Homothetic Non-CES Demand Systems with Applications to Monopolistic Competition

Kiminori Matsuyama
Northwestern University

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Part 1: Introduction

"Non-CES Aggregators: A Guided Tour," Matsuyama (ARE 2023)

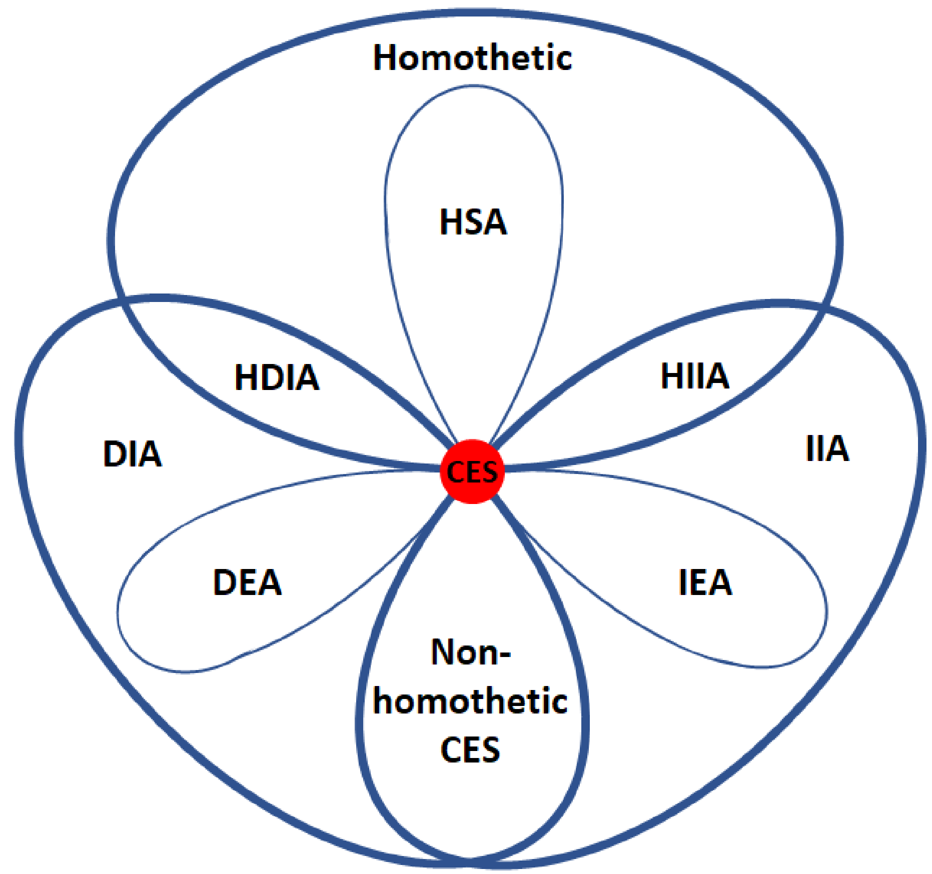
CES demand system has many knife-edge properties, which
- help to make CES tractable.
- make CES too restrictive for some applications.

Many typically look for *alternatives*, e.g., linear-quadratic, translog, but these alternatives have their own limitations.

Instead, *relaxing just a few knife-edge properties*, while keeping the rest,
- creates many different classes of demand systems, depending on which properties to relax and which ones to keep.
- CES is an intersection of these different classes.
- They're more flexible than CES without losing much of tractability of CES.

Of course, which classes to use depend on your applications.

ARE (2023) focused on applications of non-CES to inter-sectoral demand, with special emphasis on
- ✓ *Nonhomotheticity*
- ✓ *Gross complementarity and Essentiality of goods and factors.*



This Review focuses on applications of ***homothetic*** **non-CES** to demand for differentiated products within a **monopolistic competitive (MC) industry**. This necessitates some additional restrictions.

- ***Endogenous*** **range of** ***inessential*** **products**

Demand systems need to be well-defined even when some products are unavailable or not yet invented, allowing for innovation/endogenous entry/exit.

- ***Gross substitutes***: price elasticity of each product > 1 ↔ the revenue is decreasing in the price.
  To ensure that MC firms face strictly positive marginal revenue curve.

Moreover, we restrict to

- ***Continuum*** **of differentiated products**
  - To make product variety a continuous variable for tractability
  - To ensure that firms cannot affect the aggregate variables (unless they produce positive measure of products), to be consistent with the assumption of MC.

- ***Symmetric Demand System*** to focus on heterogeneity on the supply side, such as
  - productivity difference a la Melitz (2003).
  - Differential market access based on different locations, as in most trade/spatial models.
  - pricing setting a la Calvo (1983).
  - technology diffusion causes some but not all MC firms to face competitive fringes a la Judd (1985).

Note: Neither homotheticity nor symmetry are so restrictive, because one can always nest them into a nonhomothetic and/or asymmetric upper-tier demand system.

Organization of this review

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Part 4: Dixit-Stiglitz under General Homothetic Demand Systems

Part 5: Homothetic Single Aggregator (H.S.A.) Demand System

Part 6: Dixit-Stiglitz under H.S.A.

Part 7: Melitz under H.S.A.

Part 8: Other Forms of Firm Heterogeneity under H.S.A.

Appendix 1: H.S.A., HDIA and HIIA

Appendix 2: Some Parametric Families under H.S.A.

Part 2: Dixit-Stiglitz under CES: A Quick Refresher

2.1. *Symmetric* CES Demand System over *a continuum* of products *with gross substitutes*

We discuss CES in terms of demand for differentiated inputs, generated by a competitive industry that produces the output, using symmetric CES production function, $X = X(\mathbf{x})$,

Production Function	$X = X(\mathbf{x}) = Z\left[\int_{\Omega} (x_\omega)^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}}, \sigma > 1$

$Z > 0$: TFP. $\mathbf{x} = \{x_\omega; \omega \in \Omega\}$**:** the quantity vector of differentiated inputs.

$\omega \in \Omega$: the endogenous set of available differentiated input varieties; $V \equiv |\Omega|$.

Given $\mathbf{p} = \{p_\omega; \omega \in \Omega\}$, the price vector, the competitive industry chooses $\mathbf{x}$ to minimize the production cost.

Unit Cost Function	$P = P(\mathbf{p}) \equiv \min_{\mathbf{x}} \left\{\mathbf{px} \equiv \int_{\Omega} p_\omega x_\omega d\omega \,\middle|\, X(\mathbf{x}) \geq 1\right\} = \frac{1}{Z}\left[\int_{\Omega} (p_\omega)^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}}$
Demand for ω	$x_\omega = \left(\frac{p_\omega}{ZP(\mathbf{p})}\right)^{-\sigma} \frac{X(\mathbf{x})}{Z} = \frac{(p_\omega)^{-\sigma}}{(ZP(\mathbf{p}))^{1-\sigma}} E$
Budget Shares of ω	$s_\omega \equiv \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{p_\omega x_\omega}{E} = \left(\frac{p_\omega}{ZP(\mathbf{p})}\right)^{1-\sigma} = \left(\frac{Zx_\omega}{X(\mathbf{x})}\right)^{1-\frac{1}{\sigma}}$

$E \equiv P(\mathbf{p})X(\mathbf{x}) = \mathbf{px}$**:** market size or the size of this industry, treated as given.

Duality Principle: $X(\mathbf{x})$ can be recovered from $P(\mathbf{p})$.

$$X(\mathbf{x}) \equiv \min_{\mathbf{p}} \left\{\mathbf{px} \equiv \int_{\Omega} p_\omega x_\omega d\omega \,\middle|\, P(\mathbf{p}) \geq 1\right\} = Z\left[\int_{\Omega} (x_\omega)^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}}.$$

2.2. Dixit-Stiglitz Environment:

One Primary Factor of Production: "Labor" taken as numeraire.

A Continuum of Differentiated Intermediate Inputs:

Each variety is produced from "labor" and supplied exclusively by a single firm.

$\omega \in \Omega$: index of a differentiated product and the firm producing it.

Symmetry of Intermediate Input Producing MC Firms:

- The symmetric demand system: Their products enter symmetrically in the demand system.

- Each firm needs to hire $F + \psi x_\omega$ units of labor to supply x_ω units of its own product.
  - F: *Fixed cost, a combination of*
    - *the entry/innovation cost to enter the market.*
    - *overhead cost to stay in the market.*
  - ψx_ω variable labor cost of production, or "employment"; ψ: Marginal cost of production.

Free-Entry (Zero Profit): $F = \Pi_\omega$. Gross profit is just enough to cover the fix cost. No excess profit.

➔ Total Labor Demand = Market Size $L = \mathbf{px} = P(\mathbf{p})X(\mathbf{x}) = E.$

I make no assumption how this sector interacts with the rest of the economy, except that E *is given by this sector.*

2.3. Equilibrium: Dixit-Stiglitz under CES

Pricing Behavior: Max the gross profit, $\Pi_\omega = (p_\omega - \psi)x_\omega$ holding $P(\mathbf{p})$ & E, given. → $max\ (p_\omega - \psi)(p_\omega)^{-\sigma}$.

Lerner Formula:	$p_\omega\left(1-\frac{1}{\sigma}\right)=\psi$	**Equilibrium Price of** ω:	$p_\omega \equiv p = \left(\frac{\sigma}{\sigma-1}\right)\psi \equiv \mu\psi$

μ: the constant (and common) markup rate:
Under CES, the pricing rule of each firm is independent of the prices set by other firms. Strategic independence!!

Equilibrium is symmetric. $p_\omega = p$; $x_\omega = x$; $r_\omega = r = px$; $pxV = E$; $\Pi_\omega \equiv \Pi = (p-\psi)x = px/\sigma = E/\sigma V$, where $V = |\Omega|$ is product variety = the mass of firms

Free Entry-Zero Profit Condition: $\Pi = F$.

Unique Equilibrium:	$V^{eq} = \frac{E}{\sigma F}$; $\quad p^{eq} = \left(\frac{\sigma}{\sigma-1}\right)\psi = \mu\psi$; $\quad x^{eq} = \frac{(\sigma-1)F}{\psi} = \frac{F}{(\mu-1)\psi}$

2.4. Comparative Statics: 3 endogen. variables (V^{eq}, p^{eq}, x^{eq}) ; 3 exogen. (E, F, ψ). By denoting $\hat{q} \equiv \partial \ln q = \partial q/q$,

$$\widehat{V^{eq}} = \hat{E} - \hat{F};\ \widehat{p^{eq}} = \hat{\psi};\ \widehat{x^{eq}} = \hat{F} - \hat{\psi}$$

Market Size Effect: p^{eq}, x^{eq} independent of E → All the adjustments at the extensive margin.

Profit Share	$\frac{1}{\sigma} = 1 - \frac{1}{\mu}$	**Production Cost Share**	$\frac{1}{\mu} = 1 - \frac{1}{\sigma}$	**Profit/Production Cost Ratio**	$\frac{1}{\sigma-1} = \frac{\mu}{\sigma} = \mu - 1$

All independent of E/F and ψ.

2.5. Optimality of the Equilibrium Allocation under CES.

Imagine that the competitive industry could fully integrate and control all intermediate input producers. Then,

$$\max X(\mathbf{x}) = \max Z\left[\int_{\Omega}(x_\omega)^{1-\frac{1}{\sigma}}d\omega\right]^{\frac{\sigma}{\sigma-1}} \quad s.t. \quad \int_{\Omega}\psi x_\omega d\omega + VF \leq E.$$

The optimal allocation must satisfy $x_\omega = x > 0$ for $\omega \in \Omega$ and $x_\omega = 0$ for $\omega \notin \Omega$.

$$\max_{(\psi x+F)V\leq E} V^{\frac{\sigma}{\sigma-1}}(Zx) = \frac{ZF}{\psi}\max_V V^{\frac{1}{\sigma-1}}\left(\frac{E}{F}-V\right)$$

Optimum Product Variety:	$V^{op} = \frac{E}{\sigma F}.$	**Optimum Quantity of** ω:	$x^{op} = \frac{(\sigma-1)F}{\psi}$

Equilibrium is Optimal!.

A prior, we expect that MC equilibrium would not be optimal due to the presence of two sources of externalities.

- **Negative** externalities due to **the business stealing effect:** A firm, when it pays the fixed cost to create a variety, does not take into account the fact that it reduces demand for other products and their profits.→ **Too Many Varieties**
- **Positive** externalities due to **incomplete appropriability**: A firm is motivated not by the social surplus, but by the profit, which is a fraction of the social surplus. → **Too Little Varieties.**

Under CES, these two externalities happen to cancel out exactly. Not robust, but this makes CES a useful benchmark.

Unfortunately, the logic behind this result is poorly understood.

Fallacy #1. The equilibrium allocation is optimal because all the products are sold at the same markup rate, and hence the relative prices across varieties are not distorted.

Easy to see why this intuition is false. If it were correct, then the equilibrium allocation would be optimal,

- ✓ whenever all products were sold at the same markup rate; it would not have to be equal to $\sigma/(\sigma-1)$.
- ✓ In the presence of the uniform sale taxation of intermediate inputs.
- ✓ in any symmetric equilibrium. The demand system would not have to be CES.

The intuition is incorrect, because the common markup rate merely ensures that the allocation across available varieties is not distorted; it does not ensure that the equilibrium incentive to create a variety is optimal.

Fallacy #2. "The equilibrium allocation is optimal if and only if it is under CES."

Polar opposite of Fallacy #1.

Of course,
The optimality of the equilibrium under CES is *not robust*, because it must satisfy the *knife-edge* condition, as the two externalities must cancel out each other.

But,
CES is *not unique* in satisfying the *knife-edge* condition, as explained later.

Part 3: General Homothetic Demand Systems

Constant-Returns-to-Scale (CRS) Production Technologies: A General Case

Homothetic demand systems for differentiated inputs generated by symmetric CRS production technology.

CRS Production Function	Unit Cost Function
$X = X(\mathbf{x}) \equiv \min_{\mathbf{p}} \left\{ \mathbf{px} \equiv \int_{\Omega} p_\omega x_\omega d\omega \,\middle|\, P(\mathbf{p}) \geq 1 \right\}$	$P = P(\mathbf{p}) \equiv \min_{\mathbf{x}} \left\{ \mathbf{px} \equiv \int_{\Omega} p_\omega x_\omega d\omega \,\middle|\, X(\mathbf{x}) \geq 1 \right\}$

$\mathbf{x} = \{x_\omega; \omega \in \overline{\Omega}\}$**:** the input quantity vector; $\mathbf{p} = \{p_\omega; \omega \in \overline{\Omega}\}$**:** the input price vector.

$\overline{\Omega}$, a continuum of all potential input varieties; $\Omega \subset \overline{\Omega}$, the set of available input varieties, with $V \equiv |\Omega|$.

$\overline{\Omega}\backslash\Omega$: the set of unavailable varieties, $x_\omega = 0$ and $p_\omega = \infty$ for $\omega \in \overline{\Omega}\backslash\Omega$.

Inputs are *inessential*, i.e., $\overline{\Omega}\backslash\Omega \neq \emptyset$ doesn't imply $X(\mathbf{x}) = 0 \Leftrightarrow P(\mathbf{p}) = \infty$.

Duality Principle:

Either $X(\mathbf{x})$ or $P(\mathbf{p})$ can be a *primitive* if linear homogeneity, monotonicity & strict quasi-concavity are satisfied.

3.1. Demand Systems

Demand Curve (from Shepherd's Lemma)	Inverse Demand Curve
$x_\omega = \frac{\partial P(\mathbf{p})}{\partial p_\omega} X(\mathbf{x})$	$p_\omega = P(\mathbf{p}) \frac{\partial X(\mathbf{x})}{\partial x_\omega}$

From Euler's homogenous function theorem,

$$\mathbf{px} = \int_{\Omega} p_\omega x_\omega d\omega = \int_{\Omega} p_\omega \frac{\partial P(\mathbf{p})}{\partial p_\omega} X(\mathbf{x}) d\omega = \int_{\Omega} P(\mathbf{p}) \frac{\partial X(\mathbf{x})}{\partial x_\omega} x_\omega d\omega = P(\mathbf{p}) X(\mathbf{x}) = E.$$

The value of inputs is equal to the total value of output under CRS.

Budget Share of $\omega \in \Omega$:	$s_\omega \equiv \frac{p_\omega x_\omega}{E} = \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} \equiv s(p_\omega, \mathbf{p}) = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} \equiv s^*(x_\omega, \mathbf{x})$

Under general CRS, little restrictions on s_ω beyond homogeneity of degree zero in $(p_\omega, \mathbf{p})$ or in $(x_\omega, \mathbf{x})$. $\rightarrow s_\omega = s(1, \mathbf{p}/p_\omega) = s^*(1, \mathbf{x}/x_\omega)$, depends on the whole price (quantity) distribution divided by its own price (quantity).

Definition: Gross Substitutability

$$\frac{\partial \ln s(p_\omega; \mathbf{p})}{\partial \ln p_\omega} < 0 \Leftrightarrow \frac{\partial \ln s^*(x_\omega;\ \mathbf{x})}{\partial \ln x_\omega} > 0$$

Price Elasticity of Demand for $\omega \in \Omega$	$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p}) \equiv 1 - \frac{\partial \ln s(p_\omega; \mathbf{p})}{\partial \ln p_\omega} = \zeta^*(x_\omega; \mathbf{x}) \equiv \left[1 - \frac{\partial \ln s^*(x_\omega;\ \mathbf{x})}{\partial \ln x_\omega}\right]^{-1} > 1.$

Under general CRS, little restrictions on ζ_ω, beyond the homogeneity of degree zero in $(p_\omega, \mathbf{p})$ or in $(x_\omega, \mathbf{x})$. $\rightarrow \zeta_\omega = \zeta(1, \mathbf{p}/p_\omega) = \zeta^*(1, \mathbf{x}/x_\omega)$ =depends on the whole price (quantity) distribution divided by its own price (quantity).

Definition: The 2nd Law of Demand

$$\frac{\partial \ln \zeta(p_\omega; \mathbf{p})}{\partial \ln p_\omega} > 0 \Leftrightarrow \frac{\partial \ln \zeta^*(x_\omega; \mathbf{x})}{\partial \ln x_\omega} < 0.$$

Clearly, CES does not satisfy the 2nd Law.

3.2. Substitutability and Love-for-Variety Measures

Unit Quantity Vector: $\mathbf{1}_\Omega \equiv \{(1_\Omega)_\omega; \omega \in \bar{\Omega}\}$, where $(1_\Omega)_\omega \equiv \begin{cases} 1 & \text{for } \omega \in \Omega \\ 0 & \text{for } \omega \in \bar{\Omega}\backslash\Omega \end{cases}$

Unit Price Vector: $\mathbf{1}_\Omega^{-1} \equiv \left\{\left(1_\Omega^{-1}\right)_\omega; \omega \in \bar{\Omega}\right\}$, where $\left(1_\Omega^{-1}\right)_\omega \equiv \begin{cases} 1 & \text{for } \omega \in \Omega \\ \infty & \text{for } \omega \in \bar{\Omega}\backslash\Omega \end{cases}$

Note: $\int_\Omega (1_\Omega)_\omega d\omega = \int_\Omega \left(1_\Omega^{-1}\right)_\omega d\omega = |\Omega| \equiv V$. At the symmetric patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$ and $\mathbf{x} = x\mathbf{1}_\Omega$,

$$s_\omega = s(1, \mathbf{p}/p_\omega) = s^*(1, \mathbf{x}/x_\omega) = s\left(1, \mathbf{1}_\Omega^{-1}\right) = s^*(1, \mathbf{1}_\Omega) = 1/V$$

$$\zeta_\omega = \zeta(1, \mathbf{p}/p_\omega) = \zeta^*(1, \mathbf{x}/x_\omega) = \zeta\left(1, \mathbf{1}_\Omega^{-1}\right) = \zeta^*(1, \mathbf{1}_\Omega) > 1$$

Clearly, this depends only on V. Thus,

Definition: *The substitutability measure across varieties* is defined by

$$\sigma(V) \equiv \zeta\left(1; \mathbf{1}_\Omega^{-1}\right) = \zeta^*(1; \mathbf{1}_\Omega) > 1.$$

We call $\sigma'(V) > (<) 0$ the case of ***increasing (decreasing) substitutability***.

Alternatively, we can define the substitutability by the Allen-Uzawa elasticity of substitution btw ω and ω', $AES(p_\omega, p_{\omega'}, \mathbf{p})$, at the symmetric patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$. It turns out that these definitions are equivalent because

$$\sigma(V) = AES\left(p, p, p\mathbf{1}_\Omega^{-1}\right) = AES\left(1,1, \mathbf{1}_\Omega^{-1}\right).$$

In general, the 2nd Law is neither sufficient nor necessary for $\sigma'(V) > 0$.

Love-for-Variety Measure Commonly defined by the productivity gain from a higher V, holding xV

$$\left.\frac{d\ln X(\mathbf{x})}{d\ln V}\right|_{\mathbf{x}=x\mathbf{1}_\Omega, xV=const.} = \left.\frac{d\ln xX(\mathbf{1}_\Omega)}{d\ln V}\right|_{xV=const.} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0$$

Alternatively, LfV may be defined by the decline in $P(\mathbf{p})$ from a higher V, at $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$, holding p constant.

$$-\left.\frac{d\ln P(\mathbf{p})}{d\ln V}\right|_{\mathbf{p}=p\mathbf{1}_\Omega^{-1},\ p=const.} = -\frac{d\ln P\left(\mathbf{1}_\Omega^{-1}\right)}{d\ln V} > 0.$$

Both are functions of V only, and equivalent because, by applying $\mathbf{x} = x\mathbf{1}_\Omega$ and $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$ to $\mathbf{px} = P(\mathbf{p})X(\mathbf{x})$,

$$pxV = pP\left(\mathbf{1}_\Omega^{-1}\right)xX(\mathbf{1}_\Omega) \Longrightarrow -\frac{d\ln P\left(\mathbf{1}_\Omega^{-1}\right)}{d\ln V} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0.$$

Definition. *The love-for-variety measure* is defined by:

$$\mathcal{L}(V) \equiv -\frac{d\ln P\left(\mathbf{1}_\Omega^{-1}\right)}{d\ln V} = \frac{d\ln X(\mathbf{1}_\Omega)}{d\ln V} - 1 > 0.$$

We call the case of $\mathcal{L}'(V) < (>) 0$ for all $V > 0$, the case of ***diminishing (increasing) love-for-variety***.

