Love-for-Variety

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Introduction

Love-for-Variety: Productivity (utility) gains from increasing variety of intermediate inputs (consumer goods).

- A natural consequence of the convexity of the production technologies (preferences).
- Willingness to pay for new inputs (goods); Dixit-Stiglitz(1977), Krugman(1980), Ethier(1982), Romer (1987), etc.
- A central concept in economic growth (Grossman-Helpman 1993; Gancia-Zillibotti 2005, Acemoglu 2008), international trade (Helpman-Krugman 1095), and economic geography (Fujita-Krugman-Venables 1999).

But, little is known about how love-for-variety depends on the underlying production (or utility) function.

Under symmetric CES with gross substitutes: the analytical expression for love-for-variety is $1/(\sigma - 1) > 0$, where $\sigma > 1$ represents both:

✓ the (constant) elasticity of substitution across varieties

&

✓ the (constant) price elasticity of demand for each variety.

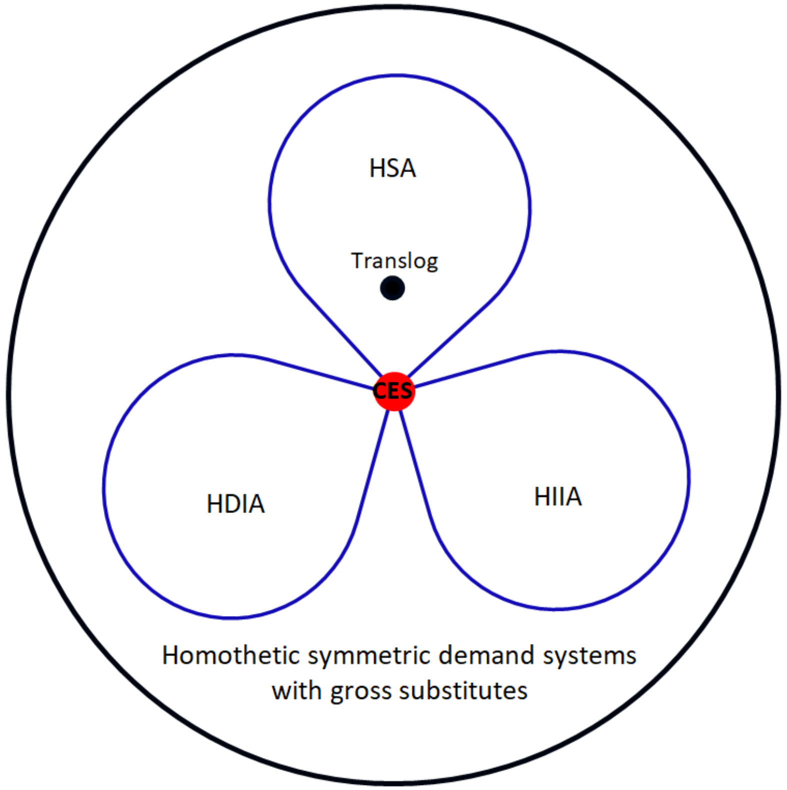
- **Appealing feature:** love-for-variety is smaller when different varieties are more substitutable and when the price elasticity of demand for each variety is higher (i.e., a larger σ).
- **Unappealing feature:** love-for-variety is independent of how many varieties are already available.

Questions:

- How does love-for-variety depend on the underlying demand structure?
- Under what conditions, should we expect love-for-variety to decline as more varieties become available?

For **general symmetric homothetic demand systems**, we define **Substitutability**, $\sigma(V)$, & **Love-for-Variety,** $\mathcal{L}(V)$.

- Both are functions of the mass of available varieties, V, only.
- We can say little about their relations without some additional restrictions.



We turn to the **3 classes of homothetic demand systems:**
H.S.A. (Homothetic Single Aggregator)
HDIA (Homothetic Direct Implicit Additivity)
HIIA (Homothetic Indirect Implicit Additivity)

- Pairwise disjoint with the sole exception of CES.
- Price elasticity is a function of a single variable.

$\zeta_\omega \equiv \zeta\left(\frac{p_\omega}{\mathcal{A}(\mathbf{p})}\right) \equiv \zeta^*\left(\frac{x_\omega}{\mathcal{A}^*(\mathbf{x})}\right)$, where $\mathcal{A}(\mathbf{p})$ or $\mathcal{A}^*(\mathbf{x})$ is linear homogeneous, a sufficient statistic that captures the cross-variety effects.

Main Results: In each of these 3 classes,

- The substitutability is increasing in V, if and only if Marshall's 2nd law of demand holds.
- Increasing (decreasing) substitutability implies diminishing (increasing) love-for-variety. The converse is not true.
- Constant love-for-variety, constant substitutability and constant price elasticity are all equivalent and occur iff CES.

The 3 classes offer a tractable way of capturing the intuition that **gains from increasing variety is diminishing, if different varieties are more substitutable when more varieties are available.**

General Symmetric Homothetic Demand Systems

General Symmetric Homothetic (Input) Demand System: A Quick Refresher of Duality Theory

Consider homothetic demand systems for differentiated inputs generated by symmetric CRS production technology.

CRS Production Function	Unit Cost Function
$X(\mathbf{x}) \equiv \min_{\mathbf{p}}\{\mathbf{px} \mid P(\mathbf{p}) \geq 1\}$	$P(\mathbf{p}) \equiv \min_{\mathbf{x}}\{\mathbf{px} \mid X(\mathbf{x}) \geq 1\}$

$\mathbf{x} = \{x_\omega; \omega \in \bar{\Omega}\}$**:** the input quantity vector; $\mathbf{p} = \{p_\omega; \omega \in \bar{\Omega}\}$**:** the input price vector.

$\bar{\Omega}$, a continuum of all potential input varieties.

$\Omega \subset \bar{\Omega}$, the set of available input varieties, with its mass denoted by $V \equiv |\Omega|$.

$\bar{\Omega}\backslash\Omega$: the set of unavailable varieties, $x_\omega = 0$ and $p_\omega = \infty$ for $\omega \in \bar{\Omega}\backslash\Omega$.

Either $X(\mathbf{x})$ or $P(\mathbf{p})$ can be a *primitive*, as long as they are linear homogeneous, monotonic & strict quasi-concave.
To study the effect of $V \equiv |\Omega|$, we assume inputs are *inessential*, i.e., $\bar{\Omega}\backslash\Omega \neq \emptyset$ does*n't* imply $X(\mathbf{x}) = 0 \Leftrightarrow P(\mathbf{p}) = \infty$.

Inverse Demand Curve	Demand Curve
$p_\omega = P(\mathbf{p})\dfrac{\partial X(\mathbf{x})}{\partial x_\omega}$	$x_\omega = \dfrac{\partial P(\mathbf{p})}{\partial p_\omega}X(\mathbf{x})$

From Euler's Homogenous Function Theorem,

$$\mathbf{px} \equiv \int_\Omega p_\omega x_\omega d\omega = P(\mathbf{p})X(\mathbf{x})$$

Budget Share of $\omega \in \Omega$:	$s_\omega \equiv \dfrac{p_\omega x_\omega}{\mathbf{px}} = \dfrac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \dfrac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} \equiv s(x_\omega, \mathbf{x}) = \dfrac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} \equiv s(p_\omega, \mathbf{p})$

Defining the Love-for-Variety Measure

Unit Quantity Vector: $\mathbf{1}_\Omega \equiv \{(1_\Omega)_\omega; \omega \in \bar{\Omega}\}$, where $(1_\Omega)_\omega \equiv \begin{cases} 1 & \text{for} \quad \omega \in \Omega \\ 0 & \text{for} \quad \omega \in \bar{\Omega}\backslash\Omega \end{cases}$

Unit Price Vector: $\mathbf{1}_\Omega^{-1} \equiv \left\{\left(1_\Omega^{-1}\right)_\omega; \omega \in \bar{\Omega}\right\}$, where $\left(1_\Omega^{-1}\right)_\omega \equiv \begin{cases} 1 & \text{for} \quad \omega \in \Omega \\ \infty & \text{for} \quad \omega \in \bar{\Omega}\backslash\Omega \end{cases}$

Note: $\int_\Omega (1_\Omega)_\omega d\omega = \int_\Omega \left(1_\Omega^{-1}\right)_\omega d\omega = |\Omega| \equiv V$.

