Non-CES Aggregators: A Guided Tour

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Introduction

Landscape of non-CES Aggregators

CES has many properties, each helps to make it tractable and restrictive at the same time.

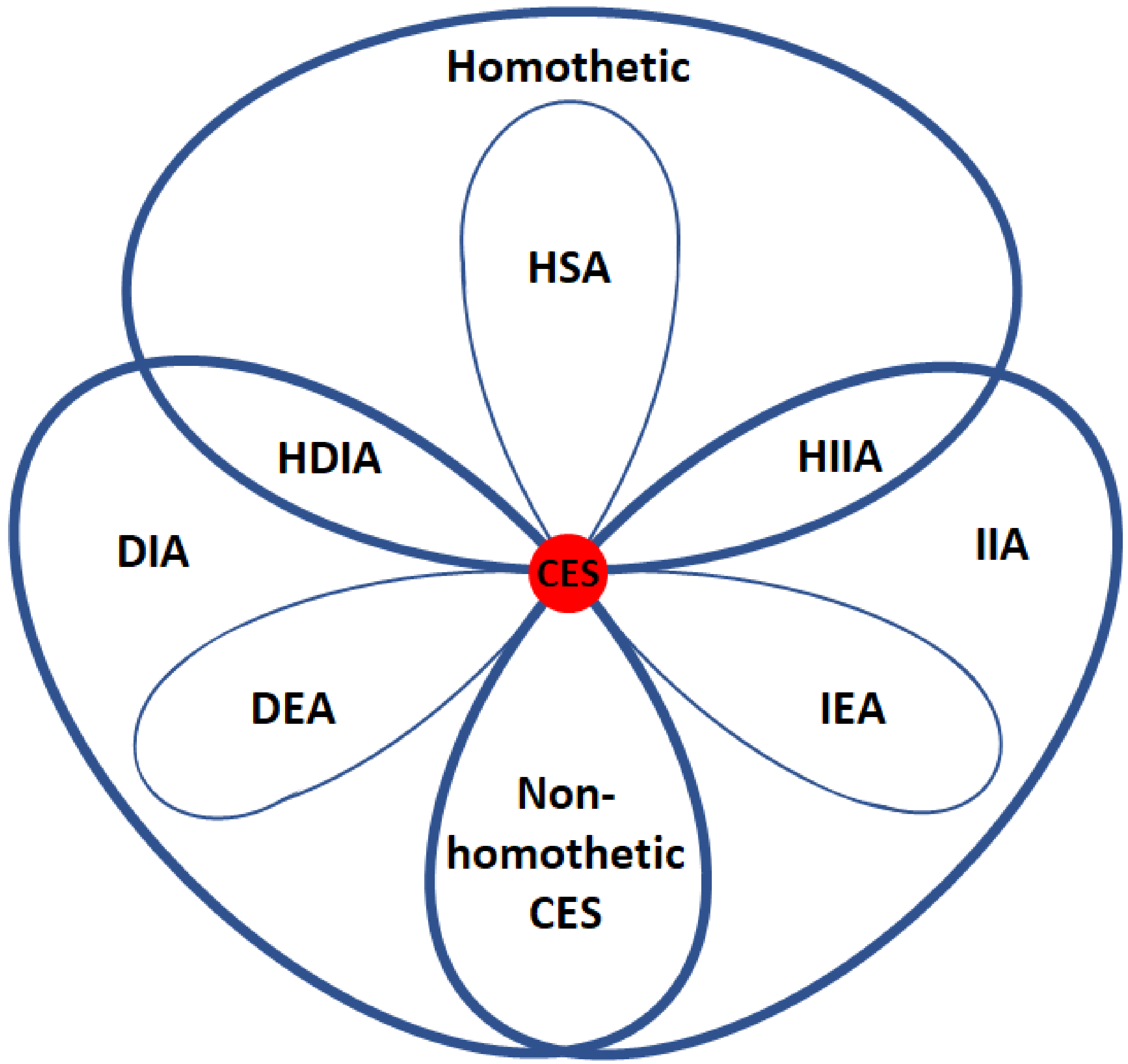
CES is an intersection of many different classes of aggregators. (Each class defined by its properties.)

Many different directions to depart from CES, depending on which properties of CES you want to keep or relax.

Which properties you want to keep or relax depend on the goal of your analysis.

My goal: A guided tour of the non-CES aggregators.

discussing their relations and the relative strengths and weaknesses.



Standard CES We all know it and love using it almost anytime we need some kinds of aggregators (preferences, production functions, externalities, etc.)

Direct Utility Function **(Production Function)**	$X(\mathbf{x}) = U(\mathbf{x}) = \left[\sum_{i=1}^{n} (\beta_i)^{\frac{1}{\sigma}} (x_i)^{1-\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$

By maximizing $U(\mathbf{x})$ subject to $\sum_{i=1}^{n} p_i x_i \leq E$ or by minimizing $\sum_{i=1}^{n} p_i x_i$ subject to $X(\mathbf{x}) \geq X$

Demand	$x_i = \frac{\beta_i (p_i)^{-\sigma} E}{\sum_{k=1}^{n} \beta_k (p_k)^{1-\sigma}} = \frac{\beta_i (p_i)^{-\sigma} E}{(P(\mathbf{p}))^{1-\sigma}} = \beta_i \left(\frac{p_i}{P(\mathbf{p})}\right)^{-\sigma} X = \beta_i \left(\frac{p_i}{P(\mathbf{p})}\right)^{-\sigma} X$
Cost-of-Living Index **(Unit Cost Function)**	$P(\mathbf{p}) = \left[\sum_{i=1}^{n} \beta_i (p_i)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} = \frac{E}{U(\mathbf{p}/E)}$
Budget Shares	$m_i \equiv \frac{p_i x_i}{E} = \beta_i \left(\frac{p_i}{P(\mathbf{p})}\right)^{1-\sigma} = \frac{p_i x_i}{P(\mathbf{p}) X(\mathbf{x})}$
Indirect Utility Function	$U\left(\frac{\mathbf{p}}{E}\right) \equiv \frac{1}{P(\mathbf{p}/E)} = \left[\sum_{i=1}^{n} \beta_i \left(\frac{E}{p_i}\right)^{\sigma-1}\right]^{\frac{1}{\sigma-1}}$

Some Special Features of Standard CES

- Income elasticity of each good is one; no *necessity* nor *luxury* (due to **Homotheticity**)
- Marginal rate of substitution btw any two goods is *independent* of the quantity of any other goods (due to **DEA**). Only the *ratio* of the quantities of the two goods matters.
- Relative demand for any two goods is *independent* of the prices of any other goods (due to **IEA**). Only the *ratio* of the prices of the two goods matter.
- Elasticities of substitution btw all pairs of factors are not only *constant* but also *identical* across all pairs.
- Either all goods are *gross complements* (m_i incr. in the rel. price) if $\sigma < 1$ or *gross substitutes* (m_i decr.) if $\sigma > 1$.
- Either all goods are *essential* ($p_i \to \infty$ implies $P(\mathbf{p}) \to \infty$) if $\sigma \leq 1$ or *inessential* if $\sigma > 1$. No essential gross subst.
- *No choke price;* demand for any good remains strictly positive at any relative price.
- *No saturation:* demand for any good grows unbounded when its relative price becomes arbitrarily low.
- For $\sigma \neq 1$, one could assume, without loss of generality, that standard CES is *symmetric*, by choosing the unit of measurement of each good appropriately.

Nested CES (Sato1967) inherit much of these features as they use CES as building blocks. Its properties dictated by how goods are partitioned into different nests.

- Elasticities of substitution across goods within the same nest are identical.
- Relative demand between two goods in the same nest is independent of the prices of a third good
- Some combinations of essentials and inessentials are ruled out.
- Essentials cannot be gross substitutes.

etc.

Three Properties of Standard CES: $\mathcal{M}[\cdot]$ is a monotone transformation.

Direct Explicit Additivity (DEA): ***Direct*** utility $U(\mathbf{x})$ is ***Explicitly Additive***

$$U(\mathbf{x}) = \mathcal{M}\left[\sum_{i=1}^{n} \bar{u}_i(x_i)\right]$$

some additional conditions on $\bar{u}_i(\cdot), i \in I = (1,2, \dots n)$ to ensure $U(\mathbf{x})$ being strictly increasing and quasi-concave.

Indirect Explicit Additivity (IEA): ***Indirect*** utility $U(\mathbf{p}/E)$ is ***Explicitly Additive***.

$$U\left(\frac{\mathbf{p}}{E}\right) = \mathcal{M}\left[\sum_{i=1}^{n} \bar{v}_i\left(\frac{p_i}{E}\right)\right]$$

some additional conditions on $\bar{v}_i(\cdot),\ i \in I$ to ensure $U(\mathbf{p}/E)$ being strictly decreasing and quasi-convex.

Homotheticity

$$U(\mathbf{x}) = \mathcal{M}[X(\mathbf{x})],$$

where $X(\lambda\mathbf{x}) = \lambda X(\mathbf{x})$ for any $\lambda > 0$.

