-wide variables, $P(\mathbf{p})$ and E, fixed. The first-order condition of the profit maximization leads to the familiar Lerner pricing formula and the markup rule:

$$p_\omega\left(1-\frac{1}{\sigma}\right) = \psi \qquad \Leftrightarrow \qquad p_\omega \equiv p = \left(\frac{\sigma}{\sigma-1}\right)\psi \equiv \mu\psi,$$

where μ is the constant and common markup rate. Thus, all firms set the same price, and the equilibrium is symmetric. By dropping the index to denote the common values, $p_\omega = p$, and $x_\omega = x$, which implies $pxV = E$. Thus, the common gross profit is $\Pi = (p - \psi)x = px/\sigma = E/\sigma V$. Finally, the free-entry/exit condition implies that the common gross profit is equal to the fixed cost in equilibrium, $E/\sigma V^{eq} = F$, so that

$$V^{eq} = \frac{E}{\sigma F};\;\; p^{eq} = \left(\frac{\sigma}{\sigma-1}\right)\psi;\;\; x^{eq} = \frac{(\sigma-1)F}{\psi}.$$

In equilibrium, the revenue $p^{eq}x^{eq} = E/V^{eq} = \sigma F$ is divided into the (gross) profit, $(p^{eq} - \psi)x^{eq} = F$, and the production cost, $\psi x^{eq} = (\sigma-1)F$ in every firm. Thus, their shares in revenue,

$$\frac{F}{p^{eq}x^{eq}} = \frac{1}{\sigma};\;\; \frac{\psi x^{eq}}{p^{eq}x^{eq}} = 1 - \frac{1}{\sigma} = \frac{1}{\mu},$$

and the profit/production cost ratio,

$$\frac{F}{\psi x^{eq}} = \frac{\mu}{\sigma} = \frac{1}{\sigma-1} = \mu - 1,$$

are all constant and independent of E/F under CES.

2.4 Comparative Statics:

By denoting the percentage change by $\hat{q} \equiv \partial \ln q = \partial q/q$, the three endogenous variables, (V^{eq}, p^{eq}, x^{eq}), respond to the three exogenous variables, (E, F, ψ), as

$$\widehat{V^{eq}} = \hat{E} - \hat{F};\; \widehat{p^{eq}} = \hat{\psi};\;\; \widehat{x^{eq}} = \hat{F} - \hat{\psi}.$$

Note that the firm behavior, p^{eq}, x^{eq}, are not affected by E, while the mass of firms, V^{eq}, responds proportionally to E. Thus, the adjustment to a market size change takes place only at the extensive margin under CES.

2.5 Optimal Allocation:

Now imagine that this sector were fully integrated and could control all intermediate inputs production. Then,

$$\max X(\mathbf{x}) = \max Z\left[\int_{\Omega} (x_\omega)^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \quad s.t. \quad \int_{\Omega} \psi x_\omega d\omega + VF \le E.$$

The optimal allocation is clearly symmetric, $x_\omega = x > 0$ for $\omega \in \Omega$, simplifying the problem to:

$$\max_{(\psi x+F)V\le E} V^{\frac{\sigma}{\sigma-1}}(Zx) = \frac{ZF}{\psi}\max_{V} V^{\frac{1}{\sigma-1}}\left(\frac{E}{F}-V\right).$$

By solving this problem, the optimal allocation is given by:

$$V^{op} = \frac{E}{\sigma F}; \quad x^{op} = \frac{(\sigma-1)F}{\psi},$$

which is identical with the equilibrium allocation.

The optimality result, though not robust, is surprising. A priori, one would expect that MC equilibrium would not be optimal due to the presence of externalities. First, there are **negative externalities** due to the **business stealing effect.** A firm, when paying the fixed cost to enter and stay with its own product, does not take into account that this action reduces demand for other products and their profits, which would suggest *excessive product variety*. On the other hand, there are **positive externalities** due to **incomplete appropriability**: A firm is motivated to produce and sell its own variety, not by the social surplus, but by the profit, which is a fraction of the social surplus. This would suggest *insufficient product variety*. As explained in Tirole (1988, Chapter 7) and Matsuyama (1995; Section 3E), these two sources of externalities cancel out each other under CES, which is why the equilibrium is optimal. This feature makes the Dixit-Stiglitz environment a useful benchmark against which the efficiency implications of departing from CES can be evaluated.[4]

Unfortunately, the logic behind the optimality result is poorly understood.

Fallacy #1. The equilibrium allocation is optimal because all the products are sold at the same markup rate, and hence the relative prices across products are not distorted.

[4] We will show later how departing from CES within the Dixit-Stiglitz environment could break the optimality. Of course, we can also break the optimality by changing the environment while keeping CES. For example, the equilibrium is no longer optimal if producing intermediate inputs needs not only "labor" but also the final good, or if the taxation is added, and so on.

It is easy to see why this is false. If this logic were correct, the equilibrium would be optimal, as long as all products were sold at the same markup rate, and it would not have to be equal to $\sigma/(\sigma-1)$. Indeed, any symmetric equilibrium would be optimal, even if the demand system were non-CES and/or in the presence of a uniform taxation on intermediate inputs. The logic is incorrect, because the common markup rate merely ensures that the allocation across available products is not distorted; it does not ensure that the equilibrium incentive to introduce another product is optimal.

Fallacy #2. The equilibrium allocation is optimal if and only if it is under CES.

This claim is the opposite of the claim in Fallacy #1. Of course, the optimality under CES is *not robust*, because it must satisfy the *knife-edge* condition, the two sources of externalities canceling out each other. However, CES is not *unique* in this respect, as explained in Section 4.

3. General Homothetic Symmetric Demand Systems[5]

Let us now assume that the industry that uses symmetric production technologies, specified either as the CRS production function, $X(\mathbf{x})$, or the unit cost function, $P(\mathbf{p})$, which are related to each other as:

$$X(\mathbf{x}) \equiv \min_{\mathbf{p}} \left\{ \mathbf{px} \equiv \int_{\Omega} p_\omega x_\omega d\omega \,\middle|\, P(\mathbf{p}) \geq 1 \right\};$$

$$P(\mathbf{p}) \equiv \min_{\mathbf{x}} \left\{ \mathbf{px} \equiv \int_{\Omega} p_\omega x_\omega d\omega \,\middle|\, X(\mathbf{x}) \geq 1 \right\},$$

where $\mathbf{x} = \{x_\omega; \omega \in \overline{\Omega}\}$, the input quantity vector, and $\mathbf{p} = \{p_\omega; \omega \in \overline{\Omega}\}$, the input price vector, are now defined over $\overline{\Omega}$, the set of *all potential* input varieties, so that $\Omega \subset \overline{\Omega}$, the set of available input varieties, with $V \equiv |\Omega|$. Thus, $\overline{\Omega}\backslash\Omega$ is the set of unavailable varieties, with $x_\omega = 0$ and $p_\omega = \infty$ for $\omega \in \overline{\Omega}\backslash\Omega$. To ensure the feasibility of production, we need to assume that inputs are ***inessential***, i.e., $\overline{\Omega}\backslash\Omega \neq \emptyset$ does not imply $X(\mathbf{x}) = 0 \Leftrightarrow P(\mathbf{p}) = \infty$. Notice that $P(\mathbf{p})$ can be derived from $X(\mathbf{x})$ and that $X(\mathbf{x})$ can be recovered from $P(\mathbf{p})$, as long as they satisfy linear homogeneity, monotonicity, and strict quasi-concavity. Thus, we could use either $X(\mathbf{x})$ or $P(\mathbf{p})$, as the primitive of this CRS production technologies.

3.1 Demand Systems

[5] This section draws heavily from Matsuyama & Ushchev (2023).

The demand curve and the inverse demand curve for $\omega \in \Omega$ are:

$$x_\omega = \frac{\partial P(\mathbf{p})}{\partial p_\omega} X(\mathbf{x}) = \frac{\partial \ln P(\mathbf{p})}{\partial p_\omega} E; \qquad p_\omega = P(\mathbf{p}) \frac{\partial X(\mathbf{x})}{\partial x_\omega} = \frac{\partial \ln X(\mathbf{x})}{\partial x_\omega} E; \quad ,$$

from either of which Euler's homogenous function theorem implies

$$\mathbf{px} = \int_\Omega p_\omega x_\omega d\omega = \int_\Omega p_\omega \frac{\partial P(\mathbf{p})}{\partial p_\omega} X(\mathbf{x}) d\omega = \int_\Omega P(\mathbf{p}) \frac{\partial X(\mathbf{x})}{\partial x_\omega} x_\omega d\omega = P(\mathbf{p}) X(\mathbf{x}) = E.$$

The budget share of $\omega \in \Omega$, $s_\omega \equiv p_\omega x_\omega / P(\mathbf{p}) X(\mathbf{x})$, can be thus written as a homogeneous function of degree zero both in price and in quantity;

$$s_\omega = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} \equiv s(p_\omega, \mathbf{p}) = s(1, \mathbf{p}/p_\omega);$$

$$s_\omega = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} \equiv s^*(x_\omega, \mathbf{x}) = s^*(1, \mathbf{x}/x_\omega).$$

From now on, we also impose **gross substitutability**,

$$\frac{\partial \ln s(p_\omega; \mathbf{p})}{\partial \ln p_\omega} < 0 \Leftrightarrow \frac{\partial \ln s^*(x_\omega;\ \mathbf{x})}{\partial \ln x_\omega} > 0.$$

This ensures that the firm selling $\omega \in \Omega$ faces the positive marginal revenue curve.[6]

The price elasticity of demand for $\omega \in \Omega$, $\zeta_\omega \equiv -\partial \ln x_\omega / \partial \ln p_\omega$, can be also written as a homogeneous function of degree zero in prices or in quantities:

$$\zeta_\omega = 1 - \frac{\partial \ln s(p_\omega; \mathbf{p})}{\partial \ln p_\omega} \equiv \zeta(p_\omega; \mathbf{p}) = \zeta(1, \mathbf{p}/p_\omega) > 1;$$

$$\zeta_\omega = \left[1 - \frac{\partial \ln s^*(x_\omega; \mathbf{x})}{\partial \ln x_\omega}\right]^{-1} \equiv \zeta^*(x_\omega; \mathbf{x}) = \zeta^*(1, \mathbf{x}/x_\omega) > 1.$$

Notice that the restriction of gross substitutability imposed above is equivalent to the restriction that the price elasticity is always greater than one. In general, the price elasticity can be increasing or decreasing in its own price. The literature typically focuses on the increasing case,

$$\frac{\partial \ln \zeta(p_\omega; \mathbf{p})}{\partial \ln p_\omega} > 0 \Leftrightarrow \frac{\partial \ln \zeta^*(x_\omega;\ \mathbf{x})}{\partial \ln x_\omega} < 0.$$

This is **Marshall's 2nd law of demand,** or **the 2nd law** for short. For the case where the price elasticity is decreasing, we say the anti-2nd law holds. Clearly, CES is the borderline case. All other examples listed in Appendix 2 satisfies the 2nd law.

[6] Under CES, $\sigma > 1$ ensures both the inessentiality and gross substitutability of inputs. In general, the inessentiality and gross substitutability are different concepts and need to be assumed separately.

Note that the budget share of $\omega \in \Omega$, s_ω, and its price elasticity of demand, ζ_ω, are both functions of $\mathbf{p}/p_\omega$ or $\mathbf{x}/x_\omega$. Of course, symmetry implies that they are invariant of permutation, but they still depend on the entire distribution of the prices (or the quantities) relative to its own price (or its own quantity), which is infinite dimensional. This suggests that the cross-variety interactions could be complicated under general homothetic demand systems.

3.2 Substitutability and Love-for-Variety Measures:

We now introduce two measures that help to characterize general homothetic symmetric demand systems. First, define the unit quantity vector,

$$\mathbf{1}_\Omega \equiv \{(1_\Omega)_\omega; \omega \in \overline{\Omega}\}, \qquad \text{where} \qquad (1_\Omega)_\omega \equiv \begin{cases} 1 & \text{for} \quad \omega \in \Omega \\ 0 & \text{for} \quad \omega \in \overline{\Omega} \backslash \Omega \end{cases}$$

which is the indicator function of Ω, and the unit price vector,

$$\mathbf{1}_\Omega^{-1} \equiv \{(1_\Omega^{-1})_\omega; \omega \in \overline{\Omega}\}, \qquad \text{where} \qquad (1_\Omega^{-1})_\omega \equiv \begin{cases} 1 & \text{for} \quad \omega \in \Omega \\ \infty & \text{for} \quad \omega \in \overline{\Omega} \backslash \Omega \end{cases}.$$

Clearly, $\int_\Omega (1_\Omega)_\omega d\omega = \int_\Omega (1_\Omega^{-1})_\omega d\omega = |\Omega| \equiv V$. Moreover, at the symmetric patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$ and $\mathbf{x} = x\mathbf{1}_\Omega$,

$$s_\omega = s(1, \mathbf{p}/p_\omega) = s^*(1, \mathbf{x}/x_\omega) = s(1, \mathbf{1}_\Omega^{-1}) = s^*(1, \mathbf{1}_\Omega) = 1/V,$$

and the price elasticity of each variety,

$$\zeta_\omega = \zeta(1, \mathbf{p}/p_\omega) = \zeta^*(1, \mathbf{x}/x_\omega) = \zeta(1, \mathbf{1}_\Omega^{-1}) = \zeta^*(1, \mathbf{1}_\Omega) > 1,$$

is a function of V only, hence can be denoted as $\sigma(V)$. Furthermore, as shown in Matsuyama & Ushchev (2023), $\zeta(1, \mathbf{1}_\Omega^{-1}) = \zeta^*(1, \mathbf{1}_\Omega)$ is equal to the Allen-Uzawa elasticity of substitution between any pair, ω and $\omega' \in \Omega$, evaluated at $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$. Hence,

Definition: *The substitutability measure across varieties* is defined by

$$\sigma(V) \equiv \zeta(1; \mathbf{1}_\Omega^{-1}) = \zeta^*(1; \mathbf{1}_\Omega) > 1,$$

where $\sigma(V) > 1$ is guaranteed by gross substitutability. If $\sigma'(V) > (<) 0$, we call the case of ***increasing (decreasing) substitutability***. In general, however, $\sigma(V)$ may be nonmonotonic in V.