Note: $\mathcal{L}(V) > 0$ is guaranteed by the strict quasi-concavity.

Example: Back to Standard CES:

$$X(\mathbf{x}) = Z\left[\int_{\Omega} x_{\omega}^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \Leftrightarrow P(\mathbf{p}) = \frac{1}{Z}\left[\int_{\Omega} p_{\omega}^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}}.$$

	Definition	CES
Budget Share	$\frac{\partial \ln P(\mathbf{p})}{\partial \ln p_{\omega}} \equiv s(p_{\omega}, \mathbf{p}) = s^*(x_{\omega}; \mathbf{x}) = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_{\omega}}$	$s_{\omega} = \left(\frac{p_{\omega}}{ZP(\mathbf{p})}\right)^{1-\sigma} = \left(\frac{Zx_{\omega}}{X(\mathbf{x})}\right)^{1-1/\sigma}$
Price Elasticity	$\zeta_{\omega} \equiv -\frac{\partial \ln x_{\omega}}{\partial \ln p_{\omega}} = \zeta(p_{\omega}; \mathbf{p}) = \zeta^*(x_{\omega}; \mathbf{x})$	$\zeta(p_{\omega}; \mathbf{p}) = \zeta^*(x_{\omega}; \mathbf{x}) = \sigma > 1$
Substitutability	$\sigma(V) \equiv \zeta\left(1; \mathbf{1}_{\Omega}^{-1}\right) = \zeta^*(1; \mathbf{1}_{\Omega})$	$\sigma(V) = \sigma > 1$
Love-for-variety	$\mathcal{L}(V) \equiv -\frac{d \ln P\left(\mathbf{1}_{\Omega}^{-1}\right)}{d \ln V} = \frac{d \ln X(\mathbf{1}_{\Omega})}{d \ln V} - 1 > 0.$	$\mathcal{L}(V) = \frac{1}{\sigma - 1} > 0.$

Under Standard CES,

- Price elasticity of demand, $\zeta(p_{\omega}; \mathbf{p}) = \zeta^*(x_{\omega}; \mathbf{x})$, is independent of **p** or **x** and equal to σ.
- Substitutability, $\sigma(V)$, is independent of V and it is equal to σ.
- LV, $\mathcal{L}(V)$, is independent of V and $\mathcal{L}(V) = \mathcal{L} = 1/(\sigma - 1)$.

Fallacy #3: $\sigma(V)$ is constant only under CES.
Fallacy #4: $\sigma'(V) > (<) 0$ iff the 2nd law (anti-2nd law) holds.
Fallacy #5: $\sigma(V)$ is an inverse measure of love-for-variety, $\mathcal{L}(V)$.
These statements may be true under some subclasses of homothetic symmetric demand systems, but not true in general. See the following (counter)example.

Example: Generalized CES with Gross Substitutes a la Benassy (1996).

$$X(\mathbf{x}) = Z(V)\left[\int_{\Omega} x_\omega^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \Leftrightarrow P(\mathbf{p}) = \frac{1}{Z(V)}\left[\int_{\Omega} p_\omega^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}}.$$

Note: $Z(V)$ allows variety to have direct externalities to TFP (or affinity)

	Definition	Generalized CES
Budget Share	$\frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} \equiv s(p_\omega, \mathbf{p}) = s^*(x_\omega; \mathbf{x}) = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega}$	$s_\omega = \left(\frac{p_\omega}{Z(V)P(\mathbf{p})}\right)^{1-\sigma} = \left(\frac{Z(V)x_\omega}{X(\mathbf{x})}\right)^{1-1/\sigma}$
Price Elasticity	$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$	$\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x}) = \sigma > 1$
Substitutability	$\sigma(V) \equiv \zeta\left(1; \mathbf{1}_\Omega^{-1}\right) = \zeta^*(1; \mathbf{1}_\Omega)$	$\sigma(V) = \sigma > 1$
Love-for-variety	$\mathcal{L}(V) \equiv -\frac{d \ln P\left(\mathbf{1}_\Omega^{-1}\right)}{d \ln V} = \frac{d \ln X(\mathbf{1}_\Omega)}{d \ln V} - 1 > 0.$	$\mathcal{L}(V) = \frac{1}{\sigma - 1} + \frac{d \ln Z(V)}{d \ln V}.$

Under Generalized CES,

- Price Elasticity, $\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$, and Substitutability, $\sigma(V)$, are not affected by $\frac{d \ln Z(V)}{d \ln V}$.
- $\frac{d \ln Z(V)}{d \ln V}$, the Benassy residual, "accounts for" the discrepancy between the LV implied by CES and the observed LV.
  - Benassy (1996) set $\frac{d \ln Z(V)}{d \ln V} = \nu - \frac{1}{\sigma-1}$, so that $\mathcal{L}(V) = \nu$ is a separate parameter independent of σ.
  - If we instead assume that $\frac{d \ln Z(V)}{d \ln V}$ is independent of σ, $\mathcal{L}(V)$ is still inversely related to σ.

Even if you believe in the Benassy residual, your estimate of its magnitude depends on the CES structure.

(Counter)Example: Geometric Mean of CES Unit Cost Function

$$P(\mathbf{p}) \equiv \exp\left[\int_1^\infty \ln P(\mathbf{p};\sigma)\, dG(\sigma)\right]. \text{ where } [P(\mathbf{p};\sigma)]^{1-\sigma} \equiv \int_\Omega p_\omega{}^{1-\sigma}\, d\omega,$$

where $G(\cdot)$ is a cdf. over $(1,\infty)$, $\int_1^\infty dG(\sigma) = 1$.

	Definition	Geometric Mean of CES Unit Cost Function
Budget Share	$s(p_\omega, \mathbf{p}) = \dfrac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega}$	$s_\omega = E_G\left[\left(\dfrac{p_\omega}{P(\mathbf{p};\sigma)}\right)^{1-\sigma}\right]$
Price Elasticity	$\zeta_\omega \equiv -\dfrac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p})$	$\zeta(p_\omega;\mathbf{p}) = E_G\left[\sigma\left(\dfrac{p_\omega}{P(\mathbf{p};\sigma)}\right)^{1-\sigma}\right] \Big/ E_G\left[\left(\dfrac{p_\omega}{P(\mathbf{p};\sigma)}\right)^{1-\sigma}\right].$
Substitutability	$\sigma(V) \equiv \zeta\left(1; \mathbf{1}_\Omega^{-1}\right)$	$\sigma(V) = E_G(\sigma) > 1$
Love-for-variety	$\mathcal{L}(V) \equiv -\dfrac{d \ln P\left(\mathbf{1}_\Omega^{-1}\right)}{d \ln V} > 0.$	$\mathcal{L}(V) = E_G\left(\dfrac{1}{\sigma - 1}\right) > 0$

- Price elasticity of demand, $\zeta(p_\omega; \mathbf{p})$, is not constant, and *violates* the 2nd Law.
- Both $\sigma(V)$ and $\mathcal{L}(V)$ are *independent* of V.
- The range of $\sigma(V)$ and $\mathcal{L}(V)$ is $0 < \frac{1}{\sigma(V)-1} \leq \mathcal{L}(V) < \infty$, where the equality holds iff G is degenerate.
- Easy to construct a parametric family of G, such that $\sigma(V)$ and $\mathcal{L}(V)$ are positively related.

(Counter)Example: Geometric Mean of CES Production Function

$$X(\mathbf{x}) \equiv \exp\left[\int_1^\infty \ln X(\mathbf{x};\sigma)\, dG(\sigma)\right], \text{where } [X(\mathbf{x};\sigma)]^{1-\frac{1}{\sigma}} \equiv \int_\Omega {x_\omega}^{1-\frac{1}{\sigma}} d\omega$$

where $G(\cdot)$ is a cdf of $\sigma \in (1,\infty)$, satisfying $\int_1^\infty dG(\sigma) = 1$.

	Definition	Geometric Mean of CES Production Function
Budget Share	$s_\omega = s^*(x_\omega;\mathbf{x}) = \dfrac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega}$	$s_\omega = E_G\left[\left(\dfrac{x_\omega}{X(\mathbf{x};\sigma)}\right)^{1-\frac{1}{\sigma}}\right]$
Price Elasticity	$\zeta_\omega \equiv -\dfrac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta^*(x_\omega;\mathbf{x})$	$\zeta^*(x_\omega;\mathbf{x}) = E_G\left[\left(\dfrac{x_\omega}{X(\mathbf{x};\sigma)}\right)^{1-\frac{1}{\sigma}}\right] \Big/ E_G\left[\dfrac{1}{\sigma}\left(\dfrac{x_\omega}{X(\mathbf{x};\sigma)}\right)^{1-\frac{1}{\sigma}}\right] > 1$
Substitutability	$\sigma(V) \equiv \zeta^*(1;\mathbf{1}_\Omega)$	$\sigma(V) = \dfrac{1}{E_G(1/\sigma)} > 1$
Love-for-variety	$\mathcal{L}(V) \equiv \dfrac{d \ln X(\mathbf{1}_\Omega)}{d \ln V} - 1 > 0$	$\mathcal{L}(V) = E_G\left(\dfrac{1}{\sigma - 1}\right) > 0$

- Price elasticity of demand, $\zeta^*(x_\omega;\mathbf{x})$, is not constant, and *violates* the 2nd Law.
- Both $\sigma(V)$ and $\mathcal{L}(V)$ are *independent* of V.
- The range of $\sigma(V)$ and $\mathcal{L}(V)$ is $0 < \frac{1}{\sigma(V)-1} \leq \mathcal{L}(V) < \infty$, where the equality holds iff G is degenerate.
- Easy to construct a parametric family of G, such that $\sigma(V)$ and $\mathcal{L}(V)$ are positively related.

In general, homotheticity imposes little restrictions on the relation btw $\zeta(p_\omega;\mathbf{p}) = \zeta^*(x_\omega;\mathbf{x})$, $\sigma(V)$, & $\mathcal{L}(V)$.

Part 4: Dixit-Stiglitz under General Homothetic Demand Systems

Fallacy #6. With the symmetric firms, the equilibrium is symmetric.

- The symmetry of the environment ensures the set of equilibrium is symmetric, not the symmetry of an equilibrium.
- Even if the symmetric equilibrium exists, it may co-exist with a symmetric set of asymmetric equilibriums.
- There may be multiple symmetric equilibriums.

4.1. Symmetric Equilibrium: *Suppose* that it exists. In such an equilibrium, the price elasticity is $\sigma(V)$. Hence,

Lerner Pricing Formula	$p\left(1-\frac{1}{\sigma(V)}\right)=\psi$
Markup Rate	$\mu(V) \equiv \frac{\sigma(V)}{\sigma(V)-1}$

with $\frac{1}{\sigma(V)}+\frac{1}{\mu(V)}=1$ and $\frac{1}{\sigma(V)-1}=\frac{\mu(V)}{\sigma(V)}=\mu(V)-1$.

Maximized Gross Profit	$\Pi=(p-\psi)x=\frac{px}{\sigma(V)}=\frac{E}{\sigma(V)}$
Free Entry-Zero Profit Condition	$\Pi=F$
Equilibrium Product Variety	$V^{eq}\sigma(V^{eq})=\frac{E}{F}$

The uniqueness requires that $V\sigma(V)$ is globally increasing in V.

Increasing substitutability $\sigma'(V)>0$ ↔ ***Procompetitive entry***, $\mu'(V)<0$, is sufficient (not necessary) for the uniqueness.

4.2. Comparative Statics: Let $\mathcal{E}_f(x) \equiv xf'(x)/f(x) = \partial \ln f(x)/\partial \ln x$ the elasticity of $f(x) > 0$, w.r.t. $x > 0$.

$$\widehat{V^{eq}} = \frac{\hat{E} - \hat{F}}{1 + \mathcal{E}_\sigma(V^{eq})}; \quad \widehat{p^{eq}} = \frac{\mathcal{E}_\mu(V^{eq})(\hat{E} - \hat{F})}{1 + \mathcal{E}_\sigma(V^{eq})} + \hat{\psi}; \quad \widehat{x^{eq}} = \frac{\mu(V^{eq})\mathcal{E}_\sigma(V^{eq})(\hat{E} - \hat{F})}{1 + \mathcal{E}_\sigma(V^{eq})} + \hat{F} - \hat{\psi}.$$

Market Size Effect: for $\mathcal{E}_\sigma(V) \gtreqless 0 \Leftrightarrow \mathcal{E}_\mu(V) \lesseqgtr 0$,

$$0 < \frac{\partial \ln V^{eq}}{\partial \ln E} = 1 - \frac{\partial \ln(p^{eq}x^{eq})}{\partial \ln E} = \frac{1}{1 + \mathcal{E}_\sigma(V^{eq})} \lesseqgtr 1; \; \frac{\partial \ln p^{eq}}{\partial \ln E} = \frac{\mathcal{E}_\mu(V^{eq})}{1 + \mathcal{E}_\sigma(V^{eq})} \lesseqgtr 0; \; \frac{\partial \ln x^{eq}}{\partial \ln E} = \frac{\mu(V^{eq})\mathcal{E}_\sigma(V^{eq})}{1 + \mathcal{E}_\sigma(V^{eq})} \gtreqless 0$$

and the profit/production cost ratio changes as

$$\frac{\partial \ln(\mu(V^{eq})/\sigma(V^{eq}))}{\partial \ln E} = \frac{\mathcal{E}_\mu(V^{eq}) - \mathcal{E}_\sigma(V^{eq})}{1 + \mathcal{E}_\sigma(V^{eq})} \lesseqgtr 0.$$

Under ***increasing substitutability ↔pro-competitive entry***, $\mathcal{E}_\sigma(V) > 0 \Leftrightarrow \mathcal{E}_\mu(V) < 0$,

- A large market size causes more firms to enter.
- If increasing product variety makes the products more substitutable, the markup rate goes down.
- The firms needs to expand and increases its revenue just to break-even.
- As each firm becomes larger/each product sold more, the mass of product variety up at a lower rate than market size.
- The profit/production cost ratio goes down.

Note: What is crucial for these results is procompetitive entry, not the 2nd Law.
Fallacy #4, which can now be restated as "*Entry is procompetitive iff the 2nd law holds.*"

4.3. Optimum under General Homothetic Demand System.

$$\max_{} X(\mathbf{x}) \quad s.t. \int_{\Omega}^{} \psi x_\omega d\omega + VF \leq E.$$

The solution satisfies $x_\omega = x > 0$ for $\omega \in \Omega$ and $x_\omega = 0$ for $\omega \notin \Omega$. Thus,

$$\max_{} X(\mathbf{x}) = \max_{V(\psi x + F) \leq E} xX(\mathbf{1}_\Omega) = \frac{F}{\psi} \max_V \frac{X(\mathbf{1}_\Omega)}{V}\left(\frac{E}{F} - V\right).$$

FOC:

$$\frac{d \ln X(\mathbf{1}_\Omega)}{d \ln V} - 1 + \frac{d \ln(E/F - V)}{d \ln V} = \mathcal{L}(V) - \frac{V}{E/F - V} = 0,$$

$$\Rightarrow \left[1 + \frac{1}{\mathcal{L}(V^{op})}\right] V^{op} = \frac{E}{F}.$$

This FOC fully characterizes the optimal variety for each L/F if LHS is strictly increasing in V^O. Thus,

$$\mathcal{E}_\mathcal{L}(V) < 1 + \mathcal{L}(V).$$

Diminishing love-for-variety, $\mathcal{L}'(V) < 0 \Leftrightarrow \mathcal{E}_\mathcal{L}(V) < 0$, is sufficient but not necessary.

4.4. Optimal vs. Equilibrium:

$$\left[1+\frac{1}{\mathcal{L}(V^{op})}\right]V^{op}=\frac{E}{F};\quad \sigma(V^{eq})V^{eq}=\left[1+\frac{\sigma(V^{eq})}{\mu(V^{eq})}\right]V^{eq}=\frac{E}{F},$$

with the LHS of each condition is strictly increasing in V^{op} and in V^{eq} respectively. Thus,

> **Proposition 1.** Assume that the symmetric equilibrium exists uniquely in the Dixit-Stiglitz environment under general homothetic symmetric demand systems. Then,
>
> $$\mathcal{L}(V) \gtreqless \frac{\mu(V)}{\sigma(V)}=\frac{1}{\sigma(V)-1} \text{ for all } V>0 \Leftrightarrow V^{eq} \lesseqgtr V^{op} \text{ for all } E/F>0.$$

SOCIAL incentive for additional product variety =
POSITIVE externalities from incomplete appropriability $\mathcal{L}(V)$
PRIVATE incentive for additional product variety =
NEGATIVE externalities from business stealing $\frac{1}{\sigma(V)-1}=\mu(V)-1=\frac{\Pi}{\psi x}$

The Optimality Condition: $\mathcal{L}(V)[\sigma(V)-1]=1.$

The optimality requires the knife-edge condition, not satisfied by ***almost all*** homothetic demand systems.

- For some classes of homothetic DSs, $V^{eq}<V^{op}$ for all $E/F>0$; e.g., the geometric means of CES.
- For some classes of homothetic DSs, $V^{eq}>V^{op}$ for all $E/F>0$; see below.
- There are also borderline classes of homothetic DS for which $V^{eq}=V^{op}$ for all $E/F>0$.

CES is one of them, but not the only one. *If Fallacy #2 were true, then* $\mathcal{L}(V)[\sigma(V)-1]=1$ *holds iff CES.*

In general, one can say little about the relation btw $\zeta(p_\omega;\mathbf{p}) = \zeta^*(x_\omega;\mathbf{x})$, $\sigma(V)$, & $\mathcal{L}(V)$.

- Whether the 2nd Law holds or not says little about the derivatives of $\sigma(V)$ and $\mathcal{L}(V)$.
- $\sigma(V)$ and $\mathcal{L}(V)$ may not be inversely related. Instead, they could be positively related.

Yet, one may think, intuitively, that, as input varieties are more substitutable,

- the price elasticity of demand for each variety become larger,
- the love-for-variety measure become smaller.

One may also think, intuitively. That the 2nd Law could be tightly connected to procompetitive entry.

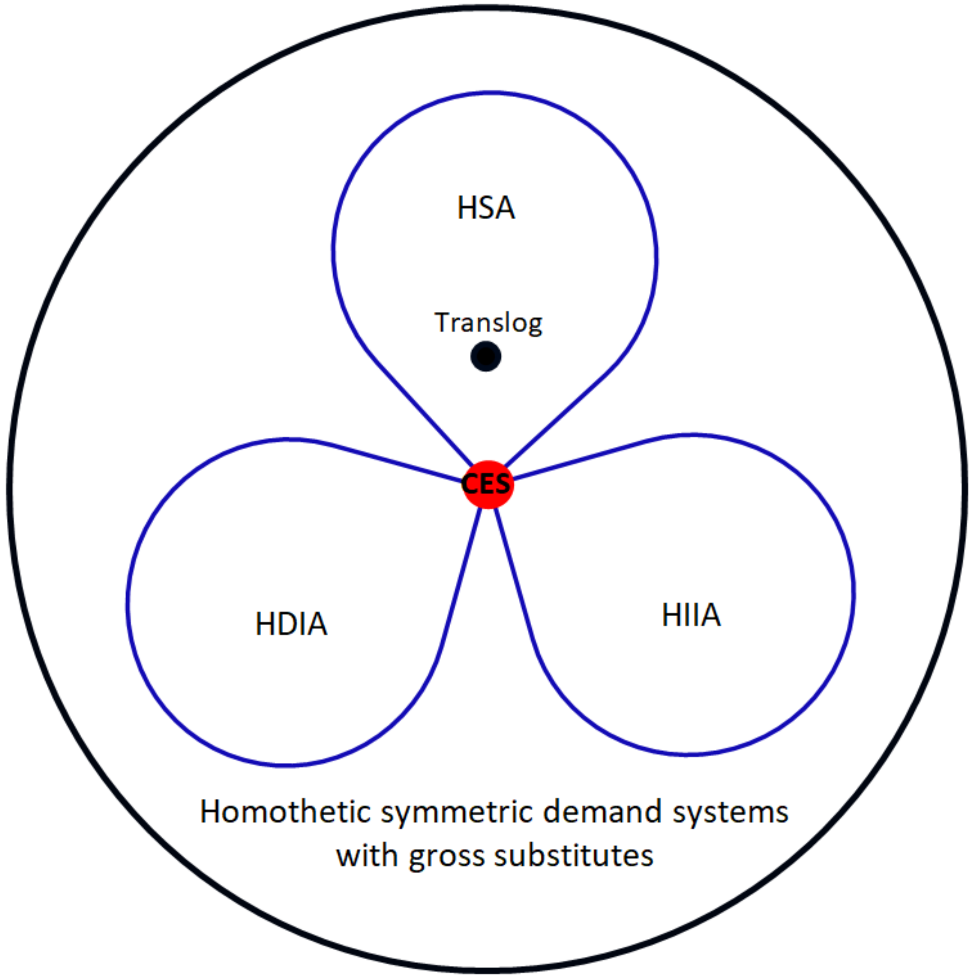
Homotheticity alone cannot capture this intuition!! Too broad.