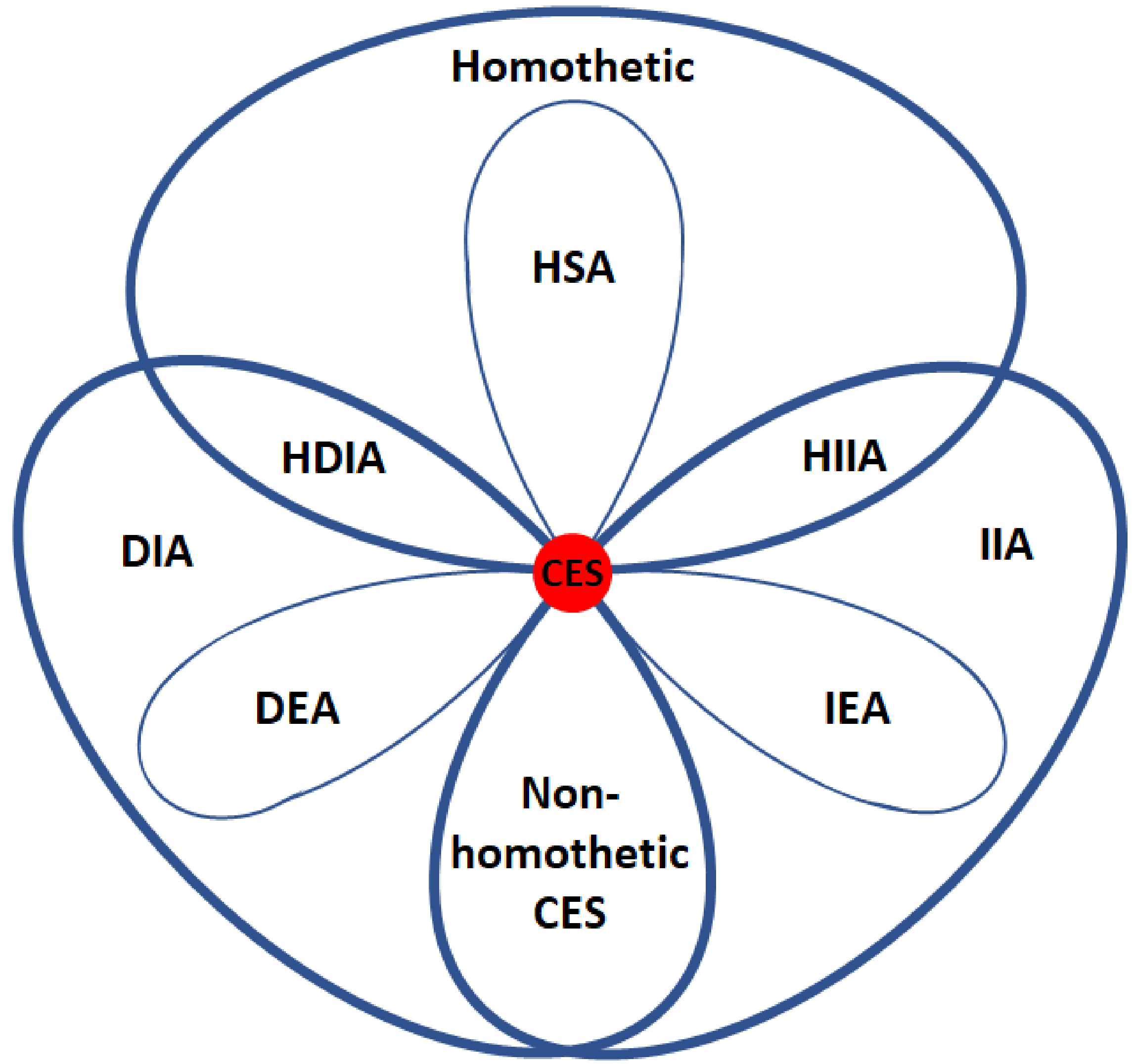
Landscape of non-CES Aggregators

DEA ∩ Homothetic = CES
Known as "Bergson's Law"

DEA ∩ IEA = CES.
Samuelson (1965)

IEA ∩ Homothetic = CES.
Berndt and Christensen (1973)

Departing from CES in the direction of DEA or IEA introduces nonhomotheticity.



Direct Explicit Additivity (DEA) & Indirect Explicit Additivity (IEA)

Direct Explicit Additivity (DEA): ***Direct*** utility $U(\mathbf{x})$ is ***Explicitly Additive***

$$U(\mathbf{x}) = \mathcal{M}\left[\sum_{i=1}^{n} \bar{u}_i(x_i)\right]$$

- Marginal rate of substitution (MRS) btw any two goods is *independent* of the quantity of any other goods, because

$$\frac{p_i}{p_j} = \frac{\partial U(\mathbf{x})/\partial x_i}{\partial U(\mathbf{x})/\partial x_j} = \frac{\bar{u}_i{}'(x_i)}{\bar{u}_j{}'(x_j)} \Longrightarrow p_i = \frac{\bar{u}_i{}'(x_i)E}{\sum_j \bar{u}_j{}'(x_j)x_j}$$

But MRS btw i & $j \in I$ is not a function of x_i/x_j, except CES.

- DEA is homothetic iff CES (Bergson's Law). Hence, any departure from CES would be nonhomothetic.

Example 1: Quasi-Linear

$$U(\mathbf{x}) = \mathcal{M}\left[x_k + \sum_{i \neq k}^{n} u_i(x_i)\right]$$

where $u_i(x_i), i \neq k$ are all strictly concave. The income elasticity of k is one, and those of $i \neq k$ are zero.

Example 2: Distance to the Bliss Points

$$U(\mathbf{x}) = -\sum_{i=1}^{n} \beta_i (b_i - x_i)^{1+\delta}$$

for $0 < x_i < b_i$ where $\delta > 0$.

Example 3: (Generalized) Stone-Geary

$$U(\mathbf{x}) = \left[\sum_{i=1}^{n} (\beta_i)^{\frac{1}{\sigma}} (x_i - \bar{x}_i)^{1-\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \Leftrightarrow U(\mathbf{x}) = \sum_{i=1}^{n} \frac{\check{\beta}_i (x_i - \bar{x}_i)^{1-1/\sigma}}{1 - 1/\sigma};\ \sigma \neq 1$$

$$\Rightarrow m_i \equiv \frac{p_i x_i}{E} = B_i(\mathbf{p}) + \frac{\Gamma_i(\mathbf{p})}{E}.$$

Notes:

- $\bar{x}_i > 0$: the subsistent level of consumption; $-\bar{x}_i > 0$ the nontransferable endowment of i.
- The budget share (or the *average* propensity to consume), m_i, is decreasing in E (a necessity) for $\Gamma_i(\mathbf{p}) > 0$ & increasing in E (a luxury) for $\Gamma_i(\mathbf{p}) < 0$,. **(i.e., non-homothetic).**
- The *marginal* propensity to consume, $B_i(\mathbf{p})$, is independent of E, which allow for aggregation across households.
- *Asymptotically homothetic*; nonhomotheticity is important only for poor households/countries. This feature
  - ➢contradicts with stable slopes of Engel's curves. e.g., Comin-Lashkari-Mestieri (2021)
  - ➢makes it difficult to fit the long-run data. e.g., Buera-Kaboski (2009)
- The price elasticity of a luxury (a necessity) is decreasing (increasing) in E.
- The key parameters, $\bar{x}_i$, are defined in quantity of good i, hence not unit-free. Indeed, one could choose a unit of each good so that $\bar{x}_i = 1, = 0,$ or -1, w.l.o.g. In other words, it cannot meaningfully distinguish more than three sectors in terms of income elasticities.

Example 4:

$$U(\mathbf{x}) = -\sum_{i=1}^{n} \tilde{\beta}_i \exp(-\alpha_i x_i)$$

- $\alpha_i x_i - \alpha_j x_j$ is independent of the expenditure
- can be viewed as a limit case of (Generalized) Stone-Geary for $\sigma \to 0$ with $\gamma_i = \left(1 - \frac{1}{\sigma}\right)/\alpha_i \to -\infty$.

Examples 2, 3, & 4, sometimes called collectively **the Pollak (1971) family** or **Linear Expenditure System (LES)**. They all imply

- the marginal propensity to consume of each good is constant
- Asymptotically homothetic.

Example 5: Houthakker (1960)'s **"direct addilog"**. Mukerji (1963) **Constant Ratio of Elasticities of Substitution (CRES).** Caron et al. (2014) **Constant Ratio of Income Elasticities (CRIE)**

$$U(\mathbf{x}) = \left[\sum_{i=1}^{n} (\beta_i)^{\frac{1}{\sigma_i}} (x_i)^{1-\frac{1}{\sigma_i}}\right]^{\frac{\sigma_0}{\sigma_0 - 1}}; \ \frac{\sigma_i - 1}{\sigma_0 - 1} > 0 \Leftrightarrow U(\mathbf{x}) = \sum_{i=1}^{n} \frac{\breve{\beta}_i (x_i)^{1-1/\sigma_i}}{1 - 1/\sigma_i}; \ \frac{\sigma_i - 1}{\sigma_j - 1} > 0$$

Let η_i (income elasticity of i) & σ_{ij} (Allen-Uzawa EoS btw i and j). Then, for any $i \neq j \neq k \in I$,

$$\sigma_{ij} = \frac{\sigma_i \sigma_j}{\bar{\sigma}}; \ \frac{\eta_i}{\eta_j} = \frac{\sigma_{ik}}{\sigma_{jk}} = \frac{\sigma_i}{\sigma_j}$$

where $\bar{\sigma} \equiv \sum_{l=1}^{n} m_l \sigma_l$ is the budget-share weighted average of $\{\sigma_l\}$.

Indeed, for all DEA,

Pigou's Law: Houthakker (1960), Goldman-Uzawa (1964), Hanoch [1975; Eq.(2.11)].

Under DEA, for any $i \neq j \neq k \in I$,

$$\frac{\eta_i}{\eta_j} = \frac{\sigma_{ik}}{\sigma_{jk}}.$$

- Bergson's Law is a special case.

- Also explains why
  - In Ex. 1 (Quasi-Linear), the income elasticities of all the goods that enter nonlinearly must be equal to zero.
  - In Ex. 3 (Stone-Geary), the relative price elasticity of luxury goods must be decreasing in the total expenditure.

- The reason behind the (well-known but counter-intuitive) result that **the optimal commodity taxation**, which taxes the goods with *lower price elasticity* more heavily, should tax the goods with *lower income elasticity* more heavily.

- Pigou's Law is rejected empirically: Deaton (1974) and many others.

- Under DEA, the effects of the income elasticity differences across goods cannot be disentangled from those of the price elasticity differences across goods.