Love-for-variety is commonly defined by the rate of productivity gain from a higher V, at $\mathbf{x} = x\mathbf{1}_\Omega$, holding xV constant,

$$\left.\frac{d \ln X(\mathbf{x})}{d \ln V}\right|_{\mathbf{x}=x\mathbf{1}_\Omega, xV=const.} = \left.\frac{d \ln xX(\mathbf{1}_\Omega)}{d \ln V}\right|_{xV=const.} = \frac{d \ln X(\mathbf{1}_\Omega)}{d \ln V} - 1.$$

Since $X(\mathbf{1}_\Omega)$ is a function of V only, so is this measure, and hence it can be denoted as $\mathcal{L}(V)$. Alternatively, love-for-variety may be defined by the decline in $P(\mathbf{p})$ from a higher V, at $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$, holding p constant.

$$-\left.\frac{d\ln P(\mathbf{p})}{d\ln V}\right|_{\mathbf{p}=p\mathbf{1}_\Omega^{-1},\ p=const.} = -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V}.$$

Since $P(\mathbf{1}_\Omega^{-1})$ is a function of V only, so is this measure. These two definitions are indeed equivalent because, by applying $\mathbf{x} = x\mathbf{1}_\Omega$ and $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$ to $\mathbf{px} = P(\mathbf{p})X(\mathbf{x})$,

$$pxV = pP(\mathbf{1}_\Omega^{-1})xX(\mathbf{1}_\Omega) \Longrightarrow -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1.$$

Hence,

Definition. *The love-for-variety measure* is defined by:

$$\mathcal{L}(V) \equiv -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0,$$

where $\mathcal{L}(V) > 0$ is guaranteed by the strict quasi-concavity of the production technologies. If $\mathcal{L}'(V) > (<)0$, we call the case of ***increasing (diminishing) love-for-variety***. In general, however, $\mathcal{L}(V)$ may be nonmonotonic in V.

Under CES,

- The price elasticity of demand is constant; $\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x}) = \sigma$.
- Substitutability is constant; $\sigma(V) = \sigma$.
- Love-for-variety is constant and inversely related to σ, as $\mathcal{L}(V) = \mathcal{L} = 1/(\sigma - 1)$.

Under general homothetic symmetric demand systems, however, we can say little about the relation between $\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$, $\sigma(V)$, and $\mathcal{L}(V)$, even though that the following claims are often made:

Fallacy #3: $\sigma(V)$ is constant only under CES.

Fallacy #4: $\sigma'(V) > (<)0$ iff the 2nd law (anti-2nd law) holds.

Fallacy #5: $\sigma(V)$ is an inverse measure of love-for-variety, $\mathcal{L}(V)$.[7]

See Matsuyama & Ushchev (2023, 2024b) for some counterexamples. Symmetry and homotheticity alone are not strong enough to impose much restriction, because the budget share

[7] Though many have derived $\sigma(V)$ for specific non-CES demand systems, I am unaware of any attempt prior to Matsuyama & Ushchev (2023) to derive $\mathcal{L}(V)$ for non-CES. I suspect that those who made this claim just take it for granted that $\mathcal{L} = 1/(\sigma - 1)$ under CES would be generalized to $\mathcal{L}(V) = 1/(\sigma(V) - 1)$ under non-CES.

and the price elasticity of each variety can depend on the entire distribution of prices across different varieties. Nevertheless, one might find that the claims made in these fallacies are appealing features for a demand system to have. Even though these claims are false in general, they are true under H.S.A., as will be shown in Section 5.

4. Dixit Stiglitz under General Homothetic Demand Systems[8]

Let us now apply the general homothetic symmetric demand system to **the Dixit-Stiglitz environment.**

Fallacy #6. With the symmetric firms, the equilibrium is symmetric.

The symmetry of the Dixit-Stiglitz environment only ensures the symmetry of the set of equilibria, not the symmetry of any equilibrium. (This is called "Symmetry-Breaking:" see Matsuyama 2008). Even if the symmetric equilibrium exists, it may co-exist with a symmetric set of asymmetric equilibriums. In such an asymmetric equilibrium, ex-ante symmetric firms pursue different pricing strategies, where some choose to have higher markup rates with smaller quantities while others choose to have lower markup rates with larger quantities, and they are indifferent between the two, so that firms become endogenously asymmetric, giving rise to endogenous price distribution.

4.1 Symmetric Equilibrium:

Nevertheless, let us proceed under *the assumption that a symmetric equilibrium exists.* Each firm chooses p_ω to maximize its gross profit,

$$\Pi_\omega = (p_\omega - \psi)x_\omega = \left(1 - \frac{\psi}{p_\omega}\right) p_\omega x_\omega = \left(1 - \frac{\psi}{p_\omega}\right) s(p_\omega, \mathbf{p})E,$$

holding E and **p,** given. The first-order condition generates the Lerner pricing formula,

$$p_\omega \left(1 - \frac{1}{\zeta(p_\omega; \mathbf{p})}\right) = \psi.$$

In any symmetric equilibrium, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$, $\zeta(p_\omega; \mathbf{p}) = \zeta(1, \mathbf{1}_\Omega^{-1}) = \sigma(V)$. Hence,

$$p_\omega \left(1 - \frac{1}{\sigma(V)}\right) = \psi \qquad \Leftrightarrow \qquad p_\omega \equiv p = \frac{\sigma(V)}{\sigma(V) - 1}\psi \equiv \mu(V)\psi,$$

where the markup rate, $\mu(V)$, which satisfies the following identities:

[8] This and next sections draw heavily from Matsuyama & Ushchev (2020a).

$$\frac{1}{\sigma(V)}+\frac{1}{\mu(V)}=1;\qquad \frac{1}{\sigma(V)-1}=\frac{\mu(V)}{\sigma(V)}=\mu(V)-1;$$

and [9]

$$\mathcal{E}_\sigma(V)=-\frac{\mathcal{E}_\mu(V)}{\mu(V)-1};\quad \mathcal{E}_\mu(V)=-\frac{\mathcal{E}_\sigma(V)}{\sigma(V)-1}.$$

The common gross profit is $\Pi=(p-\psi)x=px/\sigma(V)=E/[V\sigma(V)]$, which must be equal to the fixed cost, F. Thus, a symmetric equilibrium satisfies:

$$V^{eq}\sigma(V^{eq})=\frac{E}{F};\ p^{eq}=\mu(V^{eq})\psi;\ \ x^{eq}=\frac{\sigma(V^{eq})}{\mu(V^{eq})}\frac{F}{\psi}.$$

The uniqueness of the symmetric equilibrium for any $E/F>0$ requires that $V\sigma(V)$ is globally increasing in V. This condition can be expressed as:

$$\mathcal{E}_\sigma(V)>-1\Leftrightarrow\mathcal{E}_\mu(V)<\mu(V)-1.$$

This condition also ensures that V^{eq} is globally increasing in E/F.[10] Clearly, $\sigma'(\cdot)>0$, the case of ***increasing substitutability***, or equivalently, $\mu'(\cdot)<0$, the case of ***procompetitive entry***, is sufficient for this, but not necessary.

In the symmetric equilibrium, the profit share and the production cost share are, respectively:

$$\frac{1}{\sigma(V^{eq})};\ \frac{1}{\mu(V^{eq})},$$

and the profit/production cost ratio is:

$$\frac{\mu(V^{eq})}{\sigma(V^{eq})}=\frac{1}{\sigma(V^{eq})-1}=\mu(V^{eq})-1$$

in all firms. All of them generally vary with V^{eq}, and hence with E/F.

4.2 Comparative Statics:

Under the condition that ensures the uniqueness of the symmetric equilibrium and its stability, $1+\mathcal{E}_\sigma(V)>0$,

[9] Throughout this review, $\mathcal{E}_f(x)\equiv xf'(x)/f(x)=\partial\ln f(x)/\partial\ln x$ denotes the elasticity of a positive-valued function, $f(x)>0$, defined over a positive real number $x>0$.

[10] Locally increasing $V\sigma(V)$ in the neighborhood of a symmetric equilibrium also ensures its local stability in any adjustment process with the following property: $\dot{V}_t \gtreqless 0$ if and only if $\pi_t=E/V_t\sigma(V_t)\gtreqless F$.

$$\widehat{V^{eq}} = \frac{\hat{E} - \hat{F}}{1 + \mathcal{E}_\sigma(V^{eq})}; \quad \widehat{p^{eq}} = \frac{\mathcal{E}_\mu(V^{eq})\big(\hat{E} - \hat{F}\big)}{1 + \mathcal{E}_\sigma(V^{eq})} + \hat{\psi}; \quad \widehat{x^{eq}} = \frac{\mu(V^{eq})\mathcal{E}_\sigma(V^{eq})\big(\hat{E} - \hat{F}\big)}{1 + \mathcal{E}_\sigma(V^{eq})} + \hat{F} - \hat{\psi}.$$

Thus, for $\mathcal{E}_\sigma(V) \gtreqless 0 \Leftrightarrow \mathcal{E}_\mu(V) \lesseqgtr 0$, **the market size effect** is

$$0 < \frac{\partial \ln V^{eq}}{\partial \ln E} = 1 - \frac{\partial \ln(p^{eq}x^{eq})}{\partial \ln E} = \frac{1}{1 + \mathcal{E}_\sigma(V^{eq})} \lesseqgtr 1;$$

$$\frac{\partial \ln p^{eq}}{\partial \ln E} = \frac{\mathcal{E}_\mu(V^{eq})}{1 + \mathcal{E}_\sigma(V^{eq})} \lesseqgtr 0; \qquad \frac{\partial \ln x^{eq}}{\partial \ln E} = \frac{\mu(V^{eq})\mathcal{E}_\sigma(V^{eq})}{1 + \mathcal{E}_\sigma(V^{eq})} \gtreqless 0;$$

and the profit/production cost ratio changes as:

$$\frac{\partial \ln(\mu(V^{eq})/\sigma(V^{eq}))}{\partial \ln E} = \frac{\mathcal{E}_\mu(V^{eq}) - \mathcal{E}_\sigma(V^{eq})}{1 + \mathcal{E}_\sigma(V^{eq})} \lesseqgtr 0.$$

The intuition is easy to grasp. For example, consider the case of *increasing substitutability* $\mathcal{E}_\sigma(V) > 0$, i.e., the case of *procompetitive entry,* $\mathcal{E}_\mu(V) < 0$. In response to a market size increase, more firms enter and product variety goes up. When this makes the products more substitutable, $\mathcal{E}_\sigma(V) > 0$, the markup rate goes down, $\mathcal{E}_\mu(V) < 0$, necessitating each firm to increase the scale of operation and earn more revenue just to break even. Because each firm is larger, the masses of firms and product variety go up at a rate lower than the rate of market size increase. This also means a decline in the profit/production cost ratio.

Note that these above results depend on $\text{sgn}\{\mathcal{E}_\sigma(V)\} = -\text{sgn}\{\mathcal{E}_\mu(V)\}$, i.e., how the markup rate responds to entry, not whether the 2nd law hold or not. It is also unrelated to the property of $\mathcal{L}(V)$, which plays a crucial role in determining the optimal allocation.

4.3 Optimal Allocation: This now solves the following problem:

$$\max X(\mathbf{x}) \quad s.t. \int_\Omega \psi x_\omega d\omega + VF \le E.$$

The solution satisfies $x_\omega = x > 0$ for $\omega \in \Omega$; $x_\omega = 0$ for $\omega \notin \Omega$, simplifying the problem to:

$$\max X(\mathbf{x}) = \max_{V(\psi x + F) \le E} xX(\mathbf{1}_\Omega) = \frac{F}{\psi} \max_V \frac{X(\mathbf{1}_\Omega)}{V}\left(\frac{E}{F} - V\right).$$

From the first-order condition,

$$\frac{d \ln X(\mathbf{1}_\Omega)}{d \ln V} - 1 + \frac{d \ln(E/F - V)}{d \ln V} = \mathcal{L}(V) - \frac{V}{E/F - V} = 0,$$

the optimal variety V^{op} satisfies

$$\left[1+\frac{1}{\mathcal{L}(V^{op})}\right] V^{op} = \frac{E}{F}.$$

This condition fully characterizes V^{op} if LHS is strictly increasing, i.e.,

$$\mathcal{E}_{\mathcal{L}}(V) < 1 + \mathcal{L}(V),$$

which also ensures that V^{op} is an increasing function of E/F. This condition is clearly satisfied by $\mathcal{E}_{\mathcal{L}}(V) < 1$. In particular, the case of ***diminishing love-for-variety***, $\mathcal{L}'(V) < 0$, is sufficient (but not necessary) for this.

