# When Does Procompetitive Entry Imply Excessive Entry?*

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**Abstract**

The Dixit-Stiglitz model of monopolistic competition is widely used as a building block across many applied general equilibrium fields. Two of its remarkable features are the invariance of the markup rate and the optimality of the free-entry equilibrium. Of course, neither of these two features is robust. Departure from CES makes entry either *procompetitive* or *anticompetitive* (i.e., the markup rate either goes down or goes up as more firms enter). Departure from CES also makes entry either *excessive* or *insufficient*. But how is the condition for procompetitive vs. anticompetitive entry related to that for excessive vs. insufficient entry? To investigate this question, we extend the Dixit-Stiglitz monopolistic competition model to three classes of homothetic demand systems, which are mutually exclusive except that each of them contains CES as a knife-edge case. In all three classes, we show, among others, that entry is excessive (insufficient) when it is *globally* procompetitive (anticompetitive) and that, in the presence of the choke price, entry is procompetitive and excessive at least for a sufficiently large market size.

*Keywords:* Procompetitive vs. Anticompetitive entry; Excessive vs. Insufficient entry; Monopolistic competition; symmetric homothetic demand systems with gross substitutes

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# 1. Introduction

The monopolistic competition model with symmetric CES demand systems with gross substitutes, developed in Dixit-Stiglitz (1977, Section I), is widely used as a building block across many applied general equilibrium fields, most notably in macroeconomics and international trade. Among its remarkable features are the invariance of the markup rate and the optimality of the free-entry equilibrium.[1] Of course, neither of these two features is robust. Once we depart from the knife-edge case of CES, the markup rate charged by each firm would change, as more firms enter in response to a market size increase. The markup rate may either go down (the case of *procompetitive* entry) or go up (the case of *anticompetitive* entry). Departure from CES would also lead to the inefficiency of the free-entry equilibrium. There may be either too many firms operating and hence too much product variety being offered (the case of *excessive* entry/variety), or too few firms operating and too little product variety being offered (the case of *insufficient* entry/variety).

But how is the condition for procompetitive vs. anticompetitive entry related to that for excessive vs. insufficient entry? Of course, unless we impose some restrictions on the demand system, it is possible to have all four combinations (procompetitive-excessive, procompetitive-insufficient, anticompetitive-excessive, and anticompetitive-insufficient). After all, CES leads to the markup rate invariance and the optimality of the free-entry equilibrium for different reasons. The markup rate is invariant under CES, because each firm faces demand curve whose price elasticity is exogenously constant. It depends entirely on the *local* property of the demand *curve*, that is, it continues to hold as long as the price elasticity is constant around the point chosen by each firm. In contrast, the optimality of the free-entry equilibrium depends on the *global* property of the demand *system*. To understand this, recall that there are two sources of externalities in monopolistic competition with entry, as discussed in Tirole (1988, Chapter 7) and Matsuyama (1995; Section 3E) among many others. They are the inability of a firm to fully appropriate its social surplus its entry generates, which creates positive externalities, and its failure to account for business stealing from other firms, which creates negative externalities[2]. It turns out that

[1]To be precise, there exists an outside competitive sector in Dixit and Stiglitz (1977). Due to intersectoral distortion, there may be too little entry to the monopolistically competitive sector. Yet, they showed that the resource allocation *within* the monopolistically competitive sector is optimal at the free-entry equilibrium under CES.

[2]Mankiw and Whinston (1986) and Suzumura and Kiyono (1987) showed that such a business stealing effect causes excessive entry in a homogenous goods industry in partial equilibrium settings. Mankiw and Whinston also showed that the excessive entry result could be overturned if different firms produce imperfect substitutes.

these two sources of externalities, one positive and one negative, exactly cancel out each other under CES.

This does not mean, however, that the condition for procompetitive vs. anticompetitive entry and that for excessive vs. insufficient entry are unrelated. On the contrary, we argue there is a tight connection between the two. Departing from the knife-edge case of CES, where the equilibrium entry is optimal, in the direction of making entry procompetitive exacerbates negative externalities to other firms by reducing their profit margin, which could cause excessive entry. Likewise, departing from the knife-edge case of CES in the direction of making entry anticompetitive mitigates negative externalities to other firms by increasing their profit-margin, which could cause insufficient entry. Of course, the above argument is only partial, because making the entry procompetitive or anticompetitive might also affect positive externalities due to the other source of inefficiency. Nevertheless, it suggests that, *under some additional conditions*, procompetitive entry is excessive and anticompetitive entry is insufficient.

So, the right question to ask is: "*under which conditions* does procompetitive entry imply excessive entry and *under which conditions* does anticompetitive entry imply insufficient entry?" To investigate this question, we extend the Dixit-Stiglitz monopolistic competition model with symmetric firms under the CES demand system with gross substitutes to three classes of symmetric homothetic demand systems with gross substitutes, each named for its defining properties, *Homotheticity with a Single Aggregator* (H.S.A.), *Homotheticity with Direct Implicit Additivity* (H.D.I.A.), and *Homotheticity with Indirect Implicit Additivity* (H.I.I.A.). We have chosen these three classes for several reasons.