Indirect Explicit Additivity (IEA): ***Indirect*** utility $U(\mathbf{p}/E)$ is ***Explicitly Additive***.

$$U\left(\frac{\mathbf{p}}{E}\right) = \mathcal{M}\left[\sum_{i=1}^{n} \bar{v}_i\left(\frac{p_i}{E}\right)\right]$$

- Relative demand (RD) for any two goods is *independent* of the price of any other goods, because

$$\frac{x_i}{x_j} = \frac{\partial U(\mathbf{p}/E)/\partial p_i}{\partial U(\mathbf{p}/E)/\partial p_j} = \frac{\bar{v}_i{}'(p_i/E)}{v_j{}'(p_j/E)} > 0 \Rightarrow x_i = \frac{\bar{v}_i{}'(p_i/E)}{\sum_j (p_j/E)\bar{v}_j{}'(p_j/E)}$$

Caution: Some claimed that, with $\bar{v}_i{}'(p_i/E) < 0$ for $0 < p_i/E < z_i < \infty$; $= 0$ for $p_i/E \geq z_i$, $z_i E$ is the choke price. However, it is easy to see that $z_i < \infty$ for all i would violate the monotonicity of preferences.

But RD for i & $j \in I$ is neither independent of E, nor a function of p_i/p_j, except CES.

- IEA is homothetic iff CES. Hence, any departure from CES would be nonhomothetic.

Example 6: Houthakker (1960)'s **"indirect addilog"**; Hanoch (1975)'s Constant Differences of Elasticities of Substitution **(CDES)**, or Constant Differences of Income Elasticities (CDIE)

$$U\left(\frac{\mathbf{p}}{E}\right) = \left[\sum_{i=1}^{n} \beta_i \left(\frac{p_i}{E}\right)^{1-\sigma_i}\right]^{\frac{1}{\sigma_0 - 1}}; \frac{\sigma_i - 1}{\sigma_0 - 1} > 0 \Leftrightarrow U\left(\frac{\mathbf{p}}{E}\right) = -\sum_{i=1}^{n} \frac{\tilde{\beta}_i (p_i/E)^{1-\sigma_i}}{1 - \sigma_i}; \frac{\sigma_i - 1}{\sigma_j - 1} > 0$$

Just like DEA, IEA impose strong functional relations btw income and price elasticities.

Indirect Pigou's Law: Hanoch (1975; Eq.(3.11)). under IEA, for any $i \neq j \neq k \in I$,

$$\sigma_{ik} - \sigma_{jk} = \eta_i - \eta_j.$$

Direct Implicit Additivity (DIA), Indirect Implicit Additivity (IIA), and Nonhomothetic CES

Two Additional Properties of Standard CES: $\mathcal{M}[\cdot]$ is a monotone transformation.

Direct Implicit Additivity (DIA): ***Direct*** utility $U(\mathbf{x})$ ***Implicitly Additive***

$$\mathcal{M}\left[\sum_{i=1}^{n} \tilde{u}_i(x_i, U)\right] = const.$$

certain additional conditions on $\tilde{u}_i(\cdot,\cdot)$, $i \in I$ for strict monotonicity & strict quasi-concavity of $U(\mathbf{x})$.

DEA is a subclass of DIA, with $\tilde{u}_i(x_i, U) = \bar{u}_i(x_i)g(U)$.
Income elasticity and price elasticity differences can be separately controlled for with $\tilde{u}_i(x_i, U) = \bar{u}_i(x_i)g_i(U)$.

Indirect Implicit Additivity (IIA): ***Indirect*** utility $U(\mathbf{p}/E)$ ***Implicitly Additive***.

$$\mathcal{M}\left[\sum_{i=1}^{n} \tilde{v}_i\left(\frac{p_i}{E}, U\right)\right] = const.$$

certain additional conditions on $\tilde{v}_i(\cdot,\cdot)$, $i \in I$ for strict monotonicity & strict quasi-convexity of $U(\mathbf{p}/E)$.

IEA is a subclass of IIA, where $\tilde{v}_i\left(\frac{p_i}{E}, U\right) = \bar{v}_i\left(\frac{p_i}{E}\right)h(U)$.
Income elasticity and price elasticity differences can be separately controlled for $\tilde{v}_i\left(\frac{p_i}{E}, U\right) = \bar{v}_i\left(\frac{p_i}{E}\right)h_i(U)$.

Landscape of non-CES Aggregators

DIA ∩ IIA = Nonhomothetic CES

Homothetic ∩ DIA = HDIA

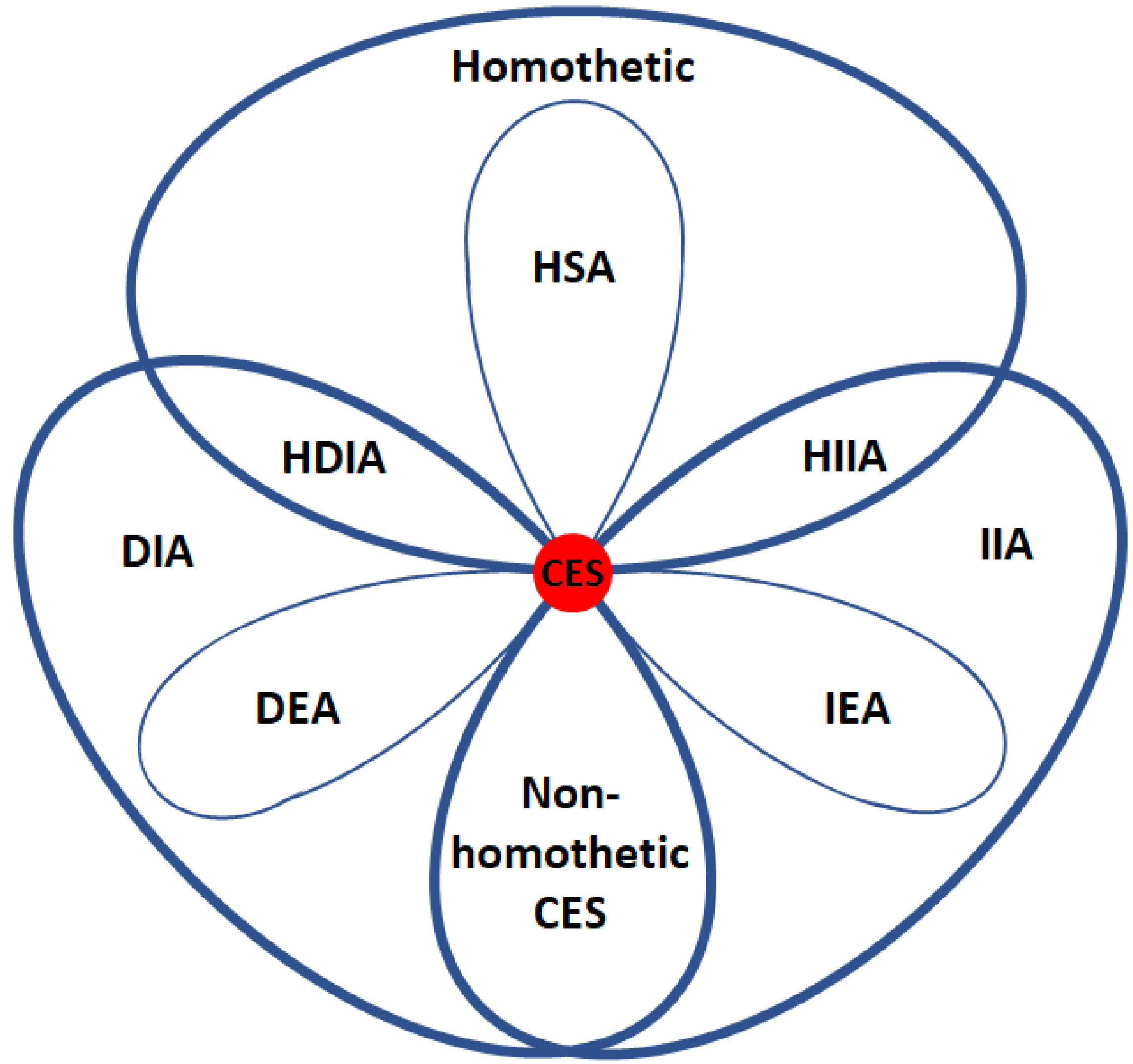
Homothetic ∩ IIA = HIIA

(Direct and Indirect) Implicit additivity allows for both

Nonhomothetic CES

&

Homothetic non-CES



Nonhomothetic CES= DIA ∩ IIA

Direct Utility $\boldsymbol{U}(\mathbf{x})$: defined implicitly by

$$\left[\sum_{i=1}^{n} (\beta_i(U))^{\frac{1}{\sigma(U)}} \left(\frac{x_i}{U}\right)^{1-\frac{1}{\sigma(U)}}\right]^{\frac{\sigma(U)}{\sigma(U)-1}} \equiv 1; \sigma(U) > 0, \neq 1.$$

$\beta_i(U) > 0,\ i \in I$ must satisfy some additional conditions to ensure strict monotonicity.

Indirect Utility $U(\mathbf{p}/E)$, defined implicitly by:

$$\left[\sum_{i=1}^{n} \beta_i(U) \left(\frac{p_i}{E}\right)^{1-\sigma(U)}\right]^{\frac{1}{1-\sigma(U)}} \equiv 1$$

Cost-of-Living Index $P(\mathbf{p}, U)$, defined implicitly by:

$$\left[\sum_{i=1}^{n} \frac{\beta_i(U)}{U^{1-\sigma(U)}} \left(\frac{p_i}{P}\right)^{1-\sigma(U)}\right]^{\frac{1}{1-\sigma(U)}} \equiv 1$$

where $U = U(\mathbf{p}/E)$ and $P = P(\mathbf{p}, U)$ satisfy $PU = E$.

In spite of CES, *nonhomothetic* if $\partial \log \beta_i(U)/\partial \log U$ depends on i and/or $\sigma(U)$ depends on U.

Example 7: Isoelastic Nonhomothetic CES: Comin-Lashkari-Mestieri (2021) & Matsuyama (2019).

$$\sigma(U) = \sigma > 0; \neq 1; \quad \beta_i(U) = \beta_i(U)^{\varepsilon_i - \sigma} \to \partial \log \beta_i(U)/\partial \log U = \varepsilon_i - \sigma$$

Direct Utility: $U(\mathbf{x})$	$\left[\sum_{i=1}^{n} (\beta_i)^{\frac{1}{\sigma}} (U)^{\frac{\varepsilon_i - \sigma}{\sigma}} (x_i)^{1-\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \equiv 1$

$(\varepsilon_i - \sigma)/(1 - \sigma) > 0$ ensures strict monotonicity and strict quasi-concavity globally.