4.4 Optimal vs. Equilibrium: By comparing the two conditions,

$$\left[1+\frac{1}{\mathcal{L}(V^{op})}\right] V^{op} = \frac{E}{F}; \;\; \sigma(V^{eq})V^{eq} = \left[1+\frac{\sigma(V^{eq})}{\mu(V^{eq})}\right] V^{eq} = \frac{E}{F},$$

with the LHS of each condition strictly increasing in V^{op} and in V^{eq} respectively, one could easily verify:

Proposition 1. Assume that the symmetric equilibrium exists uniquely in the Dixit-Stiglitz environment under general homothetic symmetric demand systems. Then,

$$\mathcal{L}(V) \gtreqless \frac{\mu(V)}{\sigma(V)} = \frac{1}{\sigma(V)-1} \text{ for all } V > 0 \Leftrightarrow V^{eq} \lesseqgtr V^{op} \text{ for all } E/F > 0.$$

The logic behind this result is simple; $\mathcal{L}(V)$ captures the social incentive to add product variety, while $[\sigma(V)-1]^{-1} = \mu(V) - 1 = \mu(V)/\sigma(V)$, the profit/production cost ratio, captures the private incentive to add product variety. In general, these two do not coincide. For some classes of demand systems, $\mathcal{L}(V)[\sigma(V)-1] > 1$, hence $V^{eq} < V^{op}$. For some other classes, $\mathcal{L}(V)[\sigma(V)-1] < 1$, hence $V^{eq} > V^{op}$. In-between, there are the borderline classes for which $\mathcal{L}(V)[\sigma(V)-1] = 1$, hence $V^{eq} = V^{op}$. CES belongs to the borderline, but not the only one. And the optimality result in any of them is not robust.[11] Moreover, though $\mathcal{L}(V) = \mu(V) - 1$ ensures the optimality of the unique symmetric equilibrium, it does not rule out the existence of a symmetric set of asymmetric equilibria, none of which is optimal.

Proposition 1 gives us the condition for evaluating the optimality of the symmetric equilibrium, if it exists uniquely. However, because homotheticity and symmetry alone impose little restriction on the relation between $\sigma(V)$ and $\mathcal{L}(V)$, "almost anything goes." Thus it is

[11] Matsuyama & Ushchev (2024a) constructed a two-parameter family of homothetic symmetric demand systems, in which the equilibrium variety is generically either excessive or insufficient, as well as there is a continuum of non-generic cases in which the equilibrium is optimal, to which CES belongs.

necessary to restrict the demand systems to make further progress. The next section introduces such a restriction in the form of H.S.A. demand systems.

5. Homothetic Single Aggregator (H.S.A.) Demand Systems

A homothetic symmetric demand system belongs to the *homothetic single aggregator* (H.S.A.) class with gross substitutes if the budget share of $\omega \in \Omega$ is strictly decreasing in its relative price, $z_\omega \equiv p_\omega / A(\mathbf{p})$, its own price, p_ω divided by a single price aggregator $A(\mathbf{p})$, which is *common* across all varieties. That is, all the cross-price effects are summarized in a single number, $A(\mathbf{p})$, or a sufficient statistic, which is the key feature of H.S.A.

5.1 Definition

Formally, a homothetic symmetric demand system belongs to H.S.A. with gross substitutes if **its budget share** of $\omega \in \Omega$ can be expressed as:

$$s_\omega = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(\mathbf{p})}\right), \text{ where } \int_\Omega s\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega \equiv 1.$$

Here, $s: \mathbb{R}_{++} \to \mathbb{R}_+$ is strictly decreasing as long as $s(z) > 0$ with $\lim_{z \to \bar{z}} s(z) = 0$, where $\bar{z} \equiv \inf\{z > 0 | s(z) = 0\}$.[12] If $\bar{z} < \infty$, $s_\omega = 0$ for $p_\omega \geq \bar{z}A(\mathbf{p})$, hence $\bar{z}A(\mathbf{p})$ is **the choked price**.

The price elasticity of demand for $\omega \in \Omega$ is, for $p_\omega < \bar{z}A(\mathbf{p})$,

$$\zeta_\omega = \zeta(p_\omega; \mathbf{p}) = 1 - \frac{z_\omega s'(z_\omega)}{s(z_\omega)} \equiv 1 - \mathcal{E}_s(z_\omega) \equiv \zeta(z_\omega) \equiv \zeta\left(\frac{p_\omega}{A(\mathbf{p})}\right) > 1,$$

with $\lim_{z \to \bar{z}} \zeta(z) = \infty$, if $\bar{z} < \infty$. The 2nd law holds iff $\zeta'(\cdot) > 0$.[13]

Note that **the budget share function,** $s(\cdot)$, is the primitive of the H.S.A. demand system. **The common price aggregator**, $A(\mathbf{p})$, is not, because it needs to be derived from $s(\cdot)$ using **the adding-up constraint**, $\int_\Omega s(p_\omega / A(\mathbf{p})) d\omega \equiv 1$. By construction, the budget share $s(z_\omega) = s(p_\omega / A(\mathbf{p}))$ adds up to one, and $A(\mathbf{p})$ is linear homogenous in $\mathbf{p}$ for any fixed Ω. It is important

[12] For $s: \mathbb{R}_{++} \to \mathbb{R}_+$, satisfying these conditions, a class of the budget share functions, $s(z;\gamma) \equiv \gamma s(z)$ for $\gamma > 0$, generate the same demand system with the same common price aggregator. We just need to renormalize the indices of varieties, as $\omega' = \gamma\omega$, so that $\int_\Omega s(p_\omega / A(\mathbf{p}); \gamma) d\omega = \int_\Omega s(p_{\omega'} / A(\mathbf{p})) d\omega' = 1$. In this sense, $s(z;\gamma) \equiv \gamma s(z)$ for $\gamma > 0$ are all equivalent. Also, a class of the budget share functions, $s(z;\beta) \equiv s(z/\beta)$ for $\beta > 0$, generate the same demand system, with $A(\mathbf{p};\beta) = A(\mathbf{p})/\beta$, because $s(p_\omega / A_\beta(\mathbf{p}); \beta) = s(p_\omega / A_\beta(\mathbf{p};\beta)) = s(p_\omega / A(\mathbf{p}))$. In this sense, $s(z;\beta) \equiv s(z/\beta)$ for $\beta > 0$ are all equivalent.

[13] Conversely, one can obtain $s(\cdot)$ as $s(z) = \gamma \exp\left[\int_{z_0}^{z} \frac{1-\zeta(\xi)}{\xi} d\xi\right]$, from any $\zeta(\cdot) > 1$, with $\lim_{z \to \bar{z}} \zeta(z) = \infty$, if $\bar{z} < \infty$.

that $A(\mathbf{p})$ is common across varieties and that both the budget share of $\omega \in \Omega$, $s(z_\omega)$ and its price elasticity $\zeta(z_\omega)$ are functions of $z_\omega \equiv p_\omega/A(\mathbf{p})$ only. Thus, all the cross-variety effects in H.S.A. are summarized by the single price aggregator, $A(\mathbf{p})$.[14]

After deriving $A(\mathbf{p})$ from $s(\cdot)$, the unit cost function, $P(\mathbf{p})$, can be derived by integrating $s_\omega = \partial \ln P(\mathbf{p})/\partial \ln p_\omega = s(p_\omega/A(\mathbf{p}))$ as:

$$cP(\mathbf{p}) = A(\mathbf{p}) \exp\left[-\int_\Omega s\left(\frac{p_\omega}{A(\mathbf{p})}\right)\Phi\left(\frac{p_\omega}{A(\mathbf{p})}\right)d\omega\right], \text{ where } \Phi(z) \equiv \frac{1}{s(z)}\int_z^{\bar{z}} \frac{s(\xi)}{\xi}d\xi > 0.$$

where $\Phi(z)$ is the productivity gain created by the product sold at the normalized price, z, and $c > 0$ is an integral constant, proportional to TFP. Clearly, $P(\mathbf{p})$ is linear homogeneous and monotonic. Moreover, Matsuyama & Ushchev (2017) showed that it is strictly quasi-concave, thereby proving the integrability (in the sense of Samuelson 1950 and Hurwicz & Uzawa 1971) of H.S.A. demand systems. It is worth emphasizing that $P(\mathbf{p})/A(\mathbf{p})$ is not constant, with the sole exception of CES.[15] $A(\mathbf{p})$ and $P(\mathbf{p})$ generally move differently in response of a change in **p.** This should make sense, because $A(\mathbf{p})$ is the inverse measure of *competitive pressures* from other products, which captures the *cross-variety interactions* in the demand system, while $P(\mathbf{p})$ is the unit cost function, which captures the *productivity consequences* of price changes; there is no reason to expect them to move together in general.[16] In other words, $A(\mathbf{p})$ in the definition of H.S.A. cannot be replaced by $P(\mathbf{p})$, contrary to:

Fallacy #7; $s_\omega = f(p_\omega/P(\mathbf{p}))$, *with* $f'(\cdot) < 0$ *defines the class of flexible homothetic demand systems, which contains CES as a special case, where* $s_\omega \propto (p_\omega/P(\mathbf{p}))^{1-\sigma}$.

This is false because $\partial \ln P(\mathbf{p})/\partial \ln p_\omega = s_\omega = f(p_\omega/P(\mathbf{p}))$ is a partial differential equation of $P(\mathbf{p})$, whose solution must take the form of $s_\omega \propto (p_\omega/P(\mathbf{p}))^{1-\sigma}$.

---

[14] Recall that, under general homothetic symmetric demand systems, the budget share of $\omega \in \Omega$, and its price elasticity depends on $\mathbf{p}/p_\omega$, the price distribution normalized by its own price, an infinite dimensional object.

[15] To see this, differentiating $\int_\Omega s(p_\omega/A(\mathbf{p}))d\omega \equiv 1$, yields

$$\frac{\partial \ln A(\mathbf{p})}{\partial \ln p_\omega} = \frac{z_\omega s'(z_\omega)}{\int_\Omega s'(z_{\omega'})z_{\omega'}d\omega'} = \frac{[\zeta(z_\omega) - 1]s(z_\omega)}{\int_\Omega [\zeta(z_{\omega'}) - 1]s(z_{\omega'})d\omega'},$$

which differs from $\partial \ln P(\mathbf{p})/\partial \ln p_\omega = s(z_\omega)$, unless $\zeta(z)$ is constant, i.e., except the case of CES.

[16] Moreover, $A(\mathbf{p})$, the "average input price", depends on the unit of measurement of inputs, but not on the unit of measurement of the final good. In contrast, $P(\mathbf{p})$ is the cost of producing one unit of the final good, when the input prices are **p**. Hence, it depends not only on the unit measurement of inputs but also on that of the final good. Furthermore, a change in TFP, while affecting $P(\mathbf{p})$, leaves the market share unaffected. This is why the H.S.A. demand system and $A(\mathbf{p})$ are independent of the integral constant, $c > 0$, and hence it cannot be determined.

5.2 Substitutability and Love-for-Variety under H.S.A.

For symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$, $z_\omega = z$ satisfies $s(z)V = 1$, and $-\ln P(\mathbf{1}_\Omega^{-1}) = \ln c + \ln z + \Phi(z)$, from which

> **Proposition 2:** Under H.S.A.,
>
> $$\sigma(V) = \zeta\left(\frac{1}{A(\mathbf{1}_\Omega^{-1})}\right) = \zeta\left(s^{-1}\left(\frac{1}{V}\right)\right) > 1.$$
>
> $$\mathcal{L}(V) \equiv -\frac{d\ln P(\mathbf{1}_\Omega^{-1})}{d\ln V} = \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) > 0.$$

Since $s^{-1}(1/V)$ is increasing in V, $\text{sgn}\{\zeta'(\cdot)\} = \text{sgn}\{\sigma'(\cdot)\}$ and $\text{sgn}\{\Phi'(\cdot)\} = \text{sgn}\{\mathcal{L}'(\cdot)\}$. In particular, increasing substitutability and procompetitive entry are equivalent to the 2nd law under H.S.A. Moreover, Matsuyama & Ushchev (2020a, 2023) show that

$$\zeta'(\cdot) \gtreqless 0, \forall z \in (z_0, \bar{z}) \Rightarrow \Phi'(z) \lesseqgtr 0, \forall z \in (z_0, \bar{z}).$$

The reverse is not true in general, except

$$\Phi'(z) = 0, \forall z \in (z_0, \bar{z}) \quad \Rightarrow \quad \zeta'(\cdot) = 0, \forall z \in (z_0, \bar{z}).$$

Thus, from Proposition 2,

> **Proposition 3:** Under H.S.A.,
>
> $$\sigma'(V) \gtreqless 0, \ \forall V \in (1/s(z_0), \infty) \quad \Rightarrow \quad \mathcal{L}'(V) \lesseqgtr 0, \quad \forall V \in (1/s(z_0), \infty),$$
>
> The reverse is not true in general, except
>
> $$\mathcal{L}'(V) = 0, \quad \forall V \in (1/s(z_0), \infty) \Rightarrow \sigma'(V) = 0, \ \forall V \in (1/s(z_0), \infty).$$

Thus, the 2nd law, increasing substitutability, and procompetitive entry are not only equivalent to each other under H.S.A. If any of them holds *globally,* it is sufficient (but not necessary) for *global* diminishing love-for variety under H.S.A.[17]

6. Dixit Stiglitz under H.S.A.

Let us now apply H.S.A. to **the Dixit-Stiglitz environment.[18]**

[17]Note that $\sigma'(\cdot) > 0$ everywhere over (V, ∞) is sufficient for $\mathcal{L}'(V) < 0$, but $\sigma'(V) > 0$ is not. This is because substitutability is a local property of the demand system, while love-for-variety depends on its global properties.