Both $X(\mathbf{1}_\Omega)$ and $P\left(\mathbf{1}_\Omega^{-1}\right)$ depend only on V. Hence, at the symmetric patterns, $\mathbf{x} = x\mathbf{1}_\Omega$ and $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$,

$$X(\mathbf{x}) = xX(\mathbf{1}_\Omega) = \frac{x}{\mathcal{Y}(V)} \equiv \frac{xV}{V\mathcal{Y}(V)}; \qquad \frac{d\ln \mathcal{Y}(V)}{d\ln V} + 1 < 0.$$

$$P(\mathbf{p}) = pP\left(\mathbf{1}_\Omega^{-1}\right) \equiv \frac{p}{\mathcal{Z}(V)}; \qquad \frac{d\ln \mathcal{Z}(V)}{d\ln V} > 0.$$

Moreover,

$$\mathbf{px} = P(\mathbf{p})X(\mathbf{x}) \Longrightarrow pxV = \frac{p}{\mathcal{Z}(V)}\frac{x}{\mathcal{Y}(V)} \Longrightarrow \frac{d\ln \mathcal{Z}(V)}{d\ln V} = -\frac{d\ln \mathcal{Y}(V)}{d\ln V} - 1 > 0.$$

Definition. *The love-for-variety measure* is defined by:

$$\mathcal{L}(V) \equiv \frac{d\ln \mathcal{Z}(V)}{d\ln V} = -\frac{d\ln \mathcal{Y}(V)}{d\ln V} - 1 > 0.$$

Price Elasticity of Demand for Each Variety and Marshall's 2nd Law

Price Elasticity of Demand for ω	$\zeta_\omega \equiv -\dfrac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega;\mathbf{p}) \equiv 1 - \dfrac{\partial \ln s(p_\omega;\mathbf{p})}{\partial \ln p_\omega} = \zeta^*(x_\omega;\mathbf{x}) \equiv \left[1 - \dfrac{\partial \ln s^*(x_\omega;\ \mathbf{x})}{\partial \ln x_\omega}\right]^{-1} > 1.$

Definition: Marshall's 2nd Law holds if

$$\frac{\partial \ln \zeta(p_\omega;\mathbf{p})}{\partial \ln p_\omega} > 0 \Leftrightarrow \frac{\partial \ln \zeta^*(x_\omega;\mathbf{x})}{\partial \ln x_\omega} < 0.$$

Defining the Measure of Substitutability Across Different Varieties

Because $\zeta(p_\omega;\mathbf{p}) = \zeta^*(x_\omega;\mathbf{x})$ is homogenous of degree zero in $\mathbf{p}$ and $\mathbf{x}$,

$$\zeta\left(1;\mathbf{1}_\Omega^{-1}\right) = \zeta\left(p;\ p\mathbf{1}_\Omega^{-1}\right) = \zeta^*(x;x\mathbf{1}_\Omega) = \zeta^*(1;\mathbf{1}_\Omega) \equiv \sigma(V).$$

Appendix A: $\sigma(V)$ is the Allen-Uzawa elasticity of substitution btw every pair of inputs at $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$ or $\mathbf{x} = x\mathbf{1}_\Omega$.

Definition: *The substitutability measure* is defined by

$$\sigma(V) \equiv \zeta\left(1;\mathbf{1}_\Omega^{-1}\right) = \zeta^*(1;\mathbf{1}_\Omega) > 1.$$

Example: Standard CES with Gross Substitutes:

$$X(\mathbf{x}) = Z\left[\int_{\Omega} x_{\omega}{}^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \quad \Leftrightarrow \quad P(\mathbf{p}) = \frac{1}{Z}\left[\int_{\Omega} p_{\omega}{}^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}},$$

where $\sigma > 1$ is the (constant) elasticity of substitution parameter and Z is the TFP parameter under Standard CES.

	Definition	**Under CES**
Price Elasticity	$\zeta_{\omega} \equiv -\frac{\partial \ln x_{\omega}}{\partial \ln p_{\omega}} = \zeta(p_{\omega}; \mathbf{p}) = \zeta^{*}(x_{\omega}; \mathbf{x})$	$\zeta(p_{\omega}; \mathbf{p}) = \zeta^{*}(x_{\omega}; \mathbf{x}) = \sigma > 1$
Substitutability	$\sigma(V) \equiv \zeta\left(1; \mathbf{1}_{\Omega}^{-1}\right) = \zeta^{*}(1; \mathbf{1}_{\Omega})$	$\sigma(V) = \sigma > 1$
Love-for-variety	$\mathcal{L}(V) \equiv \frac{d \ln z(V)}{d \ln V} = -\frac{d \ln y(V)}{d \ln V} - 1 > 0.$	$\mathcal{L}(V) = \frac{1}{\sigma - 1} > 0.$

Under Standard CES,

- Price elasticity of demand, $\zeta(p_{\omega}; \mathbf{p}) = \zeta^{*}(x_{\omega}; \mathbf{x})$, is independent of $\mathbf{p}$ or $\mathbf{x}$ and equal to σ.
- Substitutability, $\sigma(V)$, is independent of V and equal to σ.
- Love-for-variety, $\mathcal{L}(V)$, is also independent of V, and equal to a constant that is inversely related to σ.

Example: Generalized CES with Gross Substitutes a la Benassy (1996).

$$X(\mathbf{x}) = Z(V)\left[\int_\Omega {x_\omega}^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \Leftrightarrow \quad P(\mathbf{p}) = \frac{1}{Z(V)}\left[\int_\Omega {p_\omega}^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}},$$

Note: $Z(V)$ allows variety to have direct externalities to TFP.

	Definition	**Under CES**
Price Elasticity	$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$	$\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x}) = \sigma > 1$
Substitutability	$\sigma(V) \equiv \zeta\left(1; \mathbf{1}_\Omega^{-1}\right) = \zeta^*(1; \mathbf{1}_\Omega)$	$\sigma(V) = \sigma > 1$
Love-for-variety	$\mathcal{L}(V) \equiv \frac{d \ln \mathcal{z}(V)}{d \ln V} = -\frac{d \ln \mathcal{y}(V)}{d \ln V} - 1 > 0.$	$\mathcal{L}(V) = \frac{1}{\sigma - 1} + \frac{d \ln Z(V)}{d \ln V}.$

Under Generalized CES,

- Price elasticity of demand, $\zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$, is independent of $\mathbf{p}$ or $\mathbf{x}$ and equal to σ.
- Substitutability, $\sigma(V)$, is independent of V and equal to σ.
- Benassy (1996) assumed $\frac{d \ln Z(V)}{d \ln V} = \nu - \frac{1}{\sigma-1}$. Then, $\mathcal{L}(V) = \nu$ is a separate parameter independent of σ.
- If we instead assume $\frac{d \ln Z(V)}{d \ln V}$ is independent of σ, $\mathcal{L}(V)$ is still inversely related to σ.

General Homothetic DS: The relation btw $\zeta(p_\omega;\mathbf{p}) = \zeta^*(x_\omega;\mathbf{x})$, $\sigma(V)$, & $\mathcal{L}(V)$ can be complex.

- Whether Marshall's 2nd Law holds or not says little about the derivatives of $\sigma(V)$ and $\mathcal{L}(V)$.
- $\sigma(V)$ and $\mathcal{L}(V)$ could be positively related.

(Counter)Example: Weighted Geometric Mean of Standard CES with Gross Substitutes:

$$X(\mathbf{x}) \equiv \exp\left[\int_1^\infty \ln X(\mathbf{x};\sigma)\, dF(\sigma)\right], \qquad \text{where} \qquad [X(\mathbf{x};\sigma)]^{1-\frac{1}{\sigma}} \equiv \int_\Omega {x_\omega}^{1-\frac{1}{\sigma}} d\omega$$

and $F(\cdot)$ is a c.d.f. of $\sigma \in (1,\infty)$, satisfying $\int_1^\infty dF(\sigma) = 1$.