Budget Shares:	$m_i = \frac{\beta_i(U)^{\varepsilon_i - \sigma}(p_i)^{1-\sigma}}{\sum_{k=1}^{n} \beta_k(U)^{\varepsilon_k - \sigma}(p_k)^{1-\sigma}} = \frac{\beta_i(U)^{\varepsilon_i - \sigma}(p_i)^{1-\sigma}}{(E)^{1-\sigma}} = \beta_i(U)^{\varepsilon_i - 1}\left(\frac{p_i}{P}\right)^{1-\sigma}$
Indirect Utility Function:	$\left[\sum_{i=1}^{n} \beta_i(U)^{\varepsilon_i - \sigma}\left(\frac{p_i}{E}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \equiv 1$
Cost-of-Living Index:	$\left[\sum_{i=1}^{n} \beta_i \left(\frac{E}{P}\right)^{\varepsilon_i - 1} \left(\frac{p_i}{P}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \equiv 1$

From $m_i = \frac{\beta_i (U)^{\varepsilon_i - \sigma} (p_i)^{1-\sigma}}{\sum_{k=1}^{n} \beta_k (U)^{\varepsilon_k - \sigma} (p_k)^{1-\sigma}}$, one can also show:

Relative Budget Share:	$\ln\left(\frac{m_i}{m_j}\right) = \ln\left(\frac{\beta_i}{\beta_j}\right) - (\sigma - 1)\ln\left(\frac{p_i}{p_j}\right) + \left(\varepsilon_i - \varepsilon_j\right)\ln\left(\frac{E}{P}\right)$
Relative Demand:	$\ln\left(\frac{x_i}{x_j}\right) = \ln\left(\frac{\beta_i}{\beta_j}\right) - \sigma \ln\left(\frac{p_i}{p_j}\right) + \left(\varepsilon_i - \varepsilon_j\right)\ln\left(\frac{E}{P}\right)$
Income Elasticity:	$\eta_i \equiv \frac{\partial \ln x_i}{\partial \ln E} = \frac{\partial \ln x_i}{\partial \ln(E/P)} = 1 + \frac{\partial \ln m_i}{\partial \ln(E/P)} = 1 + \varepsilon_i - \sum_{k=1}^{n} m_k \varepsilon_k$

Notes:

- Income elasticities $\{\eta_i\}$ can be controlled by $\{\varepsilon_i\}$.
- Price elasticity σ, a constant parameter, chosen separately from $\{\varepsilon_i\}$.
- Double-log demand systems with the stable slopes of the Engel's curves: (e.g., Comin-Lashkari-Mestieri, 2021)

- With $\varepsilon_1 < \cdots < \varepsilon_n$,
  - a larger $U = E\ /P$ shifts $\{m_i\}$ to the right in the monotone likelihood way.
  - η_i is monotonically decreasing in $U = E/P$,
    - $\eta_1 < 1; \eta_n > 1$ for any $U = E/P > 0$.
    - $\eta_i > 1$ for a small $U = E/P > 0$ and $\eta_i < 1$ for a large $U = E/P > 0$ for $2 \leq i \leq n-1$ (with $n \geq 3$)

$$\eta_i = 1 + \varepsilon_i - \sum_{k=1}^{n} m_k \varepsilon_k \lesseqgtr 1 \Leftrightarrow \varepsilon_i \lesseqgtr \sum_{k=1}^{n} m_k \varepsilon_k$$

Whether a particular good is a luxury (i.e., $\eta_i > 1$) or a necessity (i.e., $\eta_i < 1$) depends on the household income.
- A private jet may be a luxury for most people but may be a necessity for the billionaires.
- Air-conditioners, dishwashers, or smart phones may be necessities for many, but luxuries for the poor.
- Hump-shaped budget shares of clothing & alcohol in the total expenditure.

This feature makes nonhomothetic CES well-suited for explaining the rise & fall of industry, more generally structural transformation.
In Stone-Geary or CRES = CRIE, or Almost Ideal Demand System (AIDS), the budget share is monotone in the total expenditure, and whether a good is a necessity or a luxury is independent of the household income.

Other examples of nonhomothetic preferences with this feature
Hierarchical Demand System: Matsuyama (2000, 2002)**,** Foellmi-Zweimueller (2008)**,** Buera and Kaboski (2012)

$$U(\mathbf{x}) = \sum_{j=1}^{\infty} \beta_j \min\{x_j, \hat{x}_j\}$$

where $\hat{x}_j$ is the saturation level of good j. This belongs to DEA. If β_j/p_j is monotone decreasing,

- households buy only $j \in \{1,2,\dots,J\}$ up to the saturation levels and some of $J+1$, where J is determined by

$$\sum_{j=1}^{J} p_j \hat{x}_j \leq E < \sum_{j=1}^{J+1} p_j \hat{x}_j.$$

- As E rises, they expand the range of goods purchased.
- Each good is a luxury for poor households, and a necessity for rich households.

.

Alternatively, for $\beta < 1$, $u(x_j) = \beta \min\{x_j, 1\}$, and

$$U(\mathbf{x}) = u(x_1) + u(x_1)u(x_2) + u(x_1)u(x_2)u(x_3) + u(x_1)u(x_2)u(x_3)u(x_4) + \cdots.$$

$$\Longrightarrow \frac{\partial U(\mathbf{x})}{\partial x_k} = 0, \text{if } x_j = 0 \text{ for any } j < k.$$

Demand is hierarchical for any prices, and each good is a luxury for the poor and a necessity for the rich.

These hierarchical systems have relatively easy aggregation properties due to the linearity, which comes with its own drawbacks (almost all goods are either not consumed at all or reach their saturation levels).

Homothetic and Linear Homogeneous Functions: A Quick Refresher

Why Linear Homogeneous (& Homothetic) Aggregators

- Aggregating many factors (or goods) into a composite of factors (or goods)
- Competitive Industry → CRS Production Functions → Linear Homogenous Functions
- Representative consumer justified by homothetic preferences
- Ensuring the existence of the steady state in dynamic models
- In a Multi-Tiered Demand System, assuming nonhomothetic demand systems anywhere but in the highest tier prevents us for solving the overall demand system by
  - breaking it down to smaller problems
  - solving them sequentially
  - with multi-stage budgeting procedure.
- Linear homogeneity/homotheticity useful not only for production/utility functions. Also for matching functions, externalities, etc., in order to keep the properties of a model scale-free

Homothetic and Linear Homogeneous Functions: A general case

- $X(\mathbf{x}): \mathbb{R}_+^n \to \mathbb{R}_+$ is *linear homogeneous* if $X(\lambda\mathbf{x}) = \lambda X(\mathbf{x})$ for all $\lambda > 0$.

- $H(\mathbf{x})$ is *homothetic in* $\mathbf{x} \in \mathbb{R}_+^n$ if $H(\mathbf{x}) = \mathcal{M}\big(X(\mathbf{x})\big)$, where $\mathcal{M}(\cdot)$ is a *monotone* transformation, with *linear homogeneous* $X(\mathbf{x})$.

- Conversely, any *homothetic* $H(\mathbf{x})$ can be expressed as $H(\mathbf{x}) = \mathcal{M}\big(X(\mathbf{x})\big)$, where $X(\mathbf{x})$ is determined up to a positive scalar.

- When $X(\mathbf{x})$ interpreted as a ***CRS production function*****,** one could define *its unit cost function*, or the *price index*, which is *linear homogeneous, monotone, quasi-concave in* $\mathbf{p} \in \mathbb{R}_+^n$.

$$P(\mathbf{p}) \equiv \min_{\mathbf{x}\in\mathbb{R}_+^n}\{\mathbf{px} | X(\mathbf{x}) \geq 1\}.$$

- If $X(\mathbf{x})$ is *monotone and quasi-concave*, it can be recovered from $P(\mathbf{p})$ as:

$$X(\mathbf{x}) \equiv \min_{\mathbf{p}\in\mathbb{R}_+^n}\{\mathbf{px} | P(\mathbf{p}) \geq 1\}.$$

Homothetic demands and budget shares: A general case

- Demand, $\mathbf{x}(\mathbf{p}) \equiv \underset{\mathbf{x}\in\mathbb{R}^n_+}{Argmin}\{\mathbf{px}|X(\mathbf{x}) \geq \boldsymbol{X}\}$. For a strictly quasi-concave $X(\mathbf{x})$,

$$x_i(\mathbf{p}) = \frac{\partial P(\mathbf{p})}{\partial p_i}X \Rightarrow m_i = \frac{p_i x_i(\mathbf{p})}{\mathbf{px}(\mathbf{p})} = \frac{\partial P(\mathbf{p})}{\partial p_i}\frac{p_i}{P(\mathbf{p})} = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_i}$$

For general CRS, little restrictions on m_i, beyond the homogeneity of degree zero in **p.**

- Inverse Demand, $\mathbf{p}(\mathbf{x}) \equiv \underset{\mathbf{p}\in\mathbb{R}^n_+}{Argmin}\{\mathbf{px}|P(\mathbf{p}) \geq \boldsymbol{P}\}$. For a strictly quasi-concave $P(\mathbf{p})$,

$$p_i(\mathbf{x}) = \frac{\partial X(\mathbf{x})}{\partial x_i}P \Rightarrow m_i = \frac{p_i(\mathbf{x})x_i}{\mathbf{p}(\mathbf{x})\mathbf{x}} = \frac{\partial X(\mathbf{x})}{\partial x_i}\frac{x_i}{X(\mathbf{x})} = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_i}$$

For general CRS, little restrictions on m_i, beyond the homogeneity of degree zero in **x.**

From Euler's theorem on linear homogenous functions,

$$\mathbf{px} = \sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i \frac{\partial P(\mathbf{p})}{\partial p_i} X(\mathbf{x}) = \sum_{i=1}^{n} P(\mathbf{p}) \frac{\partial X(\mathbf{x})}{\partial x_i} x_i = P(\mathbf{p})X(\mathbf{x})$$

Thus, the value of the output is equal to the total value of all inputs.

Three Properties of Standard CES: $\mathcal{M}[\cdot]$ is a monotone transformation.