In search for the additional restrictions to capture this intuition, and to ensure the uniqueness of the equilibrium, we turn to

Homothetic Single Aggregator (H.S.A.) for the remainder.

Appendix A also discusses
Homothetic Direct Implicit Additivity (HDIA)
Homothetic Indirect Implicit Additivity (HIIA)

The three are pair-wise disjoint with the sole exception of CES.



Part 5: Homothetic Single Aggregator (H.S.A.) Demand Systems

5.1. Symmetric H.S.A. (Homothetic Single Aggregator) DS with Gross Substitutes

Definition: A symmetric CRS technology, $P = P(\mathbf{p})$ is called *homothetic single aggregator* (H.S.A.) if the budget share of ω depends solely on a single variable, $z_\omega \equiv p_\omega/A$, its own price p_ω, normalized by the common price aggregator, $A = A(\mathbf{p})$.

$$s_\omega \equiv \frac{p_\omega x_\omega}{\mathbf{px}} = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(\mathbf{p})}\right), \quad \text{where} \quad \int_\Omega s\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega \equiv 1.$$

- $s: \mathbb{R}_{++} \to \mathbb{R}_+$: **the budget share function**, decreasing in the normalized price, $z_\omega \equiv p_\omega/A$ for $s(z_\omega) > 0$ with
  - $\lim_{z\to\bar{z}} s(z) = 0$. If $\bar{z} \equiv \inf\{z > 0 | s(z) = 0\} < \infty$, $\bar{z}A(\mathbf{p})$ is **the choke price.**
- $A = A(\mathbf{p})$: **the common price aggregator**, defined implicitly by **the adding-up constraint**, $\int_\Omega s(p_\omega/A) d\omega \equiv 1$.

By construction, the budget shares add up to one. $A(\mathbf{p})$ linear homogenous in $\mathbf{p}$ for a fixed Ω. A larger Ω reduces $A(\mathbf{p})$.

Some Special Cases

CES with gross substitutes $\quad s(z) = \gamma z^{1-\sigma}; \quad \sigma > 1$

Generalized Translog (GT) $\quad s(z) = \gamma \max\left\{\left(\frac{\sigma-1}{\eta}\ln\left(\frac{\bar{z}}{z}\right)\right)^\eta, 0\right\}. \quad \eta > 0;\ \bar{z} \equiv \beta e^{\frac{\eta}{\sigma-1}} < \infty$

Standard translog if $\eta = 1$. As $\eta \to \infty$, GT converges to CES with $\bar{z} \equiv \beta e^{\frac{\eta}{\sigma-1}} \to \infty$.

Constant Pass Through (CoPaTh) $\quad s(z) = \gamma \max\left\{\left[\sigma - (\sigma-1) z^{\frac{1-\rho}{\rho}}\right]^{\frac{\rho}{1-\rho}}, 0\right\} \quad \sigma > 1;\ 0 < \rho < 1$

As $\rho \nearrow 1$, CoPaTh converges to CES with $\bar{z} = \left(\frac{\sigma}{\sigma-1}\right)^{\frac{\rho}{1-\rho}} \to \infty$.

Price Elasticity:	$\zeta_\omega = \zeta(p_\omega; \mathbf{p}) = 1 - \frac{z_\omega s'(z_\omega)}{s(z_\omega)} \equiv \zeta(z_\omega) > 1$

- A function of a single variable, the normalized price, $z_\omega \equiv p_\omega / A(\mathbf{p}) \leq \bar{z}$.
- $\zeta(z_\omega) = \sigma > 1$ under CES, $s(z) = \gamma z^{1-\sigma}$.
- The 2nd law iff $\zeta'(\cdot) > 0$. For example,

Generalized Translog	$\zeta(z_\omega) = 1 - \frac{\eta}{\ln(z_\omega/\bar{z})}$
Constant Pass-Through	$\zeta(z_\omega) = \frac{\sigma}{\sigma - (\sigma-1) z_\omega{}^{(1-\rho)/\rho}} = \frac{1}{1 - (z_\omega/\bar{z})^{(1-\rho)/\rho}}; \quad \bar{z} = \left(\frac{\sigma}{\sigma-1}\right)^{\frac{\rho}{1-\rho}}$

- From any $\zeta(\cdot) > 1$, satisfying $\lim_{z \to \bar{z}} \zeta(z) = \infty$, if $\bar{z} < \infty$, one can reverse-engineer

$$s(z) = \gamma \exp\left[\int_{z_0}^{z} \frac{1 - \zeta(\xi)}{\xi} d\xi\right].$$

This can be useful for those who want to estimate H.S.A. demand systems nonparametrically.

Unit Cost Function: By integrating $\frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(\mathbf{p})}\right)$,

$$cP(\mathbf{p}) = A(\mathbf{p}) \exp\left[-\int_\Omega s\left(\frac{p_\omega}{A(\mathbf{p})}\right)\Phi\left(\frac{p_\omega}{A(\mathbf{p})}\right)d\omega\right], \quad \text{where } \Phi(z) \equiv \frac{1}{s(z)}\int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi > 0.$$

- $c > 0$ is a constant, proportional to TFP.
- $\Phi(z_\omega)$: the productivity gain from a product sold at $z_\omega = p_\omega/A$.
- $P(\mathbf{p})$: linear homogeneous, monotonic, & strictly quasi-concave, ensuring the integrability of H.S.A. [Our 2017 paper proved the integrability, iff $\zeta(z) \equiv 1 - \frac{d \ln s(z)}{d \ln z} > 0$, clearly satisfied here.]
- $A(\mathbf{p})/P(\mathbf{p})$ is not constant and depends on $\mathbf{p}$, with the sole exception of CES, because

$$\frac{\partial \ln A(\mathbf{p})}{\partial \ln p_\omega} = \frac{z_\omega s'(z_\omega)}{\int_\Omega s'(z_{\omega'})z_{\omega'}d\omega'} = \frac{[\zeta(z_\omega) - 1]s(z_\omega)}{\int_\Omega [\zeta(z_{\omega'}) - 1]s(z_{\omega'})d\omega'} \neq \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s(z_\omega),$$

unless $\zeta(z)$ is independent of z as $\zeta(z) = \sigma > 1 \Leftrightarrow s(z) = \gamma z^{1-\sigma} \Leftrightarrow \Phi(z) = 1/(\sigma - 1) > 0$.

✓ $A(\mathbf{p})$, the inverse measure of *competitive pressures*, captures *cross price effects* in DS.

✓ $P(\mathbf{p})$, the inverse measure of TFP, captures the *productivity consequences* of price changes.

Fallacy #7; $s_\omega = f(p_\omega/P(\mathbf{p}))$, *with* $f'(\cdot) < 0$ *defines the class of flexible homothetic demand systems, which contains CES as a special case, where* $s_\omega \propto (p_\omega/P(\mathbf{p}))^{1-\sigma}$.

5.2. Substitutability and Love-for Variety under H.S.A. At $\mathbf{p} = p\mathbf{1}_{\Omega}^{-1}$,

$$s(z)V == s\left(\frac{p}{A(p\mathbf{1}_{\Omega}^{-1})}\right)V = s\left(\frac{1}{A(\mathbf{1}_{\Omega}^{-1})}\right)V = 1 \Leftrightarrow z = \frac{1}{A(\mathbf{1}_{\Omega}^{-1})} = s^{-1}\left(\frac{1}{V}\right).$$

$$\frac{1}{P(\mathbf{1}_{\Omega}^{-1})} = \frac{1}{A(\mathbf{1}_{\Omega}^{-1})}\frac{A(\mathbf{1}_{\Omega}^{-1})}{P(\mathbf{1}_{\Omega}^{-1})} = cs^{-1}\left(\frac{1}{V}\right)\exp\left[\Phi\left(s^{-1}\left(\frac{1}{V}\right)\right)\right],$$

from which,

Proposition 2: Under H.S.A.,

$$\sigma(V) = \zeta\left(\frac{1}{A(\mathbf{1}_{\Omega}^{-1})}\right) = \zeta\left(s^{-1}\left(\frac{1}{V}\right)\right) = 1 - \frac{s^{-1}(1/V)s'\left(s^{-1}(1/V)\right)}{1/V} > 1;$$

$$\mathcal{L}(V) \equiv -\frac{d\ln P\left(\mathbf{1}_{\Omega}^{-1}\right)}{d\ln V} = \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) = V\int_{s^{-1}(1/V)}^{\bar{z}} \frac{s(\xi)}{\xi}\mathrm{d}\xi > 0.$$

Since $s^{-1}(1/V)$ is increasing in V,

$$\mathrm{sgn}\,\{\zeta'(\cdot)\} = \mathrm{sgn}\,\{\sigma'(\cdot)\} \quad \& \quad \mathrm{sgn}\,\{\Phi'(\cdot)\} = \mathrm{sgn}\,\{\mathcal{L}'(\cdot)\}.$$

Under H.S.A., the 2nd Law ⟺ Increasing Substitutability

By differentiating $\Phi(z) \equiv \frac{1}{s(z)}\int_z^{\bar{z}} \frac{s(\xi)}{\xi} \mathrm{d}\xi$,

$$\frac{z\Phi'(z)}{\Phi(z)} = \zeta(z) - \int_z^{\bar{z}} \zeta(\xi)w(\xi)\, d\xi, \qquad where\ w(\xi) \equiv \frac{s(\xi)/\xi}{\int_z^{\bar{z}}[s(\xi')/\xi']\, d\xi'}, \int_z^{\bar{z}} w(\xi)\, d\xi = 1$$

Thus,

$$\sigma'(V) \gtreqless 0 \Longrightarrow \mathcal{L}'(V) \lesseqgtr 0,$$

and

$$\mathcal{L}'(V) = 0 \ \Longrightarrow\ \sigma'(V) = 0.$$

From this,

Proposition 3: Under H.S.A.,

$$\sigma'(V) \gtreqless 0,\ \ \forall V \in (1/s(z_0), \infty) \quad \Longrightarrow \quad \mathcal{L}'(V) \lesseqgtr 0,\ \ \forall V \in (1/s(z_0), \infty),$$

The reverse is not true in general, except

$$\mathcal{L}'(V) = 0,\ \ \forall V \in (1/s(z_0), \infty) \ \Longrightarrow\ \sigma'(V) = 0,\ \ \forall V \in (1/s(z_0), \infty).$$

Under H.S.A., the 2nd Law (⟺ Increasing Substitutability) ⟹ Diminishing Love-for-Variety.

Part 6: Dixit-Stiglitz under H.S.A.

6.1. Equilibrium: Dixit-Stiglitz Monopolistic Competition under H.S.A.

Recall that, under general homothetic demand systems, the equilibrium may not be symmetric and may not be unique.

Profit Maximization:

$$\max_{p_\omega} \Pi_\omega = (p_\omega - \psi)x_\omega = \left(1 - \frac{\psi}{p_\omega}\right) s\left(\frac{p_\omega}{A(\mathbf{p})}\right) E$$

Firms take the value of $A = A(\mathbf{p})$ as given. From the FOC,

Lerner Pricing Formula:	$p_\omega\left[1 - \frac{1}{\zeta(p_\omega/A)}\right] = \psi \Rightarrow \frac{p_\omega}{A}\left[1 - \frac{1}{\zeta(p_\omega/A)}\right] = \frac{\psi}{A}.$

For the expositional reason,

Assumption A1: *For all* $z \in (0, \bar{z})$,

$$\frac{d}{dz}\left(z\left[1 - \frac{1}{\zeta(z)}\right]\right) = \frac{1}{\zeta(z)}\left[\zeta(z) - 1 + \frac{z\zeta'(z)}{\zeta(z)}\right] = -\frac{z}{\zeta(z)}\frac{d}{dz}\ln\left(\frac{s(z)}{\zeta(z)}\right) > 0.$$

The 2nd Law, $\zeta'(z) > 0$, is sufficient but not necessary for **A1.** **A1** ensures

- The MR curve is increasing in p_ω (decreasing in x_ω) → the profit-maximizing price is unique & increasing in ψ/A,

$$\frac{p_\omega}{A}\left(1 - \frac{1}{\zeta(p_\omega/A)}\right) = \frac{\psi_\omega}{A} \Rightarrow z_\omega = \frac{p_\omega}{A} = \tilde{Z}\left(\frac{\psi}{A}\right); \quad \tilde{Z}'(\cdot) > 0.$$

All the firms set the same price. **The Equilibrium is Symmetric!!**

- $\Pi = \max_{p_\omega} \Pi_\omega = (1 - \psi/p)s(z)E = [s(z)/\zeta(z)]E$ is decreasing in $z = \tilde{Z}(\psi/A)$ and hence in ψ/A.

Thus, the Free Entry-Zero Profit Condition, $\Pi = F$, uniquely pins down $z = \tilde{Z}(\psi/A)$ and A and V from $s(z)V = 1$.

Without **A1,** $\tilde{Z}(\cdot)$, still increasing, may be only piecewise continuous, and $\Pi = \left[s\left(\tilde{Z}(\psi/A)\right)/\zeta\left(\tilde{Z}(\psi/A)\right)\right]E$, still decreasing, may be only piecewise-differentiable, which complicates the exposition.

Equilibrium is Unique and Symmetric! And we already know that these can be expressed as:

Lerner Formula:	$p^{eq}\left(1-\frac{1}{\sigma(V^{eq})}\right)=\psi$	**Equilibrium Price:**	$p^{eq}=\frac{\sigma(V^{eq})}{\sigma(V^{eq})-1}\psi\equiv\mu(V^{eq})\psi$
Product Variety:	$V^{eq}\sigma(V^{eq})=\frac{E}{F}.$	**Equilibrium Quantity**:	$x^{eq}=\frac{(\sigma(V^{eq})-1)F}{\psi}=\frac{F}{(\mu(V^{eq})-1)\psi}$

A1 ensures that $s(z)/\zeta(z)$ is globally decreasing, and hence $V\sigma(V)$ is globally increasing, because

$$\frac{s(z)}{\zeta(z)}=\frac{1}{V\sigma(V)}.$$

6.2. Comparative Statics under H.S.A.

The results obtained for general homothetic demand system under the assumption that the equilibrium is unique and symmetric obviously carry over to this case.

The results in the case of procompetitive entry are also those in the case of the 2nd law under H.S.A., because entry is procompetitive iff the 2nd law holds under H.S.A.

6.3. Optimum vs. Equilibrium under H.S.A. Differentiating $\Phi(z) \equiv \frac{1}{s(z)}\int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi$,

$$\frac{\partial \ln \Phi(z)}{\partial \ln z} = -\frac{zs'(z)}{s(z)} - \frac{1}{\Phi(z)} = \zeta(z) - 1 - \frac{1}{\Phi(z)}.$$

Thus,

$$\mathcal{L}'(V) \lesseqgtr 0 \Leftrightarrow \Phi(z) \lesseqgtr 0 \Leftrightarrow \zeta(z) - 1 \lesseqgtr \frac{1}{\Phi(z)} \Leftrightarrow \mathcal{L}(V) \lesseqgtr \frac{1}{\sigma(V)-1} \Leftrightarrow V^o \lesseqgtr V^e.$$

Recall

SOCIAL incentive for additional product variety =
POSITIVE externalities from incomplete appropriability

$$\mathcal{L}(V) = \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) \equiv V \int_{s^{-1}\left(\frac{1}{V}\right)}^{\bar{z}} \frac{s(\xi)}{\xi} d\xi > 0$$

PRIVATE incentive for additional product variety =
NEGATIVE externalities from business stealing

$$\frac{1}{\sigma(V)-1} = \mu(V) - 1 = \frac{\Pi}{\psi x}$$

Proposition 4: In the Dixit-Stiglitz environment under H.S.A.,

$$\zeta'(z) \lesseqgtr 0 \text{ for all } z > 0 \Leftrightarrow \sigma'(V) \lesseqgtr 0 \text{ for all } V > 0$$

$$\Longrightarrow$$

$$\mathcal{L}'(V) \gtreqless 0 \text{ for all } V > 0 \Leftrightarrow V^{eq} \lesseqgtr V^{op} \text{ for all } E/F > 0.$$

Moreover,

$$\zeta(z) = const. \Leftrightarrow \sigma(V) = const. \Leftrightarrow \mathcal{L}(V) = const. \Leftrightarrow V^{eq} = V^{op} \text{ for all } E/F > 0.$$

Under H.S.A.,

The 2nd Law $\zeta(p_\omega; \mathbf{p})$ is increasing in p_ω $\zeta^*(x_\omega; \mathbf{x})$ is decreasing in x_ω	$\Longrightarrow$	**Diminishing Love-for-Variety** $\mathcal{L}'(V) < 0$ for all $V > 0$.
$\Updownarrow$		$\Updownarrow$
Increasing Substitutability **Procompetitive Entry** $\sigma'(V) < 0$ for all $V > 0$.	$\Longrightarrow$	**Excessive Product Variety** $\mathcal{L}(V)[\sigma(V) - 1] < 1$ for all $V > 0$ $\Leftrightarrow V^{eq} > V^{op}$ for all $E/F > 0$.

Notes:

You can also prove these relations under HDIA and HIIA. However, ensuring that the existence and the uniqueness of the symmetric equilibrium requires additional restrictions.

Outside of the three classes, these relations do not hold generally!!

Generalized Translog (GT): A Parametric Family of symmetric H.S.A.

$$s_\omega \equiv \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(\mathbf{p})}\right), \qquad \text{where} \int_\Omega s\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega \equiv 1.$$

with $s(z) = \gamma \max\left\{\left(\frac{\sigma-1}{\eta}\ln\left(\frac{\bar{z}}{z}\right)\right)^\eta, 0\right\}; \sigma > 1;\ \bar{z} \equiv \beta e^{\frac{\eta}{\sigma-1}} > 0;\ \gamma > 0;\ \eta > 0.$

Price Elasticity	$\zeta(z) = 1 + \frac{\eta}{\ln(\bar{z}/z)} > 1$	increasing in $z < \bar{z}$ (Marshall's 2nd law)
Substitutability	$\sigma(V) = 1 + (\sigma - 1)(\gamma V)^{1/\eta}$	increasing in V (procompetitive entry)
Love-for-variety	$\mathcal{L}(V) = \frac{1}{\sigma - 1}\frac{1}{1 + 1/\eta}(\gamma V)^{-1/\eta}$	decreasing in V (diminishing LV)
	$\mathcal{L}(V)[\sigma(V) - 1] = \frac{1}{1 + 1/\eta} < 1$	$V^{op} < V^{eq}$ for all $E/F > 0$ (excessive entry).

- Translog is a special case, $\eta = 1$.
- CES is a limit as $1/\eta \to 0$, since $s(z) \to \gamma(z/\beta)^{1-\sigma}$; $\sigma(V) \to \sigma$; $\mathcal{L}(V) = \frac{1}{\sigma-1}$; $\mathcal{L}(V)[\sigma(V) - 1] \to 1$.
- $1/\eta$ measures the distance from CES and the strength of procompetitive-ness.
- $P(\mathbf{p})$ and $A(\mathbf{p})$ do not have closed form expressions unless $1/\eta = 1$ or $1/\eta \to 0$.
- σ is a parameter, which shifts $\sigma(V)$ up and $\mathcal{L}(V)$ and down.

(Counter)Example: Geometric Means of Generalized Translogs (GTs)

$$\ln P(\mathbf{p}) = \int_1^\infty \ln P(\mathbf{p};\sigma)\, dG(\sigma) \equiv \mathbb{E}_G[\ln P(\mathbf{p};\sigma)]$$

where $G(\cdot)$ is a cdf of σ and $P(\mathbf{p};\sigma)$ is the unit cost function of the generalized translog.