First, they are all *homothetic*. Even though there have been many attempts to develop monopolistic competition models without CES, they have typically done so by making the demand system nonhomothetic. However, in order to *isolate* the efficiency implications of the markup rate being responsive to entry caused by a market size change, we need to avoid introducing the scale effect of a market size change operating through nonhomotheticity. In addition, it is desirable to maintain homotheticity for our departure from CES to be useful as a building block in applied general equilibrium models. This is because monopolistic competition with free entry is typically used as a way to endogenize sector-level productivity in multi-sector models, where its output in each sector is produced competitively with *constant returns to scale*

(CRS) technologies, which generate *homothetic* demand for differentiated inputs, which are supplied by monopolistically competitive producers.

Second, these three classes are mutually exclusive except that they all contain CES as a knife-edge case; see Figure 1, adopted from Matsuyama and Ushchev (2017)[3]. These three classes thus offer three alternative ways of departing from CES, while respecting the homotheticity requirement.

Third, they are all defined nonparametrically and hence flexible enough to allow for any downward-sloping demand curve each firm might face. In particular, each of these three demand systems allows the possibility that entry can be either procompetitive or anticompetitive, as well as the possibility that it can be either excessive or insufficient.

Fourth, in spite of such flexibility, they remain tractable. This is not only because the entry and pricing behavior of other firms affect the demand curve each firm faces only through one aggregator (under H.S.A., as the name suggests) or two aggregators (under H.D.I.A. and H.I.I.A.), but also because the entry and pricing behavior of other firms affect the price elasticity of demand each firm faces only through one aggregator under all three classes.[4] Due to this tractability, one could identify the additional restrictions on the primitives that ensure the existence and uniqueness of the symmetric free-entry equilibrium for any level of market size, and the unique equilibrium is analytically solvable. This facilitates the comparative statics and welfare analysis. In particular, the condition for procompetitive entry turns out to be equivalent to Marshall's second law of demand (i.e., the price elasticity of the demand curve each firm faces increases in its price, holding everything else, including prices of other firms, constant) under these three classes.[5] The condition for excessive entry can also be obtained explicitly, and hence can be compared with the condition for procompetitive entry.

Here are our main findings. In all three classes, entry is excessive when it is *globally* procompetitive. By "globally" procompetitive, we mean that the equilibrium markup rate is

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[3]Matsuyama and Ushchev (2017, Proposition 4) proved that these classes are pairwise-disjoint with the sole exception of CES, even without restricting to be symmetric with gross substitutes. However, in this paper, we impose these restrictions to make them applicable to the Dixit-Stiglitz environment.
[4]Under H.S.A., the effect of behaviors of competing firms on the demand curve each firm faces is summarized by only one aggregator, and hence so is the effect on its price elasticity. Under H.D.I.A. and H.I.I.A., the effect of the demand curve each firm faces is summarized by two aggregators, but only one of them can affect its price elasticity, because the other aggregator enters the demand curve multiplicatively.
[5]In general, Marshall's second law of demand is neither sufficient nor necessary for procompetitive entry, since the former is about the property of the individual demand curve, while the latter is about the property of the entire demand system.

*monotonically* decreasing in market size. Likewise, in all three classes, entry is insufficient when it is *globally* anticompetitive, that is, when the equilibrium markup rate is *monotonically* increasing in market size.[6] Between these two cases lies the borderline case of CES, where the markup rate is globally independent of market size and entry is always efficient. One important implication of these findings, as visualized in Figure 2, is that, for those who believe that procompetitive entry is the empirically relevant case, entry is excessive, which suggests that (small) regulation of entry is welfare-improving, at least in the absence of any other distortions.

We also show that entry is procompetitive and excessive for a sufficiently large market size in the presence of the choke price.[7] This is because the price elasticity goes to infinity at the choke price. This means that, as market size increases and more firms enter, each firm is forced to operate close to the choke price, that is, in the range where the price elasticity is increasing and the markup rate is decreasing in market size.

There have been many attempts to extend the Dixit-Stiglitz monopolistic competition models under CES to non-CES demand systems, starting from Dixit and Stiglitz (1977, Section II). However, as already indicated, virtually all of them have done so by making the demand system nonhomothetic.[8] Feenstra (2003) is an exception. He used symmetric homothetic translog as an alternative to CES, which exhibits the procompetitive effect with a choke-price, thereby ruling out the possibility of anticompetitive entry. Furthermore, he did not investigate how the equilibrium and optimal allocations differ from each other. Since symmetric homothetic translog is a special case of symmetric H.S.A., our analysis suggests excessive entry and hence a welfare-improving entry regulation under translog. Kimball (1995) considered the class of symmetric demand systems identical to symmetric H.D.I.A., except that he assumed an exogenous set of firms producing an exogenous set of products. By ruling out entry by assumption, he did not need to worry about imposing additional restrictions to ensure the