	Definition	Under Geometric Mean of CES
Price Elasticity	$\zeta_\omega \equiv -\dfrac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta^*(x_\omega;\mathbf{x})$	$\zeta^*(x_\omega;\mathbf{x}) = E_F\left((x_\omega)^{-\frac{1}{\sigma}}/(X(\mathbf{x};\sigma))^{1-\frac{1}{\sigma}}\right)\Big/E_F\left((x_\omega)^{-\frac{1}{\sigma}}/\sigma(X(\mathbf{x};\sigma))^{1-\frac{1}{\sigma}}\right) > 1$
Substitutability	$\sigma(V) \equiv \zeta^*(1;\mathbf{1}_\Omega)$	$\sigma(V) = \dfrac{1}{E_F(1/\sigma)} > 1$
Love-for-variety	$\mathcal{L}(V) \equiv -\dfrac{d\ln \mathcal{Y}(V)}{d\ln V} - 1 > 0$	$\mathcal{L}(V) = E_F\left(\dfrac{1}{\sigma-1}\right) > 0$

- Price elasticity of demand, $\zeta^*(x_\omega;\mathbf{x})$, is not constant, and *violates* the Marshall's 2nd Law.
- Both $\sigma(V)$ and $\mathcal{L}(V)$ are *independent* of V.
- The range of $\sigma(V)$ and $\mathcal{L}(V)$ is given by $0 < \frac{1}{\sigma(V)-1} \leq \mathcal{L}(V) < \infty$, where the equality holds iff F is degenerate.
- Easy to construct a parametric family of F, such that $\sigma(V)$ and $\mathcal{L}(V)$.

However, it is intuitive to think that, as input varieties are more substitutable,

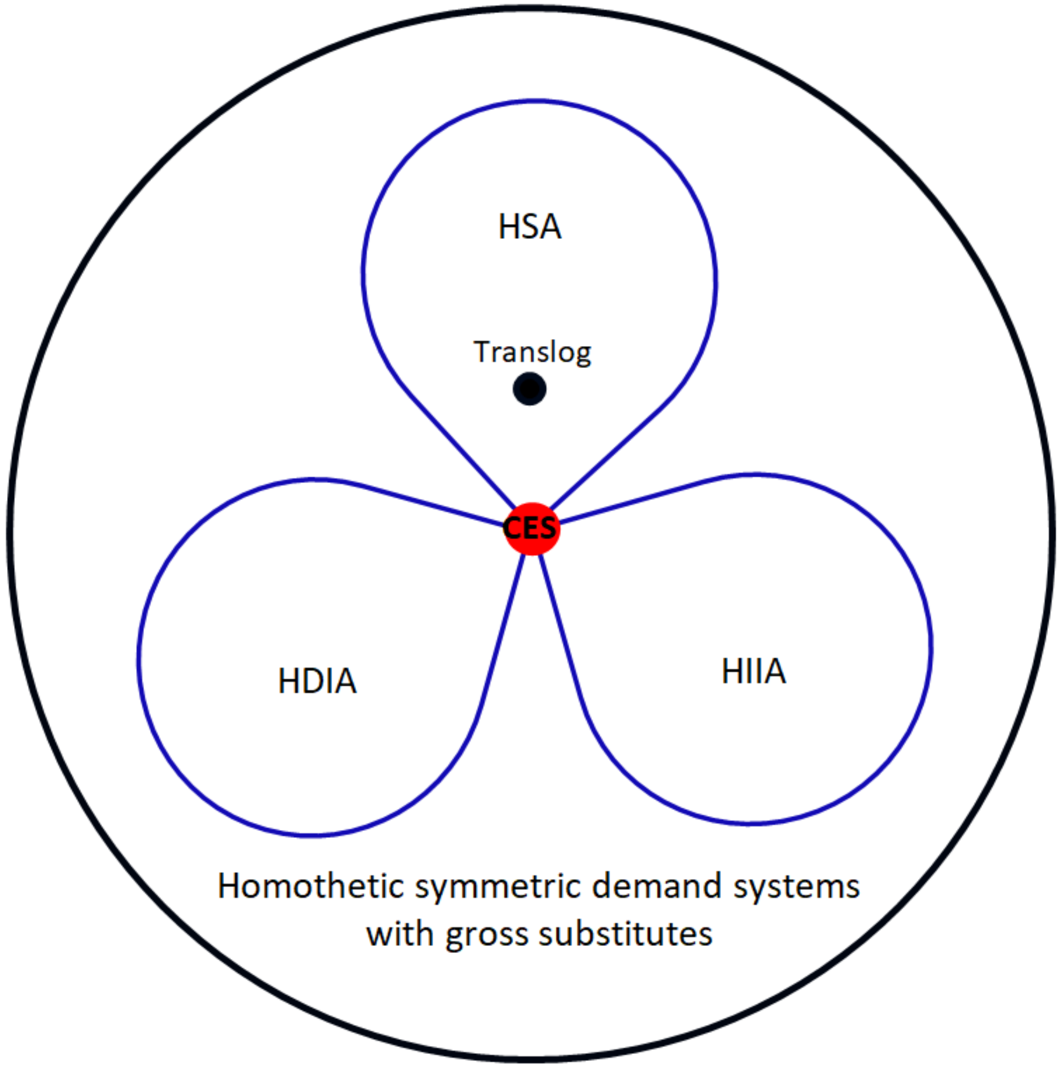
- the price elasticity of demand for each variety become larger,
- the love-for-variety measure become smaller.

Homotheticity alone cannot capture this intuition!!

In search for additional restrictions to capture this intuition, we turn to

Three Classes of Symmetric CRS Production Functions:

✓ **Homothetic Single Aggregator (H.S.A.)**
✓ **Homothetic Direct Implicit Additivity (HDIA)**
✓ **Homothetic Indirect Implicit Additivity (HIIA)**



3 Classes of Symmetric CRS Production Functions with Gross Substitutes (and Inessentiality)

Homothetic Single Aggregator (H.S.A.): Two Equivalent Definitions

$$s_\omega = s\left(\frac{p_\omega}{A(\mathbf{p})}\right) \quad \text{with} \quad \int_\Omega s\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega \equiv 1 \quad \Leftrightarrow \quad s_\omega = s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) \quad \text{with} \quad \int_\Omega s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega \equiv 1$$

$s(z) > 0$, $s'(z) < 0$ for $0 < z < \bar{z} \leq \infty$; $s(z) = 0$ for $z \geq \bar{z}$ $\qquad$ $s^*(0) = 0,\ s^*(y) > 0,\ 0 < ys^{*\prime}(y)/s^*(y) < 1$

Homothetic Direct Implicit Additivity (HDIA):

$$\int_\Omega \phi\left(\frac{Zx_\omega}{X(\mathbf{x})}\right) d\omega \equiv 1$$

$\phi(0) = 0; \phi(\infty) = \infty; \phi'(y) > 0, \phi''(y) < 0, -y\phi''(y)/\phi'(y) < 1$ for $0 < y < \infty$. $Z > 0$ is TFP.

Homothetic Indirect Implicit Additivity (HIIA):

$$\int_\Omega \theta\left(\frac{p_\omega}{ZP(\mathbf{p})}\right) d\omega \equiv 1$$

$\theta(z) > 0, \theta'(z) < 0, \theta''(z) > 0, -z\theta''(z)/\theta'(z) > 1$ for $0 < z < \bar{z} \leq \infty$ & $\theta(z) = 0$ for $z \geq \bar{z}$. $Z > 0$ is TFP.

We focus on these three classes for two reasons.

- They are pairwise disjoint with the sole exception of CES.
- Price elasticity is a function of a single variable of the form, $\zeta_\omega \equiv \zeta\left(\frac{p_\omega}{\mathcal{A}(\mathbf{p})}\right) \equiv \zeta^*\left(\frac{x_\omega}{\mathcal{A}^*(\mathbf{x})}\right)$, where $\mathcal{A}(\mathbf{p})$ or $\mathcal{A}^*(\mathbf{x})$ is a linear homogeneous aggregator of $\mathbf{p}$ or of $\mathbf{x}$, a sufficient statistic to capture the interdependence across varieties.

Homothetic Single Aggregator (H.S.A.)