Homothetic with a Single Aggregator (HSA)

$$m_i = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_i} = s_i\left(\frac{p_i}{A(\mathbf{p})}\right), \text{where} \sum_{i=1}^{n} s_i\left(\frac{p_i}{A(\mathbf{p})}\right) \equiv 1$$

or

$$m_i = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_i} = s_i^*\left(\frac{x_i}{A^*(\mathbf{x})}\right), \text{where} \sum_{i=1}^{n} s_i^*\left(\frac{x_i}{A^*(\mathbf{x})}\right) \equiv 1$$

some restrictions on $s_i(\cdot)$ or $s_i^*(\cdot), i \in I$ to ensure strict monotonicity & strict quasi-concavity of $X(\mathbf{x})$ or $P(\mathbf{p})$.

Homothetic Direct Implicit Additivity (HDIA): $\mathrm{X}(\mathbf{x})$ ***implicitly additive***

$$\mathcal{M}\left[\sum_{i=1}^{n} \phi_i\left(\frac{x_i}{X(\mathbf{x})}\right)\right] = const.$$

some restrictions on $\phi_i(\cdot), i \in I$ to ensure strict monotonicity & strict quasi-concavity of $X(\mathbf{x})$

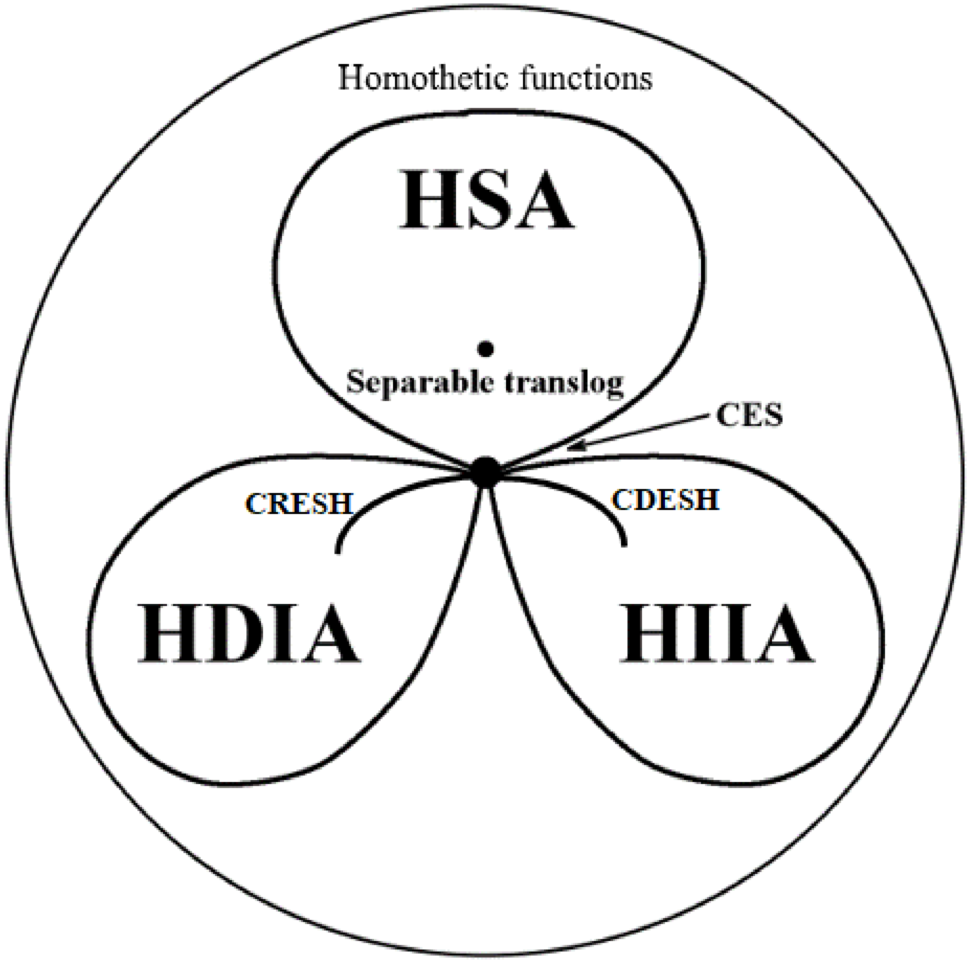
Homothetic Indirect Implicit Additivity (HIIA): $P(\mathbf{p})$ ***implicitly additive***

$$\mathcal{M}\left[\sum_{i=1}^{n} \theta_i\left(\frac{p_i}{P(\mathbf{p})}\right)\right] = const.$$

some restrictions on $\theta_i(\cdot), i \in I$ to ensure strict monotonicity & strict quasi-concavity of $P(\mathbf{p})$.

Matsuyama-Ushchev (2017) show that, if $n > 2$,

HSA ∩ HDIA = HSA ∩ HIIA = HDIA ∩ HIIA = CES.



- 3 ***alternative*** ways of departing from CES without introducing nonhomotheticity
- Contain some known homothetic functions as special cases.
- ***Tractable*** since the budget shares depend on only one (for HSA) or two aggregators (for HDIA & HIIA) for *any number* of factors
- "Gross complements" & "gross substitutes" defined naturally.
- Defined nonparametrically, thus ***flexible***. Offers a template to construct parametric families that relax some features of CES. For example,
  - Each factor has its own constant price elasticity different from others.
  - Factors can be gross substitutes and yet essential
  - Any combination of essential and of inessential factors are possible
  - A factor can be a gross substitute at some prices & a gross complement at other prices (for HDIA and HIIA)
  - Any combination of gross substitutes and gross complements (for HDIA and HIIA),

etc.

Homothetic with a Single Aggregator (HSA)

Definition: HSA Demand Systems

$$m_i = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_i} = s_i\left(\frac{p_i}{A(\mathbf{p})}\right), \qquad \text{where } \sum_{i=1}^{n} s_i\left(\frac{p_i}{A(\mathbf{p})}\right) \equiv 1$$

- $s_i: \mathbb{R}_+ \to \mathbb{R}_+$: budget share of factor i is a function of its relative price, $p_i/A(\mathbf{p})$,
- $A(\mathbf{p})$: a *common price aggregator* defined implicitly by the adding-up constraint, $\sum_{i=1}^{n} s_i\left(\frac{p_i}{A(\mathbf{p})}\right) \equiv 1$

Matsuyama-Ushchev (Proposition 1) shows that

- **The integrability:** There exists $X(\mathbf{p})$ or $P(\mathbf{p})$**,** that rationalizes this demand system, if $s_i(z_i), i \in I$ are either non-increasing in all i with $\sum_i s_i(0) > 1 > \sum_i s_i(\infty)$ or non-decreasing in all i with $\sum_i s_i(0) < 1 < \sum_i s_i(\infty)$ and satisfy

$$z_i s_i'(z_i) < s_i(z_i), \;\; s_i'(z_i) s_j'(z_j) \geq 0,$$

- For $n > 2$, $A(\mathbf{p}) = cP(\mathbf{p})$ iff HSA is a CES.

(If $n = 2$, *all* CRS functions are HSA and any linear homogeneous function can play a role of $A(\mathbf{p})$.)

Alternative (but Equivalent) Definition: HSA Inverse Demand Systems

$$\frac{p_i(\mathbf{x})x_i}{\mathbf{p}(\mathbf{x})\mathbf{x}} = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_i} = s_i^*\left(\frac{x_i}{A^*(\mathbf{x})}\right), \qquad \text{where } \sum_{i=1}^{n} s_i^*\left(\frac{x_i}{A^*(\mathbf{x})}\right) \equiv 1$$

- $s_i^*: \mathbb{R}_+ \to \mathbb{R}_+$: budget share of factor i is a function of its relative quantity, $x_i/A^*(\mathbf{x})$,
- $A^*(\mathbf{x})$: a *common quantity aggregator* defined implicitly by the adding-up constraint, $\sum_{i=1}^{n} s_i^*\left(\frac{x_i}{A^*(\mathbf{x})}\right) = 1$

Matsuyama-Ushchev (Proposition 1*) shows that

- **The integrability:** There exists $X(\mathbf{p})$ or $P(\mathbf{p})$**,** that rationalizes this demand system, if $s_i^*(y_i)$ $i \in I$ are either non-increasing in all i with $\sum_i s_i^*(0) > 1 > \sum_i s_i^*(\infty)$ or non-decreasing in all i with $\sum_i s_i^*(0) < 1 < \sum_i s_i^*(\infty)$ and satisfy

$$y_i s_i^{*\prime}(y_i) < s_i^*(y_i), \;\; s_i^{*\prime}(y_i) s_i^{*\prime}(y_j) \geq 0.$$

For $n > 2$, $X(\mathbf{x}) = cA^*(\mathbf{x})$ iff HSA is a CES

(If $n = 2$, *all* CRS functions are HSA and any linear homogeneous function can play a role of $A^*(\mathbf{x})$.)

Self-Duality of HSA demand systems and HSA inverse demand system

The two classes of HSA are *self-dual* with the one-to-one correspondence btw $s_i(z_i)$ & $s_i^*(y_i)$ defined by

$$s_i^*(y_i) \equiv s_i\left(\frac{s_i^*(y_i)}{y_i}\right) \Leftrightarrow s_i(z_i) \equiv s_i^*\left(\frac{s_i(z_i)}{z_i}\right).$$

By differentiating the above,

$$\left[1 - \frac{d\ln s_i(z_i)}{d\ln z_i}\right]\left[1 - \frac{d\ln s_i^*(y_i)}{d\ln y_i}\right] = 1,$$

Furthermore,

$$\frac{p_i x_i}{P(\mathbf{p})X(\mathbf{x})} = s_i(z_i) = s_i^*(y_i) = z_i y_i = \frac{p_i x_i}{A(\mathbf{p})A^*(\mathbf{x})}$$

$$\Longrightarrow \frac{A(\mathbf{p})}{P(\mathbf{p})} = \frac{X(\mathbf{x})}{A^*(\mathbf{x})}$$

Price Elasticity Functions: Gross Substitutes vs Gross Complements

Prince elasticity functions, $\zeta_i(z_i) = \zeta_i^*(y_i)$

$$\zeta_i(z_i) \equiv \left[1 - \frac{d\ln s_i(z_i)}{d\ln z_i}\right] = \left[1 - \frac{d\ln s_i^*(y_i)}{d\ln y_i}\right]^{-1} \equiv \zeta_i^*(y_i)$$

Gross Substitutes: $s_i'(z_i) < 0 \iff \zeta_i(z_i) = \zeta_i^*(y_i) > 1 \iff 0 < y_i s_i^{*\prime}(y_i) < s_i^*(y_i)$

Gross Complements: $0 < z_i s_i'(z_i) < s_i(z_i) \iff 0 < \zeta_i(z_i) = \zeta_i^*(y_i) < 1 \iff s_i^{*\prime}(y_i) < 0$

The integrability condition can be restated as:

$$\zeta_i(z_i) > 0;\ \ [1 - \zeta_i(z_i)][1 - \zeta_j(z_j)] \geq 0$$

or

$$\zeta_i^*(y_i) > 0;\ \ \left[1 - \frac{1}{\zeta_i^*(y_i)}\right]\left[1 - \frac{1}{\zeta_j^*(y_j)}\right] \geq 0.$$

HSA does not allow for a mixture of gross substitutes & gross complements.