[6]The qualification that the equilibrium markup rate responds monotonically is important. In all three classes, we show by means of counterexamples that, if the equilibrium markup rate responds *nonmonotonically*, entry can be procompetitive and yet insufficient or it can be anticompetitive and yet excessive in some range of parameter values.
[7]A choke price exists if demand for a product goes to zero at a finite price. There exists no choke price under CES.
[8]For example, Dixit and Stiglitz (1977, Section II) extended their monopolistic competition model to a class of demand systems, which have been further explored by Behrens and Murata (2007), Zhelobodko, Kokovin, Parenti, and Thisse (2012), Dhingra and Morrow (2019), Latzer, Matsuyama, and Parenti (2019), among many others. Though Dixit and Stiglitz called this class, "Variable Elasticity Case," the well-known Bergson's Law states that, within this class of demand systems, they are homothetic if and only if they are CES. In other words, any departure from CES within this class introduces nonhomotheticity. See Parenti, Thisse, and Ushchev (2017) and Thisse and Ushchev (2018) for more discussions on this issue with extensive references.

existence and uniqueness of the free-entry equilibrium, as we do, and he did not address any of the issues we are interested in. To the best of our knowledge, this is the first paper to offer a full characterization of the free-entry equilibrium of monopolistic competition models under symmetric H.S.A., H.D.I.A., and H.I.I.A., which we hope would be useful for many other applications.[9]

Indeed, very few have ever investigated the question of excessive vs. insufficient entry in monopolistic competition even under nonhomothetic, non-CES demand systems. Exceptions are Dixit and Stiglitz (1977, Section II), Dhingra and Morrow (2019), Nocco, Ottaviano, and Salto (2014) and Behrens, Mion, Murata, and Südekum (2020). The latter two used the parametric families of non-CES demand systems that rule out anticompetitive entry. Although the nonparametric demand system used by both Dixit and Stiglitz and Dhingra and Morrow is flexible enough to allow for the possibility of both procompetitive and anticompetitive entry, neither of them investigated how the condition for excessive vs. insufficient entry is related to the condition for procompetitive vs. anticompetitive entry. Due to the nonhomotheticity, which introduces additional effects of a market size increase, we do not expect our results to be extended to their settings.

The rest of the paper is organized as follows. In Section 2, we present what we call the Dixit-Stiglitz environment, the common setting across all three classes. Then, we turn to each of the three classes, H.S.A. (Section 3), H.D.I.A. (Section 4), and H.I.I.A. (Section 5). We start with H.S.A. because it is the easiest of the three.[10] However, these three sections are written in such a way that they can be read independently and in any order. Indeed, we have made conscious effort to keep the structure of these sections as similar as possible. In each section, we first define the class of symmetric homothetic demand systems, and explain its key properties. Then, we address the firm's behavior, impose the conditions that ensure the existence and uniqueness of symmetric free-entry equilibrium, and solve for it explicitly (Propositions 1, 4, and 7, respectively). Then,

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[9] In related work, Matsuyama and Ushchev (2020b) study *parametric* families of *asymmetric* H.S.A, H.D.I.A., and H.I.I.A., which features a constant pass-through rate, common across otherwise *heterogenous* monopolistically competitive firms without entry. The goal of that paper is to propose asymmetric demand systems for monopolistic competition that are tractable even with firms that are heterogenous in many dimensions.

[10] For example, under symmetric H.S.A., the condition that rules out asymmetric equilibria also ensures the uniqueness of symmetric free-entry equilibrium. Under symmetric H.D.I.A. and H.I.I.A., the two separate conditions are required. The cases of symmetric H.D.I.A. and H.I.I.A. also differ from each other in subtle ways. For example, the condition that ensures the existence of the unique symmetric equilibrium is enough to ensure the existence of the social optimum allocation under H.I.I.A., but not under H.D.I.A.

we conduct the comparative statics to characterize the condition for procompetitive vs. anticompetitive entry (Propositions 2, 5, and 8, respectively), and perform the welfare analysis to characterize the condition for excessive vs. insufficient entry (Propositions 3, 6, and 9, respectively). Then, we investigate the connection between the two conditions (Theorems 1, 2, and 3, respectively), and illustrate the theorems by three examples; one with global monotonicity (Examples 1, 4, and 7, respectively), one with a choke price (Examples 2, 5, and 8, respectively), and one without global monotonicity (Examples 3, 6, and 9, respectively). We conclude in Section 6. Technical proofs for some lemmas related to H.D.I.A. and H.I.I.A. are gathered in the two appendices.

## 2. The Dixit-Stiglitz Environment

Consider the economy endowed with $L$ units of the single factor of production, which we shall call "labor" and take as the numeraire. Labor is used to produce a continuum of varieties of differentiated intermediate inputs, which are in turn assembled to produce the single final good.