Symmetric H.S.A. (Homothetic Single Aggregator) DS with Gross Substitutes

Definition: A symmetric CRS technology, $P = P(\mathbf{p})$ is called *homothetic single aggregator* (H.S.A.) if the budget share of ω depends solely on a single variable, $z_\omega \equiv p_\omega / A,$ its own price p_ω, normalized by the common price aggregator, $A = A(\mathbf{p})$.

$$s_\omega \equiv \frac{p_\omega x_\omega}{\mathbf{px}} = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(\mathbf{p})}\right), \qquad \text{where} \qquad \int_\Omega s\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega \equiv 1.$$

- $s: \mathbb{R}_{++} \to \mathbb{R}_+$: **the budget share function**, decreasing in the normalized price, $z_\omega \equiv p_\omega / A$ for $s(z_\omega) > 0$ with
  - $\lim_{z \to \bar{z}} s(z) = 0$. If $\bar{z} \equiv \inf\{z > 0 | s(z) = 0\} < \infty$, $\bar{z}A(\mathbf{p})$ is **the choke price.**
- $A = A(\mathbf{p})$: **the common price aggregator**, defined implicitly by **the adding-up constraint**, $\int_\Omega s(p_\omega / A) d\omega \equiv 1$.

By construction, the budget shares add up to one. $A(\mathbf{p})$ linear homogenous in $\mathbf{p}$ for a fixed Ω. A larger Ω reduces $A(\mathbf{p})$.

Some Special Cases

CES with gross substitutes	$s(z) = \gamma z^{1-\sigma};$	$\sigma > 1$
Translog Cost Function	$s(z) = \gamma \max\{-\ln(z/\bar{z}), 0\};$	$\bar{z} < \infty$
Constant Pass Through (CoPaTh)	$s(z) = \gamma \max\left\{\left[\sigma - (\sigma - 1) z^{\frac{1-\rho}{\rho}}\right]^{\frac{\rho}{1-\rho}}, 0\right\}$	$\sigma > 1;\ 0 < \rho < 1$

As $\rho \nearrow 1$, CoPaTh converges to CES with $\bar{z} = \left(\frac{\sigma}{\sigma - 1}\right)^{\frac{\rho}{1-\rho}} \to \infty$.

Price Elasticity:	$$\zeta_\omega = \zeta(p_\omega; \mathbf{p}) = 1 - \frac{z_\omega s'(z_\omega)}{s(z_\omega)} \equiv \zeta(z_\omega) > 1$$

Notes:

- A function of a single variable, $z_\omega \equiv p_\omega / A(\mathbf{p})$.
- $\zeta(z_\omega) = \sigma > 1$ under CES, $s(z) = \gamma z^{1-\sigma}$.
- Marshall's 2nd law iff $\zeta'(\cdot) > 0$, e.g., $\zeta(z_\omega) = 1 - \frac{1}{\ln(z_\omega/\bar{z})}$ for translog; $= \frac{\sigma}{\sigma-(\sigma-1)z_\omega{}^{(1-\rho)/\rho}} = \frac{1}{1-(z_\omega/\bar{z})^{(1-\rho)/\rho}}$ for CoPaTh.

Unit Cost Function: By integrating $\frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(\mathbf{p})}\right)$,

$$\ln\left[\frac{A(\mathbf{p})}{cP(\mathbf{p})}\right] = \int_\Omega s\left(\frac{p_\omega}{A(\mathbf{p})}\right)\Phi\left(\frac{p_\omega}{A(\mathbf{p})}\right)d\omega\,, \quad \text{where} \quad \Phi(z) \equiv \frac{1}{s(z)}\int_z^{\bar{z}} \frac{s(\xi)}{\xi}\mathrm{d}\xi > 0.$$

where $c > 0$ is a constant, proportional to TFP.

Notes:

- $P(\mathbf{p})$: linear homogeneous, monotonic, and strictly quasi-concave, ensuring the integrability of H.S.A.
- $A(\mathbf{p})/P(\mathbf{p})$ is not constant and depends on $\mathbf{p}$, with the sole exception of CES, because

$$\frac{\partial \ln A(\mathbf{p})}{\partial \ln p_\omega} = \frac{z_\omega s'(z_\omega)}{\int_\Omega s'(z_{\omega'})z_{\omega'}d\omega'} = \frac{[\zeta(z_\omega)-1]s(z_\omega)}{\int_\Omega [\zeta(z_{\omega'})-1]s(z_{\omega'})d\omega'} \neq \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s(z_\omega),$$

unless $\zeta(z)$ is independent of z or $s(z) = \gamma z^{1-\sigma}$ with $\zeta(z) = \sigma > 1$.

For symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$,

$$1 = s\left(\frac{p_\omega}{A(\mathbf{p})}\right)V = s\left(\frac{p}{A(p\mathbf{1}_\Omega^{-1})}\right)V = s\left(\frac{1}{A(\mathbf{1}_\Omega^{-1})}\right)V \Rightarrow z_\omega = \frac{p_\omega}{A(\mathbf{p})} = \frac{1}{A(\mathbf{1}_\Omega^{-1})} = s^{-1}\left(\frac{1}{V}\right).$$

Hence,

	Definition	**Under H.S.A.**
Price Elasticity	$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$	$\zeta_\omega \equiv \zeta\left(\frac{p_\omega}{A(\mathbf{p})}\right) > 1,$
Substitutability	$\sigma(V) \equiv \zeta\left(1; \mathbf{1}_\Omega^{-1}\right) = \zeta^*(1; \mathbf{1}_\Omega)$	$\sigma(V) = \zeta\left(s^{-1}(1/V)\right) > 1$
Love-for-variety	$\mathcal{L}(V) \equiv \frac{d \ln z(V)}{d \ln V} = -\frac{d \ln \mathcal{y}(V)}{d \ln V} - 1 > 0.$	$\mathcal{L}(V) = \Phi\left(s^{-1}(1/V)\right) > 0.$

Notes:

- At symmetric price patterns,

$$\ln\left[\frac{A(\mathbf{p})}{cP(\mathbf{p})}\right] = \ln\left[\frac{A\left(\mathbf{1}_\Omega^{-1}\right)}{cP\left(\mathbf{1}_\Omega^{-1}\right)}\right] = \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right) = \mathcal{L}(V).$$

- Since $s^{-1}(1/V)$ is increasing in V,

$$\sigma(V) = \zeta\left(s^{-1}\left(\frac{1}{V}\right)\right)$$

implies that Marshall's 2nd law, $\zeta'(\cdot) > 0$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$, under H.S.A.

$$\sigma(V) = \zeta\left(s^{-1}\left(\frac{1}{V}\right)\right); \ \mathcal{L}(V) = \Phi\left(s^{-1}\left(\frac{1}{V}\right)\right), \qquad \text{where} \qquad \zeta(z) \equiv 1 - \frac{zs'(z)}{s(z)}; \ \ \Phi(z) \equiv \frac{1}{s(z)}\int_z^{\bar{z}} \frac{s(\xi)}{\xi}\mathrm{d}\xi.$$

Lemma 1:

$$\zeta'(z) \gtreqless 0, \forall z \in (z_0, \bar{z}) \implies \Phi'(z) \lesseqgtr 0, \forall z \in (z_0, \bar{z}).$$

Furthermore,

$$\zeta'(z) = 0 \iff \Phi'(z) = 0 \iff \text{CES}.$$

From this,

Proposition 1

$$\zeta'(z) \gtreqless 0, \forall z \in (z_0, \bar{z}) \Leftrightarrow \sigma'(V) \gtreqless 0, \forall V \in (1/s(z_0), \infty)$$

$$\Longrightarrow$$

$$\Phi'(z) \lesseqgtr 0, \forall z \in (z_0, \bar{z}) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (1/s(z_0), \infty).$$

Furthermore,

$$\zeta'(z) = 0 \Leftrightarrow \sigma'(V) = 0 \Leftrightarrow \Phi'(z) = 0 \Leftrightarrow \mathcal{L}'(V) = 0 \Leftrightarrow \text{CES}.$$

Thus, under H.S.A.,

- Marshall's 2nd Law, $\zeta'(\cdot) > 0$ for all $z < \bar{z}$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$ for all V.
- Increasing (decreasing) substitutability implies diminishing (increasing) love-for-variety. The converse is not true.
- Constant love-for-variety, constant substitutability, and constant price elasticity are all equivalent and occur iff CES.