[18] In addition to Matsuyama & Ushchev (2020a,b, 2022a,b, and 2024a), recent applications of H.S.A. to monopolistic competition include Baqaee et. al. (2024), Fujiwara & Matsuyama (2022), and Grossman et. al. (2023). Trottner (2023) applies H.S.A. to both monopolistic and monopsonic competition among firms with two-sided market power.

6.1 Equilibrium:

Holding E and $A = A(\mathbf{p})$ fixed, each firm chooses p_ω (hence $z_\omega \equiv p_\omega/A$) to maximize its gross profit,

$$(p_\omega - \psi)x_\omega = \left(1 - \frac{\psi}{p_\omega}\right)p_\omega x_\omega = \left(1 - \frac{\psi}{p_\omega}\right)s\left(\frac{p_\omega}{A(\mathbf{p})}\right)E = \left(1 - \frac{\psi/A}{z_\omega}\right)s(z_\omega)E.$$

The first-order condition can be written as the Lerner pricing formula, normalized by A, as:

$$z_\omega\left[1 - \frac{1}{\zeta(z_\omega)}\right] = \frac{\psi}{A}.$$

In what follows, let us assume for the expositional purpose.[19]

> **Assumption A1:** *For all* $z \in (0, \overline{z})$,
> $$\frac{d}{dz}\left(z\left[1 - \frac{1}{\zeta(z)}\right]\right) > 0.$$

Clearly, the 2nd law, $\zeta'(z) > 0$, is sufficient but not necessary for **A1.** **A1** states that, for any $A = A(\mathbf{p})$, the marginal revenue of each firm is strictly increasing in p_ω (i.e., decreasing in x_ω). Under **A1**, the LHS of the normalized Lerner formula is strictly increasing, it can be inverted to express the profit-maximizing z_ω as:

$$z_\omega = \frac{p_\omega}{A} = \tilde{Z}\left(\frac{\psi}{A}\right); \quad \tilde{Z}'(\cdot) > 0.$$

Thus, under **A1**, **the equilibrium is symmetric**, so that $p_\omega = p$ and $z_\omega = z$ satisfying:

$$z = \frac{p}{A} = \frac{p}{A(\mathbf{p})} = \frac{1}{A(\mathbf{1}_\Omega^{-1})} = s^{-1}\left(\frac{1}{V}\right).$$

Moreover, **A1** is equivalent to:

$$\frac{d}{dz}\ln\left(\frac{s(z)}{\zeta(z)}\right) < 0 \Leftrightarrow \frac{d}{dV}\ln V\sigma(V) > 0,$$

so that the maximized gross profit of each firm,

$$\left(1 - \frac{\psi}{p}\right)s(z)E = \left(1 - \frac{\psi/A}{z}\right)s(z)E = \frac{s(z)}{\zeta(z)}E = \frac{E}{V\sigma(V)}$$

[19] Even without **A1,** the profit maximizing z_ω would be strictly increasing and the maximized profit $\Pi_\omega = s(z_\omega)E/\zeta(z_\omega)$ would be strictly decreasing in the normalized cost ψ/A. However, z_ω could become piecewise-continuous (i.e., it would jump up at some values of ψ/A), and Π_ω could become piecewise-differentiable, which would complicate the exposition of comparative static analysis.

is strictly decreasing in z and in V. Hence, the free-entry condition, $\pi = F$, uniquely pins down ψ/A, z, and V. Thus, **the equilibrium is not only symmetric but also unique under A1.** From Section 4,

$$V^{eq}\sigma(V^{eq}) = \frac{E}{F};\ p^{eq} = \mu(V^{eq})\psi;\ \ x^{eq} = \frac{(\sigma(V^{eq})-1)F}{\psi} = \frac{F}{(\mu(V^{eq})-1)\psi},$$

and the profit share and the production cost share in the revenue in all firms are equal to

$$\frac{1}{\sigma(V^{eq})};\ \frac{1}{\mu(V^{eq})}.$$

and the ratio of the profit to the production cost is equal to:

$$\frac{\mu(V^{eq})}{\sigma(V^{eq})} = \frac{1}{\sigma(V^{eq})-1} = \mu(V^{eq}) - 1$$

in all firms. They all vary with V^{eq}, and hence with E/F under non-CES H.S.A.

6.2 Comparative Statics: Clearly, the results obtained in Section 4 for general homothetic demand system under the assumption that the symmetric equilibrium exists uniquely carry over to this case. Moreover, the comparative statics results for $\mathcal{E}_\sigma(V) \gtreqless 0 \Leftrightarrow \mathcal{E}_\mu(V) \lesseqgtr 0$ carry over for $\zeta'(z) \gtreqless 0$, because they are equivalent under H.S.A.

6.3 Optimal vs. Equilibrium: Differentiating $\Phi(z) \equiv \left[\int_z^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi\right] / s(z)$ yields

$$\frac{\partial \ln \Phi(z)}{\partial \ln z} = -\frac{zs'(z)}{s(z)} - \frac{1}{\Phi(z)} = \zeta(z) - 1 - \frac{1}{\Phi(z)}.$$

Thus,

$$\Phi'(z) \lesseqgtr 0 \iff \zeta(z) - 1 \lesseqgtr \frac{1}{\Phi(z)}.$$

Since $\sigma(V) = \zeta\big(s^{-1}(1/V)\big)$, $\mathcal{L}(V) = \Phi\big(s^{-1}(1/V)\big)$, and $s^{-1}(1/V)$ is increasing in V, the above equivalence translates into

$$\mathcal{L}'(V) \lesseqgtr 0 \iff \mathcal{L}(V) \lesseqgtr \frac{1}{\sigma(V)-1} = \mu(V) - 1 = \frac{\mu(V)}{\sigma(V)}.$$

Thus, from Propositions 1, 2, and 3,

Proposition 4: In the Dixit-Stiglitz environment under H.S.A.,

$$\zeta'(z) \lesseqgtr 0 \text{ for all } z > 0 \Leftrightarrow \sigma'(V) \lesseqgtr 0 \text{ for all } V > 0$$

$$\Longrightarrow$$

$$\mathcal{L}'(V) \gtreqless 0 \text{ for all } V > 0 \Leftrightarrow V^{eq} \lesseqgtr V^{op} \text{ for all } E/F > 0.$$

Moreover,

$$\zeta(z) = const. \;\Leftrightarrow\; \sigma(V) = const. \Leftrightarrow \mathcal{L}(V) = const. \;\Leftrightarrow\; V^{eq} = V^{op} \text{ for all } E/F,$$

Thus, under H.S.A., equilibrium variety is excessive (insufficient) if and only if love-for-variety is diminishing (increasing), for which globally increasing (decreasing) substitutability or equivalently, the 2nd law (the anti-2nd law) is sufficient. Moreover, CES is the only H.S.A. demand system in which substitutability is constant, love-for-variety is constant, and the equilibrium is optimal.

7. Melitz under H.S.A.[20]

Let us now depart from the Dixit-Stiglitz environment and introduce heterogeneity across firms and their differentiated inputs.

7.1 The Environment

Consider what I shall call **the Melitz (2003) Environment.** As before, there exists a single primary factor of production, called simply as "labor" and taken as numeraire. Each differentiated input variety, $\omega \in \Omega$, is produced from "labor" and sold exclusively by a single MC firm, also indexed by $\omega \in \Omega$, and their products enter symmetrically in the demand system. Moreover, the firms are ex-ante identical before they enter the market. However, unlike the Dixit-Stiglitz environment, they become ex-post heterogenous in their marginal cost of production. More specifically, each firm pays F_e units of "labor" to enter the market, which is the sunk cost of entry. Upon entry, each firm draws its marginal cost of production, ψ_ω from the common cdf, $G(\psi)$, with the density function, $g(\psi) = G'(\psi) > 0$ over the support, $\left(\underline{\psi}, \bar{\psi}\right) \subseteq (0, \infty)$. Then, firm ω needs to hire $F + \psi_\omega x_\omega$ units of "labor" to produce x_ω units of its own product, where F is the overhead cost, the fixed cost of production, which is not sunk. Thus, upon discovering its marginal cost, ψ_ω, firm ω calculates its gross profit, $\Pi(\psi_\omega)$, and chooses to stay in the market if $\Pi(\psi_\omega) \geq F$ and to exit if $\Pi(\psi_\omega) < F$. Finally, there is free-entry to the market. Ex-ante identical firms enter until their expected gross profit is equal to the entry cost;

[20] This section draws heavily from Matsuyama & Ushchev (2022b).

$F_e = \int_{\underline{\psi}}^{\overline{\psi}} \max\{\Pi(\psi) - F, 0\} dG(\psi)$. This ensures no excess profit in equilibrium, so that the total demand for "labor" in this sector is equal to $L = \mathbf{px} = P(\mathbf{p})X(\mathbf{x}) = E$. Let us now apply H.S.A. to **the Melitz environment.**[21]

7.2 Pricing Behavior, Markup and Pass-Through Rates Across Firms:

Knowing its marginal cost, ψ_ω, and holding E and $A = A(\mathbf{p})$ fixed, firm ω chooses p_ω (hence $z_\omega \equiv p_\omega/A$) to maximize,

$$(p_\omega - \psi)x_\omega = \left(1 - \frac{\psi_\omega/A}{z_\omega}\right) s(z_\omega)E,$$

whose first-order condition is given by:

$$z_\omega \left[1 - \frac{1}{\zeta(z_\omega)}\right] = \frac{\psi_\omega}{A}.$$

Under **A1**, this can be inverted as $p_\omega/A = z_\omega = \tilde{Z}(\psi_\omega/A)$, $\tilde{Z}'(\cdot) > 0$. Thus, all the firms that share the same ψ_ω set the same price. This means that we can identify firms only by their marginal cost, ψ, so that we reindex them by ψ. Their profit-maximizing **normalized price** satisfies

$$\frac{p_\psi}{A} = z_\psi = \tilde{Z}\left(\frac{\psi}{A}\right), \quad \tilde{Z}'(\cdot) > 0.$$

The price elasticity of demand at the point ψ-firms operate and **their markup rate** can both expressed as functions of ψ/A:[22]

$$\zeta(z_\psi) = \zeta\left(\tilde{Z}\left(\frac{\psi}{A}\right)\right) \equiv \sigma\left(\frac{\psi}{A}\right); \quad \mu\left(\frac{\psi}{A}\right) \equiv \frac{\sigma(\psi/A)}{\sigma(\psi/A) - 1}.$$

which are related with the following identities:

$$\frac{1}{\sigma(\psi/A)} + \frac{1}{\mu(\psi/A)} = 1; \quad \mathcal{E}_\sigma\left(\frac{\psi}{A}\right) = -\frac{\mathcal{E}_\mu(\psi/A)}{\mu(\psi/A) - 1}; \quad \mathcal{E}_\mu\left(\frac{\psi}{A}\right) = -\frac{\mathcal{E}_\sigma(\psi/A)}{\sigma(\psi/A) - 1}.$$

The pass-through rate is also a function of ψ/A:

[21] Melitz under CES is a special case of Melitz under H.S.A. Melitz under HDIA or HIIA is not analytically tractable without some additional assumptions (e.g., the presence of the choke price combined with zero overhead cost, as in Arkolakis et.al. 2019). One could say very little under general homothetic symmetric demand systems.

[22] Notice some abuse of notations here. Until the previous section, $\sigma(\cdot)$ and $\mu(\cdot)$ are both functions of V, denoting the common values across symmetric firms. In this section, $\sigma(\cdot)$ and $\mu(\cdot)$ are both functions of ψ/A, denoting the price elasticity and the markup rate of ψ-firms. This should not cause any confusion.

$$\rho_\psi \equiv \frac{\partial \ln p_\psi}{\partial \ln \psi} = \mathcal{E}_{\tilde{Z}}\left(\frac{\psi}{A}\right) \equiv \rho\left(\frac{\psi}{A}\right) = \frac{1}{1+\mathcal{E}_{1-1/\zeta}\left(\tilde{Z}(\psi/A)\right)} = 1 + \mathcal{E}_\mu\left(\frac{\psi}{A}\right) > 0.$$

Note that $\sigma(\cdot), \mu(\cdot), \rho(\cdot)$ are all functions of the *normalized cost, ψ/A, only*. This means that, for non-CES H.S.A., market size E affects the pricing behaviors of firms only through its effects on $A = A(\mathbf{p})$. (They are constant under CES, $\sigma(\cdot) = \sigma;\ \mu(\cdot) = \sigma/(\sigma-1) = \mu; \rho(\cdot) = 1$.) Moreover, $A = A(\mathbf{p})$ enters only as the divisor of ψ. This means that a decline in A, more competitive pressures, act like a uniform decline in productivity across firms.