$A(\mathbf{p})$ versus $P(\mathbf{p})$

$$\frac{\partial \ln P(\mathbf{p})}{\partial \ln p_i} = s_i\left(\frac{p_i}{A(\mathbf{p})}\right), \qquad \text{where } \sum_{i=1}^{n} s_i\left(\frac{p_i}{A(\mathbf{p})}\right) \equiv 1\,;$$

$$\ln P(\mathbf{p}) = \ln A(\mathbf{p}) + \sum_{i=1}^{n} \int_{c_1}^{p_i/A(\mathbf{p})} \frac{s_i(\xi)}{\xi} d\xi\,;$$

$$\frac{\partial \ln A(\mathbf{p})}{\partial \ln p_i} = \frac{\left[1 - \zeta_i\left(\frac{p_i}{A}\right)\right] \cdot s_i\left(\frac{p_i}{A}\right)}{\sum_{k=1}^{n}\left[1 - \zeta_k\left(\frac{p_k}{A}\right)\right] \cdot s_k\left(\frac{p_k}{A}\right)}; \quad \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_i} = s_i\left(\frac{p_i}{A}\right).$$

Notes:

- For $n > 2, P(\mathbf{p})/A(\mathbf{p}) \neq c$ for any $c > 0$, unless CES
  - $A(\mathbf{p})$, the inverse measure of *competitive pressures*, captures *cross price effects* in the demand system
  - $P(\mathbf{p})$, the inverse measure of TFP, captures the *productivity consequences* of price changes

- Similarly for $A^*(\mathbf{x})$ vs. $X(\mathbf{x})$.

Example 8: CES as a Special Case of HSA

$$s_i(z_i) = \beta_i z_i^{1-\sigma} \Leftrightarrow s_i^*(y_i) = \beta_i^{\frac{1}{\sigma}} y_i^{1-\frac{1}{\sigma}}; \qquad \sigma > 0, \neq 1; \;\; \beta_i > 0, \quad \sum_{i=1}^{n} \beta_i = 1$$

$$\zeta_i(z_i) = \zeta_i^*(y_i) = \sigma > 0$$

$$A(\mathbf{p}) = \left(\sum_{i=1}^{n} \beta_i p_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}} = ZP(\mathbf{p}) \Leftrightarrow X(\mathbf{x}) = Z\left(\sum_{i=1}^{n} \beta_i^{\frac{1}{\sigma}} x_i^{1-\frac{1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} = ZA^*(\mathbf{x})$$

- Z is TFP.
- $A(\mathbf{p})$ and $A^*(\mathbf{x})$ both independent of TFP, *true in general;* $A(\mathbf{p})/P(\mathbf{p}) = X(\mathbf{x})/A^*(\mathbf{x}) = Z$ is constant, *true iff CES*.
- $s_i(z_i) = \beta_i z_i^{1-\sigma}$ is strictly positive for any $z_i < \infty \Leftrightarrow s_i^{*\prime}(0) = \infty$. *No choke price*.
- $s_i(z_i) = \beta_i z_i^{1-\sigma}$ is strictly decreasing and $s_i^*(y_i) = \beta_i^{\frac{1}{\sigma}} y_i^{1-\frac{1}{\sigma}}$ is strictly increasing for $\sigma > 1$; *Gross substitutes*.
- $s_i(z_i) = \beta_i z_i^{1-\sigma}$ is strictly increasing and $s_i^*(y_i) = \beta_i^{\frac{1}{\sigma}} y_i^{1-\frac{1}{\sigma}}$ is strictly decreasing for $\sigma < 1$; *Gross complements*

Essential vs. Inessential under HSA

Definition: For any $X(\mathbf{x})$ and $P(\mathbf{p})$, we define
- Factor i is *essential* (or *indispensable*) if $x_i = 0$ implies $X(\mathbf{x}) = 0$ (or equivalently, if $p_i \to \infty$ implies $P(\mathbf{p}) \to \infty$).
- Factor i is *inessential* (or *dispensable*), otherwise.

Under General HSA Factor i is *inessential if and only if*

$$s_i(\infty) + \sum_{k\neq i} s_k(0) > 1 \quad \& \quad \int_c^{\infty} \frac{s_i(\xi)}{\xi} \mathrm{d}\xi < \infty.$$

- Strictly increasing $s_i(\cdot)$, i.e., gross complements, implies essential.
- Strictly decreasing $s_i(\cdot)$, i.e., gross substitutes, have four possible cases:
  - $\lim_{z_i\to\infty} s_i(z_i) = s_i(\infty) > 0$, so that $\int_{c_1}^{\infty}(s_i(\xi)/\xi)\mathrm{d}\xi = \infty$, which means essential.
  - $s_i(z_i) > 0$ for $z_i < \infty$; $\lim_{z_i\to\infty} s_i(z_i) = 0$, $\int_{c_1}^{\infty}(s_i(\xi)/\xi)\mathrm{d}\xi = \infty$, which means essential.
  - $s_i(z_i) > 0$ for $z_i < \infty$; $\lim_{z_i\to\infty} s_i(z_i) = 0$, $\int_{c_1}^{\infty}(s_i(\xi)/\xi)\mathrm{d}\xi < \infty$.
  - $s_i(z_i) = 0$ for $z_i \geq \bar{z}_i$ for a finite $\bar{z}_i$ (the choke price); $\int_{c_1}^{\infty}(s_i(\xi)/\xi)\mathrm{d}\xi < \infty$.

In the 3rd and 4th case, $s_i(\infty) + \sum_{k\neq i} s_k(0) > 1$ would imply inessential.

Under CES with $\sigma > 1$, only the 3rd case with $s_i(\infty) + \sum_{k\neq i} s_k(0) > 1$ is possible.

Example 9: A Hybrid of Cobb-Douglas and CES under HSA

$$s_i(z_i) = \varepsilon\alpha_i + (1-\varepsilon)\beta_i {z_i}^{1-\sigma}; \;\; 0 < \varepsilon < 1,\ \alpha_i \geq 0,\ \ \beta_i > 0,\ \ \sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} \beta_k = 1$$

$$\zeta_i(z_i) = \frac{\varepsilon\alpha_i + \sigma(1-\varepsilon)\beta_i {z_i}^{1-\sigma}}{\varepsilon\alpha_i + (1-\varepsilon)\beta_i {z_i}^{1-\sigma}};\ A(\mathbf{p}) = \left(\sum_{i=1}^{n} \beta_i p_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}};\ \ P(\mathbf{p}) = \frac{1}{Z}\left(\prod_{i=1}^{n} p_i^{\alpha_i}\right)^{\varepsilon} \left(A(\mathbf{p})\right)^{1-\varepsilon}$$

$A(\mathbf{p})$ is independent of ε, $P(\mathbf{p})$ depends on ε. Not nested CES, because $\alpha_i\beta_i \neq 0$ for some i.

Example 9*: A Convex Combination of Cobb-Douglas and CES under HSA

$$s_i^*(y_i) = \varepsilon\alpha_i + (1-\varepsilon)\beta_i^{1/\sigma} {y_i}^{1-1/\sigma}; \;\; 0 < \varepsilon < 1,\ \alpha_i \geq 0,\ \ \beta_i > 0,\ \ \sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} \beta_k = 1$$

$$\zeta_i^*(y_i) = \frac{\varepsilon\alpha_i + (1-\varepsilon)\beta_i^{1/\sigma} y^{1-1/\sigma}}{\varepsilon\alpha_i + (1/\sigma)(1-\varepsilon)\beta_i^{1/\sigma} y^{1-1/\sigma}};\ A^*(\mathbf{x}) = \left(\sum_{i=1}^{n} \beta_i^{1/\sigma} x_i^{1-1/\sigma}\right)^{\frac{1}{1-1/\sigma}};\ \ X(\mathbf{x}) = Z\left(\prod_{i=1}^{n} x_i^{\alpha_i}\right)^{\varepsilon} \left(A^*(\mathbf{x})\right)^{1-\varepsilon}$$

$A^*(\mathbf{x})$ is independent of ε, $X(\mathbf{x})$ depends on ε. Not nested CES, because $\alpha_i\beta_i \neq 0$ for some i.

Two Hybrids of Cobb-Douglas and CES (Continue…)

- When $\sigma > 1$, all factors are gross substitutes.

- Factor i is *essential* if $\alpha_i > 0$ (and *inessential* if $\alpha_i = 0$).

- **Implication**: consider a model of international trade where each country produces the single nontradable consumption good using with tradeable factors under HSA

  - Trade elasticity is $\sigma > 1$. With a small ε, the demand system can be approximated by CES.

  - Under CES ($\varepsilon = 0$), autarky would lead to a small welfare loss with a moderately large $\sigma > 1$.

  - For an arbitrarily small but positive $\varepsilon > 0$, the welfare loss of autarky, measured in the cost-of-living index, is *infinity* if a country has no domestic supply of an essential factor.

More broadly, when gross substitutes are essential (with their price elasticities converging to one as they get scarcer).

A caution against assessing the impacts of large changes, say sanctions or pandemic-induced lockdowns, by using the empirical evidence obtained by local changes as "disciplines," under the straitjacket of CES.