Homothetic Direct Implicit Additivity (HDIA)

Symmetric HDIA (Homothetic Directly Implicitly Additive) DS with Gross Substitutes

Definition: A symmetric CRS technology, $X = X(\mathbf{x}) \equiv Z\hat{X}(\mathbf{x})$ is called *homothetic with direct implicit additivity* (HDIA) with gross substitutes if it can be defined implicitly by:

$$\int_{\Omega} \phi\left(\frac{Zx_\omega}{X(\mathbf{x})}\right) d\omega = \int_{\Omega} \phi\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right) d\omega \equiv 1,$$

where $\phi(\cdot): \mathbb{R}_+ \to \mathbb{R}_+$ is independent of $Z > 0$, C^3, with $\phi(0) = 0; \phi(\infty) = \infty; \phi'(y) > 0, \phi''(y) < 0, -y\phi''(y)/\phi'(y) < 1\ \forall y \in (0, \infty)$.

- By construction, $\hat{X}(\mathbf{x})$ is independent of $Z > 0$, TFP.
- If $\phi'(0) < \infty$, the choke price is $B(\mathbf{p})\phi'(0)$. If $\phi'(0) = \infty$, no choke price.
- CES with gross substitutes: $\phi(y) = (y)^{1-1/\sigma}$, $(\sigma > 1)$.
- CoPaTh: $\phi(y) = \int_0^{y} \left(1 + \frac{1}{\sigma-1}(\xi)^{\frac{1-\rho}{\rho}}\right)^{\frac{\rho}{\rho-1}} d\xi$, $0 < \rho < 1$, converging to CES with $\rho \nearrow 1$.
- An extension of the Kimball (1995) aggregator in the sense that Ω is not fixed and $V \equiv |\Omega|$ is a variable.

Inverse Demand Curve:	$\frac{p_\omega}{B(\mathbf{p})} = \phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right) = \phi'\left(\frac{Zx_\omega}{X(\mathbf{x})}\right)$	**Demand Curve:**	$\frac{Zx_\omega}{X(\mathbf{x})} = \frac{x_\omega}{\hat{X}(\mathbf{x})} = (\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)$
Unit Cost Function:	$P(\mathbf{p}) = \frac{1}{Z}\hat{P}(\mathbf{p}) \equiv \frac{1}{Z}\int_{\Omega} p_\omega (\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) d\omega$		

where $B(\mathbf{p})$ and $\hat{P}(\mathbf{p})$ are both independent of $Z > 0$ and

$$\int_{\Omega} \phi\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right) d\omega \equiv 1.$$

Budget Share:	$s_\omega \equiv \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{p_\omega}{\hat{P}(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) = \frac{x_\omega}{C^*(\boldsymbol{x})}\phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right),$

where

$$C^*(\mathbf{x}) \equiv \int_\Omega x_\omega \phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right) d\omega$$

satisfying the identity

$$\frac{\hat{P}(\mathbf{p})}{B(\mathbf{p})} = \int_\Omega \frac{p_\omega}{B(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) d\omega = \int_\Omega \phi'\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right)\frac{x_\omega}{\hat{X}(\mathbf{x})} d\omega = \frac{C^*(\mathbf{x})}{\hat{X}(\mathbf{x})}.$$

Budget share under HDIA: A function of the two relative prices, $p_\omega/\hat{P}(\mathbf{p})$ & $p_\omega/B(\mathbf{p})$, or of the two relative quantities, $x_\omega/\hat{X}(\mathbf{x})$ & $x_\omega/C^*(\mathbf{x})$, unless $\hat{P}(\mathbf{p})/B(\mathbf{p}) = C^*(\mathbf{x})/\hat{X}(\mathbf{x})$ is a constant, which occurs iff CES.

Price Elasticity:	$\zeta_\omega = \zeta^*(x_\omega; \mathbf{x}) = -\frac{\phi'(y_\omega)}{y_\omega \phi''(y_\omega)} \equiv \zeta^D(y_\omega) = \zeta^D\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right) = \zeta(p_\omega; \mathbf{p}) > 1$

Notes:

- Price Elasticity, unlike the budget share, is a function of a single variable, $y_\omega \equiv x_\omega/\hat{X}(\mathbf{x})$ or $\phi'(y_\omega) = p_\omega/B(\mathbf{p})$.
- $\zeta^D(y_\omega) = \sigma > 1$ under CES, $\phi(y) = (y)^{1-1/\sigma}$
- Marshall's 2nd law iff $\zeta^{D'}(\cdot) < 0$, satisfied by $\zeta^D(y) = 1 + (\sigma - 1)(y)^{\frac{\rho-1}{\rho}}$ under CoPaTh.

For symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$,

$$\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right)V = 1 \implies \frac{1}{\hat{X}(\mathbf{1}_\Omega)} = \phi^{-1}\left(\frac{1}{V}\right).$$

Hence,

	Definition	**Under HDIA**
Price Elasticity	$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$	$\zeta_\omega = \zeta^D\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right) = \zeta^D\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right) > 1,$
Substitutability	$\sigma(V) \equiv \zeta(1; \mathbf{1}_\Omega^{-1}) = \zeta^*(1; \mathbf{1}_\Omega)$	$\sigma(V) = \zeta^D(\phi^{-1}(1/V)) > 1$
Love-for-variety	$\mathcal{L}(V) \equiv \frac{d \ln z(V)}{d \ln V} = -\frac{d \ln \mathcal{y}(V)}{d \ln V} - 1 > 0.$	$\mathcal{L}(V) = \frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/V))} - 1 > 0.$

where

$$0 < \mathcal{E}_\phi(\mathcal{y}) \equiv \frac{\mathcal{y}\phi'(\mathcal{y})}{\phi(\mathcal{y})} < 1.$$

Notes:

- At symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$,

$$\frac{\hat{P}(\mathbf{1}_\Omega^{-1})}{B(\mathbf{1}_\Omega^{-1})} = \frac{C^*(\mathbf{1}_\Omega)}{\hat{X}(\mathbf{1}_\Omega)} = \int_\Omega \mathcal{E}_\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right)\phi\left(\frac{1}{\hat{X}(\mathbf{1}_\Omega)}\right)d\omega = \mathcal{E}_\phi\left(\phi^{-1}\left(\frac{1}{V}\right)\right) \Rightarrow \frac{B(\mathbf{1}_\Omega^{-1})}{\hat{X}(\mathbf{1}_\Omega^{-1})} = \frac{\hat{X}(\mathbf{1}_\Omega)}{C^*(\mathbf{1}_\Omega)} = \mathcal{L}(V) + 1.$$

- Since $\phi^{-1}(1/V)$ is decreasing in V,

$$\sigma(V) = \zeta^D\left(\phi^{-1}(1/V)\right)$$

implies that Marshall's 2nd law, $\zeta^{D\prime}(\cdot) < 0$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$, under HDIA.

$$\sigma(V) = \zeta^D\left(\phi^{-1}(1/V)\right);\ \mathcal{L}(V) = \frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/V))} - 1, \qquad \text{where} \qquad \zeta^D(y) \equiv -\frac{\phi'(y)}{y\phi''(y)};\ \mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)}$$

Hence,

Lemma 2:

$$\zeta^{D'}(y) \lesseqgtr 0, \forall y \in (0, y_0) \implies \mathcal{E}_\phi'(y) \lesseqgtr 0, \forall y \in (0, y_0).$$

Furthermore,

$$\zeta^{D'}(y) = 0 \iff \mathcal{E}_\phi'(y) = 0 \iff \text{CES}.$$

From this,

Proposition 2:

$$\zeta^{D'}(y) \lesseqgtr 0\ \forall y \in (0, y_0) \Leftrightarrow \sigma'(V) \gtreqless 0, \forall V \in (1/\phi(y_0), \infty)$$

$$\implies$$

$$\mathcal{E}_\phi'(y) \lesseqgtr 0, \forall y \in (0, y_0) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (1/\phi(y_0), \infty).$$

Furthermore,

$$\zeta^{D'}(y) = 0 \iff \sigma'(V) = 0 \Leftrightarrow \mathcal{E}_\phi'(y) = 0 \iff \mathcal{L}'(V) = 0 \iff \text{CES}.$$

Thus, under HDIA,

- Marshall's 2nd Law, $\zeta^{D\prime}(\cdot) < 0$ for all $y > 0$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$ for all V.
- Increasing (decreasing) substitutability implies diminishing (increasing) love-for-variety. The converse is not true.
- Constant love-for-variety, constant substitutability, and constant price elasticity are all equivalent and occur iff CES.