### *2.1. Competitive Final Goods Producers and Their Demand for Intermediate Inputs*

The final good is produced competitively by using CRS technology, given by $X = X(\mathbf{x})$, where $\mathbf{x} = \{x_\omega;\ \omega \in \Omega\}$ is a quantity vector of intermediate inputs, with $\omega$ being the index of a particular input variety, and $\Omega$ being the set of input varieties available. It is assumed that $X(\mathbf{x})$ satisfies linear homogeneity, strict monotonicity, quasi-concavity, and symmetry, for each $\Omega$. The unit cost function corresponding to $X = X(\mathbf{x})$ can be obtained by:

$$P = P(\mathbf{p}) \equiv \min_{\mathbf{x}} \left\{ \mathbf{px} = \int_\Omega p_\omega x_\omega d\omega \,\middle|\, X(\mathbf{x}) \geq 1 \right\}, \tag{1}$$

where $\mathbf{p} = \{p_\omega;\ \omega \in \Omega\}$ is a price vector of intermediate inputs, and $P(\mathbf{p})$ also satisfies linear homogeneity, strict monotonicity, quasi-concavity, and symmetry, for each $\Omega$. Conversely, starting from any linear homogeneous, strictly monotonic, quasi-concave and symmetric $P(\mathbf{p})$, one could recover the underlying linear homogenous, strictly monotonic, quasi-concave and symmetric production function as follows:

$$X = X(\mathbf{x}) \equiv \min_{\mathbf{p}} \left\{ \mathbf{px} = \int_\Omega p_\omega x_\omega d\omega \,\middle|\, P(\mathbf{p}) \geq 1 \right\}. \tag{2}$$

Thus, either $X = X(\mathbf{x})$ or $P = P(\mathbf{p})$ can be used as a primitive of this CRS technology.

As is well-known from the duality theory, the cost minimization by competitive producers generates the demand curve and the inverse demand curve for each input,

$$x_\omega = X(\mathbf{x})\frac{\partial P(\mathbf{p})}{\partial p_\omega}; \qquad p_\omega = P(\mathbf{p})\frac{\partial X(\mathbf{x})}{\partial x_\omega},$$

from either of which one could show, using Euler's theorem on linear homogeneous functions,

$$\mathbf{px} = \int_\Omega p_\omega x_\omega d\omega = P(\mathbf{p})X(\mathbf{x}).$$

Furthermore, the market share of each input can be expressed as

$$\frac{p_\omega x_\omega}{\mathbf{px}} = \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega},$$

and the condition for a pair of inputs to be *gross substitutes* (i.e., the Hicks-Allen elasticity of substitution between the two is greater than one) can be written as:

$$-\frac{\partial \ln\left(\frac{x_{\omega_1}}{x_{\omega_2}}\right)}{\partial \ln\left(\frac{p_{\omega_1}}{p_{\omega_2}}\right)} = -\frac{\partial \ln\left[\frac{\partial P(\mathbf{p})/\partial p_{\omega_1}}{\partial P(\mathbf{p})/\partial p_{\omega_2}}\right]}{\partial \ln\left(\frac{p_{\omega_1}}{p_{\omega_2}}\right)} = -\frac{\partial \ln\left(\frac{x_{\omega_1}}{x_{\omega_2}}\right)}{\partial \ln\left[\frac{\partial X(\mathbf{x})/\partial x_{\omega_1}}{\partial X(\mathbf{x})/\partial x_{\omega_2}}\right]} > 1.$$

### *2.2.Monopolistically Competitive Differentiated Intermediate Inputs Producers*

There is a continuum of intermediate input producing firms, also indexed by $\omega \in \Omega$, each producing a single variety of its own. They share the same IRS technology: producing $x > 0$ units of input requires $\psi x + F$ units of labor, where $F > 0$ is the fixed cost of entry, and $\psi > 0$ the marginal cost of production. (Recall that labor is taken as the *numeraire*.) Being monopolistically competitive, each firm sets its price and/or its quantity to maximize profit, subject to the downward-sloping demand curve it faces with the aggregate variables taken as given. There is free entry/exit, so that the maximized profit is equal to the fixed cost of entry, $F$, and hence the net profit is equal to zero in equilibrium. Thus, for each active firm $\omega \in \Omega$, $p_\omega x_\omega = \psi x_\omega + F$ holds, and hence

$$P(\mathbf{p})X(\mathbf{x}) = \mathbf{px} = \int_\Omega p_\omega x_\omega d\omega = \int_\Omega (\psi x_\omega + F) d\omega = \psi \int_\Omega x_\omega d\omega + VF = L,$$

where $V \equiv |\Omega|$ is the Lebesgue measure of $\Omega$. Thus, the aggregate market size is given by the total labor supply, $P(\mathbf{p})X(\mathbf{x}) = L$.

### *2.3.CES Benchmark*

The above setup is ubiquitous as a building block in many applied general equilibrium fields, particularly in international trade and macroeconomics (both in business cycles and economic growth). In addition, the vast majority of studies in these literatures assumes the assembly technology of the final good to be symmetric CES with gross substitutes:

$$X = X(\mathbf{x}) = Z\left[\int_{\Omega} {x_\omega}^{1-\frac{1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}} \Leftrightarrow P = P(\mathbf{p}) = \frac{1}{Z}\left[\int_{\Omega} {p_\omega}^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}},$$

which implies

$$\frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \left(\frac{x_\omega}{X(\mathbf{x})/Z}\right)^{1-\frac{1}{\sigma}} = \left(\frac{p_\omega}{ZP(\mathbf{p})}\right)^{1-\sigma},$$

where $\sigma > 1$ is the (exogenous and constant) elasticity of substitution between each pair of inputs, and $Z > 0$ a productivity parameter.