Moreover, it is straightforward to verify:

$$\zeta'(\cdot) \gtreqless 0 \Leftrightarrow \mathcal{E}_\sigma(\cdot) \gtreqless 0 \Leftrightarrow \mathcal{E}_\mu(\cdot) \lesseqgtr 0 \Leftrightarrow \rho(\cdot) \lesseqgtr 1.$$

Under the 2nd law, $\zeta'(\cdot) > 0$, high-ψ firms set lower markup rates, and their pass-through rates are less than one (**incomplete pass-through**). Under H.S.A., the 2nd law, $\zeta'(\cdot) > 0$, also implies that more competitive pressures, a lower A, force all firms to lower their markup rates, regardless of their marginal cost, ψ.

The 2nd law alone does not say how the pass-through rate varies across firms or how it responds to more competitive pressures. Motivated by some evidence that more productive firms have lower pass-through rates, let us introduce **the Strong (Weak) 3rd law,**[23]

$$\mathcal{E}_{1-1/\zeta}{}'(\cdot) < (\leq) 0,$$

which implies $\rho'(\cdot) > (\geq) 0$ and $\mathcal{E}_\mu{}'(\cdot) > (\geq) 0$. Among the parametric families listed in Appendix 2, Generalized Translog violates even the weak 3rd law; CoPaTh features a constant pass-through rate, hence satisfying the weak (but not strong) 3rd law. PEM/FIM satisfies the Strong 3rd law.

Fallacy #8. "Translog is flexible, as it can approximate any homothetic symmetric demand system."

Symmetric translog (Feenstra 2003) belongs to Generalized Translog. Its budget share function can be expressed as $s(z) = -\max\{\ln z, 0\}$ without loss of generality, which has no parameter to fit the data. It is highly tractable, which explains its popularity, but it has no flexibility.[24]

[23] The 1st law of demand states that a higher price reduces demand, restricting the 1st derivative of the demand curve. The 2nd law states that a higher price increases the price elasticity, restricting the 2nd derivative. We call this law--a higher price reduces the rate of change in the price elasticity-- the 3rd law because it restricts the 3rd derivative.
[24] Some agree with me about non-flexibility of symmetric translog. For example, Edmond et. al. (2023, p.1623) wrote "…Kimball … is more flexible than … symmetric translog … and is better able to match our calibration

Moreover, it violates even the weak 3rd law, thus inconsistent with the evidence that more productive firms have lower pass-through rates.[25]

Under the Strong 3rd law, high-ψ firms have higher pass-through rates, and more competitive pressures, a lower A, causes the pass-through rate to go up across all firms. The strong 3rd law is also equivalent to

$$\frac{\partial^2 \ln \mu(\psi/A)}{\partial\psi\partial(1/A)} > 0.$$

That is, $\mu(\psi/A)$ is log-supermodular[26] in ψ and $1/A$. Because $\mu(\psi/A)$ is decreasing in ψ under the 2nd law, this means that more competitive pressures cause a proportionately smaller decline in the markup rate for high-ψ firms, thus a smaller dispersion of the markup rates across firms.

7.3 Revenue, Gross Profit, & Employment Across Firms:

They can be all written as functions of ψ/A, multiplied by market size E, because:

$$R_\psi = s(z_\psi)E = s\left(\tilde{Z}\left(\frac{\psi}{A}\right)\right)E \equiv r\left(\frac{\psi}{A}\right)E;$$

$$\Pi_\psi = \frac{r(\psi/A)}{\sigma(\psi/A)}E \equiv \pi\left(\frac{\psi}{A}\right)E;$$

$$\psi x_\psi = \frac{r(\psi/A)}{\mu(\psi/A)}E \equiv \ell\left(\frac{\psi}{A}\right)E.$$

Moreover, they vary according to:

$$\frac{\partial \ln R_\psi}{\partial \ln \psi} = \frac{\partial \ln R_\psi}{\partial \ln(1/A)} = \mathcal{E}_r\left(\frac{\psi}{A}\right) = \mathcal{E}_s\left(\tilde{Z}\left(\frac{\psi}{A}\right)\right)\mathcal{E}_{\tilde{Z}}\left(\frac{\psi}{A}\right) = \left[1-\sigma\left(\frac{\psi}{A}\right)\right]\rho\left(\frac{\psi}{A}\right) < 0;$$

targets. But … translog … is more tractable than … Kimball … and leads to sharp analytic results." In this respect, I argue that H.S.A. dominates both Kimball and symmetric translog, because it is as flexible as Kimball and as tractable as symmetric translog.

[25] I am not sure why some people believe that translog is flexible. Maybe it is because translog (without symmetry restriction) offers local 2nd-order approximation to any unit cost function, which may be good enough for studying the impacts of small shocks in a competitive economy, where all firms are price takers. But it is not good enough when firms make price-setting and entry decisions, because these decisions depend on the global properties and the 3rd derivatives of the unit cost function. Perhaps it is analogous to the widespread use of the quadratic function in early days of portfolio theory, "because it offers local 2nd-order approximation to any risk-averse utility function," in spite of its counterfactual implication that the rich invest a larger fraction of the wealth to the safe asset, until Arrow (1971) pointed out that the portfolio choice depends on how the Arrow-Pratt measures of absolute and relative risk aversion vary with consumption, which hinge on the 3rd derivatives.

[26] A positive-value function, $f(x,y) > 0$, is log-supermodular in x and y if $\partial^2 \ln f(x,y)/\partial x\partial y > 0$ and log-submodular in x and y if $\partial^2 \ln f(x,y)/\partial x\partial y < 0$. Costinot & Vogel (2015) offer an accessible exposition of their properties.

$$\frac{\partial \ln \Pi_\psi}{\partial \ln \psi} = \frac{\partial \ln \Pi_\psi}{\partial \ln(1/A)} = \mathcal{E}_\pi\left(\frac{\psi}{A}\right) = \mathcal{E}_r\left(\frac{\psi}{A}\right) - \mathcal{E}_\sigma\left(\frac{\psi}{A}\right) = 1 - \sigma\left(\frac{\psi}{A}\right) < 0;$$

$$\frac{\partial \ln(\psi x_\psi)}{\partial \ln \psi} = \frac{\partial \ln(\psi x_\psi)}{\partial \ln(1/A)} = \mathcal{E}_\ell\left(\frac{\psi}{A}\right) = \mathcal{E}_r\left(\frac{\psi}{A}\right) - \mathcal{E}_\mu\left(\frac{\psi}{A}\right) = 1 - \sigma\left(\frac{\psi}{A}\right)\rho\left(\frac{\psi}{A}\right),$$

all of which are independent of market size E, and depend solely on ψ/A, through $\sigma(\cdot)$ and $\rho(\cdot)$. [Under CES, $\sigma(\cdot) = \sigma$ and $\rho(\cdot) = 1$, so that $\mathcal{E}_r(\cdot) = \mathcal{E}_\pi(\cdot) = \mathcal{E}_\ell(\cdot) = 1 - \sigma < 0$.] This means that, for non-CES H.S.A., market size E affects the relative firm size in revenue, gross profit, and employment only through its effects on $A = A(\mathbf{p})$. (Under CES, the relative firm size never changes.) Moreover, $A = A(\mathbf{p})$ enters only as the divisor of ψ; a decline in A thus acts like a uniform decline in firm heterogeneity.

Note also that $R_\psi = r(\psi/A)E$ and $\Pi_\psi = \pi(\psi/A)E$ are both strictly decreasing in ψ/A, but $\ell(\psi/A)L$, may be nonmonotonic in ψ/A, because $1 - \sigma(\cdot)\rho(\cdot)$ may change its sign. Under the 2nd and the weak 3rd law, $\sigma(\cdot)\rho(\cdot)$ is strictly increasing, and one can show that $\ell(\psi/A)L$ is hump-shaped in ψ/A. Moreover, the profit is log-submodular in ψ and $1/A$ under the 2nd law,

$$\frac{\partial^2 \ln \Pi_\psi}{\partial\psi\partial(1/A)} = \sigma'\left(\frac{\psi}{A}\right) < 0,$$

while $R_\psi = r(\psi/A)E$ and $\ell(\psi/A)E$ are log-submodular in ψ and $1/A$ under the 2nd and weak 3rd laws.

$$\frac{\partial^2 \ln R_\psi}{\partial\psi\partial(1/A)} = \left[1 - \sigma\left(\frac{\psi}{A}\right)\right]\rho'\left(\frac{\psi}{A}\right) - \sigma'\left(\frac{\psi}{A}\right)\rho\left(\frac{\psi}{A}\right) < 0;$$

$$\frac{\partial^2 \ln(\psi x_\psi)}{\partial\psi\partial(1/A)} = -\sigma'\left(\frac{\psi}{A}\right)\rho\left(\frac{\psi}{A}\right) - \sigma\left(\frac{\psi}{A}\right)\rho'\left(\frac{\psi}{A}\right) < 0.$$

Since R_ψ and Π_ψ are both decreasing in ψ, this implies that more competitive pressures cause a proportionately larger decline in the revenue and the profit among high-ψ firms, hence a larger dispersion in revenue and profit across firms.

Up to now, we looked at how different firms respond to a change in competitive pressures, A. Of course, $A = A(\mathbf{p})$ is endogenous, so that it can change only in response to some changes in exogenous variables, such as the entry cost, F_e, the overhead cost, F, and market size, E. To understand this, let us now turn to:

7.4 Equilibrium:

Let us assume $F + F_e < \pi(0)E$. This ensures that a positive measure of firms always enter, because otherwise $A = A(\mathbf{p}) \to \infty$, and firms could earn enough gross profit to cover both the entry cost and the overhead cost, regardless of their marginal costs. An equilibrium is characterized by the following three conditions:

Cutoff Rule: Firms choose to stay if $\psi < \psi_c$ and to exit if $\psi > \psi_c$, where ψ_c is the cutoff level of the marginal cost, determined by:

$$\pi\left(\frac{\psi_c}{A}\right)E = F.$$

Figure 1 depicts the cutoff rule as the ray from the origin, whose slope, $\psi_c/A = \pi^{-1}(F/E)$, is decreasing in F/E. A smaller market size/overhead cost ratio thus causes a tougher selection, a smaller ψ_c, causing more firms to exit for a given A.

Free-Entry Condition: Expected gross profit is equal to the entry cost,

$$F_e = \int_{\underline{\psi}}^{\psi_c} \left[\pi\left(\frac{\psi}{A}\right)E - F\right] dG(\psi).$$

Figure 1 depicts this condition as the C-shaped curve, downward-sloping below the cutoff rule, upward-sloping above it and vertical at the intersection. The curve shifts to the left, as the entry cost declines, which causes A to decline.

As Figure 1 illustrates, these two equilibrium conditions alone fully determine $A = A(\mathbf{p})$ and ψ_c uniquely, ensuring the existence and the uniqueness of the equilibrium. Moreover, a mild technical condition ensures the interior solution, $0 < G(\psi_c) < 1$, which responds continuously to F_e/E and F/E.

With $A = A(\mathbf{p})$ and ψ_c pinned down by these two conditions, we can calculate the mass of entering firms, M, and that of active firms, $V = MG(\psi_c)$, from:[27]

Adding-up Constraint: This can be written as:

$$\int_{\Omega} s\left(\frac{p_\omega}{A}\right) d\omega = M\int_{\underline{\psi}}^{\psi_c} r\left(\frac{\psi}{A}\right) dG(\psi) = 1,$$

from which the mass of active firms (hence the mass of product variety) is:

[27]One of the advantages of H.S.A. is that the equilibrium is solved recursively. For HDIA and HIIA, both of which have two price aggregators, all the equilibrium three conditions need to be solved simultaneously, which make ensuring the existence and uniqueness of the equilibrium and the comparative static exercises challenging.

$$V = MG(\psi_c) = \left[\int_{\underline{\psi}}^{\psi_c} r\left(\frac{\psi}{A}\right)\frac{dG(\psi)}{G(\psi_c)}\right]^{-1} = \left[\int_{\underline{\xi}}^{1} r\left(\pi^{-1}\left(\frac{F}{E}\right)\xi\right)d\tilde{G}(\xi;\psi_c)\right]^{-1},$$

where $\tilde{G}(\xi;\psi_c) \equiv G(\psi_c\xi)/G(\psi_c)$ is the cdf of $\xi \equiv \psi/\psi_c$, defined for $\underline{\xi} \equiv \underline{\psi}/\psi_c < \xi \leq 1$. Thus, the selection, ψ_c, affects the equilibrium product variety, V, through its effect on $\tilde{G}(\xi;\psi_c)$. It turns out that a lower ψ_c (a tougher selection) shifts $\tilde{G}(\xi;\psi_c)$ to the right (left) in the sense of Monotone Likelihood Ratio if $\mathcal{E}_g'(\psi) < (>)0$. Pareto distributed productivity, $G(\psi) = (\psi/\bar{\psi})^{\kappa}$, is the borderline case, $\mathcal{E}_g'(\psi) = 0$, in which $\tilde{G}(\xi;\psi_c)$ is independent of ψ_c. Since Fréchet, Weibull, and Lognormal distributions all satisfy $\mathcal{E}_g'(\psi) < 0$, a lower ψ_c (a tougher selection) shifts $\tilde{G}(\xi;\psi_c)$ to the right. However, there is some evidence for $\mathcal{E}_g'(\psi) > 0$, which suggests that a lower ψ_c (a tougher selection) shifts $\tilde{G}(\xi;\psi_c)$ to the left.