Homothetic Indirect Implicit Additivity (HIIA)

Symmetric HIIA (Homothetic Indirectly Implicitly Additive) DS with Gross Substitutes

Definition: A symmetric CRS technology, $P = P(\mathbf{p}) = \hat{P}(\mathbf{p})/Z$, is called *homothetic with indirect implicit additivity* (HIIA) if it can be defined implicitly by: $$\int_{\Omega} \theta\left(\frac{p_\omega}{ZP(\mathbf{p})}\right) d\omega = \int_{\Omega} \theta\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) d\omega = 1,$$ where $\theta: \mathbb{R}_{++} \to \mathbb{R}_{+}$ is independent of $Z > 0$, C^3, with $\theta(z) > 0$, $\theta'(z) < 0$, $\theta''(z) > 0$, $-z\theta''(z)/\theta'(z) > 1$, for $\theta(z) > 0$ with $\lim_{z\to 0} \theta(z) = \infty$ and $\lim_{z\to \bar{z}} \theta(z) = 0$, where $\bar{z} \equiv \inf\{z > 0 \vert \theta(z) = 0\}$.

- By construction, $\hat{P}(\mathbf{p})$ is independent of $Z > 0$, TFP.
- If $\bar{z} < \infty$, $\hat{P}(\mathbf{p})\bar{z} = ZP(\mathbf{p})\bar{z}$ is the choke price. If $\bar{z} = \infty$, no choke price.
- CES with gross substitutes: $\theta(z) = (z)^{1-\sigma}$, $(\sigma > 1)$.
- CoPaTh: $\theta(z) = \sigma^{\frac{\rho}{1-\rho}} \int_{z/\bar{z}}^{1} \left((\xi)^{\frac{\rho-1}{\rho}} - 1\right)^{\frac{\rho}{1-\rho}} d\xi$ for $z < \bar{z} = \left(\frac{\sigma}{\sigma-1}\right)^{\frac{\rho}{1-\rho}}$; $0 < \rho < 1$, converging to CES as $\rho \nearrow 1$.

Inverse Demand Curve:	$\frac{p_\omega}{ZP(\mathbf{p})} = \frac{p_\omega}{\hat{P}(\mathbf{p})} = (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)$	**Demand Curve:**	$\frac{x_\omega}{B^*(\mathbf{x})} = -\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) = -\theta'\left(\frac{p_\omega}{ZP(\mathbf{p})}\right) > 0$
Production Function:	$X(\mathbf{x}) = Z\hat{X}(\mathbf{x}) \equiv Z\int_{\Omega} (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right) x_\omega d\omega$		

where $\hat{X}(\mathbf{x})$ and $B^*(\mathbf{x})$ are both independent of $Z > 0$ and

$$\int_{\Omega} \theta\left((-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\right) d\omega \equiv 1.$$

Budget Share:	$\frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\boldsymbol{x})}\right)\frac{x_\omega}{\hat{X}(\mathbf{x})} = -\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)\frac{p_\omega}{C(\mathbf{p})}$

where

$$C(\mathbf{p}) \equiv -\int_\Omega \theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)p_\omega d\omega > 0$$

satisfying the identity,

$$\frac{C(\mathbf{p})}{\hat{P}(\mathbf{p})} = \int_\Omega \frac{p_\omega}{\hat{P}(\mathbf{p})}\left[-\theta'\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)\right]d\omega = \int_\Omega (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\frac{x_\omega}{B^*(\mathbf{x})}d\omega = \frac{\hat{X}(\mathbf{x})}{B^*(\mathbf{x})}.$$

Budget share under HIIA: A function of two relative prices, $p_\omega/\hat{P}(\mathbf{p})$ and $p_\omega/C(\mathbf{p})$, or of two relative quantities, $x_\omega/\hat{X}(\mathbf{x})$ and $x_\omega/B^*(\mathbf{x})$, unless $C(\mathbf{p})/\hat{P}(\mathbf{p}) = \hat{X}(\mathbf{x})/B^*(\mathbf{x})$ is a constant, which occurs iff CES.

Price Elasticity:	$\zeta_\omega = \zeta(p_\omega; \mathbf{p}) = -\frac{z_\omega\theta''(z_\omega)}{\theta'(z_\omega)} \equiv \zeta^I(z_\omega) = \zeta^I\left((-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\right) = \zeta^*(x_\omega;\, \mathbf{x}) > 1$

Notes:

- Price Elasticity, unlike the budget share, is a function of a single variable, $z_\omega \equiv p_\omega/\hat{P}(\mathbf{p})$ or $x_\omega/B^*(\mathbf{x}) = -\theta'(z_\omega)$.
- $\zeta^I(z_\omega) = \sigma > 1$ under CES, $\theta(z) = (z)^{1-\sigma}$, $(\sigma > 1)$.
- Marshall's 2[nd] law iff $\zeta^{I\prime}(z_\omega) > 0$, satisfied by $\zeta^I(z_\omega) = \frac{\sigma}{\sigma-(\sigma-1)(z_\omega)^{(1-\rho)/\rho}} = \frac{1}{1-(z_\omega/\bar{z})^{(1-\rho)/\rho}}$ under CoPaTh.

For symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$,

$$\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)V = 1 \implies \frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})} = \theta^{-1}(1/V).$$

Hence,

	Definition	**Under HIIA**
Price Elasticity	$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$	$\zeta_\omega \equiv \zeta^I\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right) = \zeta^I\left((-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\right) > 1$
Substitutability	$\sigma(V) \equiv \zeta(1; \mathbf{1}_\Omega^{-1}) = \zeta^*(1; \mathbf{1}_\Omega)$	$\sigma(V) = \zeta^I\left(\theta^{-1}(1/V)\right) > 1$
Love-for-variety	$\mathcal{L}(V) \equiv \frac{d \ln z(V)}{d \ln V} = -\frac{d \ln y(V)}{d \ln V} - 1 > 0.$	$\mathcal{L}(V) = \frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/V)\right)} > 0.$

where

$$\mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} > 0.$$

Notes:

- At symmetric price patterns, $\mathbf{p} = p\mathbf{1}_\Omega^{-1}$,

$$\frac{C\left(\mathbf{1}_\Omega^{-1}\right)}{\hat{P}(\mathbf{1}_\Omega^{-1})} = \frac{\hat{X}(\mathbf{1}_\Omega)}{B^*(\mathbf{1}_\Omega)} = \int_\Omega \mathcal{E}_\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)\theta\left(\frac{1}{\hat{P}(\mathbf{1}_\Omega^{-1})}\right)d\omega = \mathcal{E}_\theta\left(\theta^{-1}\left(\frac{1}{V}\right)\right) \implies \mathcal{L}(V) = \frac{\hat{P}\left(\mathbf{1}_\Omega^{-1}\right)}{C\left(\mathbf{1}_\Omega^{-1}\right)} = \frac{B^*(\mathbf{1}_\Omega)}{\hat{X}(\mathbf{1}_\Omega)}$$

- Since $\theta^{-1}(1/V)$ is increasing in V,

$$\sigma(V) = \zeta^I\left(\theta^{-1}(1/V)\right)$$

implies that Marshall's 2nd law, $\zeta^{I\prime}(\cdot) > 0$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$, under HIIA.

$$\sigma(V) = \zeta^I\big(\theta^{-1}(1/V)\big);\ \mathcal{L}(V) = \frac{1}{\mathcal{E}_\theta\big(\theta^{-1}(1/V)\big)}, \qquad \text{where} \qquad \zeta^I(z) \equiv -\frac{z\theta''(z)}{\theta'(z)};\ \mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)}.$$

Hence,

Lemma 3:

$$\zeta^{I'}(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \quad \Longrightarrow \quad \mathcal{E}'_\theta(z) \gtreqless 0, \forall z \in (z_0, \overline{z}).$$

Furthermore,

$$\zeta^{I'}(z) = 0 \quad \Leftrightarrow \quad \mathcal{E}'_\theta(z) = 0 \Leftrightarrow \text{CES}.$$

From this,

Proposition 3:

$$\zeta^{I'}(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \Leftrightarrow \sigma'(V) \gtreqless 0, \forall V \in (1/\theta(z_0), \infty)$$

$$\Longrightarrow$$

$$\mathcal{E}'_\theta(z) \gtreqless 0, \forall z \in (z_0, \overline{z}) \Leftrightarrow \mathcal{L}'(V) \lesseqgtr 0, \forall V \in (1/\theta(z_0), \infty).$$

Furthermore,

$$\zeta^{I'}(z) = 0 \Leftrightarrow \sigma'(V) = 0 \Leftrightarrow \mathcal{E}'_\theta(z) = 0 \Leftrightarrow \mathcal{L}'(V) = 0 \Leftrightarrow \text{CES}.$$

Under HIIA,

- Marshall's 2nd Law, $\zeta^{I'}(\cdot) < 0$ for all $z < \overline{z}$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$ for all V.
- Increasing (decreasing) substitutability implies diminishing (increasing) love-for-variety. The converse is not true.
- Constant love-for-variety, constant substitutability, and constant price elasticity are all equivalent and occur iff CES.