Substitutability	$\sigma(V) = \mathbb{E}_G\left[1 + (\sigma - 1)(\gamma V)^{1/\eta}\right] = 1 + (\gamma V)^{1/\eta}\, \mathbb{E}_G(\sigma - 1)$	increasing in V (procompetitive entry)
Love-for-variety	$\mathcal{L}(V) = \mathbb{E}_G\left[\frac{1}{\sigma - 1}\frac{\eta}{1+\eta}(\gamma V)^{-1/\eta}\right] = \frac{\eta}{1+\eta}(\gamma V)^{-1/\eta}\mathbb{E}_G\left(\frac{1}{\sigma - 1}\right)$	decreasing in V (diminishing LV)

Thus,

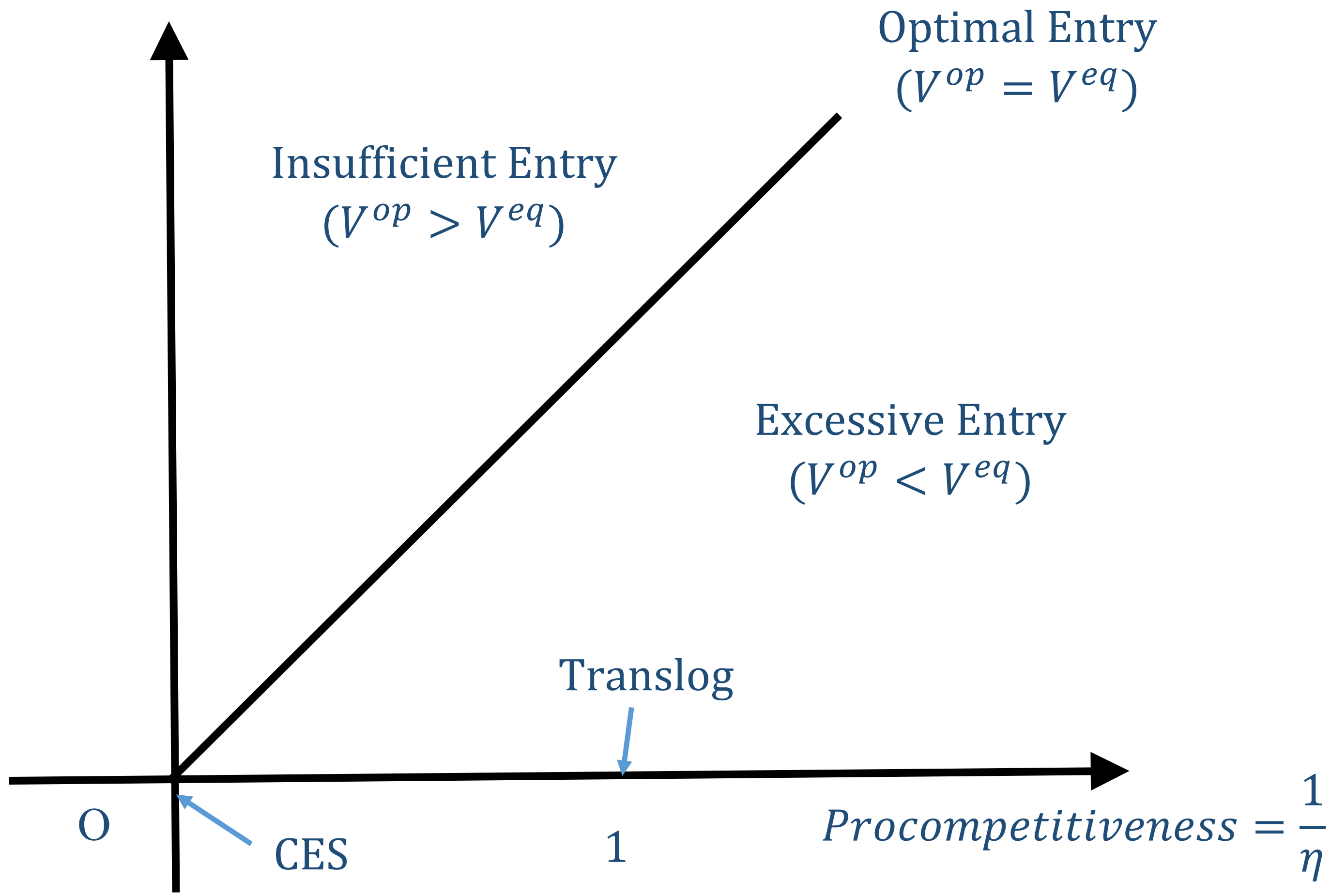
$$V^{op} \lesseqgtr V^{eq} \Leftrightarrow \mathcal{L}(V)[\sigma(V) - 1] = \frac{\eta}{1+\eta}\mathbb{E}_G\left(\frac{1}{\sigma-1}\right)\mathbb{E}_G(\sigma - 1) = \frac{\eta}{1+\eta}\frac{\mathrm{AM}_G(\sigma-1)}{\mathrm{HM}_G(\sigma-1)} = \frac{\eta}{1+\eta}\frac{\mathrm{AM}_G\left(\frac{1}{\sigma-1}\right)}{\mathrm{HM}_G\left(\frac{1}{\sigma-1}\right)} \lesseqgtr 1$$

$$\Leftrightarrow 0 < \frac{\mathrm{AM}_G(\sigma-1)}{\mathrm{HM}_G(\sigma-1)} - 1 = \frac{\mathrm{AM}_G\left(\frac{1}{\sigma-1}\right)}{\mathrm{HM}_G\left(\frac{1}{\sigma-1}\right)} - 1 \lesseqgtr \frac{1}{\eta}.$$

- LHS: the elasticity dispersion; RHS the strength of procompetitive-ness.
- Optimality under CES is a limit case, where both LHS and RHS are zero.

Procompetitiveness fosters Excessive Entry; Elasticity Dispersion fosters Insufficient Entry.

$$Elasticity\ Dispersion = \frac{\mathrm{AM}_G(\sigma - 1)}{\mathrm{HM}_G(\sigma - 1)} - 1$$



Part 7: Melitz under H.S.A.

7.1. The Melitz Environment:

One Primary Factor of Production: "Labor" taken as numeraire.

A Continuum of Differentiated Intermediate Inputs:
Each variety is produced (and sold exclusively) by a single firm using "labor".
$\omega \in \Omega$ is both the index of a differentiated product, as well as that of the firm producing it.

Asymmetric Intermediate Input Producing MC Firms:

- The symmetric demand system: Their products enter symmetrically in the demand system.
- All firms have the same **sunk cost of entry**,F_e. But they become *ex-post* heterogeneous in their marginal cost.
    - Entry allows firm ω to learn its marginal cost of production, ψ_ω, randomly drawn from the cdf, $G(\psi)$, with the pdf, $g(\psi) = G'(\psi) > 0$ over the support, $\left(\underline{\psi}, \bar{\psi}\right) \subseteq (0, \infty)$.
- All firms have the same fixed "overhead" cost of production, F. **The overhead cost is not sunk**; firms don't have to pay if they choose not to stay in the market.
- Upon learning its marginal cost,, firm ω calculate its gross profit, $\Pi(\psi_\omega)$.
    - If $\Pi(\psi_\omega) < F$, the firms exit immediately without paying the overhead, F.
    - If $\Pi(\psi_\omega) \geq F$, the firms stay by paying F.

Free Entry: Ex-ante identical firms enter until $F_e = \int_{\underline{\psi}}^{\bar{\psi}} \max\{\Pi(\psi) - F, 0\} dG(\psi)$.

This ensures no excess profit, and the total demand for "labor" in this sector is equal to $L = \mathbf{px} = P(\mathbf{p})X(\mathbf{x}) = E$.

7.2. Pricing: Markup & Pass-Through Rates. Given E & $A = A(\mathbf{p})$, firm ω chooses p_ω and hence $z_\omega = p_\omega/A$.

$$\max(p_\omega - \psi_\omega)x_\omega = \max_{z_\omega}\left(1 - \frac{\psi_\omega/A}{z_\omega}\right)s(z_\omega)E.$$

Lerner Pricing Formula

$$z_\omega\left[1 - \frac{1}{\zeta(z_\omega)}\right] = \frac{\psi_\omega}{A}.$$

Under **A1,** LHS is monotonically increasing in z_ω → firms with the same ψ set the same price → $p_\omega = p_\psi$.

Normalized Price: $$\frac{p_\psi}{A} \equiv z_\psi = \tilde{Z}\left(\frac{\psi}{A}\right) \in (\psi/A, \bar{z}), \tilde{Z}'(\cdot) > 0;$$

Price Elasticity: $$\zeta(z_\psi) = \zeta\left(\tilde{Z}\left(\frac{\psi}{A}\right)\right) \equiv \sigma\left(\frac{\psi}{A}\right) > 1;$$ **Markup Rate:** $$\mu_\psi \equiv \frac{p_\psi}{\psi} = \frac{\sigma(\psi/A)}{\sigma(\psi/A) - 1} \equiv \mu\left(\frac{\psi}{A}\right) > 1.$$

with

$$\frac{1}{\sigma(\psi/A)} + \frac{1}{\mu(\psi/A)} = 1; \qquad \mathcal{E}_\sigma\left(\frac{\psi}{A}\right) = -\frac{\mathcal{E}_\mu(\psi/A)}{\mu(\psi/A) - 1}; \quad \mathcal{E}_\mu\left(\frac{\psi}{A}\right) = -\frac{\mathcal{E}_\sigma(\psi/A)}{\sigma(\psi/A) - 1}$$

Pass-Through Rate: $$\rho_\psi \equiv \frac{\partial \ln p_\psi}{\partial \ln \psi} = \mathcal{E}_{\tilde{Z}}\left(\frac{\psi}{A}\right) \equiv \rho\left(\frac{\psi}{A}\right) = \frac{1}{1 + \mathcal{E}_{1-1/\zeta}\left(\tilde{Z}(\psi/A)\right)} = 1 + \mathcal{E}_\mu\left(\frac{\psi}{A}\right) > 0$$

are all functions of the *normalized cost,* ψ/A, *only*.

- Market size E affects the pricing behaviors of firms only through its effects on A.
- More competitive pressures, a lower A, act like a uniform decline in firm productivity.

Easy to verify:

$$\zeta'(z) \gtreqless 0 \Leftrightarrow \mathcal{E}_\sigma\left(\frac{\psi}{A}\right) \gtreqless 0 \Leftrightarrow \mathcal{E}_\mu\left(\frac{\psi}{A}\right) \lesseqgtr 0 \Leftrightarrow \rho\left(\frac{\psi}{A}\right) \lesseqgtr 1.$$

- Under CES, $\sigma(\cdot) = \sigma;\ \mu(\cdot) = \sigma/(\sigma-1) = \mu;\ \rho(\cdot) = 1$. More competitive pressures, $1/A \uparrow$, has no effect.
- The 2nd law implies that more competitive pressures, $1/A \uparrow$, force all firms to reduce their markup rates.

The equivalence of the 2nd law and incomplete pass-through is general and not specific to H.S.A., as neither depend on the cross-variety interaction. However, it hinges on the assumption that MC firms are price-takers in the "labor" market.

$$\zeta'(z) > 0 \Leftrightarrow \rho\left(\frac{\psi}{A}\right) < 1$$

Definitions: Laws of Demand

	General Case:	H.S.A.
1st Law	$\zeta(p_\omega;\mathbf{p}) \equiv -\frac{\partial \ln D(p_\omega;\mathbf{p})}{\partial \ln p_\omega} > 0$	$\zeta(z_\omega) = 1 - \frac{z_\omega s'(z_\omega)}{s(z_\omega)} > 0.$
2nd Law	$\frac{\partial \ln \zeta(p_\omega;\mathbf{p})}{\partial \ln p_\omega} = \frac{\partial}{\partial \ln p_\omega}\left[-\frac{\partial \ln D(p_\omega;\mathbf{p})}{\partial \ln p_\omega}\right] > 0$	$\mathcal{E}_\zeta(z_\omega) \equiv \frac{z_\omega \zeta'(z_\omega)}{\zeta(z_\omega)} > 0.$
Strong (Weak) 3rd Law	$\frac{\partial}{\partial p_\omega}\left[\frac{\partial \ln \zeta(p_\omega;\mathbf{p})}{\partial \ln p_\omega}\right] = \frac{\partial}{\partial p_\omega}\left[\frac{\partial}{\partial \ln p_\omega}\left[-\frac{\partial \ln D(p_\omega;\mathbf{p})}{\partial \ln p_\omega}\right]\right] < (\leq) 0$	$\mathcal{E}_{\zeta/(\zeta-1)}{}'(z_\omega) > (\geq) 0$ $\Leftrightarrow \mathcal{E}_{1-1/\zeta}{}'(z_\omega) < (\leq) 0.$

The 2nd Law: Implications on the Markup and Pass-Through Rates.

Assumption A2 (The 2nd Law): *For all* $z \in (0, \bar{z})$, $\zeta'(z) > 0$.

- high ψ-firms face high price elasticity, set lower markup rates, and incomplete pass-through.
- More competitive pressures, $1/A \uparrow \Longrightarrow \sigma(\psi/A) \uparrow$ and $\mu(\psi/A) \downarrow$ across all firms.
- $\rho(\psi/A)$ could go either way.

The 3rd Law: Implications on the Markup and Pass-Through Rates

Assumption A3 (A3): *For all* $z \in (0, \bar{z})$, $.\mathcal{E}_{1-1/\zeta}{}'(z_\omega) < (\leq) 0$

Under the Strong 3rd law,

$$\mathcal{E}_{1/\mu}{}'\left(\frac{\psi}{A}\right) < 0 \Leftrightarrow \mathcal{E}_{\mu}{}'\left(\frac{\psi}{A}\right) = \rho'\left(\frac{\psi}{A}\right) > 0 \Leftrightarrow \frac{\partial^2 \ln \mu(\psi/A)}{\partial\psi\partial(1/A)} > 0.$$

- high ψ-firms have higher pass-through rates.
- More competitive pressures, $1/A \uparrow \Longrightarrow \rho(\cdot) \uparrow$ across all firms.
- $\mu(\psi/A)$ is log-supermodular in ψ and $1/A$.

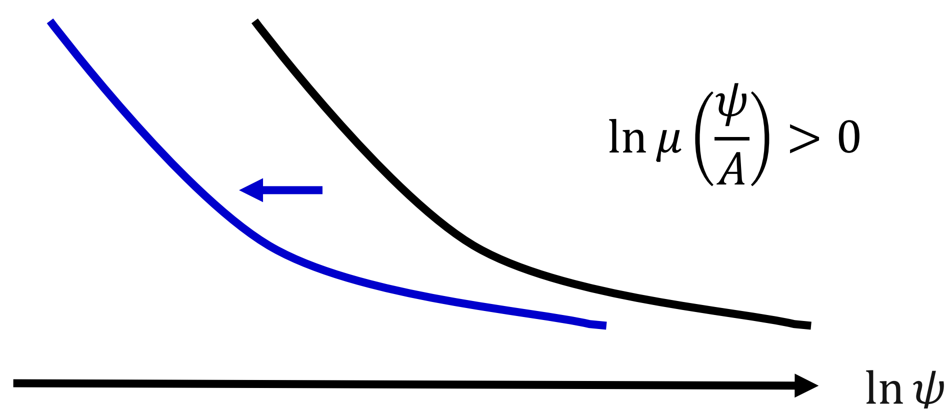
  → The rates of decline in the markup rates caused by more competitive pressures, $1/A \uparrow$, under the 2nd law, are smaller among high ψ-firms.

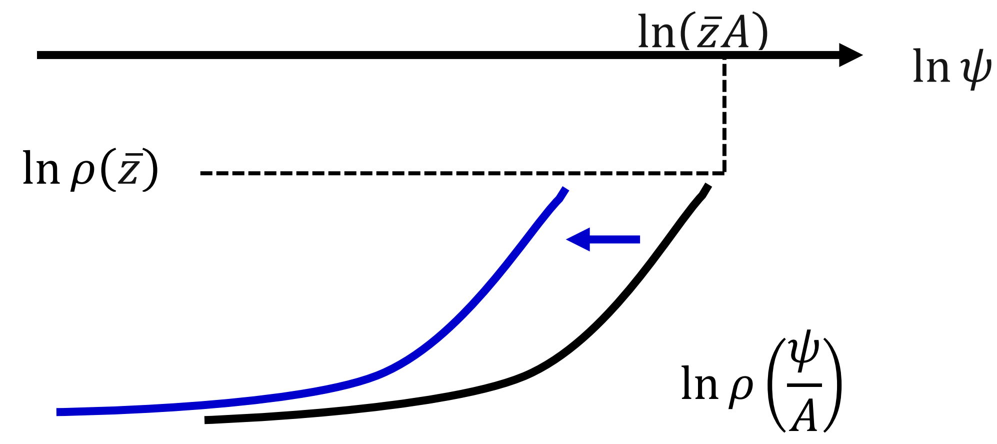
  → A small dispersion of the markup rates across firms

  → mitigating the misallocation caused by the markup rate heterogeneity.

A Graphical Illustration: The Effects of More Competitive Pressures, $1/A \uparrow$, under the 2nd and Strong 3rd law.

Log-markup rate is decreasing (2nd) and convex (3rd). Log-pass-through rate is negative (2nd) and increasing (3rd).





Fallacy #8. "Translog is flexible, as it can approximate any homothetic symmetric demand system."

Translog is a special case of H.S.A., whose market share function can be written after normalization as $s(z) = -\max\{\ln z, 0\}$, which has no parameter to fit the data.

Translog offers at best the 2nd-order local approximation to the unit cost function. But the firm pricing behaviors depend on the 3rd derivatives and their entry/exit decision depend on its global properties.

Translog violates even the weak 3rd law, and hence inconsistent with the evidence that more productive firms have lower pass-through rates.

7.3. Revenue, Profit, & Employment

Revenue	(Gross) Profit	(Variable) Employment
$R_\psi = s(z_\psi)E = s\left(\tilde{Z}\left(\frac{\psi}{A}\right)\right)E \equiv r\left(\frac{\psi}{A}\right)E$	$\Pi_\psi = \frac{r(\psi/A)}{\sigma(\psi/A)}E \equiv \pi\left(\frac{\psi}{A}\right)E$	$\psi x_\psi = \frac{r(\psi/A)}{\mu(\psi/A)}E \equiv \ell\left(\frac{\psi}{A}\right)E$

They are all functions of ψ/A, multiplied by **market size** E.

- More competitive pressures, $1/A \uparrow$, act like a uniform decline in firm productivity.

Revenue	(Gross) Profit	(Variable) Employment
$\mathcal{E}_r\left(\frac{\psi}{A}\right) = -\left[\sigma\left(\frac{\psi}{A}\right) - 1\right]\rho\left(\frac{\psi}{A}\right) < 0$	$\mathcal{E}_\pi\left(\frac{\psi}{A}\right) = 1 - \sigma\left(\frac{\psi}{A}\right) < 0$	$\mathcal{E}_\ell\left(\frac{\psi}{A}\right) = 1 - \sigma\left(\frac{\psi}{A}\right)\rho\left(\frac{\psi}{A}\right) \lesseqgtr 0$

Their elasticities depend solely on $\sigma(\cdot)$ and $\rho(\cdot)$.

- Market size affects the firm size distribution in profit, revenue and employment only via its effects on A.

Under CES, $r(\cdot)/\pi(\cdot) = \sigma$; $r(\cdot)/\ell(\cdot) = \mu = \sigma/(\sigma - 1) \Longrightarrow \mathcal{E}_r(\cdot) = \mathcal{E}_\pi(\cdot) = \mathcal{E}_\ell(\cdot) = 1 - \sigma < 0$.

- Revenue $r(\psi/A)E$ and profit $\pi(\psi/A)E$ are always strictly decreasing in ψ/A.
- Employment $\ell(\psi/A)E$ may be nonmonotonic in ψ/A; hump-shaped under the 2nd & the weak 3rd laws.

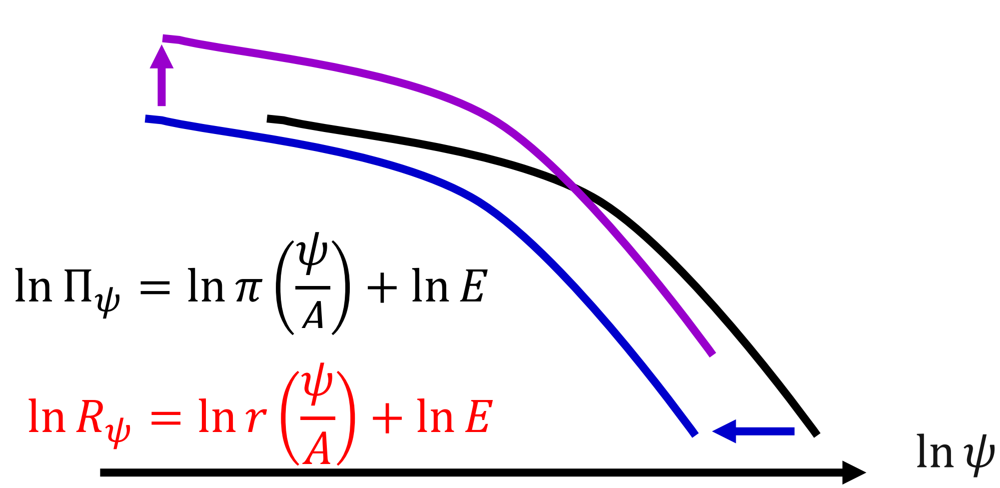
Revenue	(Gross) Profit	(Variable) Employment
$\frac{\partial^2 \ln R_\psi}{\partial\psi\partial(1/A)} = \left[1 - \sigma\left(\frac{\psi}{A}\right)\right]\rho'\left(\frac{\psi}{A}\right) - \sigma'\left(\frac{\psi}{A}\right)\rho\left(\frac{\psi}{A}\right)$ Negative under the 2nd & weak 3rd laws	$\frac{\partial^2 \ln \Pi_\psi}{\partial\psi\partial(1/A)} = -\sigma'\left(\frac{\psi}{A}\right)$ Negative under the 2nd law	$\frac{\partial^2 \ln(\psi x_\psi)}{\partial\psi\partial(1/A)} = -\sigma'\left(\frac{\psi}{A}\right)\rho\left(\frac{\psi}{A}\right) - \sigma\left(\frac{\psi}{A}\right)\rho'\left(\frac{\psi}{A}\right)$ Negative under the 2nd & the weak 3rd laws

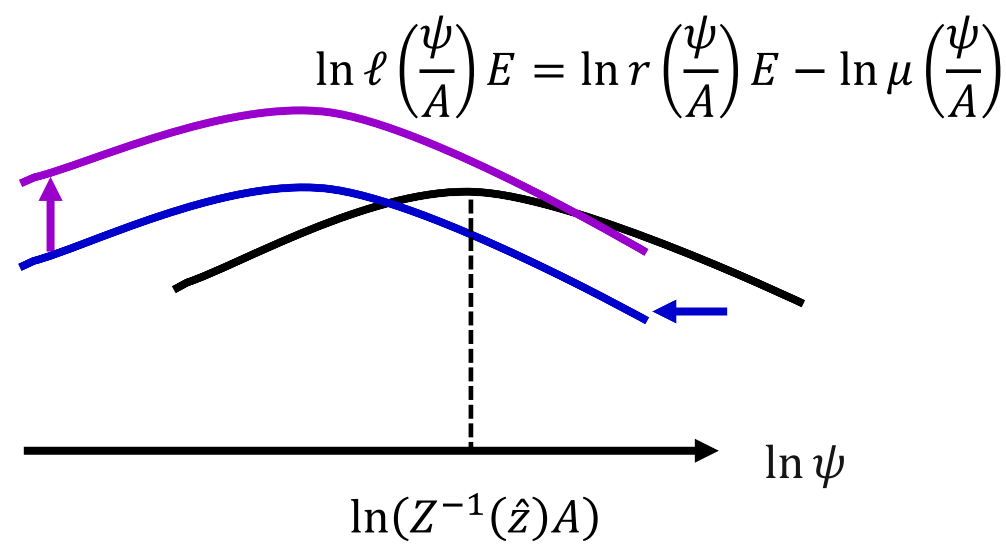
Log-submodularity of the profit (under the 2nd) and of the revenue and employment (under the 2nd and weak 3rd).