7.5 Average Markup Rate, Profit and Production Cost Across Active Firms:

Except under CES, heterogeneous firms differ in their markup rates, $\mu(\psi/A)$, so they differ also in the gross profit and the production cost shares in revenue

$$\frac{\pi(\psi/A)}{r(\psi/A)} = \frac{1}{\sigma(\psi/A)}; \qquad \frac{\ell(\psi/A)}{r(\psi/A)} = \frac{1}{\mu(\psi/A)}$$

and the profit/production cost ratio,

$$\frac{\pi(\psi/A)}{\ell(\psi/A)} = \frac{\mu(\psi/A)}{\sigma(\psi/A)} = \frac{1}{\sigma(\psi/A)-1} = \mu(\psi/A) - 1.$$

It turns out that comparative statics requires comparing the profit, the revenue and the employment of the firms at the cutoff with those of the industry average. Let

$$\mathbb{E}_1(f) \equiv \frac{\int_{\underline{\psi}}^{\psi_c} f\left(\frac{\psi}{A}\right)dG(\psi)}{\int_{\underline{\psi}}^{\psi_c} dG(\psi)}; \qquad \mathbb{E}_w(f) \equiv \frac{\int_{\underline{\psi}}^{\psi_c} f\left(\frac{\psi}{A}\right)w\left(\frac{\psi}{A}\right)dG(\psi)}{\int_{\underline{\psi}}^{\psi_c} w\left(\frac{\psi}{A}\right)dG(\psi)}$$

denote, respectively, the unweighted average of $f(\cdot)$ and the $w(\cdot)$-weighted average of $f(\cdot)$ across the active firms, which are related as follows:

$$\frac{\mathbb{E}_1(f)}{\mathbb{E}_1(w)} = \mathbb{E}_w\left(\frac{f}{w}\right) = \frac{1}{\mathbb{E}_f(w/f)}.$$

For example, by applying this formula, we could have

$$\frac{\mathbb{E}_1(\pi)}{\mathbb{E}_1(r)} = \mathbb{E}_r\left(\frac{\pi}{r}\right) = \mathbb{E}_r\left(\frac{1}{\sigma}\right) = \frac{1}{\mathbb{E}_\pi(r/\pi)} = \frac{1}{\mathbb{E}_\pi(\sigma)}.$$

That is, the sector-level profit share is equal to the revenue-weighted arithmetic mean, and the profit-weighted harmonic mean, of the profit shares across active firms. Likewise,

$$\frac{\mathbb{E}_1(\ell)}{\mathbb{E}_1(r)} = \mathbb{E}_r\left(\frac{\ell}{r}\right) = \mathbb{E}_r\left(\frac{1}{\mu}\right) = \frac{1}{\mathbb{E}_\ell(r/\ell)} = \frac{1}{\mathbb{E}_\ell(\mu)}$$

That is, the sector-level production cost share is equal to the revenue-weighted arithmetic mean, and the employment-weighted harmonic mean, of the production cost shares across active firms.

7.6. Comparative Statics (The Effects on Selection and Competitive Pressures):

By totally differentiating the cutoff rule and the free-entry condition with respect to F_e, E, and F, their effects on $A = A(\mathbf{p})$ and ψ_c are

$$\hat{A} = \frac{\mathbb{E}_1(\pi)}{\mathbb{E}_1(\ell)}\left[(1-f_x)\widehat{\left(\frac{F_e}{E}\right)} + f_x\widehat{\left(\frac{F}{E}\right)}\right]; \quad \widehat{\psi_c} = \frac{\mathbb{E}_1(\pi)}{\mathbb{E}_1(\ell)}\left[(1-f_x)\widehat{\left(\frac{F_e}{E}\right)} + (f_x-\delta)\widehat{\left(\frac{F}{E}\right)}\right];$$

where

$$\frac{\mathbb{E}_1(\pi)}{\mathbb{E}_1(\ell)} = \frac{1}{\mathbb{E}_\pi(\sigma)-1} = \mathbb{E}_\ell(\mu) - 1 > 0;$$

$$f_x \equiv \frac{FG(\psi_c)}{F_e + FG(\psi_c)} = \frac{\pi(\psi_c/A)}{\mathbb{E}_1(\pi)} < 1; \quad \delta \equiv \frac{\mathbb{E}_\pi(\sigma)-1}{\sigma(\psi_c/A)-1} = \frac{\pi(\psi_c/A)}{\ell(\psi_c/A)}\frac{\mathbb{E}_1(\ell)}{\mathbb{E}_1(\pi)} \equiv f_x\frac{\mathbb{E}_1(\ell)}{\ell(\psi_c/A)} > 0.$$

Let us look at each shock separately. First, consider **the entry cost, F_e:**

$$\frac{\partial \ln A}{\partial \ln F_e} = \frac{\partial \ln \psi_c}{\partial \ln F_e} = (1-f_x)\frac{\mathbb{E}_1(\pi)}{\mathbb{E}_1(\ell)} > 0.$$

A decline in F_e shifts the C-shaped curve to the left in Figure 1 and leads to a decline in A (more competitive pressures) and a decline in ψ_c (a tougher selection).

Next, consider **market size, E:**

$$\frac{\partial \ln A}{\partial \ln E} = -\frac{\mathbb{E}_1(\pi)}{\mathbb{E}_1(\ell)} < 0; \quad \frac{\partial \ln \psi_c}{\partial \ln E} = -(1-\delta)\frac{\mathbb{E}_1(\pi)}{\mathbb{E}_1(\ell)}.$$

A higher E shifts the C-shaped curve to the left and the cutoff rule counter-clockwise in Figure 1.[28] This always leads to a lower A, but a lower ψ_c iff $\delta < 1$, i.e., $\mathbb{E}_{\pi}(\sigma) < \sigma(\psi_c/A)$, which holds under the 2nd law.[29]

Finally, consider **the overhead cost, F:**

$$\frac{\partial \ln A}{\partial \ln F} = f_x \frac{\mathbb{E}_1(\pi)}{\mathbb{E}_1(\ell)} > 0; \quad \frac{\partial \ln \psi_c}{\partial \ln F} = (f_x - \delta)\frac{\mathbb{E}_1(\pi)}{\mathbb{E}_1(\ell)}.$$

A decline in F also shifts the C-shaped curve to the left and makes the cutoff rule steeper in Figure 1. This always leads to a lower A, but to a lower ψ_c iff $\delta < f_x$, i.e., $\mathbb{E}_1(\ell) < \ell(\psi_c/A)$, which holds if more productive firms employ less, which occurs under the 2nd and the weak 3rd laws when F is sufficiently high but not under CES.

7.7 Comparative Statics (The Effects on Firm Size Distributions in Revenue and Profit):

A change in F_e and a change in F both affect the revenue, $R_\psi = r(\psi/A)E$, and the profit, $\Pi_\psi = \pi(\psi/A)E$, across firms only through their effect on A. Thus, we already know that a decline in F_e or in F causes the profit and the revenue of all firms to decline, and the negative effects are proportionately larger among high ψ-firms in the profit (under the 2nd law) and in the revenue (under the 3rd law), thereby causing a larger dispersion in the profit and in the revenue.

For an increase in market size, E, we also need to take into account the direct positive effect of E, in addition to the indirect negative effect through the decline in A. The direct positive effect is uniform across all firms. Under CES, the indirect negative effect is also uniform, so that these two effects cancel out. Under the 2nd law, however, the decline in A caused by an increase in E causes the profit distribution more skewed toward low ψ-firms. Because of this, the combined effect is that the profit is up among low ψ-firms, down among middle ψ-firms, and

[28] Since $F > 0$, the cutoff rule implies $\pi(\psi_c/A) > 0$, hence $\psi_c/A < \tilde{Z}(\psi_c/A) < \bar{z}$. If $F = 0$ and the choke price exists, $\bar{z} < \infty$, the cutoff rule is $\pi(\psi_c/A) = 0$, so that $\psi_c/A = \tilde{Z}(\psi_c/A) = \bar{z}$. Hence, a change in E does not affect the cutoff rule, and the result is the same with a change in F_e.

[29] Under CES, $\delta = 1$, hence the cutoff does not change. The profit at the cutoff is always equal to F, and with the constant markup rate, the revenue and the employment at the cutoff are also unaffected. Moreover, we know that firm size distribution is not affected by a change in A. Thus, all firms are unaffected. Thus, the only effect of a change in E under CES is a proportional change in $V = MG(\psi_c)$.

high ψ-firms are forced to exit (a decline in ψ_c). Under the 2nd and the weak 3rd laws, the combined effect on the revenue is similar, possibly except when the overhead cost F is large.[30]

Comparative Statics (Composition Effects on Average Markup and Pass-Through Rates): Under the 2nd law, more competitive pressures, a low A, has the procompetitive effect, such that the markup rate $\mu(\psi/A)$ to decline for each firm, but the firm size distribution shifts toward low-ψ firms with higher markup rates. Likewise, under the strong 3rd law, a low A has the procompetitive effect such that the pass-through rate $\rho(\psi/A)$ to increase for each firm, but the firm size distribution shifts toward low-ψ firms with lower pass-through rates. Due to **the composition effect** working against the procompetitive effect, how more competitive pressures affect the average rates in the industry depend on whether the elasticity of the density function, $\mathcal{E}_g(\cdot)$, is globally increasing or globally decreasing.[31] For a change in F_e, which keeps ψ_c/A intact, the composition effect dominates the procompetitive effect and the average rates move in the *opposite* direction from the firm level rates iff $\mathcal{E}_g'(\cdot) < 0$. Thus, under the 2nd law, an entry cost decline, which causes all firms to lower their markup rates, ends up increasing the average markup rate by shifting the firm size distribution toward more productive, high markup firms. In contract, the procompetitive effect dominates the composition effect and the average rates move in the *same* direction with the firm level rates iff $\mathcal{E}_g'(\cdot) > 0$, satisfied, e.g., by Fréchet, Weibull, and Log-normal. And $\mathcal{E}_g'(\cdot) = 0$ (i.e., Pareto-distributed productivity) is the knife-edge case where there is no change in the average rates. For a change in E or in F, for which $d\ln\psi_c/d\ln A < 1$ hold, $\mathcal{E}_g'(\cdot) > 0$ is a necessary condition for the composition effect to dominate, while $\mathcal{E}_g'(\cdot) \leq 0$ is a sufficient condition for the procompetitive effect to dominate.

7.8 Comparative Statics (The Effects on TFP):

The logic above can be also applied to the impact of more competitive pressures on TFP, because $\ln(A/cP) = \mathbb{E}_r[\Phi \circ \tilde{Z}]$ is the revenue-weighted average of $\Phi\left(\tilde{Z}(\psi/A)\right)$ across active

[30] This qualification is necessary because the markup rate goes down for all firms, so that the cutoff firms need to earn a higher revenue to earn enough profit to cover the overhead cost, F. Hence, when F is large, the revenue of all the firms that stay may increase.

[31] The following results hold for any industry average of $f(\psi/A) = \mu(\psi/A)$ or $f(\psi/A) = \rho(\psi/A)$, of the form, $I \equiv \mathcal{M}^{-1}\left(\mathbb{E}_w\left(\mathcal{M}(f)\right)\right)$, where $\mathcal{M}: \mathbb{R}_+ \to \mathbb{R}$ is a monotone transformation and $w(\psi/A)$ is a weighted function. All Hölder means, including the arithmetic, $I = \mathbb{E}_w(f)$, geometric, $\ln I = \mathbb{E}_w(\ln f)$, harmonic, $1/I = \mathbb{E}_w(1/f)$, are all special cases, and the weight function, $w(\psi/A)$, can be the profit, the revenue, and the employment.

firms and $\zeta'(\cdot) \gtreqless 0 \Longrightarrow \Phi \circ \tilde{Z}'(\cdot) \lesseqgtr 0$. Under the 2nd law, $\zeta'(\cdot) > 0$ implies $\Phi'(\cdot) < 0$, hence $\Phi\left(\tilde{Z}(\psi/A)\right)$ is decreasing in ψ/A. This implies, for example, that a change in F_e, which keeps ψ_c/A intact, $\frac{d\ln P}{d\ln A} \gtreqless 1$ iff $\mathcal{E}_g'(\cdot) \gtreqless 0$.

7.9 Comparative Statics (The Effects on M and $V = MG(\psi_c)$):

The effect of the mass of entrants, M, is simple. It immediately follows from the adding-up constraint that M increases when hit by any shock that causes a decline in A and a decline in ψ_c. For the mass of active firms (hence product variety), $V = MG(\psi_c)$, M and $G(\psi_c)$ move in the opposite direction. The overall effect depends on whether the elasticity of the cumulative distribution function, $\mathcal{E}_G(\cdot)$, is globally increasing or globally decreasing.[32] A decline in F_e, which keeps ψ_c/A intact, causes $MG(\psi_c)$ to increase iff $\mathcal{E}_G'(\cdot) < 0$; and decline iff $\mathcal{E}_G'(\cdot) > 0$. Again, $\mathcal{E}_G'(\cdot) = 0$ (i.e., Pareto-distributed productivity) is the knife-edge case where $MG(\psi_c)$ remains unchanged. For an increase in E or a decline in F, $\mathcal{E}_G'(\cdot) > 0$ is necessary for $MG(\psi_c)$ to go down and $\mathcal{E}_G'(\cdot) \leq 0$ is sufficient for $MG(\psi_c)$ to go up.