Example 10: "Separable" Translog Unit Cost Function as a Special Case of HSA

$$P(\mathbf{p}) = \frac{1}{Z}\exp\left[\sum_{i=1}^{n}\delta_i \ln p_i - \frac{1}{2}\sum_{i,j=1}^{n}\gamma_{ij}\ln p_i \ln p_j\right]$$

$\delta_i > 0$; (γ_{ij}) is symmetric and non-negative semidefinite, normalized as $\sum_{j=1}^{n}\delta_j = 1$, $\sum_{j=1}^{n}\gamma_{ij} = 0$.

- In general, this is not HSA But, under the "separability" condition,

$$\gamma_{ij} = \begin{cases}\gamma\beta_i(1-\beta_i), i = j \\ -\gamma\beta_i\beta_j, i \neq j\end{cases} \qquad \sum_{i=1}^{n}\beta_i = 1\,;\ \gamma \geq 0,$$

it is HSA with

$$s_i\left(\frac{p_i}{A(\mathbf{p})}\right) = \max\left\{\delta_i - \gamma\beta_i \ln\frac{p_i}{A(\mathbf{p})}, 0\right\}$$

If $\gamma = 0$, Cobb-Douglas; if $\gamma > 0$, *gross substitutes* with *the choke prices*, $\bar{z}_i A(\mathbf{p}) = \exp\left(\frac{\delta_i}{\gamma\beta_i}\right)A(\mathbf{p})$ *and inessential.*

- For $p_i < \bar{z}_i A(\mathbf{p})$ for all i, $A(\mathbf{p})$ is the *weighted geometric mean* of prices:

$$\ln A(\mathbf{p}) = \sum_{i=1}^{n}\beta_i \ln p_i$$

$$P(\mathbf{p}) = Z\cdot\exp\left\{\sum_{i=1}^{n}\delta_i \ln p_i - \frac{\gamma}{2}\left[\sum_{i=1}^{n}\beta_i(\ln p_i)^2 - \left(\sum_{i=1}^{n}\beta_i \ln p_i\right)^2\right]\right\} \neq A(\mathbf{p}).$$

Example 10*: "Separable" translog production function as a Special Case of HSA

$$X(\mathbf{x}) = Z \exp\left[\sum_{i=1}^{n} \delta_i \ln x_i - \frac{1}{2}\sum_{i,j=1}^{n} \gamma_{ij} \ln x_i \ln x_j\right]$$

$\delta_i > 0$; (γ_{ij}) is symmetric and non-negative semidefinite, normalized as $\sum_{j=1}^{n} \delta_j = 1$, $\sum_{j=1}^{n} \gamma_{ij} = 0$

- In general, this is not HSA But, under the "separability" condition,

$$\gamma_{ij} = \begin{cases} \gamma\beta_i(1-\beta_i), i = j \\ -\gamma\beta_i\beta_j, i \neq j \end{cases} \qquad \sum_{i=1}^{n} \beta_i = 1\,;\ \gamma \geq 0.$$

it is HSA with

$$s_i^*\left(\frac{x_i}{A^*(\mathbf{x})}\right) = \max\left\{\delta_i - \gamma\beta_i \ln\frac{x_i}{A^*(\mathbf{x})}, 0\right\}.$$

If $\gamma = 0$, Cobb-Douglas; if $\gamma > 0$, *gross complements* with *the saturation point,* $\bar{y}_i A^*(\mathbf{x}) = \exp\left(\frac{\delta_i}{\gamma\beta_i}\right) A^*(\mathbf{x})$, *essential*

- For $x_i < \bar{y}_i A^*(\mathbf{x})$, $A^*(\mathbf{x})$ is the *weighted geometric mean* of quantities:

$$\ln A^*(\mathbf{x}) = \sum_{i=1}^{n} \beta_i \ln x_i$$

$$X(\mathbf{x}) = Z \cdot \exp\left\{\sum_{i=1}^{n} \delta_i \ln x_i - \frac{\gamma}{2}\left[\sum_{i=1}^{n} \beta_i (\ln x_i)^2 - \left(\sum_{i=1}^{n} \beta_i \ln x_i\right)^2\right]\right\} \neq A^*(\mathbf{x})$$

Example 11: HSA with Constant but Different Price Elasticities

$$s_i\left(\frac{p_i}{A(\mathbf{p})}\right) = \beta_i\left(\frac{p_i}{A(\mathbf{p})}\right)^{1-\sigma_i} \Leftrightarrow s_i^*\left(\frac{x_i}{A^*(\mathbf{x})}\right) = \beta_i^{\frac{1}{\sigma_i}}\left(\frac{x_i}{A^*(\mathbf{x})}\right)^{1-\frac{1}{\sigma_i}},$$

where either $\sigma_i \leq 1$ for all i, or $\sigma_i \geq 1$ for all i, and $A(\mathbf{p})$ and $A^*(\mathbf{x})$ are given implicitly by

$$\sum_{i=1}^{n}\beta_i\left(\frac{p_i}{A(\mathbf{p})}\right)^{1-\sigma_i} = \sum_{i=1}^{n}\beta_i^{1/\sigma_i}\left(\frac{x_i}{A^*(\mathbf{x})}\right)^{1-1/\sigma_i} = 1.$$

- Elasticity of substitution btw each pair is *not* constant, unless $\sigma_i = \sigma$ for all i.
- $\zeta_i\left(\frac{p_i}{A(\mathbf{p})}\right) = \zeta_i^*\left(\frac{x_i}{A^*(\mathbf{x})}\right) = \sigma_i$.
  - Holding $A(\mathbf{p})$ or $A^*(\mathbf{x})$ fixed, the price elasticity of each factor is constant but different.
  - For a large n, the impact of a change in p_i on $A(\mathbf{p})$ and the impact of a change in x_i on $A^*(\mathbf{x})$ are negligible. → The price elasticity is approximately constant but different, converging to σ_i, as $n \to \infty$.
- This example can isolate the role of price elasticity differences across factors, unlike
  - Example 5, $X(\mathbf{x}) = \left[\sum_{i=1}^{n}(\beta_i)^{\frac{1}{\sigma_i}}(x_i)^{1-\frac{1}{\sigma_i}}\right]^{\frac{\sigma_0}{\sigma_0-1}}$, direct addilog = CRES =CRIE, is not homothetic.
  - Example 6, $U(\mathbf{p}/E) = \left[\sum_{i=1}^{n}\beta_i(E/p_i)^{\sigma_i-1}\right]^{\frac{1}{\sigma_0-1}}$ indirect addilog = CDES =CDIE, is not homothetic.

Homothetic Direct Implicit Additivity (HDIA)

&

Homothetic Indirect Implicit Additivity (HIIA)

Definition: Homothetic Direct Implicit Additivity (HDIA)

- $X(\mathbf{x})$ is *homothetic with direct implicit additivity* (HDIA) if defined implicitly as

$$\sum_{i=1}^{n} \phi_i\left(\frac{x_i}{X(\mathbf{x})}\right) = 0$$

$\phi_i: \mathbb{R}_+ \to \mathbb{R}$; strictly increasing, and strictly concave, and satisfy

$$\sum_{i=1}^{n} \phi_i(0) < 0 < \sum_{i=1}^{n} \phi_i(\infty).$$

- Cobb-Douglas and CES are special cases:

$$\phi_i(y_i) = \alpha_i \ln\left(\frac{Z y_i}{\alpha_i}\right) \Longrightarrow X(\mathbf{x}) = Z \prod_{i=1}^{n} \left(\frac{x_i}{\alpha_i}\right)^{\alpha_i}$$

$$\phi_i(y_i) = \beta_i \frac{(Z y_i/\beta_i)^{1-1/\sigma} - 1}{1 - 1/\sigma} \Longrightarrow X(\mathbf{x}) = Z\left(\sum_{i=1}^{n} \beta_i^{1/\sigma} x_i^{1-1/\sigma}\right)^{\frac{1}{1-1/\sigma}}$$

If $\sigma > 1, \phi_i(y_i)$ is unbounded from above, bounded from below; and $0 < -\frac{y_i \phi_i''(y_i)}{\phi_i'(y_i)} = 1/\sigma < 1$;

If $\sigma < 1$, $\phi_i(y_i)$ is unbounded from below, bounded from above; and $-\frac{y_i \phi_i''(y_i)}{\phi_i'(y_i)} = 1/\sigma > 1$.

Demand System under HDIA

$$\frac{\partial \ln P(\mathbf{p})}{\partial \ln p_i} = \frac{p_i}{P(\mathbf{p})}({\phi_i}')^{-1}\left(\frac{p_i}{B(\mathbf{p})}\right), \qquad where \sum_{k=1}^{n} \phi_k\left(({\phi_k}')^{-1}\left(\frac{p_k}{B(\mathbf{p})}\right)\right) \equiv 0; \quad P(\mathbf{p}) = \sum_{k=1}^{n} p_k(\phi_k')^{-1}\left(\frac{p_k}{B(\mathbf{p})}\right);$$

Inverse Demand System under HDIA

$$\frac{\partial \ln X(\mathbf{x})}{\partial \ln x_i} = \frac{x_i}{C^*(\mathbf{x})}\phi_i'\left(\frac{x_i}{X(\mathbf{x})}\right), \quad where\ C^*(\mathbf{x}) \equiv \sum_{k=1}^{n} x_k \phi_k'\left(\frac{x_k}{X(\mathbf{x})}\right),$$

Notes:

$$\frac{P(\mathbf{p})}{B(\mathbf{p})} = \sum_{k=1}^{n} \frac{p_k}{B(\mathbf{p})}(\phi_k')^{-1}\left(\frac{p_k}{B(\mathbf{p})}\right) = \sum_{k=1}^{n} \phi_k'\left(\frac{x_k}{X(\mathbf{x})}\right)\frac{x_k}{X(\mathbf{x})} = \frac{C^*(\mathbf{x})}{X(\mathbf{x})}$$

- For $n > 2$, $P(\mathbf{p})/B(\mathbf{p}) = C^*(\mathbf{x})/X(\mathbf{x}) = c > 0$, iff HDIA is a CES.

$$-\frac{\partial \ln(x_i/X(\mathbf{x}))}{\partial \ln(p_i/B(\mathbf{p}))} = -\frac{{\phi_i}'(y_i)}{y_i {\phi_i}''(y_i)} \equiv \zeta_i^D(y_i) > 0.$$

We call factor- i as a *gross substitute* (*gross complement*) when $\zeta_i^D(y_i) > (<)1$.