- With both R_ψ & Π_ψ decreasing in ψ, more competitive pressures, $1/A \uparrow$, cause a proportionately larger decline in the revenue & profit among high-ψ firms, hence a larger dispersion in the revenue & profit across firms.

Cross-Sectional Implications of More Competitive Pressures, $A\downarrow$ under the 2nd and the weak 3rd laws.

Profit, Π_ψ, decreasing & concave in $\ln\psi$ under the 2nd **Revenue,** R_ψ, decreasing & concave in $\ln\psi$ under 2nd & weak 3rd $A\downarrow$ with L fixed a parallel-shift left → larger down at higher ψ. $A\downarrow$ due to $E\uparrow$, a parallel shift up, a single-crossing.	**Employment**, hump-shaped & concave in $\ln\psi$ under the 2nd & weak 3rd. $A\downarrow$ with L fixed a parallel-shift left, a single-crossing. $A\downarrow$ due to $E\uparrow$, a parallel shift up.





In summary, more competitive pressures ($A\downarrow$)
- $\mu(\psi/A)\downarrow$ under A2 & $\rho(\psi/A)\uparrow$ under strong A3
- Profit, Revenue, Employment become more concentrated among the more productive.

7.4. Equilibrium: Existence & Uniqueness: Assume $F + F_e < \pi(0)E$.

Cutoff Rule: Stay if $\psi \leq \psi_c$; exit if $\psi > \psi_c$, where

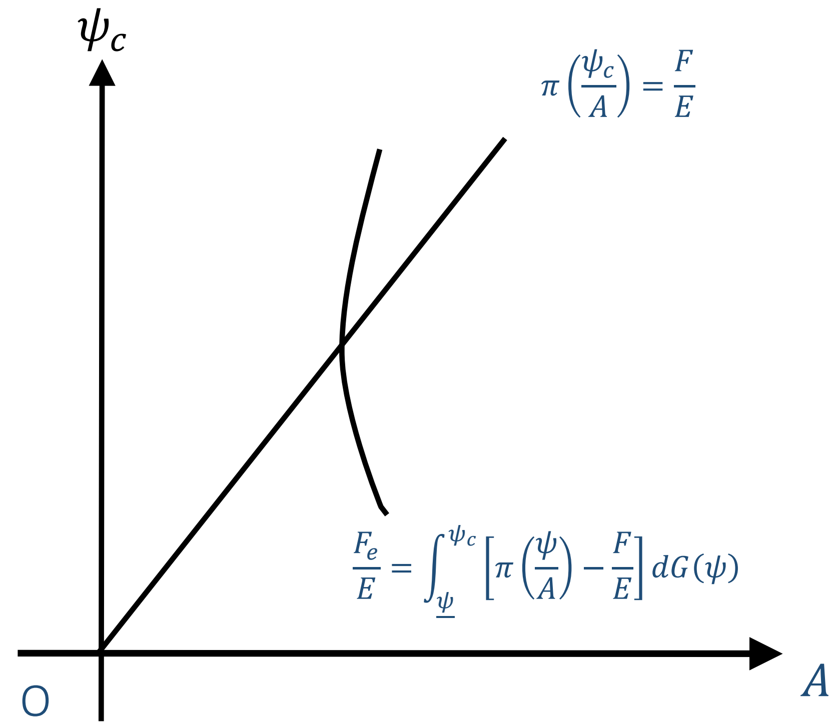
$$\max_{\psi_c} \int_{\underline{\psi}}^{\psi_c} \left[\pi\left(\frac{\psi}{A}\right)E - F\right] dG(\psi) \Longrightarrow \pi\left(\frac{\psi_c}{A}\right)E = F$$

positively-sloped. $A \downarrow$ (more competitive pressures) $\Longrightarrow \psi_c \downarrow$ (tougher selection)
rotate clockwise, as $F/E \uparrow$ (higher overhead/market size) $\Longrightarrow \psi_c/A \downarrow$.

Free Entry Condition:

$$F_e = \int_{\underline{\psi}}^{\psi_c} \left[\pi\left(\frac{\psi}{A}\right)E - F\right] dG(\psi)$$

shift to the left as $F_e \downarrow$ (lower entry cost) $\Longrightarrow A \downarrow$ (more competitive pressures).



$A = A(\mathbf{p})$ & ψ_c: uniquely determined, respond continuously to F_e/E & F/E in the interior, $0 < G(\psi_c) < 1$ for

$$0 < \frac{F_e}{E} < \int_{\underline{\psi}}^{\bar{\psi}} \left[\pi\left(\pi^{-1}\left(\frac{F}{E}\right)\frac{\psi}{\bar{\psi}}\right) - \frac{F}{E}\right] dG(\psi),$$

which holds for a sufficiently small $F_e > 0$ with no further restrictions on $G(\cdot)$ and $s(\cdot)$.

- This unique existence proof applies also to Melitz under CES.)
- An industry-wide cost shock, λ, of the form, $G(\lambda\psi)$, change A & ψ_c proportionately, such that that the distribution of ψ/A across active firms won't change. In particular, it has no effect on the markup & pass-through rate distributions.

Adding-Up (Resource) Constraint:

$$1 \equiv \int_{\Omega} s\left(\frac{p_\omega}{A}\right) d\omega = M \int_{\underline{\psi}}^{\psi_c} r\left(\frac{\psi}{A}\right) dG(\psi)$$

from which the mass of active firms(hence, of product variety) = the measure of $|\Omega| = V$ is,

$$V = MG(\psi_c) = \left[\int_{\underline{\psi}}^{\psi_c} r\left(\frac{\psi}{A}\right) \frac{dG(\psi)}{G(\psi_c)}\right]^{-1} = \left[\int_{\underline{\xi}}^{1} r\left(\pi^{-1}\left(\frac{F}{E}\right)\xi\right) d\tilde{G}(\xi;\psi_c)\right]^{-1} > 0$$

where

$$\tilde{G}(\xi;\psi_c) \equiv \frac{G(\psi_c\xi)}{G(\psi_c)}$$

is the cdf of $\xi \equiv \psi/\psi_c$, conditional on $\underline{\xi} \equiv \underline{\psi}/\psi_c < \xi \leq 1$.

Lemma 1: $\mathcal{E}_g'(\psi) < 0 \Longrightarrow \mathcal{E}_G'(\psi) < 0$; $\mathcal{E}_g'(\psi) \geq 0 \Longrightarrow \mathcal{E}_G'(\psi) \geq 0$ with some boundary conditions.
Lemma 2: A lower ψ_c shifts $\tilde{G}(\xi;\psi_c)$ to the right (left) in MLR if $\mathcal{E}_g'(\psi) < (>)0$ and in FSD if $\mathcal{E}_G'(\psi) < (>)0$.

- Some evidence for $\mathcal{E}_g'(\psi) > 0 \Longrightarrow \psi_c \downarrow$ (tougher selection) shifts $\tilde{G}(\xi;\psi_c)$ to the left.
- Pareto-distributed productivity $\Leftrightarrow$ Power-distributed marginal cost are self-similar.
  $G(\psi) = (\psi/\bar{\psi})^\kappa \Longrightarrow \mathcal{E}_g'(\psi) = \mathcal{E}_G'(\psi) = 0 \Longrightarrow \tilde{G}(\xi;\psi_c)$ is independent of ψ_c.
- Fréchet, Weibull, Lognormal; $\mathcal{E}_g'(\psi) < 0 \Longrightarrow \mathcal{E}_G'(\psi) < 0 \Longrightarrow \psi_c \downarrow$ (tougher selection) shifts $\tilde{G}(\xi;\psi_c)$ to the right.

Notice that there is no feedback effect from M or V on A & ψ_c.

Equilibrium can be solved recursively under H.S.A.!! Not under HDIA/HIIA. (See Appendix 1).

Revisiting Melitz (2003) under CES: $s(z) = \gamma z^{1-\sigma}$

Pricing:

$$\mu\left(\frac{\psi}{A}\right) = \frac{\sigma}{\sigma - 1} > 1 \Rightarrow \rho\left(\frac{\psi}{A}\right) = 1$$

$$\Longrightarrow \mathcal{E}_r\left(\frac{\psi}{A}\right) = \mathcal{E}_\pi\left(\frac{\psi}{A}\right) = \mathcal{E}_\ell\left(\frac{\psi}{A}\right) = 1 - \sigma < 0.$$

Relative firm size, in revenue, profit, employment, unchanged across equilibriums.

Cutoff Rule:

$$c_0 E\left(\frac{\psi_c}{A}\right)^{1-\sigma} = F,$$

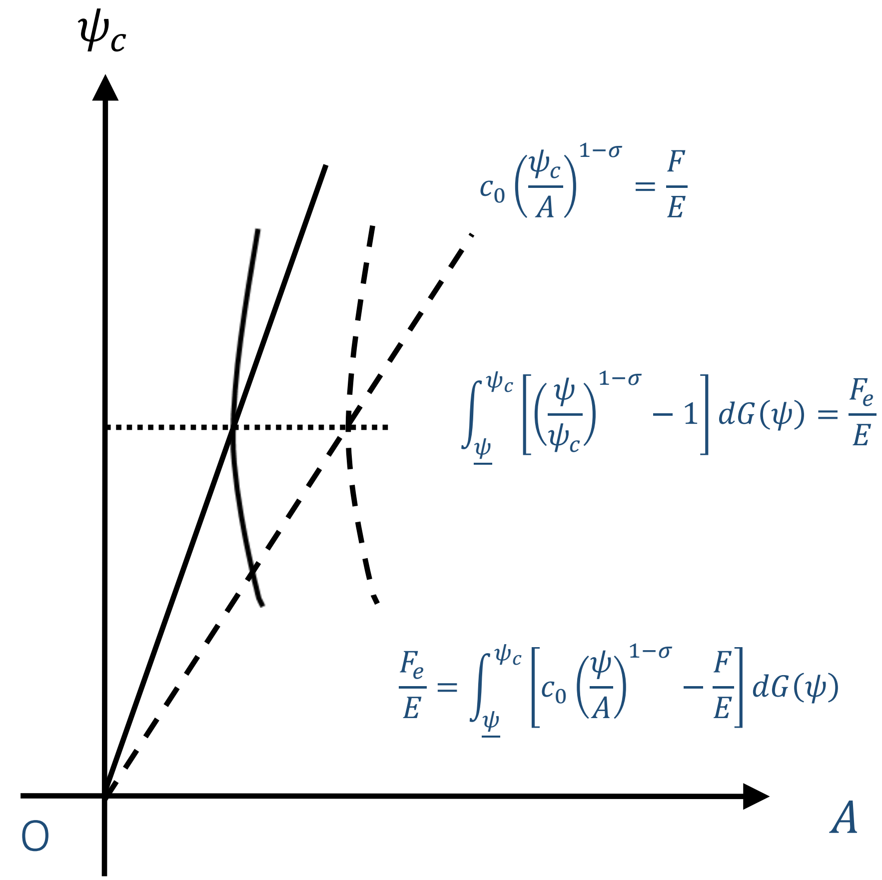
Free Entry Condition:

$$\int_{\underline{\psi}}^{\psi_c}\left[c_0 E\left(\frac{\psi}{A}\right)^{1-\sigma} - F\right] dG(\psi) = F_e,$$

with $c_0 > 0$. As E changes, the intersection moves along

$$\int_{\underline{\psi}}^{\psi_c}\left[\left(\frac{\psi}{\psi_c}\right)^{1-\sigma} - 1\right] dG(\psi) = \frac{F_e}{F}$$

horizontal, i.e., independent of A, hence of E.



Adding-Up (Resource) Constraint: $M[F_e + G(\psi_c)F] = E/\sigma.$

Aggregate entry and overhead cost are equal to the aggregate profit.

Under CES,

- $E \uparrow$ keeps ψ_c unaffected; increases both M and $MG(\psi_c)$ *proportionately;* All adjustments at ***the extensive margin.***
- $F_e \downarrow$ reduces ψ_c; increases M; increases (decreases) $MG(\psi_c)$ if $\mathcal{E}_G'(\psi) < (>)0$; $MG(\psi_c)$ unaffected under Pareto.
- $F \downarrow$ increases ψ_c; increases $MG(\psi_c)$; increases (decreases) M if $\mathcal{E}_G'(\psi) < (>)0$; M unaffected under Pareto.

7.5. Aggregate Production Cost and Profit Shares and TFP

To describe the comparative statics, it is useful to introduce the following notations:

The $w(\cdot)$-weighted average of $f(\cdot)$ among the active firms, $\psi \in (\underline{\psi}, \psi_c)$	$\mathbb{E}_w(f) \equiv \dfrac{\int_{\underline{\psi}}^{\psi_c} f\left(\frac{\psi}{A}\right) w\left(\frac{\psi}{A}\right) dG(\psi)}{\int_{\underline{\psi}}^{\psi_c} w\left(\frac{\psi}{A}\right) dG(\psi)}.$
The unweighted average of $f(\cdot)$ among the active firms, $\psi \in (\underline{\psi}, \psi_c)$	$\mathbb{E}_1(f) \equiv \dfrac{\int_{\underline{\psi}}^{\psi_c} f\left(\frac{\psi}{A}\right) dG(\psi)}{\int_{\underline{\psi}}^{\psi_c} dG(\psi)}.$

$$\Longrightarrow \mathbb{E}_w\left(\frac{f}{w}\right) = \frac{\mathbb{E}_1(f)}{\mathbb{E}_1(w)} = \left[\mathbb{E}_f\left(\frac{w}{f}\right)\right]^{-1}.$$

Then,

Aggregate TFP	$\ln\left(\frac{X}{E}\right) = \ln\left(\frac{1}{P}\right) = \ln\left(\frac{c}{A}\right) + \mathbb{E}_r[\Phi \circ Z]$
Aggregate Production Cost Share (Average inverse markup rate)	$\dfrac{\mathbb{E}_1(\ell)}{\mathbb{E}_1(r)} = \mathbb{E}_r\left(\dfrac{1}{\mu}\right) = 1 - \left[\mathbb{E}_\pi\left(\dfrac{\mu}{\mu-1}\right)\right]^{-1} = \dfrac{1}{\mathbb{E}_\ell(\mu)}$
Aggregate Profit Share (Average inverse price elasticity)	$\dfrac{\mathbb{E}_1(\pi)}{\mathbb{E}_1(r)} = \mathbb{E}_r\left(\dfrac{1}{\sigma}\right) = \dfrac{1}{\mathbb{E}_\pi(\sigma)} = 1 - \left[\mathbb{E}_\ell\left(\dfrac{\sigma}{\sigma-1}\right)\right]^{-1}$

by applying the above formulae to $\pi(\cdot)/r(\cdot) = 1 - \ell(\cdot)/r(\cdot) = 1/\sigma(\cdot) = 1 - 1/\mu(\cdot)$,

7.6. Comparative Statics: Effects of F_e, E, and F on Competitive Pressures, A, and Firm Selection, ψ_c

By totally differentiating the cutoff rule and the free entry condition,

$$\begin{bmatrix} d\ln A \\ \square \\ d\ln\psi_c \end{bmatrix} = \frac{\mathbb{E}_1(\pi)}{\mathbb{E}_1(\ell)} \begin{bmatrix} 1-f_x & \square & f_x \\ \square & \square & \square \\ 1-f_x & \square & f_x-\delta \end{bmatrix} \begin{bmatrix} d\ln(F_e/E) \\ \square \\ d\ln(F/E) \end{bmatrix}$$

where

$$\frac{\mathbb{E}_1(\pi)}{\mathbb{E}_1(\ell)} = \frac{1}{\mathbb{E}_\pi(\sigma)-1} = \{\mathbb{E}_r[\mu^{-1}]\}^{-1} - 1 = \mathbb{E}_\ell(\mu) - 1 > 0;$$

The average profit/average labor cost ratio among the active firms

$$f_x \equiv \frac{FG(\psi_c)}{F_e + FG(\psi_c)} = \frac{\pi(\psi_c/A)}{\mathbb{E}_1(\pi)} < 1;$$

The share of the overhead in the total expected fixed cost = to the profit of the cut-off firm relative to the average profit among the active firms

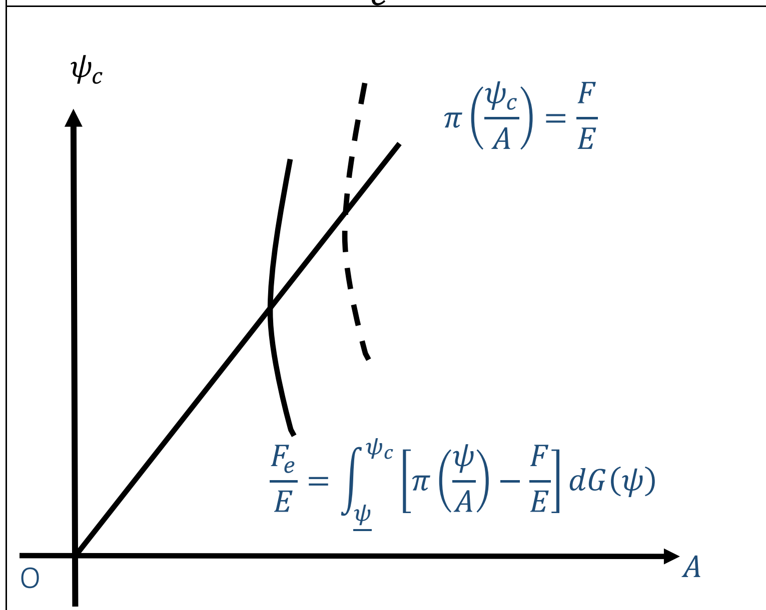
$$\delta \equiv \frac{\mathbb{E}_\pi(\sigma)-1}{\sigma(\psi_c/A)-1} = \frac{\pi(\psi_c/A)}{\ell(\psi_c/A)}\frac{\mathbb{E}_1(\ell)}{\mathbb{E}_1(\pi)} \equiv f_x\frac{\mathbb{E}_1(\ell)}{\ell(\psi_c/A)} > 0.$$

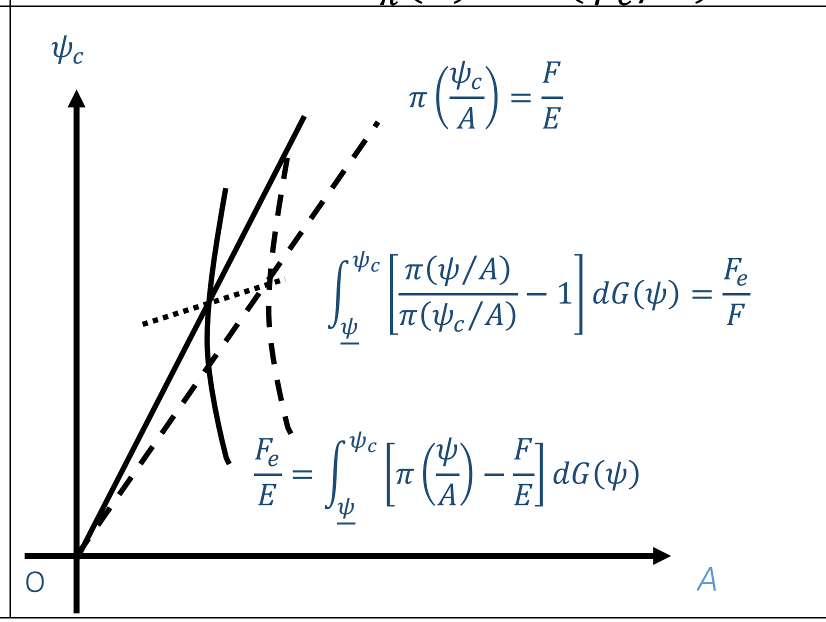
The profit/labor cost ratio of the cut-off firm to the average profit/average labor cost ratio among the active firms.

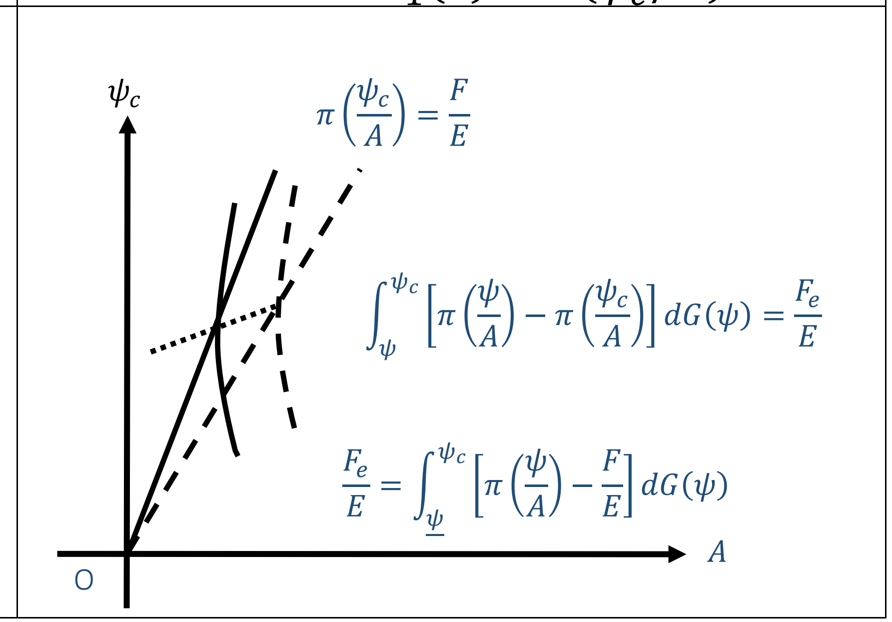
	A	ψ_c/A	ψ_c
F_e	$\frac{dA}{dF_e} > 0$	$\frac{d(\psi_c/A)}{dF_e} = 0$	$\frac{d\psi_c}{dF_e} > 0$
E	$\frac{dA}{dE} < 0$	$\frac{d(\psi_c/A)}{dE} > 0$	$\frac{d\psi_c}{dE} < 0 \Leftrightarrow \mathbb{E}_\pi(\sigma) < \sigma\left(\frac{\psi_c}{A}\right)$, which holds globally if $\sigma'(\cdot) > 0$.
F	$\frac{dA}{dF} > 0$	$\frac{d(\psi_c/A)}{dF} < 0$	$\frac{d\psi_c}{dF} > 0 \Leftrightarrow \mathbb{E}_1(\ell) < \ell\left(\frac{\psi_c}{A}\right)$, which holds globally if $\ell'(\cdot) > 0$

.