7.10 Sorting of Heterogenous Firms across Markets:

As an application, let us consider a multi-market extension of Melitz under H.S.A. Imagine that there are $J \geq 2$ markets, indexed by $j = 1,2,\dots,J$, and their market sizes, $E_1 > \cdots > E_j > \cdots > E_J > 0$, are the only exogenous source of heterogeneity across markets. The primary factor of production, "labor," is full mobile, equalizing its price across the markets, so that we can still use it as *numeraire*. As before, each MC firm pays $F_e > 0$ to draw its marginal cost $\psi \sim G(\psi)$. After learning its ψ, each firm now decides which market to enter and produce with an overhead cost, $F > 0$, or exit without producing in any market. Firms sell their products at the profit-maximizing prices in the market they enter.

The unique equilibrium under the 2nd law is characterized by $A_1 < A_2 < \cdots < A_J$, and $\underline{\psi} = \psi_0 < \psi_1 < \psi_2 < \cdots < \psi_J = \psi_c < \overline{\psi}$, with firms $\psi \in (\psi_{j-1}, \psi_j)$ entering market-j. The intuition is simple. The ratio of the profit of ψ-firms in market-$(j-1)$ relative to that in market-j is $[\pi(\psi/A_{j-1})/\pi(\psi/A_j)]\,[E_{j-1}/E_j]$. For both markets to attract some firms, this ratio must be

[32] Globally increasing (decreasing) $\mathcal{E}_G(\cdot)$ is a weaker condition than globally increasing (decreasing) $\mathcal{E}_g(\cdot)$.

greater than one for some firms and less than one for others, which implies $A_j < A_{j+1}$, i.e., more competitive pressures in larger markets. The log-submodularity of $\pi(\psi/A)$ in ψ and $1/A$ under the 2nd law means that this ratio is decreasing in ψ, which means that more productive firms sort themselves into larger markets.

Because of more competitive pressures in larger markets, firms are forced to set lower markup rates as they move to larger markets under the 2nd law. But larger markets attract more productive firms with high markup rates. Due to this composition effect, the average markup rate can be higher in larger markets. If the strong 3rd law also holds, firms set higher pass-through rates in larger markets, but larger markets attract more productive firms with lower pass-through rate firms. Due to this composition effect, the average pass-through rate may be lower in larger markets. These results should provide a caution against testing the 2nd and 3rd laws by comparing the average markup/pass-through rates in cross-section of cities.

8. Other Forms of Firm Heterogeneity under H.S.A.

One of the advantages of H.S.A. is its analytical tractability when used in monopolistic competition with entry/exit and heterogeneous firms. The Melitz-type firm heterogeneity in productivity is one class of monopolistic competition models, in which different firms, despite that their products enter symmetrically in the demand system, could set different prices. However, it is not the only one. Even when the firms share the same productivity, different firms could set different prices if they face the different constraints. I discuss two examples.

8.1 Sticky Prices:

In New Keynesian macroeconomics, they often model sticky prices by imposing some constraints on the pricing behaviors of monopolistically competitive firms; see Gali (2005). For example, under the Rotemberg (1982) pricing rule, symmetric firms always set the same price, as in the Dixit-Stiglitz environment, but they need to pay the adjustment cost that is increasing the price change, so that the price adjusts sluggishly. Under the Calvo (1983) pricing rule, only a fraction of firms is randomly given the opportunities to reset their prices at each moment, so that individual prices can jump infrequently, but the "average" price adjusts sluggishly, and at any point of time, the firms are heterogeneous in their prices. Most models in this literature assume a fixed set of firms with no entry and use CES demand systems. Exceptions include Bilbiie et. al. (2008) and Bilbiie et. al. (2014), which considered entry/exit under CES and translog with the

Rotemberg pricing. Recently, Fujiwara & Matsuyama (2022) replaced CES and translog with H.S.A. One of its main findings is that a higher entry cost and resulting market concentration causes a flattening of the Phillips curve under the 2nd law with the Rotemberg pricing and under the 3rd law with the Calvo pricing. Fujiwara & Matsuyama (2022) also considered HDIA and HIIA, but a full general equilibrium analysis was feasible only under H.S.A.

8.2 Technology Diffusion and Competitive Fringes

Up to now, it has been assumed that each MC firm is the sole producer of its own product, and its market power is constrained only by the price elasticity of the demand curve it faces. In some cases, however, firms may be constrained by the presence of competitive fringes. For example, in the dynamic monopolistic competition models by Judd (1985) and Matsuyama (1999), each firm pays the innovation cost to enter with its own product for which it enjoys monopoly power only temporarily due to technology diffusion. After the loss of its monopoly power, competitive fringes force the innovator to sell its product at the marginal cost. Thus, different products are priced differently, depending on how recently they are introduced. This causes synchronization and endogenous fluctuation of innovation activities under some conditions.[33] But these conditions are independent of market size in Judd and Matsuyama, both of which use CES demand system and features the exogenously constant markup rate by the innovators while they enjoy the monopoly power. Matsuyama & Ushchev (2022a) replaced CES by H.S.A. in the Judd model to allow for the 2nd law and procompetitive entry. The Judd model under H.S.A. remains analytically tractable and we were able to demonstrate how a large market size makes endogenous fluctuations of innovation activities more likely. Matsuyama & Ushchev (2022c) considered the Judd model under HDIA, an extension of the Kimball (1995) aggregator that allows its product range to vary endogenously. This "Judd meets Kimball" model is not analytically tractable, and we were able to solve only numerically.

[33] This is because a potential innovator needs to enter when the market for its product is large enough to recover the innovation cost. If an innovator chooses to enter when others do, most of the competing products are monopolistically priced. If an innovator enters after an innovation wave, it competes against products that are mostly priced competitively. This generates an incentive to innovate when others innovate, which creates an innovation wave. This in turn cause the market to become too saturated, and innovation stops for a while, until the growth of the economy or obsolescence of the existing products make innovation profitable again.

Appendix 1: HDIA and HIIA Demand Systems

This Appendix discusses two other classes of homothetic symmetric demand systems with gross substitutes and their key properties: see Matsuyama & Ushchev (2020a, 2023) for detail.

HDIA Demand System:

A homothetic symmetric demand system belongs to the *homothetic direct implicit additivity* **(HDIA)** class with gross substitutes, if it is generated by the cost minimization of the competitive industry whose CRS production function, $X(\mathbf{x}) = Z\hat{X}(\mathbf{x})$, can be expressed as:

$$\mathcal{M}\left[\int_\Omega \phi\left(\frac{Zx_\omega}{X(\mathbf{x})}\right)d\omega\right] \equiv \mathcal{M}\left[\int_\Omega \phi\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right)d\omega\right] \equiv 1.$$

where $\mathcal{M}[\cdot]$ is a monotone transformation; and $\phi(\cdot): \mathbb{R}_+ \to \mathbb{R}_+$, satisfies $\phi(0) = 0; \phi(\infty) = \infty;$ $\phi'(y) > 0 > \phi''(y), -y\phi''(y)/\phi'(y) < 1$ for $0 < y < \infty.$ Here, $\phi(\cdot)$ is independent of $Z > 0,$ the TFP parameter, and hence so is $\hat{X}(\mathbf{x})$. Its unit cost function, $P(\mathbf{p}) = \hat{P}(\mathbf{p})/Z$, is given by

$$P(\mathbf{p}) \equiv \min_{\mathbf{x}}\left\{\mathbf{px} \equiv \int_\Omega p_\omega x_\omega d\omega \,\middle|\, X(\mathbf{x}) \geq 1\right\}$$

$$\equiv \min_{\mathbf{x}}\left\{\mathbf{px} \equiv \int_\Omega p_\omega x_\omega d\omega \,\middle|\, Z\hat{X}(\mathbf{x}) \geq 1\right\} \equiv \hat{P}(\mathbf{p})/Z.$$

Its budget share of $\omega \in \Omega$ can be expressed as:

$$s_\omega = s(p_\omega; \mathbf{p}) = \frac{p_\omega}{\hat{P}(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right),$$

where $B(\mathbf{p})$ is derived from the adding-up constraint,

$$\int_\Omega \frac{p_\omega}{\hat{P}(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)d\omega \equiv 1,$$

and hence it is linear homogeneous in **p** for any fixed Ω. Note that the two aggregators, $\hat{P}(\mathbf{p})$ and $B(\mathbf{p})$, enter in the budget share function of HDIA, which summarizes all the information needed to keep track of the cross-variety interactions, except when $\hat{P}(\mathbf{p})/B(\mathbf{p}) = ZP(\mathbf{p})/B(\mathbf{p})$ is constant, which can occur iff it is CES. This also implies that CES is the only common element of HDIA and H.S.A. **Price elasticity of demand** for $\omega \in \Omega$ can be expressed as:

$$\zeta_\omega = \zeta(p_\omega; \mathbf{p}) = -\frac{p_\omega/B(\mathbf{p})}{(\phi')^{-1}(p_\omega/B(\mathbf{p}))\phi''\left((\phi')^{-1}(p_\omega/B(\mathbf{p}))\right)} > 1,$$

in which only one aggregator, $B(\mathbf{p})$, enters. Thus, the cross-variety interactions in the pricing rule operates only through $B(\mathbf{p})$, while the cross-variety interactions in the profit, revenue, and employment operate through $B(\mathbf{p})$ and $\hat{P}(\mathbf{p})$.

HIIA Demand System:

A homothetic symmetric demand system belongs to the *homothetic indirect implicit additivity* **(HIIA)** class with gross substitutes if it is generated by the competitive industry whose unit cost function, $P(\mathbf{p}) = \hat{P}(\mathbf{p})/Z$, can be expressed as:

$$\mathcal{M}\left[\int_{\Omega} \theta\left(\frac{p_\omega}{ZP(\mathbf{p})}\right) d\omega\right] \equiv \mathcal{M}\left[\int_{\Omega} \theta\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) d\omega\right] \equiv 1,$$

where $\mathcal{M}[\cdot]$ is a monotone transformation; and $\theta(\cdot): \mathbb{R}_{++} \to \mathbb{R}_{+}$, satisfies $\theta(z) > 0$, $\theta'(z) < 0 < \theta''(z) > 0$, $-z\theta''(z)/\theta'(z) > 1$ for $0 < z < \bar{z} \leq \infty$ & $\theta(z) = 0$ for $z \geq \bar{z}$. Here, $\theta(\cdot)$ is independent of $Z > 0$, the TFP parameter, and hence so is $\hat{P}(\mathbf{p})$. Its production function, $X(\mathbf{x}) = Z\hat{X}(\mathbf{x})$ is given by:

$$X(\mathbf{x}) \equiv \min_{\mathbf{p}}\left\{\mathbf{px} \equiv \int_{\Omega} p_\omega x_\omega d\omega \,\middle|\, P(\mathbf{p}) \geq 1\right\}$$

$$\equiv \min_{\mathbf{p}}\left\{\mathbf{px} \equiv \int_{\Omega} p_\omega x_\omega d\omega \,\middle|\, \hat{P}(\mathbf{p}) \geq Z\right\} \equiv Z\hat{X}(\mathbf{x}).$$

Its budget share of $\omega \in \Omega$ can be expressed as:

$$s_\omega = s(p_\omega; \mathbf{p}) = \frac{p_\omega}{C(\mathbf{p})}\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right),$$

where $C(\mathbf{p})$ is derived from the adding-up constraint,

$$\int_{\Omega} \frac{p_\omega}{C(\mathbf{p})}\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) d\omega \equiv 1.$$

and hence it is linear homogeneous in $\mathbf{p}$ for any fixed Ω. Note that the two aggregators, $\hat{P}(\mathbf{p})$ and $C(\mathbf{p})$, enter in the budget share function of HIIA, which summarizes all the information needed to keep track of the cross-variety interactions, except when $\hat{P}(\mathbf{p})/C(\mathbf{p}) = ZP(\mathbf{p})/C(\mathbf{p})$ is constant, which can occur iff it is CES. This also implies that CES is the only common element of HIIA and H.S.A. (Though the proof is bit more involved, CES is the only common element of HDIA and HIIA, and hence H.S.A., HDIA and HIIA are pairwise disjoint with the sole exception of CES.) **Price elasticity of demand** for $\omega \in \Omega$ can be expressed as:

$$\zeta_\omega = \zeta(p_\omega; \mathbf{p}) = -\frac{\left(p_\omega/\hat{P}(\mathbf{p})\right)\theta''\left(p_\omega/\hat{P}(\mathbf{p})\right)}{\theta'\left(p_\omega/\hat{P}(\mathbf{p})\right)} > 1,$$

in which only one aggregator, $\hat{P}(\mathbf{p})$, enters. Thus, the cross-variety interactions in the pricing rule operates only through $\hat{P}(\mathbf{p})$, while the cross-variety interactions in the profit, revenue, and employment operate through both $C(\mathbf{p})$ and $\hat{P}(\mathbf{p})$.