It is well-known (and will be verified later in this paper) that the CES assumption has some strong implications in this setup. First, it guarantees the equilibrium is unique and symmetric, $p_\omega = p$ and $x_\omega = x$ for all $\omega \in \Omega$. Second, at this unique equilibrium,

- each firm sells its own variety at the (common) exogenous markup rate; in particular, it is independent of market size, $L$;
- the equilibrium allocation is optimal; in particular, the equilibrium mass of firms that enter (and that of input varieties offered) is optimal.

Of course, neither of these two results, the market size neutrality on the markup rate, and the optimality of equilibrium entry, is robust. Depending on how we depart from the knife-edge CES assumption, we could have either the case of procompetitive entry or the case of anticompetitive entry, in which the markup rate goes either down or up in response to entry caused by a market size increase, as well as the case of excessive entry or the case of insufficient entry. But how are the cases of procompetitive or anticompetitive entry related to the cases of excessive or insufficient entry? We explore this question, using three alternative classes of CRS technologies, which are pairwise disjoint except that each contains CES as a knife-edge case.

## 3. Dixit-Stiglitz under H.S.A.

### *3.1.H.S.A. Demand System*

We call a symmetric CRS technology, $X = X(\mathbf{x})$ or $P = P(\mathbf{p})$, *homothetic with a single aggregator* (H.S.A.) if the market share of any input $\omega$, as a function of $\mathbf{p}$, can be written as:

$$\frac{p_\omega x_\omega}{\mathbf{px}} = \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{p_\omega x_\omega}{L} = \frac{\partial \ln P(\mathbf{p})}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(\mathbf{p})}\right). \tag{3}$$

Here, $s: \mathbb{R}_{++} \to \mathbb{R}_+$ is the *market share function,* and it is assumed to be twice continuously differentiable and *strictly decreasing* as long as $s(z) > 0$, with $\lim_{z\to 0} s(z) = \infty$ and $\lim_{z\to \bar{z}} s(z) = 0$, where $\bar{z} \equiv \inf\{z > 0 | s(z) = 0\}$, which can be finite or infinite, and $A(\mathbf{p})$ is linear homogenous in $\mathbf{p}$, defined implicitly and uniquely by

$$\int_\Omega s\left(\frac{p_\omega}{A(\mathbf{p})}\right) d\omega = 1, \tag{4}$$

which ensures, by construction, that the market shares of all inputs are added up to one.[11] By integrating eq.(3), one could verify that the unit cost function, $P(\mathbf{p})$, is related to $A(\mathbf{p})$, as:

$$\ln\left(\frac{P(\mathbf{p})}{A(\mathbf{p})}\right) = const. - \int_\Omega \left[\int_{p_\omega/A(\mathbf{p})}^{\bar{z}} \frac{s(\xi)}{\xi} d\xi\right] d\omega, \tag{5}$$

and it satisfies the linear homogeneity, monotonicity, quasi-concavity and symmetry and so does the production function, $X(\mathbf{x}) = L/P(\mathbf{p})$.[12]

Eqs.(3)-(4) state that the market share of any input $\omega$ is decreasing in its *relative price*, which is defined as its own price, $p_\omega$, divided by the *common price aggregator,* $A(\mathbf{p})$. Notice that $A(\mathbf{p})$ is independent of $\omega$. Thus, it is a *common* measure of the "toughness" of competition for all varieties, as it captures "the average price" against which the relative prices of *all* inputs are measured. In other words, one could keep track of all the cross-price effects in the demand system by looking at a single aggregator, $A(\mathbf{p})$, which is the key feature of H.S.A.[13] The monotonicity of $s(\cdot)$, combined with the assumptions, $\lim_{z\to 0} s(z) = \infty$ and $\lim_{z\to \bar{z}} s(z) = 0$, ensures that $A(\mathbf{p})$ is defined uniquely by eq.(4), for any $V \equiv |\Omega|$, the Lebesgue measure of $\Omega$.

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[11] For any market share function, $s: \mathbb{R}_{++} \to \mathbb{R}_+$, satisfying the above conditions, a class of the market share functions, $s_\lambda(z) \equiv s(\lambda z)$ for $\lambda > 0$, generates the same demand system, with $A_\lambda(\mathbf{p}) = \lambda A(\mathbf{p})$, because $s_\lambda(p_\omega/A_\lambda(\mathbf{p})) = s(\lambda p_\omega/A_\lambda(\mathbf{p})) = s(p_\omega/A(\mathbf{p}))$. In this sense, $s_\lambda(z) \equiv s(\lambda z)$ for $\lambda > 0$ are all equivalent.