$F_e \downarrow$	$E \uparrow$ under $\mathbb{E}_\pi(\sigma) < \sigma(\psi_c/A)$	$F \downarrow$ under $\mathbb{E}_1(\ell) < \ell(\psi_c/A)$







Limit Case of $F \to 0$ with $\bar{z} < \infty$.

Cutoff Rule:	$\pi\left(\frac{\psi_c}{A}\right) = 0 \Leftrightarrow \frac{\psi_c}{A} = \bar{z} = \pi^{-1}(0)$
Free Entry Condition:	$\frac{F_e}{E} = \int_{\underline{\psi}}^{\psi_c} \pi\left(\bar{z}\frac{\psi}{\psi_c}\right) dG(\psi) = \int_{\underline{\psi}}^{\bar{z}A} \pi\left(\frac{\psi}{A}\right) dG(\psi).$

A and ψ_c: uniquely determined as functions of F_e/E with the interior solution, $0 < G(\psi_c) < 1$ for

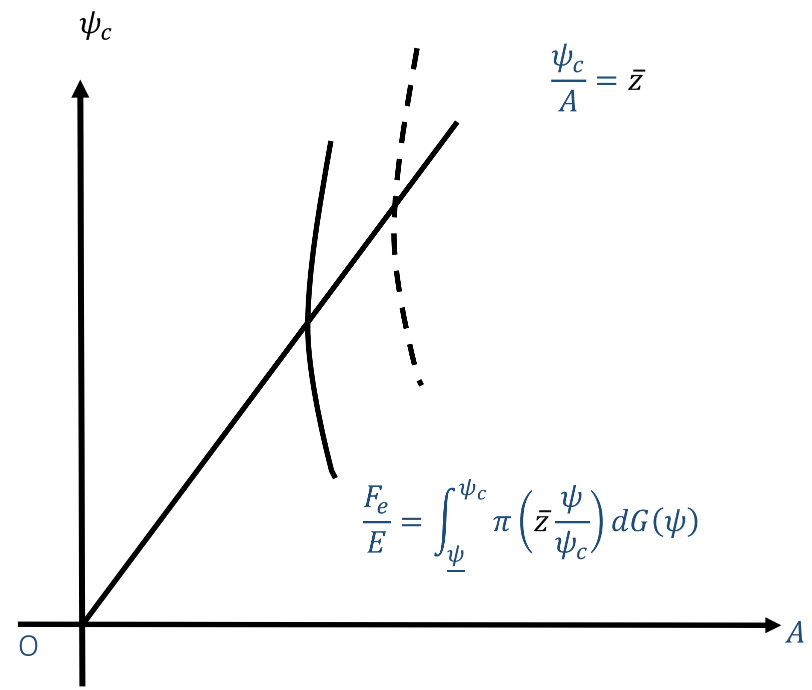
$$0 < \frac{F_e}{E} < \int_{\underline{\psi}}^{\bar{\psi}} \pi\left(\bar{z}\frac{\psi}{\bar{\psi}}\right) dG(\psi).$$

$$\frac{d\psi_c}{\psi_c} = \frac{dA}{A} = \frac{1}{\mathbb{E}_\pi(\sigma) - 1} \frac{d(F_e/E)}{F_e/E}$$

$$\frac{dM}{d(F_e/E)} < 0; \quad \mathcal{E}'_G(\psi) \lesseqgtr 0 \Leftrightarrow \frac{d[MG(\psi_c)]}{d(F_e/E)} \lesseqgtr 0$$

$E \uparrow$ is isomorphic to $F_e \downarrow$.

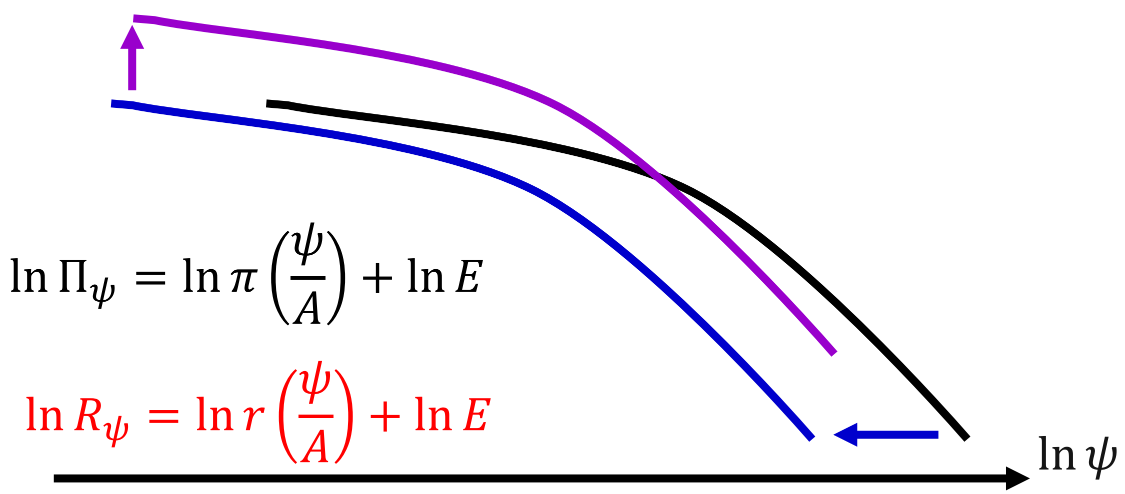
While simple and tractable, this prevents us from dealing with the effect of the overhead cost, F.



7.7. Comparative Statics: Market Size Effects on Firm Size Distributions in Profit and Revenue

7a	7b
7a: Under the 2[nd], there exists a unique $\psi_0 \in (\underline{\psi}, \psi_c)$ such that $\sigma(\psi_0/A) = \mathbb{E}_\pi(\sigma)$ with $$\frac{d\ln\Pi_\psi}{d\ln E} > 0 \Leftrightarrow \sigma\left(\frac{\psi}{A}\right) < \mathbb{E}_\pi(\sigma) \text{ for } \psi \in \left(\underline{\psi}, \psi_0\right),$$ and $$\frac{d\ln\Pi_\psi}{d\ln E} < 0 \Leftrightarrow \sigma\left(\frac{\psi}{A}\right) > \mathbb{E}_\pi(\sigma) \text{ for } \psi \in (\psi_0, \psi_c).$$	**7b:** Under the 2[nd] & **weak 3rd,** there exists $\psi_1 > \psi_0$, such that $$\frac{d\ln R_\psi}{d\ln E} > 0 \text{ for } \psi \in \left(\underline{\psi}, \psi_1\right).$$ Furthermore, $\psi_1 \in (\psi_0, \psi_c)$ and $$\frac{d\ln R_\psi}{d\ln E} < 0 \text{ for } \psi \in (\psi_1, \psi_c),$$ for a sufficiently small F.

In short, more productive firms expand in absolute terms, while less productive firms shrink.



Note:

$\psi_0 < \psi_c$ always hold since $\pi(\psi_c/A)E = F$ in equilibrium, so that the old and new cutoff firms earn the same profit. On the other hand, $\psi_1 > \psi_c$ can happen, since the new cutoff firms must earn higher revenue than the old cutoff firms, so it is possible that every surviving firm might earn higher revenue. This can be ruled out if F is small enough.

7.8. Comparative Statics: Average Markup and Pass-Through Rates (The Composition Effect)

- Under 2nd , $A\downarrow$ causes $\mu(\psi/A)\downarrow$ for each ψ, but distribution shifts toward low-ψ firms with higher $\mu(\psi/A)$.
- Under strong 3rd, $A\downarrow$ causes $\rho(\psi/A)\uparrow$ for each ψ, but distribution shifts toward low-ψ firms with lower $\rho(\psi/A)$.

Assume that $\mathcal{E}_g'(\cdot)$ does not change its sign and $\underline{\psi}=0$. Consider a shock to F_e, E, and/or F, which affects competitive pressures, i.e., $dA\neq 0$. Then, the response of any weighted generalized mean of any monotone function, $f(\psi/A)>0$, defined by

$$I\equiv \mathcal{M}^{-1}\left(\mathbb{E}_w\big(\mathcal{M}(f)\big)\right)$$

with a monotone transformation $\mathcal{M}:\mathbb{R}_+\to\mathbb{R}$ and a weighting function, $w(\psi/A)>0$, satisfies:

	$f'(\cdot)>0$	$f'(\cdot)=0$	$f'(\cdot)<0$
$\mathcal{E}_g'(\cdot)>0$	$\dfrac{d\ln(\psi_c/A)}{d\ln A}\geq 0\Rightarrow\dfrac{d\ln I}{d\ln A}>0$	$\dfrac{d\ln I}{d\ln A}=0$	$\dfrac{d\ln(\psi_c/A)}{d\ln A}\geq 0\Rightarrow\dfrac{d\ln I}{d\ln A}<0$
$\mathcal{E}_g'(\cdot)=0$ (Pareto)	$\dfrac{d\ln(\psi_c/A)}{d\ln A}\gtreqless 0\Leftrightarrow\dfrac{d\ln I}{d\ln A}\gtreqless 0$	$\dfrac{d\ln I}{d\ln A}=0$	$\dfrac{d\ln(\psi_c/A)}{d\ln A}\gtreqless 0\Leftrightarrow\dfrac{d\ln I}{d\ln A}\lesseqgtr 0$
$\mathcal{E}_g'(\cdot)<0$	$\dfrac{d\ln(\psi_c/A)}{d\ln A}\leq 0\Rightarrow\dfrac{d\ln I}{d\ln A}<0$	$\dfrac{d\ln I}{d\ln A}=0$	$\dfrac{d\ln(\psi_c/A)}{d\ln A}\leq 0\Rightarrow\dfrac{d\ln I}{d\ln A}>0$

Moreover, if $\mathcal{E}_g'(\cdot)=\frac{d\ln(\psi_c/A)}{d\ln A}=0$, $d\ln I/d\ln A=0$ for any $f(\psi/A)$, monotonic or not. Furthermore, $\mathcal{E}_g'(\cdot)$ can be replaced with $\mathcal{E}_G'(\cdot)$ in all the above statements for $w(\psi/A)=1$, i.e., the unweighted averages.

The arithmetic, $I=\big(\mathbb{E}_w(f)\big)$, geometric, $I=\exp[\mathbb{E}_w(\ln f)]$, harmonic, $I=\big(\mathbb{E}_w(f^{-1})\big)^{-1}$, means are special cases. The weight function, $w(\psi/A)$, can be profit, revenue, and employment.

Corollary

a) Entry Cost: $f'(\cdot)\mathcal{E}_g'(\cdot) \gtreqless 0 \Leftrightarrow \frac{d\ln I}{d\ln F_e} = \frac{d\ln I}{d\ln A}\frac{d\ln A}{d\ln F_e} \gtreqless 0.$

b) Market Size: *If* $\mathcal{E}_g'(\cdot) \leq 0$*, then,* $f'(\cdot) \gtreqless 0 \Longrightarrow \frac{d\ln I}{d\ln E} = \frac{d\ln I}{d\ln A}\frac{d\ln A}{d\ln E} \gtreqless 0.$

c) Overhead Cost: *If* $\mathcal{E}_g'(\cdot) \leq 0$*, then,* $f'(\cdot) \gtreqless 0 \Longrightarrow \frac{d\ln I}{d\ln F} = \frac{d\ln I}{d\ln A}\frac{d\ln A}{d\ln F} \lesseqgtr 0.$

Furthermore, $\mathcal{E}_g'(\cdot)$ *can be replaced with* $\mathcal{E}_G'(\cdot)$ *for* $w(\psi/A) = 1$*, i.e., the unweighted averages.*

For the entry cost, $\frac{d\ln(\psi_c/A)}{d\ln A} = 0.$

- $\mathcal{E}_g'(\cdot) > 0$; sufficient & necessary for the composition effect to dominate:
  - The average markup & pass-through rates move in the *opposite* direction from the firm-level rates
- $\mathcal{E}_g'(\cdot) = 0$ (Pareto); a knife-edge. $A \downarrow\rightarrow$ no change in average markup and pass-through.
- $\mathcal{E}_g'(\cdot) < 0$; sufficient & necessary for the procompetitive effect to dominate:
  The average markup & pass-through rates move in the *same* direction from the firm-level rates

For market size and the overhead cost, $\frac{d\ln(\psi_c/A)}{d\ln A} < 0$

- $\mathcal{E}_g'(\cdot) > 0$; necessary for the composition effect to dominate:
- $\mathcal{E}_g'(\cdot) \leq 0$; sufficient for the procompetitive effect to dominate:

7.9. Comparative Statics: Impact on P/A

$$\ln\left(\frac{A}{cP}\right) = \mathbb{E}_r[\Phi \circ Z];$$

$$\zeta'(\cdot) \gtreqless 0 \implies \Phi'(\cdot) \lesseqgtr 0 \iff \Phi \circ Z'(\cdot) \lesseqgtr 0.$$

Assume $\underline{\psi} = 0$, *and neither* $\zeta'(\cdot)$ *nor* $\mathcal{E}_g'(\cdot)$ *change the signs. Consider a shock to* F_e, E, *and/or* F, *which affects competitive pressures, i.e.,* $dA \neq 0$. *Then, the response of* P/A *satisfies:*

	$\zeta'(\cdot) > 0$ (A2)	$\zeta'(\cdot) = 0$ (CES)	$\zeta'(\cdot) < 0$
$\mathcal{E}_g'(\cdot) > 0$	$\frac{d\ln(\psi_c/A)}{d\ln A} \geq 0 \implies \frac{d\ln(P/A)}{d\ln A} > 0$	$\frac{d\ln(P/A)}{d\ln A} = 0$	$\frac{d\ln(\psi_c/A)}{d\ln A} \geq 0 \implies \frac{d\ln(P/A)}{d\ln A} < 0$
$\mathcal{E}_g'(\cdot) = 0$ (Pareto)	$\frac{d\ln(\psi_c/A)}{d\ln A} \gtreqless 0 \iff \frac{d\ln(P/A)}{d\ln A} \gtreqless 0$	$\frac{d\ln(P/A)}{d\ln A} = 0$	$\frac{d\ln(\psi_c/A)}{d\ln A} \gtreqless 0 \iff \frac{d\ln(P/A)}{d\ln A} \lesseqgtr 0$
$\mathcal{E}_g'(\cdot) < 0$	$\frac{d\ln(\psi_c/A)}{d\ln A} \leq 0 \implies \frac{d\ln(P/A)}{d\ln A} < 0$	$\frac{d\ln(P/A)}{d\ln A} = 0$	$\frac{d\ln(\psi_c/A)}{d\ln A} \leq 0 \implies \frac{d\ln(P/A)}{d\ln A} > 0$

7.10. Comparative Statics: Masses of Entrant M and Active Firms, $V = MG(\psi_c)$

Proposition 9: *Assume that* $\mathcal{E}_G'(\cdot)$ *does not change its sign and* $\underline{\psi} = 0$. *Consider a shock to* F_e, F, *and/or* E, *which affects competitive pressures, i.e.,* $dA \neq 0$. *Then, the response of the mass of active firms,* $V = MG(\psi_c)$, *is as follows*:

$$If\ \mathcal{E}_G'(\cdot) > 0, \qquad \frac{d\ln(\psi_c/A)}{d\ln A} \geq 0 \Rightarrow \frac{d\ln[MG(\psi_c)]}{d\ln A} > 0;$$

$$If\ \mathcal{E}_G'(\cdot) = 0, \qquad \frac{d\ln(\psi_c/A)}{d\ln A} \gtreqless 0 \Leftrightarrow \frac{d\ln[MG(\psi_c)]}{d\ln A} \gtreqless 0;$$

$$If\ \mathcal{E}_G'(\cdot) < 0, \qquad \frac{d\ln(\psi_c/A)}{d\ln A} \leq 0 \Rightarrow \frac{d\ln[MG(\psi_c)]}{d\ln A} < 0.$$

Corollary 1 of Proposition 9

a) Entry Cost: $\mathcal{E}_G'(\cdot) \gtreqless 0 \Leftrightarrow \frac{d\ln[MG(\psi_c)]}{d\ln F_e} = \frac{d\ln[MG(\psi_c)]}{d\ln A}\frac{d\ln A}{d\ln F_e} \gtreqless 0.$

b) Market Size: $\mathcal{E}_G'(\cdot) \leq 0 \Rightarrow \frac{d\ln[MG(\psi_c)]}{d\ln L} = \frac{d\ln[MG(\psi_c)]}{d\ln A}\frac{d\ln A}{d\ln E} > 0.$

c) Overhead Cost: $\mathcal{E}_G'(\cdot) \leq 0 \Rightarrow \frac{d\ln[MG(\psi_c)]}{d\ln F} = \frac{d\ln[MG(\psi_c)]}{d\ln A}\frac{d\ln A}{d\ln F} < 0.$

For a decline in the entry cost,
$\mathcal{E}_g'(\cdot) > 0$ *sufficient & necessary* for $MG(\psi_c) \downarrow$; $\mathcal{E}_g'(\cdot) = 0$, no effect; $\mathcal{E}_g'(\cdot) < 0$; *sufficient & necessary* for $MG(\psi_c) \uparrow$
For market size and the overhead cost
$\mathcal{E}_g'(\cdot) > 0$ *necessary* for $MG(\psi_c) \downarrow$; $\mathcal{E}_g'(\cdot) \leq 0$ *sufficient* for $MG(\psi_c) \uparrow$

Impact of Competitive Pressures on Unit Cost/TFP

By combining Corollary 2 of Proposition 8 and Corollary 1 of Proposition,

Corollary 2 of Proposition 9: *Assume* $\underline{\psi} = 0$*, and neither* $\zeta'(\cdot)$ *nor* $\mathcal{E}_g'(\cdot)$ *change the signs. Consider a shock to* F_e*,* E*, and/or* F*, which affects competitive pressures, i.e.,* $dA \neq 0$*. Then, the response of* P *satisfies:*

	$\zeta'(\cdot) > 0$ (A2)	$\zeta'(\cdot) = 0$ (CES)	$\zeta'(\cdot) < 0$
$\mathcal{E}_g'(\cdot) > 0$	$\frac{d \ln P}{d \ln A} > 1 \; for \; F_e$	$\frac{d \ln P}{d \ln A} = 1$	?
$\mathcal{E}_g'(\cdot) = 0$ (Pareto)	$\frac{d \ln P}{d \ln A} = 1 \; for \; F_e$ $0 < \frac{d \ln P}{d \ln A} < 1 \; for \; F \; or \; E;$	$\frac{d \ln P}{d \ln A} = 1$	$\frac{d \ln P}{d \ln A} = 1 \; for \; F_e$ $\frac{d \ln P}{d \ln A} > 1 \; for \; F \; or \; E$
$\mathcal{E}_g'(\cdot) < 0$	$0 < \frac{d \ln P}{d \ln A} < 1$	$\frac{d \ln P}{d \ln A} = 1$	$\frac{d \ln P}{d \ln A} > 1$

7.11. A Multi-Market Extension: Sorting of Heterogeneous Firms:

***J* markets,** $j = 1,2, \dots, J$, with market size L_j.

Possible Interpretations

- Identical Households with Cobb-Douglas preferences, $\sum_{j=1}^{J} \beta_j \ln X_j$ with $\sum_{j=1}^{J} \beta_j = 1$. Then, $L_j = \beta_j L$.
- J types of consumers, with L_j equal to the total income of type-j consumers. "Types" can be their "tastes" or "locations", etc.

Assume

- Market size is the only exogenous source of heterogeneity across markets: Index them as $L_1 > L_2 >, \dots, > L_J > 0$.
- Labor is fully mobile, equalizing the wage across the markets. We continue to use it as the *numeraire*.
- Firm's marginal cost, ψ, is independent of the market it chooses.
  - Each firm pays $F_e > 0$ to draw its marginal cost $\psi \sim G(\psi)$.
  - Knowing its ψ, each firm decides which market to enter and produce with an overhead cost, $F > 0$, or exit without producing.
  - Firms sell their products at the profit-maximizing prices in the market they enter.

Equilibrium Condition:

$$F_e = \int_{\underline{\psi}}^{\overline{\psi}} \max\{\Pi_\psi - F, 0\}\, dG(\psi) = \int_{\underline{\psi}}^{\overline{\psi}} \max\left\{\max_{1\le j\le J}\{\Pi_{j\psi}\} - F, 0\right\} dG(\psi),$$

where

$$\Pi_{j\psi} \equiv \frac{s\left(Z(\psi/A_j)\right)}{\zeta\left(Z(\psi/A_j)\right)} L_j \equiv \frac{r(\psi/A_j)}{\sigma(\psi/A_j)} L_j = \pi\left(\frac{\psi}{A_j}\right) L_j.$$

Equilibrium Characterization under A2

Larger markets are more competitive:

$$0 < A_1 < A_2 < \cdots < A_J < \infty, \text{where } M\int_{\psi_{j-1}}^{\psi_j} r\left(\frac{\psi}{A_j}\right) dG(\psi) = 1.$$

Note: Because $\pi(\cdot)$ is strictly decreasing, this implies $\pi(\psi/A_1) < \pi(\psi/A_2) < \cdots < \pi(\psi/A_J)$ for all ψ.