To sum up, under HDIA and HIIA, the cross-variety interactions in the budget shares operate through two aggregators, and those in the price elasticities operate through one aggregator. This offers a significant reduction in the dimensionality of the problem, compared to general homothetic symmetric demand systems, where those interactions depend on the entire distribution of the prices.

However, the presence of the two-aggregators creates more room for the multiplicity and non-existence of the equilibrium and complications when conducting comparative statics, particularly with endogenous product variety.[34] In contrast, it is straightforward to ensure the existence of the unique equilibrium and to conduct comparative statics under H.S.A., due to its single aggregator property. This is the reason why departing from CES within H.S.A. is far more tractable.[35]

[34] In applications, macroeconomists use almost exclusively the Kimball (1995) aggregator with a fixed product range and no entry, while trade economists use almost exclusively symmetric translog by Feenstra (2003). I thought for years that this is due to the lack of communication between the two fields. Perhaps I was wrong. Unlike macroeconomics who are more concerned with short-run fluctuations, endogenous product variety is important for trade economists, and they might have already tried Kimball with endogenous product variety (i.e., HDIA) and found it too hard.

[35] Another advantage of H.S.A., pointed out by Kasahara & Sugita (2020), is that the market share (in revenue) function is the primitive of H.S.A., hence it can be readily identified with the typical firm-level data, which contain revenue, but not the output.

Appendix 2: Some Parametric Families of H.S.A.

Example 1: CES. This corresponds to $s(z) = \gamma(z/\beta)^{1-\sigma}$, for $\sigma > 1$, and $\beta, \gamma > 0$, from which $\zeta(z) = \sigma > 1$. It is the only H.S.A. in which any of $\zeta(\cdot)$, $\Phi(\cdot)$, $\sigma(\cdot), \mathcal{L}(\cdot)$, $P(\mathbf{p})/A(\mathbf{p})$ is constant.

Example 2: Generalized Translog. Originally developed by Matsuyama and Ushchev (2022a) to bridge the gap between CES and translog. See also Matsuyama and Ushchev (2022b). It is also used in Matsuyama & Ushchev (2024a) as a building block to construct a parametric family of homothetic demand systems under which the equilibrium variety is optimal. It corresponds to, for $\sigma > 1$ and $\beta, \eta, \gamma > 0$,

$$s(z) = \gamma\left(-\frac{\sigma-1}{\eta}\ln\left(\frac{z}{\bar{z}}\right)\right)^{\eta};\ z < \bar{z} \equiv \beta e^{\frac{\eta}{\sigma-1}},$$

from which

$$\zeta(z) = 1 - \frac{\eta}{\ln(z/\bar{z})} > 1.$$

This family is called Generalized Translog, because it contains symmetric translog (Feenstra 2003) as a special case with $\eta = 1$. CES is the limit case, as $\eta \to \infty$, while holding $\beta > 0$ and $\sigma > 1$ fixed, with $\bar{z} \equiv \beta e^{\frac{\eta}{\sigma-1}} \to \infty;\ \zeta(z) \to \sigma;\ s(z) \to \gamma(z/\beta)^{1-\sigma}$. It features the choke price, the 2nd law, increasing substitutability & diminishing love-for-variety. However,

$$\mathcal{E}_{1-1/\zeta}(z) = \frac{1}{\eta - \ln(z/\bar{z})}$$

is strictly increasing in z for all $z \in (0, \bar{z})$, violating even the weak form of the 3rd law.

Example 3: Constant Pass-Through (CoPaTh). Developed by Matsuyama & Ushchev (2020b) without the symmetry restriction, its symmetric version has been applied by Matsuyama & Ushchev (2022a, 2022b) and Fujiwara & Matsuyama (2022). This corresponds to, for $0 < \rho < 1$, $\sigma > 1$, $\beta, \gamma > 0$,

$$s(z) = \gamma\sigma^{\frac{\rho}{1-\rho}}\left[1 - \left(\frac{z}{\bar{z}}\right)^{\frac{1-\rho}{\rho}}\right]^{\frac{\rho}{1-\rho}};\ z < \bar{z} \equiv \beta\left(\frac{\sigma}{\sigma-1}\right)^{\frac{\rho}{1-\rho}}.$$

$$\Rightarrow 1 - \frac{1}{\zeta(z)} = \left(\frac{z}{\bar{z}}\right)^{\frac{1-\rho}{\rho}} \Rightarrow \mathcal{E}_{1-1/\zeta}(z) = \frac{1-\rho}{\rho}.$$

This family is called CoPaTh, because it implies that the pass-through rate is equal to ρ, a parameter. CES is the limit case, as $\rho \nearrow 1$, while holding $\beta > 0$ and $\sigma > 1$ fixed, with $\bar{z} \equiv \beta\left(\frac{\sigma}{\sigma-1}\right)^{\frac{\rho}{1-\rho}} \to \infty$, $\zeta(z) \to \sigma$; $s(z) \to \gamma(z/\beta)^{1-\sigma}$. It features the choke price, the 2nd law, increasing substitutability & diminishing love-for-variety, and the weak (but not strong) form of the 3rd law.

Example 4: Power Elasticity of Markup Rate (PEM)/Fréchet Inverse Markup Rate (FIM). Developed by Matsuyama & Ushchev (2022b) and applied by Fujiwara & Matsuyama (2022). This corresponds to, for $\kappa > 0$ and $\lambda > 0$

$$s(z) = \exp\left[\int_{z_0}^{z} \frac{c}{c - \exp\left[-\frac{\kappa\bar{z}^{-\lambda}}{\lambda}\right]\exp\left[\frac{\kappa\xi^{-\lambda}}{\lambda}\right]} \frac{d\xi}{\xi}\right]$$

$$1 - \frac{1}{\zeta(z)} = c\exp\left[\frac{\kappa\bar{z}^{-\lambda}}{\lambda}\right]\exp\left[-\frac{\kappa z^{-\lambda}}{\lambda}\right] < 1$$

$$\Rightarrow \mathcal{E}_{1-1/\zeta}(z) = \kappa z^{-\lambda}.$$

This family is called PEM, because the elasticity of markup rate is a power function of z and FIR because the inverse markup rate is proportional to Fréchet distribution function. CES is the limit case for $\kappa \to 0$; $\bar{z} = \infty$; $c = 1 - \frac{1}{\sigma}$; CoPaTh is also the limit case for $\bar{z} < \infty$; $c = 1$; $\kappa = \frac{1-\rho}{\rho} > 0$, and $\lambda \to 0$. It features the choke price, the 2nd law, increasing substitutability & diminishing love-for-variety, and the strong form of the 3rd law.

Literature Cited:

Arkolakis C, Costinot A, Donaldson D, Rodríguez-Clare A. 2019. The elusive pro-competitive effects of trade. *The Review of Economic Studies.* 86 (1): 46–80.

Arrow, KJ. 1971. *Essays in the Theory of Risk-Bearing,* based on his 1963 Yrjö Jahnsson lectures, Chicago, Markham Pub. Co.

Baqaee D, Farhi E, Sangani K. 2024. The Darwinian returns to scale. *Review of Economic Studies*. 91(3): 1373-1405.

Behrens K, Murata Y. 2007. General equilibrium models of monopolistic competition: A new approach," *Journal of Economic Theory.* 136: 776-787.

Bertoletti P, Etro F. 2017. Monopolistic competition when income matters. *The Economic Journal.* 127: 1217-1243.

Bilbiie FO, Ghironi F, Melitz MJ. 2007. Monetary policy and business cycles with endogenous entry and product variety, in *NBER Macroeconomics Annual 2007*, Vol. 22, 299–353.

Bilbiie FO, Fujiwara I, Ghironi F. 2014. Optimal monetary policy with endogenous entry and product variety. *Journal of Monetary Economics*. 64: 1–20.

Calvo, GA. 1983. Staggered prices in a utility-maximizing framework. *Journal of Monetary Economics.* 12(3): 383–398.

Costinot, A, Vogel J. 2015. Beyond Ricardo: assignment models in international trade. *Annual Review of Economics.* 7: 31-62.

Dixit AK, Stiglitz JE. 1977. Monopolistic competition and optimal product diversity. *American Economic Review,* 67(3): 297-308.

Edmond C, Midrigan V, Xu DY. 2023. How costly are markups? *Journal of Political Economy.* 131(7): 1676-1730.

Feenstra RC. 2003. A homothetic utility function for monopolistic competition models without constant price elasticity. *Economics Letters* 78: 79-86.

Fujiwara I, Matsuyama K. 2022. *Competition and the Phillips curve.* CEPR DP #17521-3.

Gali J. 2015. *Monetary policy, inflation, and the business cycle: An introduction to the new Keynesian framework and its applications* Second edition. Princeton University Press.

Grossman, GM, Helpman E, Luillier H. 2023. Supply chain resilience: should policy promote diversification or reshoring. *Journal of Political Economy.* 131(12): 3462-3496.

Hurwicz L, Uzawa H. 1971. On the integrability of demand functions. In Chipman et al. *Preferences, utility, and demand: A Minnesota symposium*. New York: Harcourt, Brace, Jovanovich. Ch.6, 114–148.

Judd, KL. 1985. On the performance of patents. *Econometrica.* 53: 567-586.

Kasahara H, Sugita Y. 2020. Nonparametric identification of production function, total factor productivity, and markup from revenue data. UBC and Hitotsubashi.

Kimball M. 1995. The quantitative analytics of the basic neomonetarist model. *Journal of Money, Credit and Banking.* 27(4): 1241–77.

Matsuyama K. 1995. Complementarities and cumulative processes in models of monopolistic competition. *Journal of Economic Literature* 33: 701-729.

Matsuyama K. 1999. Growing through cycles. *Econometrica* 67: 335-347.

Matsuyama K. 2008. Symmetry-breaking. In Blume et al. *The new Palgrave dictionary of economics*, 2nd Edition, Palgrave Macmillan.

Matsuyama K. 2023. Non-CES aggregators: a guided tour. *Annual Review of Economics,* vol.15. 235-265.

Matsuyama K, Ushchev P. 2017. *Beyond CES: three alternative classes of flexible homothetic demand systems.* CEPR-DP #12210.

Matsuyama K, Ushchev P. 2020a. *When does procompetitive entry imply excessive entry?* CEPR-DP #14991.

Matsuyama K, Ushchev P. 2020b. *Constant pass-through*. CEPR-DP #15475.

Matsuyama K, Ushchev P. 2022a. Destabilizing effects of market size in the dynamics of innovation. *Journal of Economic Theory* 200: 105415.

Matsuyama K, Ushchev P. 2022b. *Selection and sorting of heterogeneous firms through competitive pressures*. CEPR-DP #17092-2.

Matsuyama K, Ushchev P. 2022c. Destabilizing effects of market size in the dynamics of innovation: Judd meets Kimball. presented at MDEF 2022, Urbino.

Matsuyama K, Ushchev P. 2023. *Love-for-variety*. CEPR DP #18184-2.

Matsuyama K, Ushchev P. 2024a. *Optimum product diversity under geometric means of generalized translog.* unpublished.

Matsuyama K, Ushchev P. 2024b. A note on geometric and power means of CES. unpublished.

Melitz MJ. 2018. Competitive effects of trade: theory and measurement. *Review of World Economics* 154 (1):1–13.

Melitz MJ. 2003. The impact of trade on intra-industry reallocations, and aggregate industry productivity. *Econometrica* 71(6): 1695–1725.

Melitz MJ, Ottaviano GIP. 2008. Market size, trade, and productivity. *The Review of Economic Studies,* 75(1): 295–316.

Mrázová M, Neary JP. 2017. Not so demanding: demand structure and firm behavior. *American Economic Review.* 107(12): 3835–3874

Ottaviano GIP, Tabuchi T, Thisse J-F. 2002. Agglomeration and trade revisited. *International Economic Review,* 43:409–435.

Parenti M, Ushchev P, Thisse J-F. 2017. Toward a theory of monopolistic competition. *Journal of Economic Theory* 167:86-115.

Rotemberg JJ. 1982. Sticky prices in the United States. *Journal of Political Economy* 90(6): 1187-1211.

Samuelson PA. 1950. The problem of integrability in utility theory. *Economia* 17 (68):355–385.

Thisse, JF, Ushchev P. 2018. Monopolistic competition without apology. In Cochon et al. *Handbook of game theory and industrial organization* Vol. I, Edgar Publication.

Tirole J. 1988. *The theory of industrial organization*, MIT Press.

Trottner F. 2023. Unbundling market power. UC San Diego.

Zhelobodko E, Kokovin S, Parenti M, Thisse J-F. 2012. Monopolistic competition: beyond the constant elasticity of substitution. *Econometrica* 80(6): 2765-2784.

List of Abbreviations Used

CES	Constant Elasticity of Substitution
CoPaTh	Constant Pass-Through
CRS	Constant Returns to Scale
DEA	Direct Explicit Additivity
FIM	Fréchet Inverse Markup Rate
HDIA	Homothetic Direct Implicit Additivity
HIIA	Homothetic Indirect Implicit Additivity
H.S.A.	Homothetic Single Aggregator
IEA	Indirect Explicit Additivity
MC	Monopolistically Competitive/Monopolistic Competition
PEM	Power Elasticity of Markup Rate
TFP	Total Factor Productivity

Figure 1: Melitz under H.S.A.; The cutoff rule and the free entry condition.