[12] See Matsuyama and Ushchev (2017; Proposition 1-i)), which proved the existence of the underlying CRS production functions for more general cases, including the cases of asymmetry and gross complementarity.

[13] On the other hand, the assumption that $s(\cdot)$, is independent of $\omega$ is not a defining feature of H.S.A.; this is due to the symmetry of the production technology. More generally, H.S.A. class of demand systems is defined by the property that the market share of $\omega$ is given by $s_\omega(p_\omega/A(\mathbf{p}))$, where $A(\mathbf{p})$ is the unique solution to $\int_\Omega s_\omega(p_\omega/A(\mathbf{p}))d\omega = 1$. Note that $s_\omega(\cdot)$ depends on $\omega$ but $A(\cdot)$ does not.

The assumption that $s(\cdot)$ is strictly decreasing means that inputs are *gross substitutes.* To see this, one could show from eq.(3) that the elasticity of substitution between a pair of inputs, $\omega_1$ and $\omega_2$, evaluated at the same price, is

$$-\frac{\partial \ln(x_{\omega_1}/x_{\omega_2})}{\partial \ln(p_{\omega_1}/p_{\omega_2})}\bigg|_{p_{\omega_1}=p_{\omega_2}=p} = \zeta\left(\frac{p}{A(\mathbf{p})}\right) > 1$$

where $\zeta: (0, \bar{z}) \to (1, \infty)$ is defined by:

$$\zeta(z) \equiv 1 - \frac{zs'(z)}{s(z)} > 1.$$

Note that $\zeta(\cdot)$ is continuously differentiable for $z \in (0, \bar{z})$, and $\lim_{z\to\bar{z}} \zeta(z) = \infty$ if $\bar{z} < \infty$. Conversely, from any continuously differentiable $\zeta: (0, \bar{z}) \to (1, \infty)$, satisfying $\lim_{z\to\bar{z}} \zeta(z) = \infty$ if $\bar{z} < \infty$, one could recover the market share function as follows:

$$s(z) = \exp\left[\int_{z_0}^{z} \frac{1-\zeta(\xi)}{\xi} d\xi\right],$$

where $z_0 \in (0, \bar{z})$ is a constant. [14] Hence, we could also use $\zeta(\cdot)$ as a primitive of symmetric H.S.A. with gross substitutes, instead of the market share function, $s(\cdot)$.

Note also that we allow for the possibility of $\bar{z} < \infty$, that is, the existence of the choke (relative) price; if $\bar{z} = \infty$, the choke price does not exist and demand for each input always remains positive for any positive price vector.

Symmetric CES with gross substitutes is a special case of H.S.A., generated by $s(z) = \gamma z^{1-\sigma}$ ($\sigma > 1$). In this case, $P(\mathbf{p}) = cA(\mathbf{p})$, where $c > 0$ is a constant, and $\bar{z} = \infty$, so the choke price does not exist. Symmetric translog is another special case, generated by $s(z) = \max\{\gamma \ln(\bar{z}/z), 0\}$, with the choke price, $\bar{z} < \infty$. In this case, $P(\mathbf{p}) \neq cA(\mathbf{p})$ for any constant $c$. It turns out that, with the sole exception of CES, the RHS of eq.(5) depends on $\mathbf{p}$, and hence $P(\mathbf{p}) \neq cA(\mathbf{p})$ for any constant $c$, as will be shown in the Corollary 2 of Lemma 2.[15] This should not come as a total surprise. After all, $A(\mathbf{p})$ captures the *cross-price effects* in the demand

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[14]This constant implies that $s(\cdot)$ is determined up to a positive scalar multiplier. However, $\gamma s(z)$ with $\gamma > 0$ generates the same H.S.A. technology. All we need is to renormalize the indexation of varieties, as $\int_\Omega \gamma s(p_\omega/A)d\omega = \int_\Omega \gamma s(p_{\omega'}/A)d\omega' = 1$, with $\omega' = \gamma\omega$.

[15]This holds also for asymmetric H.S.A., as well as H.S.A. with gross complements. See Matsuyama and Ushchev (2017; Proposition 1-iii))

system, while $P(\mathbf{p})$ captures the *productivity (or welfare) effects* of price changes; there is no reason to think that they should move together in general.

### *3.2.Profit Maximization By Input Producing Firms under H.S.A.*

The profit of firm $\omega \in \Omega$ is given by $\pi_\omega = (p_\omega - \psi)x_\omega - F$, which can be written, using eq.(3), as:

$$\pi_\omega = \left(1 - \frac{\psi/A(\mathbf{p})}{z_\omega}\right) s(z_\omega)L - F,$$

where $z_\omega \equiv p_\omega/A(\mathbf{p})$ is its relative price. Firm $\omega$ chooses its relative price $z_\omega$ to maximize $\pi_\omega$, taking the aggregate variables, $A(\mathbf{p})$, as given. The FOC is

$$z_\omega\left(1 - \frac{1}{\zeta(z_\omega)}\right) = \frac{p_\omega}{A(\mathbf{p})}\left(1 - \frac{1}{\zeta(z_\omega)}\right) = \frac{\psi}{A(\mathbf{p})}, \tag{6}$$

and the SOC is

$$\zeta(z_\omega) - 1 + \frac{z_\omega \zeta'(z_\omega)}{\zeta(z_\omega)} > 0.$$

In what follows, we impose the following assumption to ensure that the FOC is sufficient for the global optimum.