More productive firms self-select into larger markets (Positive Assortative Matching)

Firms with $\psi \in (\psi_{j-1}, \psi_j)$ enter market-j and those with $\psi \in (\psi_J, \infty)$ do not enter any market, where

$$0 \le \underline{\psi} = \psi_0 < \psi_1 < \psi_2 < \cdots < \psi_J < \overline{\psi} \le \infty \quad \text{where } \frac{\pi(\psi_j/A_j)L_j}{\pi(\psi_j/A_{j+1})L_{j+1}} = 1 \text{ for } 1 \le j \le J-1; \quad \pi\left(\frac{\psi_J}{A_J}\right)L_J \equiv F$$

Note: ψ_j-firms are indifferent btw entering Market-j & entering Market-$(j+1)$.

Free Entry Condition:

$$\sum_{j=1}^{J} \int_{\psi_{j-1}}^{\psi_j} \left\{\pi\left(\frac{\psi}{A_j}\right) L_j - F\right\} dG(\psi) = F_e$$

Mass of Firms in Market-*j*:

$$M[G(\psi_j) - G(\psi_{j-1})] > 0$$

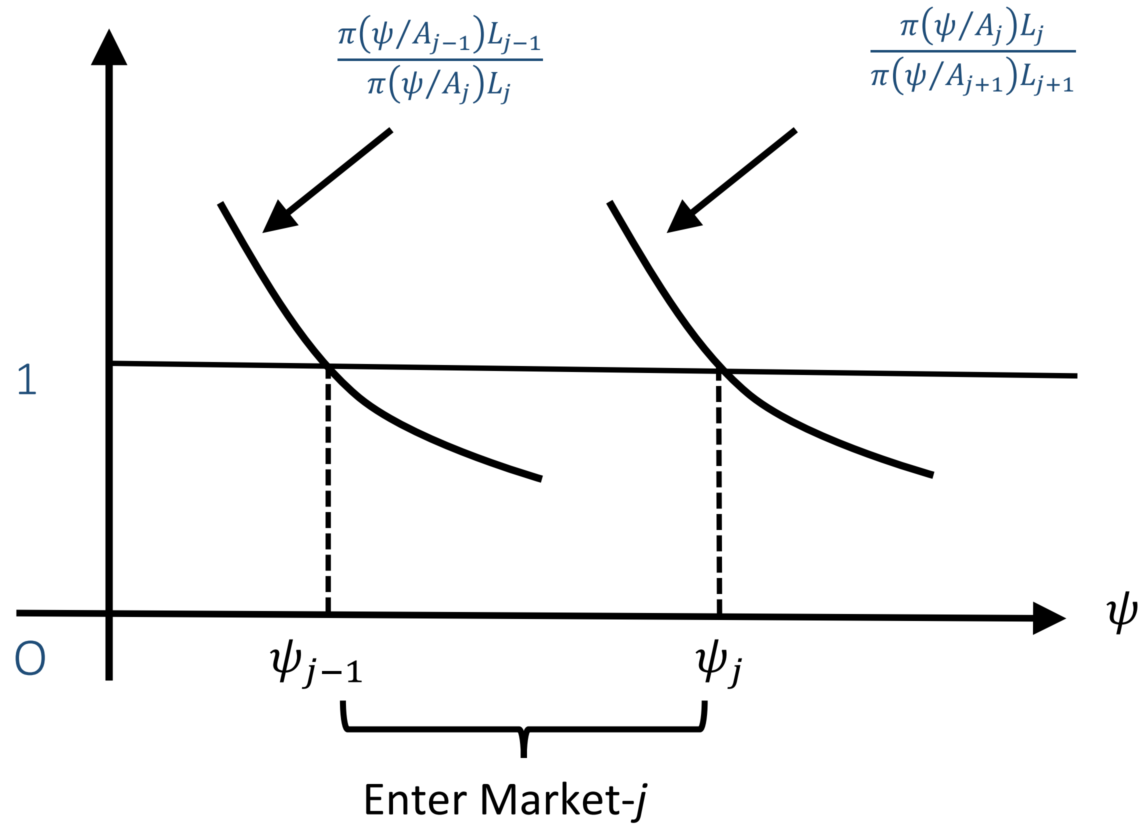
Logic Behind Sorting

$L_j > L_{j+1} \Longrightarrow A_j < A_{j+1}$. Otherwise, no firm would enter $j+1$.

$\Longrightarrow \frac{\pi(\psi/A_j)}{\pi(\psi/A_{j+1})}$ strictly decreasing in ψ

due to strict log-supermodularity of $\pi(\psi/A)$ under **A2**

$$\Longrightarrow \left[\frac{\Pi_{j\psi}}{\Pi_{(j+1)\psi}} = \frac{\pi(\psi/A_j)L_j}{\pi(\psi/A_{j+1})L_{j+1}} \gtreqless 1 \Leftrightarrow \psi \lesseqgtr \psi_j\right]$$



Under CES, $\frac{\pi(\psi/A_j)L_j}{\pi(\psi/A_{j+1})L_{j+1}}$ is independent of ψ.

$\Longrightarrow \frac{\Pi_{j\psi}}{\Pi_{(j+1)\psi}} = \frac{\pi(\psi/A_j)L_j}{\pi(\psi/A_{j+1})L_{j+1}} = 1$ in equilibrium.

$\Longrightarrow$ Firms indifferent across all markets.

$\Longrightarrow$ Distribution of firms across markets is indeterminate.

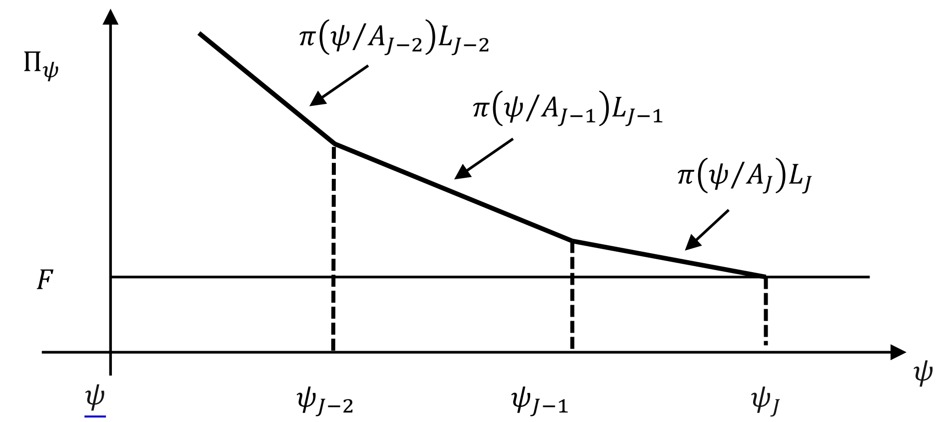
Our mechanism generates sorting through competitive pressures. As such,

- complementary to agglomeration-economies-based mechanisms offered by Gaubert (2018) and Davis-Dingel (2019)
- justifies the equilibrium selection criterion used by Baldwin-Okubo (2006), which use CES, as a limit argument.

Cross-Sectional, Cross-Market Implications:

Profits: Under A2

$$L_j > L_{j+1} \Longrightarrow A_j < A_{j+1} \Longrightarrow \left[\frac{\pi(\psi/A_j)L_j}{\pi(\psi/A_{j+1})L_{j+1}} \gtreqless 1 \Leftrightarrow \psi \lesseqgtr \psi_j\right]$$

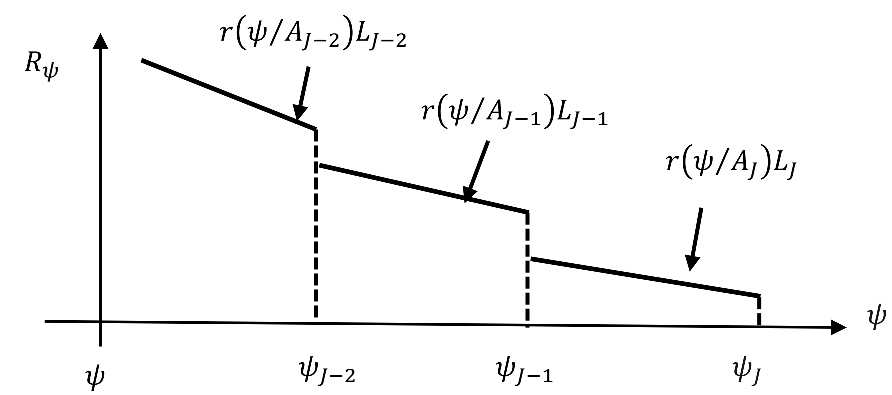


$\Pi_\psi = \max_j \left\{\pi\left(\frac{\psi}{A_j}\right)L_j\right\}$, the upper-envelope of $\pi(\psi/A_j)L_j$, is continuous and decreasing in ψ, with the kinks at ψ_j.

Continuous, since the lower markup rate in Market-j cancels out its larger market size, keeping ψ_j-firms indiffierent btw Market-j & Market-$(j+1)$.

Revenues: Under A2

$$\frac{r(\psi_j/A_j)L_j}{r(\psi_j/A_{j+1})L_{j+1}} = \frac{\sigma(\psi_j/A_j)\pi(\psi_j/A_j)L_j}{\sigma(\psi_j/A_{j+1})\pi(\psi_j/A_{j+1})L_{j+1}} = \frac{\sigma(\psi_j/A_j)}{\sigma(\psi_j/A_{j+1})} > 1$$

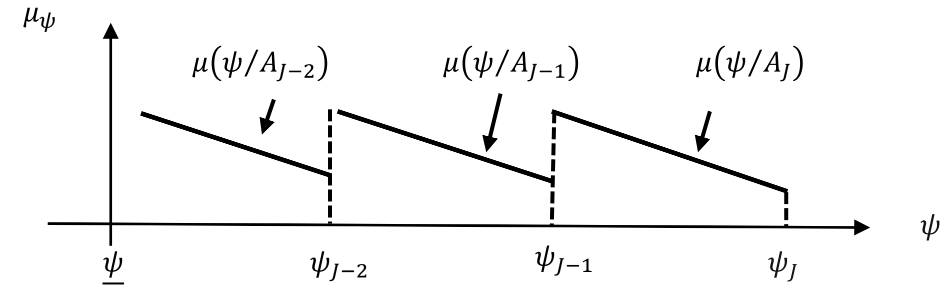


R_ψ: continuously decreasing in ψ within each market; jumps down at ψ_j.
With the markup rate lower in Market-j, ψ_j-firms need to earn higher revenue to keep them indiffierent btw Market-j & and Market-$(j+1)$.

Markup Rates: Under A2

$$L_j > L_{j+1} \Rightarrow A_j < A_{j+1} \Rightarrow \sigma\left(\frac{\psi_j}{A_j}\right) > \sigma\left(\frac{\psi_j}{A_{j+1}}\right) \Leftrightarrow \mu\left(\frac{\psi_j}{A_j}\right) < \mu\left(\frac{\psi_j}{A_{j+1}}\right)$$

μ_ψ: continuously decreasing in ψ within each market but jumps up at ψ_j.

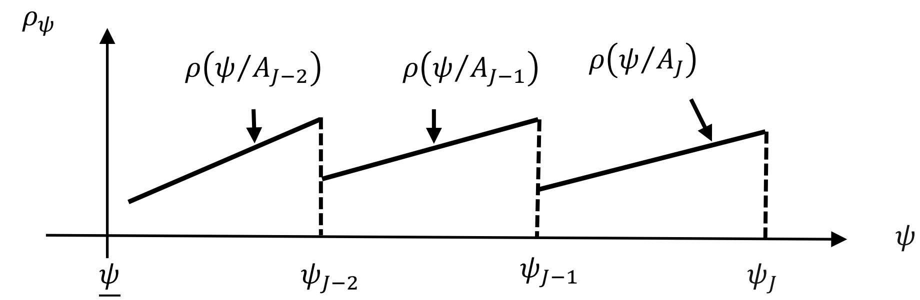


- The average markup rates may be *higher* in larger (and hence more competitive) markets.
- The average markup rates in all markets may *go up*, even if all markets become more competitive ($A_j \downarrow$).

Pass-Through Rates: Under A2 and the strong A3

$$L_j > L_{j+1} \Rightarrow A_j < A_{j+1} \Rightarrow \rho\left(\frac{\psi_j}{A_j}\right) > \rho\left(\frac{\psi_j}{A_{j+1}}\right)$$

ρ_ψ: continuously increasing in ψ within each market but jumps down at ψ_j.



- The average pass-through rates may be *lower* in larger (and hence more competitive) markets.
- The average pass-through rates in all markets go *down* even if all markets become more competitive ($A_j \downarrow$).

Average Markup and Pass-Through Rates in a Multi-Market Model: The Composition Effect

Proposition 11a: *Suppose A2 and* $G(\psi) = \left(\psi/\overline{\psi}\right)^{\kappa}$. *There exists a sequence,* $L_1 > L_2 > \cdots > L_J > 0$, *such that, in equilibrium, any weighted generalized mean of* $f\left(\psi/A_j\right)$ *across firms operating at market-j are increasing (decreasing) in* j *even though* $f(\cdot)$ *is increasing (decreasing) and hence* $f\left(\psi/A_j\right)$ *is decreasing (increasing) in* j.

Corollary of Proposition 11a: An example with $G(\psi) = \left(\psi/\overline{\psi}\right)^{\kappa}$, such that the average markup rates are *higher* (and the average pass-through rates are *lower* under Strong A3) in larger markets.

Proposition 11b: *Suppose A2 and* $G(\psi) = \left(\psi/\overline{\psi}\right)^{\kappa}$. *Then, a change in* F_e *keeps*

i) *the ratios* $a_j \equiv \psi_{j-1}/\psi_j$ *and* $b_j \equiv \psi_j/A_j$

and

ii) *any weighted generalized mean of* $f\left(\psi/A_j\right)$ *across firms operating at market-j, for any weighting function* $w\left(\psi/A_j\right)$,

unchanged for all $j = 1,2,\ldots,J$.

Corollary of Proposition 11b: $F_e \downarrow$ and $G(\psi) = \left(\psi/\overline{\psi}\right)^{\kappa}$ offers a knife-edge case, where the average markup and pass-through rates of all markets remain unchanged.

A caution against testing A2/A3 by comparing the average markup & pass-through rates across space and time.

7.12. International/Interregional Trade with Differential Market Access

- Two symmetric markets, characterized by market size E, and "labor" supply at the price equal to one, ensuring the same level of competitive pressures, A.

- After paying F_e, & learning ψ_ω, firm ω can produce its product at home & sell to both markets.
  - The overhead cost, $F > 0$ and the marginal cost of selling to the home market, ψ_ω.
  - The overhead cost, $F > 0$ and the marginal cost of selling to the export market, $\tau\psi_\omega > \psi_\omega$. **Iceberg cost, $\boldsymbol{\tau > 1}$.**

Cutoff Rules: Firm ω sells to both markets iff $\psi_\omega \leq \psi_{xc} < \psi_c$; only to the home market iff $\psi_{xc} < \psi_\omega \leq \psi_c$, where

$$F \equiv \pi\left(\frac{\psi_c}{A}\right)E \equiv \pi\left(\frac{\tau\psi_{xc}}{A}\right)E.$$

Free-Entry Condition:

$$F_e = \int_{\underline{\psi}}^{\psi_c}\left[\pi\left(\frac{\psi}{A}\right)E - F\right]dG(\psi) + \int_{\underline{\psi}}^{\psi_{xc}}\left[\pi\left(\frac{\tau\psi}{A}\right)E - F\right]dG(\psi).$$

These two conditions jointly pin down the equilibrium value of $\psi_c \equiv \tau\psi_{xc} \equiv \pi^{-1}(F/E)A$ by:

$$\frac{F_e}{E} = \int_{\underline{\psi}}^{\psi_c}\left[\pi\left(\frac{\psi}{\psi_c}\pi^{-1}\left(\frac{F}{E}\right)\right) - \frac{F}{E}\right]dG(\psi) + \int_{\underline{\psi}}^{\psi_c/\tau}\left[\pi\left(\frac{\tau\psi}{\psi_c}\pi^{-1}\left(\frac{F}{E}\right)\right) - \frac{F}{E}\right]dG(\psi).$$

After solving for $\psi_c \equiv \tau\psi_{xc} \equiv \pi^{-1}(F/E)A$, the mass of entering firms, M, and hence those of active firms $MG(\psi_c)$, and of exporting firms, $MG(\psi_{xc})$, are pinned down by:

Adding-Up (Resource) Constraint:

$$M\left[\int_{\underline{\psi}}^{\psi_c} r\left(\frac{\psi}{A}\right) dG(\psi) + \int_{\underline{\psi}}^{\psi_{xc}} r\left(\frac{\tau\psi}{A}\right) dG(\psi)\right] = 1.$$

Comparative Statics: The Effect of Globalization: A Reduction in $\tau > 1$.

- A decline in ψ_c and an increase in $\psi_{xc} = \psi_c/\tau$. → $G(\psi_c)$ falls, $G(\psi_{xc})$ rises, and $G(\psi_{xc})/G(\psi_c)$ rises.
- A decline in A and an increase in A/τ. →
  - $r(\psi_\omega/A)$ & $\pi(\psi_\omega/A)$ decline, $r(\tau\psi_\omega/A)$ & $\pi(\tau\psi_\omega/A)$ rise.
  - $\mu(\psi_\omega/A)$ declines and $\mu(\tau\psi_\omega/A)$ rises **under the 2nd law.**
  - $\rho(\psi_\omega/A)$ rises and $\rho(\tau\psi_\omega/A)$ declines **under the Strong 3rd law.**

Part 8: Other Forms of Firm Heterogeneity

8.1 Sticky Prices in New Keynesian (NK) Macro

Two Pricing Rules
Rotemberg (1982): Symmetric firms set the same price, but must pay the adjustment cost increasing the price change.
Calvo (1983): A fraction of firms randomly given the chances to reset their prices at each moment. Individual prices jump infrequently. The "average" price adjusts sluggishly. Firms heterogenous in their prices.

Most NK models: a fixed set of firms (no entry) and CES demand systems. An exogenous slope of the Phillips curve.
Fujiwara & Matsuyama (2022): allow entry and H.S.A, endogenous slope of the Phillips curve.
A higher entry cost → market concentration → a flattening of the Phillips curve for

- The 2nd law + Rotemberg
- The Strong 3rd law + Calvo. (Under translog + Calvo, a higher entry cost leads to a steeper Philips curve.)

FM also considered HDIA and HIIA, but a full GE analysis feasible only under H.S.A.

8.2 Technology Diffusion and Competitive Fringes

MC firms enjoy only the temporary monopoly. After the loss of monopoly power, they are forced priced competitively. Different MC firms set price differently. **Possibility of endogenous innovation cycles**

Judd (1985) and Matsuyama (1999) under CES:
The condition for the instability of the steady state and endogenous innovation cycles independent of market size.
Matsuyama and Ushchev (2022) under H.S.A.
A larger market size causes the instability of the steady state and endogenous innovation cycles under the 2nd law and procompetitive entry.
Matsuyama and Ushchev (2024) under HDIA. Similar result,but shown only numerically.

Appendix 1: H.S.A., HDIA and HIIA

3 Classes of (Symmetric) CES Production Functions with Gross Substitutes (and Inessentiality)

Homothetic Single Aggregator (H.S.A.): Two Equivalent Definitions

$s_\omega = \dfrac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s\left(\dfrac{p_\omega}{A(\mathbf{p})}\right)$ with $\int_\Omega s\left(\dfrac{p_\omega}{A(\mathbf{p})}\right) d\omega \equiv 1$ $s(z) > 0 > s'(z)$ for $0 < z < \bar{z} \le \infty$; $s(z) = 0$ for $z \ge \bar{z}$. $s(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+$, thus $A(\mathbf{p})$, independent of $Z > 0$, TFP.	$\Longleftrightarrow$	$s_\omega = \dfrac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} = s^*\left(\dfrac{x_\omega}{A^*(\mathbf{x})}\right)$ with $\int_\Omega s^*\left(\dfrac{x_\omega}{A^*(\mathbf{x})}\right) d\omega \equiv 1$ $s^*(0) = 0$, $s^*(y) > 0$, $0 < y {s^*}'(y)/s^*(y) < 1$. $s^*(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+$, thus $A^*(\mathbf{x})$, independent of $Z > 0$, TFP.

$Z > 0$, TFP, shows up when we integrate the definition of H.S.A. to obtain $P(\mathbf{p})$ or $X(\mathbf{x})$.