Definition: Homothetic Indirect Implicit Additivity (HIIA)

- $P(\mathbf{p})$ is *homothetic with indirect implicit additivity* (HIIA) if defined implicitly as

$$\sum_{i=1}^{n} \theta_i\left(\frac{p_i}{P(\mathbf{p})}\right) = 0$$

$\theta_i: \mathbb{R}_+ \to \mathbb{R}$ are strictly increasing and strictly concave, and satisfy

$$\sum_{i=1}^{n} \theta_i(0) < 0 < \sum_{i=1}^{n} \theta_i(\infty)$$

- Cobb-Douglas and CES are special cases where

$$\theta_i(z_i) = \alpha_i \log\left(\frac{z_i}{Z}\right) \Rightarrow P(\mathbf{p}) = \frac{1}{Z}\prod_{i=1}^{n} p_i^{\alpha_i}$$

$$\theta_i(z_i) = \beta_i \frac{(z_i/Z)^{1-\sigma} - 1}{1-\sigma} \Rightarrow P(\mathbf{p}) = \frac{1}{Z}\left(\sum_{i=1}^{n} \beta_i p_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

If $\sigma > 1, \theta_i(z_i)$ is unbounded from below; bounded from above; and $-\frac{z_i \theta_i{}''(z_i)}{\theta_i{}'(z_i)} = \sigma > 1$;

If $\sigma < 1, \theta_i(z_i)$ is unbounded from above; bounded from below; and $0 < -\frac{z_i \theta_i{}''(z_i)}{\theta_i{}'(z_i)} = \sigma < 1$;

Demand System under HIIA

$$\frac{\partial \ln P(\mathbf{p})}{\partial \ln p_i} = \frac{p_i}{C(\mathbf{p})}\theta_i'\left(\frac{p_i}{P(\mathbf{p})}\right), \qquad where\ C(\mathbf{p}) \equiv \sum_{k=1}^{n} p_k\theta_k'\left(\frac{p_k}{P(\mathbf{p})}\right);$$

Inverse Demand System under HIIA

$$\frac{\partial \ln X(\mathbf{x})}{\partial \ln x_i} = \frac{x_i}{X(\mathbf{x})}(\theta_i{}')^{-1}\left(\frac{x_i}{B^*(\mathbf{x})}\right), where\ \sum_{k=1}^{n}\theta_k\left((\theta_k{}')^{-1}\left(\frac{x_k}{B^*(\mathbf{x})}\right)\right) = 0; X(\mathbf{x}) = \sum_{k=1}^{n} x_k(\theta_k')^{-1}\left(\frac{x_k}{B^*(\mathbf{x})}\right),$$

Notes:

$$\frac{X(\mathbf{x})}{B^*(\mathbf{x})} = \sum_{k=1}^{n}\frac{x_k}{B^*(\mathbf{x})}(\theta_k')^{-1}\left(\frac{x_k}{B^*(\mathbf{x})}\right) = \sum_{k=1}^{n}\theta_i{}'\left(\frac{p_i}{P(\mathbf{p})}\right)\frac{p_i}{P(\mathbf{p})} = \frac{C(\mathbf{p})}{P(\mathbf{p})}$$

- For $n > 2$, $C(\mathbf{p})/P(\mathbf{p}) = X(\mathbf{x})/B^*(\mathbf{x}) = c > 0$, iff HIIA is a CES.

$$-\frac{\partial \ln(x_i/B^*(\mathbf{x}))}{\partial \ln(p_i/P(\mathbf{p}))} = -\frac{z_i\theta_i{}''(z_i)}{\theta_i{}'(z_i)} \equiv \zeta_i^I(z_i) > 0.$$

We call factor- i as a *gross substitute* (*gross complement*) if $\zeta_i^I(z_i) > (<)1$.

Essential vs Inessential Factors under HDIA

- Factor i is *essential* iff $\phi_i(0) + \sum_{k\neq i}^{n} \phi_k(y_k) < 0$ for all $y_k > 0$. [Under CES, this condition *always* holds for $\sigma \leq 1$, since $\phi_i(y_i)$ is unbounded from below, but *never* hold for $\sigma > 1$, since $\phi_i(y_i)$ is unbounded from above.]
- Let $\phi_i(y_i) = \beta_i g(y_i)$, $0 < \beta_i < 1$; $g(y_i)$ strictly increasing & concave with $-\infty < g(0) < 0 < g(\infty) < \infty$.
  - $\zeta_i^D(y_i) = -\frac{g'(y_i)}{g''(y_i)y_i} > 0$ can be arbitrary, except $y_i \to 0$ and $y_i \to \infty$.
  - Yet, Factors $i = 1, \dots, j$ are *essential* & Factors $i = j+1, \dots, n$ are *inessential*, if $\beta_i > 0$ is decreasing in i and

$$\frac{\beta_j}{1-\beta_j} > -\frac{g(\infty)}{g(0)} > \frac{\beta_{j+1}}{1-\beta_{j+1}} > 0.$$

Essential vs Inessential Factors under HIIA

- Factor i is *essential* iff $\theta_i(\infty) + \sum_{k\neq i}^{n} \theta_k(z_k) > 0$ for all $z_k > 0$. [Under CES, this condition *always* holds for $\sigma \leq 1$, since $\theta_i(z_i)$ is unbounded from above, but *never* hold for $\sigma > 1$, since $\theta_i(z_i)$ is unbounded from below.]
- Let $\theta_i(z_i) = \beta_i g(z_i), 0 < \beta_i < 1$; $g(z_i)$ strictly increasing & concave with $-\infty < g(0) < 0 < g(\infty) < \infty$.
  - $\zeta_i^I(z_i) = -\frac{g''(z_i)z_i}{g'(z_i)} > 0$ can be arbitrary, except $z_i \to 0$ and $z_i \to \infty$.
  - Yet, Factors $i = 1, \dots, j$ are *essential* and Factors $i = j+1, \dots, n$ are *inessential*, if $\beta_i > 0$ is decreasing in i and

$$\frac{\beta_j}{1-\beta_j} > -\frac{g(0)}{g(\infty)} > \frac{\beta_{j+1}}{1-\beta_{j+1}} > 0.$$

Example 12: A Hybrid of Cobb-Douglas and CES under HDIA

$$\phi_i(y_i) = \varepsilon\alpha_i \log\left(\frac{Zy_i}{\alpha_i}\right) + (1-\varepsilon)\beta_i \frac{(Zy_i/\beta_i)^{1-1/\sigma} - 1}{1-1/\sigma} \Longrightarrow \zeta_i^D(y_i) = \frac{\varepsilon\alpha_i + (1-\varepsilon)\beta_i(Zy_i/\beta_i)^{1-1/\sigma}}{\varepsilon\alpha_i + (1/\sigma)(1-\varepsilon)\beta_i(Zy_i/\beta_i)^{1-1/\sigma}}$$

where $0 < \varepsilon < 1$, $\alpha_i \geq 0$, $\beta_i > 0$, $\sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} \beta_k = 1$.

Example 14: A Hybrid of Cobb-Douglas and CES under HIIA

$$\theta_i(z_i) = \varepsilon\alpha_i \log\left(\frac{z_i}{Z}\right) + (1-\varepsilon)\beta_i \frac{(z_i/Z)^{1-\sigma} - 1}{1-\sigma} \Longrightarrow \zeta_i^I(z_i) = \frac{\varepsilon\alpha_i + \sigma(1-\varepsilon)\beta_i(z_i/Z)^{1-\sigma}}{\varepsilon\alpha_i + (1-\varepsilon)\beta_i(z_i/Z)^{1-\sigma}},$$

where $0 < \varepsilon < 1$, $\alpha_i \geq 0$, $\beta_i > 0$, $\sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} \beta_k = 1$

Similar to Example 9 and Example 9* under HSA

Example 13: HDIA with Constant but Different Price Elasticities

$$\phi_i(y_i) = \beta_i \frac{(Zy_i/\beta_i)^{1-1/\sigma_i} - 1}{1 - 1/\sigma_i} \Longrightarrow \zeta_i^D(y_i) = \sigma_i$$

CRESH (Hanoch 1971) is a special case of this example, where $\sigma_i > 1$ for at least some i.

Example 15: HIIA with Constant but Different Price Elasticities

$$\theta_i(z_i) = \beta_i \frac{(z_i/Z)^{1-\sigma_i} - 1}{1 - \sigma_i} \Longrightarrow \zeta_i^I(z_i) = \sigma_i$$

This corresponds to CDESH (what Hanoch 1975 called "homothetic CDE").

In both examples,

- Elasticity of substitution btw each pair is *not* constant, unless $\sigma_i = \sigma$ for all i.
- Holding the aggregators fixed, the price elasticity of each factor is constant but different.
- For a large n, the impacts of a change in p_i or in x_i on the aggregators are negligible. → The price elasticity is approximately constant but different, converging to σ_i, as $n \to \infty$.
- These examples can isolate the role of price elasticity differences across factors. Unlike Examples 5 and 6, they are homothetic. Unlike Example 11, also homothetic, no need to impose neither $\sigma_i \leq 1$ for all i nor $\sigma_i \geq 1$ for all i.

An Important Topic Missing in this Review

Applications of non-CES demand systems to monopolistic competition (MC)

For this, we need to address a whole set of additional issues.

- Redefine the demand systems over *a continuum of* product varieties (to ensure MC firms to no power to affect the aggregate).
- Restrict to the case of *gross substitutes* (to ensure MC firms to face positive *marginal revenue* curves)
- Restrict further for monotonicity of *marginal revenue curves* (to ensure the pricing decision of MC firms to be well-behaved)
- Restrict to the case of inessentials (to allow for entry and exit and for endogenous product variety)
- Restrict further to ensure the existence (and uniqueness) of free-entry equilibriums.

These considerations change the relative merits of different classes of non-CES.