**Assumption S1:** *For all* $z \in (0, \bar{z})$,

$$\frac{d}{dz}\left(z\left[1 - \frac{1}{\zeta(z)}\right]\right) = \frac{1}{\zeta(z)}\left[\zeta(z) - 1 + \frac{z\zeta'(z)}{\zeta(z)}\right] > 0.$$

*Or equivalently, for all* $z \in (0, \bar{z})$,

$$\frac{d}{dz}\ln\left(\frac{s(z)}{\zeta(z)}\right) = -\frac{1}{z}\left[\zeta(z) - 1 + \frac{z\zeta'(z)}{\zeta(z)}\right] < 0.$$

Under **S1**, the LHS of eq.(6) is strictly increasing in $z_\omega$. Hence, eq.(6) gives the unique profit-maximizing price for each firm. Thus, all firms set the same price, $p_\omega = p$, or $z_\omega = z$, and produce the same amount, $x_\omega = x$. Hence, under **S1**, asymmetric equilibria do not exist. Note also that $s(z)/\zeta(z)$ is strictly decreasing under **S1**; this ensures the uniqueness of the symmetric equilibrium, as will be seen below.

### *3.3.Symmetric Free-Entry Equilibrium under H.S.A.*

*A symmetric free-entry equilibrium* under H.S.A. satisfies the following conditions:

**H.S.A. integral condition**, given by eq. (4) under symmetry:

$$s(z)V = 1. \tag{7}$$

**Firm's pricing formula**, given by FOC, eq.(6) under symmetry:

$$1 - \frac{\psi}{p} = \frac{1}{\zeta(z)} \tag{8}$$

**Zero-profit (free-entry) condition**:

$$(p - \psi)x = F \tag{9}$$

**Resource constraint:**

$$(\psi x + F)V = L. \tag{10}$$

Note that, from eq.(9) and eq.(10),

$$pxV = PX = L. \tag{11}$$

By combining eqs.(7), (8), (9) and (11),

$$\frac{F}{L} = \frac{(p-\psi)x}{L} = \left(1 - \frac{\psi}{p}\right)\frac{px}{L} = \frac{1}{\zeta(z)}\frac{1}{V} = \frac{s(z)}{\zeta(z)}.$$

Under **S1**, RHS of this equation is strictly decreasing in $z \in (0, \bar{z})$. Furthermore, $\lim_{z\to 0} s(z)/\zeta(z) = \infty$ and $\lim_{z\to\bar{z}} s(z)/\zeta(z) = 0$. Hence, *for each* $L/F > 0$, the equilibrium value of $z$, $z^E \in (0, \bar{z})$, is uniquely pinned down by

$$\frac{s(z^E)}{\zeta(z^E)} = \frac{F}{L}, \tag{12}$$

and $z^E$ is increasing in $L/F$. By inserting this value into eqs.(7), (8), and (9),

$$V^E = \frac{1}{s(z^E)} = \frac{1}{\zeta(z^E)}\frac{L}{F},$$

$$p^E = \frac{\zeta(z^E)\psi}{\zeta(z^E) - 1} > 0, \tag{13}$$

$$x^E = \frac{[\zeta(z^E) - 1]}{\psi}F > 0,$$

from which one could also show

$$\frac{1}{A^E} = \frac{z^E}{p^E} = \frac{z^E}{\psi}\left(1 - \frac{1}{\zeta(z^E)}\right) = \frac{z^E}{\psi}\left(1 - \frac{F/L}{s(z^E)}\right),$$

and, using eq.(5),

$$\ln\frac{X^E}{L} = \ln\frac{1}{P^E} = \ln\frac{z^E}{\psi}\left(1-\frac{F/L}{s(z^E)}\right)+\frac{1}{s(z^E)}\int_{z^E}^{\bar{z}}\frac{s(\xi)}{\xi}d\xi + const.$$

Thus, we have shown:

**Proposition 1**. *Under **S1**, no asymmetric equilibria exist. Furthermore, there exists a unique symmetric free-entry equilibrium under H.S.A. for each* $L/F > 0$*, given by eq.*(12) *and eq.*(13).

### *3.4.Comparative Statics under H.S.A.: Procompetitive versus Anticompetitive*

We now turn to the comparative statics.