Homothetic Direct Implicit Additivity (HDIA): $X(\mathbf{x}) \equiv Z\hat{X}(\mathbf{x})$ defined by

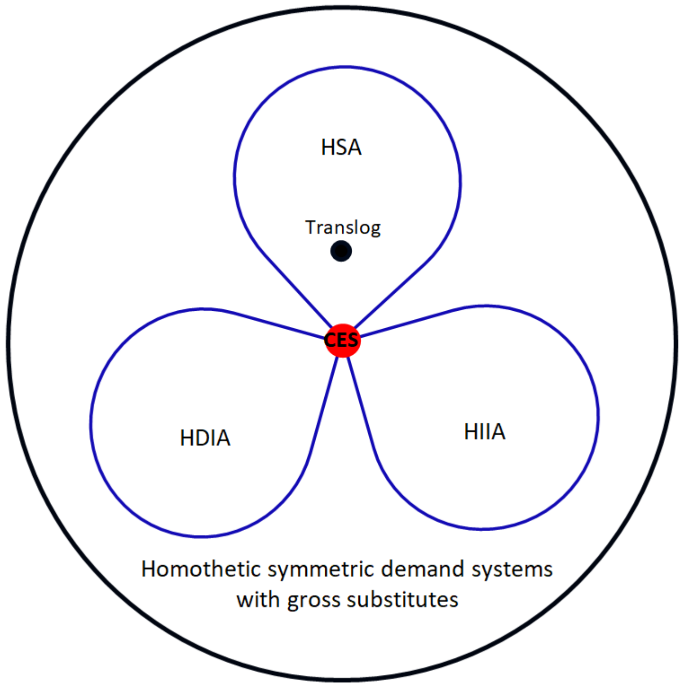
$$\mathcal{M}\left[\int_\Omega \phi\left(\frac{Z x_\omega}{X(\mathbf{x})}\right) d\omega\right] \equiv \mathcal{M}\left[\int_\Omega \phi\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right) d\omega\right] \equiv 1.$$

$\phi(0) = 0$; $\phi(\infty) = \infty$; $\phi'(y) > 0 > \phi''(y), -y\phi''(y)/\phi'(y) < 1$ for $0 < y < \infty$.
$\phi(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+$, thus $\hat{X}(\mathbf{x})$, is independent of $Z > 0$, TFP.

Homothetic Indirect Implicit Additivity (HIIA): $P(\mathbf{p}) \equiv \hat{P}(\mathbf{p})/Z$ defined by

$$\mathcal{M}\left[\int_\Omega \theta\left(\frac{p_\omega}{Z P(\mathbf{p})}\right) d\omega\right] \equiv \mathcal{M}\left[\int_\Omega \theta\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) d\omega\right] \equiv 1$$

$\theta(z) > 0, \theta'(z) < 0 < \theta''(z) > 0, -z\theta''(z)/\theta'(z) > 1$ for $0 < z < \bar{z} \le \infty$ &
$\theta(z) = 0$ for $z \ge \bar{z}$. $\theta(\cdot)\colon \mathbb{R}_{++} \to \mathbb{R}_+$, thus $\hat{P}(\mathbf{p})$, is independent of $Z > 0$, TFP.



The 3 classes are pairwise disjoint with the sole exception of CES.

Budget Shares: One Reason Why HDIA, HIIA, and H.S.A. Are Tractable.

CES	$s_\omega = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = f\left(\frac{p_\omega}{P(\mathbf{p})}\right) \Leftrightarrow s_\omega \propto \left(\frac{p_\omega}{P(\mathbf{p})}\right)^{1-\sigma}$; $s_\omega = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} = f^*\left(\frac{x_\omega}{X(\mathbf{x})}\right) \Leftrightarrow s_\omega \propto \left(\frac{x_\omega}{X(\mathbf{x})}\right)^{1-\frac{1}{\sigma}}$	
H.S.A.	$s_\omega = s\left(\frac{p_\omega}{A(\mathbf{p})}\right) = s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right)$	$\frac{P(\mathbf{p})}{A(\mathbf{p})} = \frac{A^*(\mathbf{x})}{X(\mathbf{x})} \neq const.$ unless CES
HDIA Kimball	$s_\omega = \frac{p_\omega}{\hat{P}(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) = \frac{x_\omega}{C^*(\mathbf{x})}\phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right)$	$\frac{\hat{P}(\mathbf{p})}{B(\mathbf{p})} = \frac{C^*(\mathbf{x})}{\hat{X}(\mathbf{x})} \neq const.,$ unless CES
HIIA	$s_\omega = \frac{p_\omega}{C(\mathbf{p})}\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) = \frac{x_\omega}{\hat{X}(\mathbf{x})}(\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)$	$\frac{C(\mathbf{p})}{\hat{P}(\mathbf{p})} = \frac{\hat{X}(\mathbf{x})}{B^*(\mathbf{x})} \neq const.,$ unless CES

$A(\mathbf{p}), B(\mathbf{p}), C(\mathbf{p})$, defined implicitly by the adding up constraint, $\int_\Omega s_\omega d\omega \equiv 1$. Clearly, they are all linear homogenous in $\mathbf{p}$;

$A^*(\mathbf{x}), B^*(\mathbf{x}), C^*(\mathbf{x})$, defined implicitly by the adding up constraint, $\int_\Omega s_\omega d\omega \equiv 1$. Clearly, they are all linear homogenous in $\mathbf{x}$.

Significant reduction in the dimensionality:

The cross-variety effect is captured in one aggregator (under H,S.A.) or in two aggregators (under HDIA or HIIA).

Price Elasticity: Another Reason Why HDIA, HIIA, and H.S.A. Are Tractable.

H.S.A.:	$\int_{\Omega} s_\omega\left(\frac{p_\omega}{A(\mathbf{p})}\right)d\omega \equiv 1$	→ **Price Elasticity**	$\zeta(z_\omega) \equiv 1 - \frac{z_\omega s'(z_\omega)}{s(z_\omega)} > 1$
	$\int_{\Omega} s_\omega^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right)d\omega \equiv 1$	→ **Price Elasticity**	$\zeta^*(y_\omega) \equiv \left[1 - \frac{y_\omega s^{*\prime}(y_\omega)}{s^*(y_\omega)}\right]^{-1} > 1$

is a function of a single variable, $z_\omega \equiv p_\omega/A(\mathbf{p})$, or $y_\omega \equiv x_\omega/A^*(\mathbf{x})$. Comparative statics hinge on its derivative.
CES if $\zeta(z_\omega) = \zeta^*(y_\omega) = const.$
The 2nd Law, if $\zeta'(z_\omega) > 0 \Leftrightarrow \zeta^{*\prime}(y_\omega) < 0$, which also informs us of the cross-variety effect.

HDIA: Kimball	$\mathcal{M}\left[\int_{\Omega} \phi\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right)d\omega\right] \equiv 1$	→ **Price Elasticity**	$\zeta^D(\mathscr{y}_\omega) \equiv -\frac{\phi'(\mathscr{y}_\omega)}{\mathscr{y}_\omega \phi''(\mathscr{y}_\omega)} > 1$

is a function of a single variable, $\mathscr{y}_\omega \equiv x_\omega/\hat{X}(\mathbf{x})$. Comparative statics hinge on its derivative.
CES if $\zeta^D(\mathscr{y}_\omega) = const.$ The 2nd Law if $\zeta^{D\prime}(\mathscr{y}_\omega) < 0$, which also informs us of the cross-variety effect.

HIIA:	$\mathcal{M}\left[\int_{\Omega} \theta\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)d\omega\right] \equiv 1$	→ **Price Elasticity**	$\zeta^I(\mathscr{z}_\omega) \equiv -\frac{\mathscr{z}_\omega \theta''(\mathscr{z}_\omega)}{\theta'(\mathscr{z}_\omega)} > 1$

is a function of a single variable, $\mathscr{z}_\omega \equiv p_\omega/\hat{P}(\mathbf{p})$, Comparative statics hinge on its derivative.
CES if $\zeta^I(\mathscr{z}_\omega) = const.$ The 2nd Law, if $\zeta^{I\prime}(\mathscr{z}_\omega) > 0$, which also informs us of the cross-variety effect.

Price Elasticity, Substitutability, & Love-for-Variety: Yet another Reason Why These 3 Classes Are Tractable.

H.S.A.	$\zeta_\omega = \zeta\left(\frac{p_\omega}{A(\mathbf{p})}\right)$	$\sigma(V) = \zeta\left(s^{-1}\left(\frac{1}{V}\right)\right)$	$\mathcal{L}(V) = \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right)$

where $\zeta(z) \equiv 1 - \frac{zs'(z)}{s(z)} > 1$ and $\Phi(z) \equiv \frac{1}{s(z)}\int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi > 0$.

HDIA	$\zeta_\omega = \zeta^D\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right)$	$\sigma(V) = \zeta^D\left(\phi^{-1}\left(\frac{1}{V}\right)\right)$	$\mathcal{L}(V) = \frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/V))} - 1$

where $\zeta^D(y) \equiv -\frac{\phi'(y)}{y\phi''(y)} > 1$ and $0 < \mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)} < 1$.

HIIA	$\zeta_\omega = \zeta^I\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)$	$\sigma(V) = \zeta^I\left(\theta^{-1}\left(\frac{1}{V}\right)\right)$	$\mathcal{L}(V) = \frac{1}{\mathcal{E}_\theta(\theta^{-1}(1/V))}$

where $\zeta^I(z) \equiv -\frac{z\theta''(z)}{\theta'(z)} > 1$ and $\mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} > 0$.

In each of these 3 classes, one could show

- $\sigma'(V) > 0$ if and only if the 2nd law of demand holds.
- $\sigma'(V) \gtreqless 0 \Rightarrow \mathcal{L}'(V) \lesseqgtr 0$. The converse is not true in general. But,
- $\mathcal{L}'(V) = 0 \Leftrightarrow \sigma'(V) = 0 \Leftrightarrow \zeta_\omega = const.$, which occurs iff CES.
- $\mathcal{L}'(V) \lesseqgtr 0 \Leftrightarrow V^{op} \lesseqgtr V^{eq}$.

In particular, The 2nd Law $\Leftrightarrow$ Increasing Substitutability (Procompetitive Entry) $\Rightarrow$ Diminishing Love-for Variety $\Leftrightarrow$ Excessive Entry

What is the relative advantage of the three classes?

H.S.A. has advantage over HDIA and HIIA, because

- Translog is a special case.

- **The revenue share functions, $s_\omega(\cdot)$ or $s^*_\omega(\cdot)$, are the primitive of H.S.A.** and hence it can be readily identified by typical firm level data, which has revenues but not output. See Kasahara-Sugita (2020).

- When applied to monopolistic competition with free-entry, it is easier under H.S.A. to ensure the existence and uniqueness of equilibrium, to characterize the equilibrium and to conduct comparative statics, because

  - For H.S.A., the interaction across products operates through only one aggregator in each sector.
    - An easy characterization of the free-entry equilibrium.
  - For HDIA and HIIA, the interaction across products operates through two aggregators in each sector, creating more room for the *multiplicity* and *non-existence* of equilibrium.

In short, H.S.A. is

- as tractable as translog, which is its special case;
- as flexible as the Kimball aggregator, which is HDIA without endogenous range of differentiated products.

Appendix 2: Some Parametric Families of H.S.A.

Three Parametric Families of H.S.A.: An Overview

Generalized Translog For $\eta > 0, \sigma > 1$	$s(z) = \gamma\left(-\frac{\sigma-1}{\eta}\ln\left(\frac{z}{\bar{z}}\right)\right)^{\eta}$; $z < \bar{z} \equiv \beta e^{\frac{\eta}{\sigma-1}}$	$1 - \frac{1}{\zeta(z)} = \frac{\eta}{\eta - \ln\left(\frac{z}{\bar{z}}\right)} \Rightarrow \begin{matrix}\mathcal{E}_\mu(\cdot) < 0\\ \mathcal{E}'_\mu(\cdot) = \rho'(\cdot) < 0\end{matrix}$ satisfying A2; violating A3.

Translog is the special case where $\eta = 1$. CES is the limit case, as $\eta \to \infty$, while holding $\beta > 0$ and $\sigma > 1$ fixed.

Constant Pass-Through (CoPaTh) For $0 < \rho < 1, \sigma > 1$	$s(z) = \gamma\sigma^{\frac{\rho}{1-\rho}}\left[1 - \left(\frac{z}{\bar{z}}\right)^{\frac{1-\rho}{\rho}}\right]^{\frac{\rho}{1-\rho}}$; $\bar{z} \equiv \beta\left(\frac{\sigma}{\sigma-1}\right)^{\frac{\rho}{1-\rho}}$	$1 - \frac{1}{\zeta(z)} = \left(\frac{z}{\bar{z}}\right)^{\frac{1-\rho}{\rho}} \Rightarrow \begin{matrix}\mathcal{E}_\mu(\cdot) < 0\\ \mathcal{E}'_\mu(\cdot) = \rho'(\cdot) = 0\end{matrix}$; satisfying A2 & weak A3; violating strong A3

CES is the limit case, as $\rho \to 1$, while holding $\beta > 0$ and $\sigma > 1$ fixed.

Power Elasticity of Markup Rate (PEM)/ Fréchet Inverse Markup Rate (FIM) For $\kappa \geq 0$ and $\lambda > 0$	$s(z) = \exp\left[\int_{z_0}^{z} \frac{c}{c - \exp\left[-\frac{\kappa\bar{z}^{-\lambda}}{\lambda}\right]\exp\left[\frac{\kappa\xi^{-\lambda}}{\lambda}\right]}\frac{d\xi}{\xi}\right]$	$1 - \frac{1}{\zeta(z)} = c\exp\left[\frac{\kappa\bar{z}^{-\lambda}}{\lambda}\right]\exp\left[-\frac{\kappa z^{-\lambda}}{\lambda}\right] \Rightarrow \mathcal{E}_\mu(\cdot) < 0; \mathcal{E}'_\mu(\cdot) = \rho'(\cdot) > 0$ satisfying A2 and strong A3 for $\kappa > 0$ and $\lambda > 0$.

CES for $\kappa = 0$; $\bar{z} = \infty$; $c = 1 - \frac{1}{\sigma}$; CoPaTh for $\bar{z} < \infty$; $c = 1$; $\kappa = \frac{1-\rho}{\rho} > 0$, and $\lambda \to 0$.

Generalized Translog:

$$s(z) = \gamma\left(1 - \frac{\sigma - 1}{\eta}\ln\left(\frac{z}{\beta}\right)\right)^{\eta} = \gamma\left(-\frac{\sigma - 1}{\eta}\ln\left(\frac{z}{\bar{z}}\right)\right)^{\eta};\ z < \bar{z} \equiv \beta e^{\frac{\eta}{\sigma-1}}$$

$$\Rightarrow \zeta(z) = 1 + \frac{\sigma - 1}{1 - \frac{\sigma - 1}{\eta}\ln\left(\frac{z}{\beta}\right)} = 1 - \frac{\eta}{\ln\left(\frac{z}{\bar{z}}\right)} > 1$$

$$\Rightarrow \eta z\zeta'(z) = [\zeta(z) - 1]^2 \Rightarrow \frac{z\zeta'(z)}{[\zeta(z) - 1]\zeta(z)} = \frac{1}{\eta}\left[1 - \frac{1}{\zeta(z)}\right] = \frac{1}{\eta - \ln\left(\frac{z}{\bar{z}}\right)}$$

satisfying the 2nd law but violating even the weak 3rd law.

- CES is the limit case, as $\eta \to \infty$, while holding $\beta > 0$ and $\sigma > 1$ fixed, so that $\bar{z} \equiv \beta e^{\frac{\eta}{\sigma-1}} \to \infty$.
- Translog is the special case where $\eta = 1$.
- $z = \tilde{Z}\left(\frac{\psi}{A}\right)$ is given as the inverse of $\frac{\eta z}{\eta - \ln(z/\bar{z})} = \frac{\psi}{A}$;
- If $\eta \geq 1$, employment is globally decreasing in z;
- If $\eta < 1$, employment is hump-shaped with the peak, given by $\hat{z}/\bar{z} = \frac{\hat{\psi}}{(1-\eta)\bar{z}A} = \exp\left[-\frac{\eta^2}{1-\eta}\right] < 1$, decreasing in η.

Constant Pass-Through (CoPaTh): Matsuyama-Ushchev (2020b). For $0 < \rho < 1$, $\sigma > 1$, $\bar{z} \equiv \beta\left(\frac{\sigma}{\sigma-1}\right)^{\frac{\rho}{1-\rho}}$

$$s(z) = \gamma\sigma^{\frac{\rho}{1-\rho}}\left[1-\left(\frac{z}{\bar{z}}\right)^{\frac{1-\rho}{\rho}}\right]^{\frac{\rho}{1-\rho}} \Rightarrow 1-\frac{1}{\zeta(z)} = \left(\frac{z}{\bar{z}}\right)^{\frac{1-\rho}{\rho}} < 1 \Rightarrow \mathcal{E}_{1-1/\zeta}(z) = -\mathcal{E}_{\zeta/(\zeta-1)}(z) = \frac{1-\rho}{\rho} > 0$$

satisfying the 2nd law and the weak (but not strong) 3rd law. Then, for $\psi/A < \bar{z}$,

$$p_\psi = (\bar{z}A)^{1-\rho}(\psi)^\rho; \qquad \tilde{Z}\left(\frac{\psi}{A}\right) = (\bar{z})^{1-\rho}\left(\frac{\psi}{A}\right)^\rho;$$

$$\sigma\left(\frac{\psi}{A}\right) = \frac{1}{1-(\psi/\bar{z}A)^{1-\rho}}; \qquad \rho\left(\frac{\psi}{A}\right) = \rho$$

$$r\left(\frac{\psi}{A}\right) = \gamma\sigma^{\frac{\rho}{1-\rho}}\left[1-\left(\frac{\psi}{\bar{z}A}\right)^{1-\rho}\right]^{\frac{\rho}{1-\rho}}; \ \pi\left(\frac{\psi}{A}\right) = \gamma\sigma^{\frac{\rho}{1-\rho}}\left[1-\left(\frac{\psi}{\bar{z}A}\right)^{1-\rho}\right]^{\frac{1}{1-\rho}}; \ \ell\left(\frac{\psi}{A}\right)$$

$$= \gamma\sigma^{\frac{\rho}{1-\rho}}\left(\frac{\psi}{\bar{z}A}\right)^{1-\rho}\left[1-\left(\frac{\psi}{\bar{z}A}\right)^{1-\rho}\right]^{\frac{\rho}{1-\rho}}$$

with

- a constant pass-through rate, $0 < \rho < 1$.
- Employment hump-shaped with $\hat{z}/\bar{z} = (1-\rho)^{\frac{\rho}{1-\rho}} > \hat{\psi}/\bar{z}A = (1-\rho)^{\frac{1}{1-\rho}}$, both decreasing in ρ.
- CES is the limit case, as $\rho \to 1$, while holding $\beta > 0$ and $\sigma > 1$ fixed, so that $\sigma(\psi/A) \to \sigma$; $\bar{z} \equiv \beta\left(\frac{\sigma}{\sigma-1}\right)^{\frac{\rho}{1-\rho}} \to \infty$.

Power Elasticity of Markup Rate (Fréchet Inverse Markup Rate): For $\kappa \geq 0$ and $\lambda > 0$.

$$s(z) = \exp\left[\int_{z_0}^{z} \frac{c}{c - \exp\left[-\frac{\kappa \bar{z}^{-\lambda}}{\lambda}\right] \exp\left[\frac{\kappa \xi^{-\lambda}}{\lambda}\right]} \frac{d\xi}{\xi}\right]$$

with either $\bar{z} = \infty$ and $c \leq 1$ or $\bar{z} < \infty$ and $c = 1$. Then,

$$1 - \frac{1}{\zeta(z)} = c \exp\left[\frac{\kappa \bar{z}^{-\lambda}}{\lambda}\right] \exp\left[-\frac{\kappa z^{-\lambda}}{\lambda}\right] < 1 \Rightarrow \mathcal{E}_{1-1/\zeta}(z) = -\mathcal{E}_{\zeta/(\zeta-1)}(z) = \kappa z^{-\lambda}$$

satisfying the 2nd law and the strong 3rd law for $\kappa > 0$ and $\lambda > 0$.

CES for $\kappa = 0$; $\bar{z} = \infty$; $c = 1 - \frac{1}{\sigma}$; CoPaTh for $\bar{z} < \infty$; $c = 1$; $\kappa = \frac{1-\rho}{\rho} > 0$, and $\lambda \to 0$.

- $\rho\left(\frac{\psi}{A}\right) = \frac{1}{1+\kappa(z_\psi)^{-\lambda}}$, with $z_\psi = \tilde{Z}\left(\frac{\psi}{A}\right)$ given implicitly by $c \exp\left[\frac{\kappa \bar{z}^{-\lambda}}{\lambda}\right] z_\psi \exp\left[-\frac{\kappa (z_\psi)^{-\lambda}}{\lambda}\right] \equiv \frac{\psi}{A}$,
- $\frac{\partial^2 \ln \rho(\psi/A)}{\partial A \partial \psi} \lesseqgtr 0 \Leftrightarrow (\kappa)^{\frac{1}{\lambda}} \gtreqless z_\psi = \tilde{Z}\left(\frac{\psi}{A}\right) \Leftrightarrow \frac{\psi}{A} \lesseqgtr (\kappa)^{\frac{1}{\lambda}} c \exp\left[\frac{\kappa \bar{z}^{-\lambda} - 1}{\lambda}\right]$; Log-sub(super)modular among more (less) efficient firms. In particular, if $\bar{z} < (\kappa)^{\frac{1}{\lambda}}$, $\frac{\partial^2 \ln \rho(\psi/A)}{\partial A \partial \psi} < 0$ for all $\psi/A < \tilde{Z}(\psi/A) < \bar{z} < \infty$.
- Employment hump-shaped with the peak at $\hat{z} = \tilde{Z}\left(\frac{\hat{\psi}}{A}\right) < \bar{z}$, given implicitly by

$$c\left(1 + \frac{\hat{z}^{\lambda}}{\kappa}\right) \exp\left[-\frac{\kappa \hat{z}^{-\lambda}}{\lambda}\right] \exp\left[\frac{\kappa \bar{z}^{-\lambda}}{\lambda}\right] = 1 \Leftrightarrow \left(1 + \frac{\hat{z}^{\lambda}}{\kappa}\right) \hat{z} = \frac{\hat{\psi}}{A}.$$