Summing Up

Question: How does love-for-variety (gains from increasing variety) depend on the underlying demand structure?

We define **Price Elasticity, Substitutability, Love-for-Variety** for general symmetric homothetic demand systems.

- Substitutability, $\sigma(V)$, Love-for-Variety, $\mathcal{L}(V)$, are both functions of the mass of available varieties, V, only.
- We can say little about their relations, unless we impose additional restrictions.

We turn to H.S.A., HDIA, and HIIA, under which the price elasticity can be written as a function of a single variable.

	CES	H.S.A.	HDIA	HIIA
Price Elasticity: $\zeta_\omega = \zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$.	σ	$\zeta\left(\frac{p_\omega}{A(\mathbf{p})}\right)$	$\zeta^D\left(\frac{x_\omega}{\hat{X}(\mathbf{x})}\right)$	$\zeta^I\left(\frac{p_\omega}{\hat{P}(\mathbf{p})}\right)$
Substitutability: $\sigma(V) = \zeta\left(1; \mathbf{1}_\Omega^{-1}\right) = \zeta^*(1; \mathbf{1}_\Omega)$.	σ	$\zeta\left(s^{-1}\left(\frac{1}{V}\right)\right)$	$\zeta^D\left(\phi^{-1}\left(\frac{1}{V}\right)\right)$	$\zeta^I\left(\theta^{-1}\left(\frac{1}{V}\right)\right)$
Love-for-variety: $\mathcal{L}(V) \equiv \frac{d\ln z(V)}{d\ln V} = -\frac{d\ln \mathcal{Y}(V)}{d\ln V} - 1.$	$\frac{1}{\sigma-1}$	$\Phi\left(s^{-1}\left(\frac{1}{V}\right)\right)$	$\frac{1}{\mathcal{E}_\phi(\phi^{-1}(1/V))} - 1$	$\frac{1}{\mathcal{E}_\theta\left(\theta^{-1}(1/V)\right)}$

In each of these three classes,

- The substitutability is increasing in V, if and only if Marshall's 2nd law of demand holds.
- Increasing (decreasing) substitutability implies diminishing (increasing) love-for-variety. The converse is not true.
- Constant love-for-variety, constant substitutability and constant price elasticity are equivalent and occur iff CES.

Thus, they offer a tractable way of capturing the intuition that **gains from increasing variety is diminishing, if different varieties are more substitutable when more varieties are available.**

Appendices

Appendix C: An Alternative (and Equivalent) Definition of H.S.A.

Definition: A symmetric CRS technology, X= $X(\mathbf{x})$ is called *homothetic single aggregator* (H.S.A.) if the budget share of ω depends solely on a single variable, $y_\omega \equiv x_\omega/A^*$, its own quantity x_ω, normalized by the common quantity aggregator, $A^* = A^*(\mathbf{x})$. $$s_\omega \equiv \frac{p_\omega x_\omega}{\mathbf{px}} = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} = s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right), \quad \text{where} \quad \int_\Omega s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega \equiv 1.$$

- $s^*: \mathbb{R}_{++} \to \mathbb{R}_+$: the *budget share function,* in $y_\omega \equiv x_\omega/A^*$ with $0 < \mathcal{E}_{s^*}(y) \equiv \frac{d \ln s^*(y)}{d \ln y} < 1$, $s^*(0) = 0$, $s^*(\infty) = \infty$.
- $A^* = A^*(\mathbf{x})$: **the common quantity aggregator**, defined by **the adding-up constraint**, $\int_\Omega s^*(x_\omega/A^*) d\omega \equiv 1$. By construction, the budget shares add up to one. $A^*(\mathbf{x})$ linear homogenous in $\mathbf{x}$ for a fixed Ω. A larger Ω increases A^*.

Price Elasticity:	$\zeta_\omega = \zeta^*(x_\omega; \mathbf{x}) = \left[1 - \frac{d \ln s^*(y_\omega)}{d \ln y_\omega}\right]^{-1} \equiv \zeta^*(y_\omega) > 1,$

Notes:

- Also a function of a single variable, $y_\omega \equiv x_\omega/A^*(\mathbf{x})$.
- $\zeta^*(y) = \sigma > 1$ under CES, $s^*(y) = \gamma^{1/\sigma}(y)^{1-1/\sigma}$.
- Marshall's 2nd law, $\partial\zeta(x_\omega; \mathbf{x})/\partial x_\omega < 0$, holds iff $\zeta^{*\prime}(\cdot) < 0$.
- The choke price exists iff $\lim_{y\to 0} s^{*\prime}(y) < \infty$, which implies $\lim_{y\to 0} \frac{d \ln s^*(y)}{d \ln y} = 1$ and hence $\lim_{y\to 0} \zeta^*(y) = \infty$. For example, translog corresponds to $s^*(y)$, defined implicitly by $s^* \exp(s^*/\gamma) \equiv \bar{z}y$, for $\bar{z} < \infty$.

Production Function: By integrating $= \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} = s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right)$,

$$\ln\left[\frac{X(\mathbf{x})}{c^* A^*(\mathbf{x})}\right] = \int_\Omega s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right)\Phi^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right)d\omega,$$

where

$$\Phi^*(y) \equiv \frac{1}{s^*(y)}\int_0^y \frac{s^*(\xi^*)}{\xi^*}d\xi^* = \frac{\int_0^y [s^*(\xi^*)/\xi^*]\,d\xi^*}{\int_0^y [s^*(y)/y]\,d\xi^*} > 1,$$

and $c^* > 0$ is a constant, proportional to TFP. $\Phi^*(y) > 1$ follows from $\mathcal{E}_{s^*}(y) \equiv \frac{d \ln s^*(y)}{d \ln y} < 1$ implying that $s^*(y)/y$ is decreasing in y.

Notes:

- $X(\mathbf{x})$, linear homogeneous, monotonic, and strictly quasi-concave, ensuring the integrability of H.S.A.
- $X(\mathbf{x})/A^*(\mathbf{x})$is not constant and depends on $\mathbf{x}$, with the sole exception of CES, because

$$\frac{\partial \ln A^*(\mathbf{x})}{\partial \ln x_\omega} = \frac{y_\omega {s^*}'(y_\omega)}{\int_\Omega {s^*}'(y_{\omega'})y_{\omega'}d\omega'} = \frac{\left[1 - \frac{1}{\zeta^*(y_\omega)}\right]s^*(y_\omega)}{\int_\Omega \left[1 - \frac{1}{\zeta^*(y_{\omega'})}\right]s^*(y_{\omega'})d\omega'} \neq \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} = s^*(y_\omega),$$

unless $\zeta^*(y)$ is independent of y or $s^*(y) = \gamma^{1/\sigma}(y)^{1-1/\sigma}$ with $\zeta^*(y) = \sigma > 1$.

For symmetric quantity patterns, $\mathbf{x} = x\mathbf{1}_\Omega$,

$$1 = s^*\left(\frac{x}{A^*(x\mathbf{1}_\Omega)}\right)V = s^*\left(\frac{1}{A^*(\mathbf{1}_\Omega)}\right)V \Rightarrow y_\omega \equiv \frac{1}{A^*(\mathbf{1}_\Omega)} = s^{*-1}\left(\frac{1}{V}\right).$$

Hence,

	Definition	**Under H.S.A.**
Price Elasticity	$\zeta_\omega \equiv -\frac{\partial \ln x_\omega}{\partial \ln p_\omega} = \zeta(p_\omega; \mathbf{p}) = \zeta^*(x_\omega; \mathbf{x})$	$\zeta_\omega \equiv \zeta^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) > 1$
Substitutability	$\sigma(V) \equiv \zeta\left(1; \mathbf{1}_\Omega^{-1}\right) = \zeta^*(1; \mathbf{1}_\Omega)$	$\sigma(V) = \zeta^*\left(s^{*-1}(1/V)\right) > 1$
Love-for-variety	$\mathcal{L}(V) \equiv \frac{d \ln z(V)}{d \ln V} = -\frac{d \ln y(V)}{d \ln V} - 1 > 0.$	$\mathcal{L}(V) = \Phi^*\left(s^{*-1}(1/V)\right) - 1 > 0.$

Notes:

- At the symmetric quantity patterns,

$$\ln\left[\frac{X(\mathbf{x})}{c^*A^*(\mathbf{x})}\right] = \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right) = \mathcal{L}(V) + 1.$$

- Since $s^{*-1}(1/V)$ is decreasing in V,

$$\sigma(V) = \zeta^*\left(s^{*-1}\left(\frac{1}{V}\right)\right)$$

implies that Marshall's 2nd law, $\zeta^{*\prime}(\cdot) < 0$, is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$.

$$\sigma(V) = \zeta^*\left(s^{*-1}\left(\frac{1}{V}\right)\right); \; \mathcal{L}(V) = \Phi^*\left(s^{*-1}\left(\frac{1}{V}\right)\right) - 1, \qquad \text{where} \quad \zeta^*(y) \equiv \left[1 - \frac{d\ln s^*(y)}{d\ln y}\right]^{-1}; \; \Phi^*(y) \equiv \frac{1}{s^*(y)}\int_0^y \frac{s^*(\xi^*)}{\xi^*}d\xi^*.$$

Lemma 1*

$$\zeta^{*\prime}(y) \lesseqqgtr 0, \forall y \in (0, y_0) \Longrightarrow \Phi^{*\prime}(y) \gtreqqless 0, \forall y \in (0, y_0).$$

Furthermore,

$$\zeta^{*\prime}(y) = 0 \iff \Phi^{*\prime}(y) = 0 \iff \text{CES}.$$

From this,

Proposition 1*

$$\zeta^{*\prime}(y) \lesseqqgtr 0, \forall y \in (0, y_0) \Leftrightarrow \sigma'(V) \gtreqqless 0, \forall V \in (1/s^*(y_0), \infty)$$

$$\Longrightarrow$$

$$\Phi^{*\prime}(y) \gtreqqless 0, \forall y \in (0, y_0) \Leftrightarrow \mathcal{L}'(V) \lesseqqgtr 0, \forall V \in (1/s^*(y_0), \infty)$$

Furthermore,

$$\zeta^{*\prime}(y) = 0 \iff \sigma'(V) = 0 \iff \Phi^{*\prime}(y) = 0 \iff \mathcal{L}'(V) = 0 \iff \text{CES}.$$

Thus, under H.S.A.,

- Marshall's 2nd Law, $\zeta^{*\prime}(\cdot) < 0$ for all $y > 0$ is equivalent to increasing substitutability, $\sigma'(\cdot) > 0$ for all V.
- Increasing (decreasing) substitutability implies diminishing (increasing) love-for-variety. The converse is not true.
- Constant love-for-variety, constant substitutability, and constant price elasticity are all equivalent and occur iff CES.

Equivalence of the Two Definitions of H.S.A.

Under the isomorphism given by the one-to-one mapping btw $s(z) \leftrightarrow s^*(y)$, defined by:

$$s^*(y) = s\left(\frac{s^*(y)}{y}\right); \qquad s(z) = s^*\left(\frac{s(z)}{z}\right).$$

From this,

$$\zeta^*(y) \equiv \left[1 - \frac{d\ln s^*(y)}{d\ln y}\right]^{-1} = \zeta(z) \equiv 1 - \frac{d\ln s(z)}{d\ln z} > 1,$$

$$0 < \mathcal{E}_{s^*}(y) \equiv \frac{d\ln s^*(y)}{d\ln y} < 1 \Leftrightarrow \mathcal{E}_s(z) \equiv \frac{d\ln s(z)}{d\ln z} < 0.$$

$y_\omega \equiv x_\omega / A^*(\mathbf{x})$, and $z_\omega \equiv p_\omega / A(\mathbf{p})$, are negatively related as

$$z_\omega = \frac{s^*(y_\omega)}{y_\omega} \Leftrightarrow y_\omega = \frac{s(z_\omega)}{z_\omega},$$

$$\frac{dy_\omega}{y_\omega} = -\zeta(z_\omega)\frac{dz_\omega}{z_\omega} \Leftrightarrow \frac{dz_\omega}{z_\omega} = -\frac{1}{\zeta^*(y_\omega)}\frac{dy_\omega}{y_\omega}$$

and

$$\frac{z_\omega \zeta'(z_\omega)}{y_\omega {\zeta^*}'(y_\omega)} = -\zeta(z_\omega) = -\zeta^*(y_\omega) < 0.$$

If $\lim_{y\to 0} s^{*\prime}(y) < \infty$, $\lim_{y\to 0} \zeta^*(y) = \infty$ and the (normalized) choke price is:

$$\lim_{y\to 0} \frac{s^*(y)}{y} = \lim_{y\to 0} s^{*\prime}(y) = \bar{z} \equiv \inf\{z > 0 | s(z) = 0\} < \infty$$

Moreover,

$$\frac{p_\omega x_\omega}{A(\mathbf{p})A^*(\mathbf{x})} = y_\omega z_\omega = s(z_\omega) = s^*(y_\omega) = \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})}$$

hence we have the identity,

$$c \exp\left[\int_\Omega s(z_\omega)\Phi(z_\omega)d\omega\right] = \frac{A(\mathbf{p})}{P(\mathbf{p})} = \frac{X(\mathbf{x})}{A^*(\mathbf{x})} = c^* \exp\left[\int_\Omega s^*(y_\omega)\Phi^*(y_\omega)\, d\omega\right]$$

which is a constant iff CES.

Furthermore, using

$$s(\xi) = s^*(\xi^*) = \xi\xi^* \to \frac{d\xi^*}{\xi^*} = \left[\frac{\xi s'(\xi)}{s(\xi)} - 1\right]\frac{d\xi}{\xi} \to s^*(\xi^*)\frac{d\xi^*}{\xi^*} = \left[s'(\xi) - \frac{s(\xi)}{\xi}\right]d\xi$$

$$\xi = z \leftrightarrow \xi^* = y;\ \xi = \bar{z} \leftrightarrow \xi^* = 0,$$

$$\Phi^*(y) - \Phi(z) \equiv \frac{1}{s^*(y)} \int_0^y \frac{s^*(\xi^*)}{\xi^*} d\xi^* - \frac{1}{s(z)} \int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi = \frac{1}{s(z)} \int_{\bar{z}}^{z} \left[s'(\xi) - \frac{s(\xi)}{\xi} \right] d\xi - \frac{1}{s(z)} \int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi = 1.$$

Since

$$w(\xi) \equiv \frac{s(\xi)/\xi}{\int_z^{\bar{z}} [s(\xi')/\xi'] \, d\xi'} \Leftrightarrow s(z)\Phi(z)w(\xi) = \frac{s(\xi)}{\xi}$$

$$w^*(\xi^*) \equiv \frac{s^*(\xi^*)/\xi^*}{\int_0^y \left[s^*\left(\xi^{*\prime}\right)/\xi^{*\prime}\right] d\xi^{*\prime}} \Leftrightarrow s^*(y)\Phi^*(y)w^*(\xi^*) = \frac{s^*(\xi^*)}{\xi^*},$$

this implies

$$\frac{\xi w(\xi)}{\xi^* w^*(\xi^*)} = \frac{\Phi^*(y)}{\Phi(z)} = 1 + \frac{1}{\Phi(z)} = \frac{\Phi^*(y)}{\Phi^*(y) - 1},$$

$$\frac{c}{c^*} = \exp\left[\int_\Omega [s^*(y_\omega)\Phi^*(y_\omega) - s(z_\omega)\Phi(z_\omega)] d\omega \right] = \exp\left[\int_\Omega s(z_\omega) d\omega \right] = e.$$

and

$$\mathcal{L}(V) = \Phi\left(s^{-1}(1/V)\right) = \Phi^*\left(s^{*-1}(1/V)\right) - 1.$$