**Proposition 2**. *Assume **S1**. At the unique symmetric equilibrium in monopolistic competition under H.S.A., given by eq.*(12) *and eq.*(13),

*Procompetitive:*

$$\zeta'(z^E) > 0 \Rightarrow \frac{\partial p^E}{\partial L} < 0;\ 0 < \frac{\partial \ln V^E}{\partial \ln L} < 1;\ \frac{\partial x^E}{\partial L} > 0;$$

*Neutral (CES):*

$$\zeta'(z^E) = 0 \Rightarrow \frac{\partial p^E}{\partial L} = 0;\ \frac{\partial \ln V^E}{\partial \ln L} = 1;\ \frac{\partial x^E}{\partial L} = 0;$$

*Anticompetitive:*

$$\zeta'(z^E) < 0 \Rightarrow \frac{\partial p^E}{\partial L} > 0;\ \frac{\partial \ln V^E}{\partial \ln L} > 1;\ \frac{\partial x^E}{\partial L} < 0.$$

**Proof:** Since eq.(12) implies $\partial z^E/\partial L > 0$ under **S1**, this follows from eq.(13). ■

It is well-known that, in the knife-edge case of CES, the market size effect is neutral on the markup rate ($\partial p^E/\partial L = 0$) and the mass of firms increases proportionally ($\partial \ln V^E/\partial \ln L = 1$) without any effect on the firm size ($\partial x^E/\partial L = 0$). Thus, the expansion takes place only at the extensive margin. In the case of $\zeta'(z^E) > 0$, the market size effect is procompetitive, ($\partial p^E/\partial L < 0$) i.e., an increase in $L$ reduces the markup rate. This forces each firm to operate at a larger scale in order to break even ($\partial x^E/\partial L = 0$), and hence some expansion also takes place at the intensive margin ($\partial \ln V^E/\partial \ln L < 1$). In the opposite case of $\zeta'(z^E) < 0$, the market size effect is anticompetitive, i.e., the markup rate increases in response to an increase in $L$, which causes a more-than-proportionate increase in the mass of firms and forces each firm to operate at a smaller scale.

It should be also pointed out that, when the condition for the procompetitive effect holds globally, $\zeta'(\cdot) > 0$, it automatically implies **S1.** However, the opposite does not hold. Hence, **S1** does not rule out the anticompetitive case, $\zeta'(\cdot) < 0$.

### *3.5. Welfare Analysis under H.S.A.: Excessive versus Insufficient*

We now turn to the welfare analysis under H.S.A.. Because $X = X(\mathbf{x})$ is strictly quasi-concave in the interior and symmetric, the optimal allocation that maximizes $X = X(\mathbf{x})$ must be symmetric, $x_\omega = x$ for all $\omega \in \Omega$. Or equivalently, the optimal allocation that minimizes $P = P(\mathbf{p})$ must be symmetric, $p_\omega = p$ for all $\omega \in \Omega$. Hence, from eq.(5), the optimal allocation can be obtained by choosing $z = p/A$ to maximize

$$\ln\frac{X}{L} = \ln\frac{1}{P} = \ln\frac{1}{A} + V\int_{z}^{\bar{z}}\frac{s(\xi)}{\xi}d\xi + const.$$

subject to the constraints, eq.(7), eq.(10), and eq.(11) which can be combined to yield:

$$\frac{1}{A} = \frac{z}{\psi}\left(1 - \frac{F/L}{s(z)}\right) \tag{14}$$

Hence, the optimal allocation can be obtained by choosing $z$ to maximize:

$$\ln\frac{X}{L} = \ln\frac{1}{P} = W(z) \equiv \ln\frac{z}{\psi}\left(1 - \frac{F/L}{s(z)}\right) + \Phi(z) + const. \tag{15}$$

where $W(z)$ is the objective function and

$$\Phi(z) \equiv \frac{1}{s(z)}\int_{z}^{\bar{z}}\frac{s(\xi)}{\xi}d\xi. \tag{16}$$

The following lemma shows that $z^E$, the equilibrium value of $z$, given by eq.(12) generally fails to maximize the RHS of eq.(15). Indeed, it maximizes the RHS of eq.(14) instead. In other words, *the unique symmetric equilibrium minimizes* $A = A(\mathbf{p})$, not $P = P(\mathbf{p})$.

**Lemma 1**. *Under **S1**, eq.(14) is unimodal, and reaches the maximum at* $z = z^E$.

**Proof**. Differentiating the RHS of eq.(14) yields

$$1 - \frac{F/L}{s(z)} + z\frac{F/L}{\left(s(z)\right)^2}s'(z) = 1 - \frac{F/L}{s(z)}\left(1 - \frac{zs'(z)}{s(z)}\right) = 1 - \frac{F}{L}\frac{\zeta(z)}{s(z)}.$$

From **S1**, this is strictly decreasing in $z$, and hence the RHS of eq.(14) is strictly concave and reaches its maximum with respect to $z$ when

$$1 - \frac{F}{L}\frac{\zeta(z)}{s(z)} = 0$$

which is equivalent to eq.(12), satisfied if and only if $z = z^E$. This completes the proof. ■

Lemma 1 states that the equilibrium allocation maximizes only the first term of $W(z)$ in eq. (15). The second term, $\Phi(z)$, given in eq.(16), represents externalities that are ignored in a decentralized equilibrium. To understand the property of this externality term, notice that it can be rewritten as:

$$1 + \frac{1}{\Phi(z)} \equiv 1 + \frac{s(z)}{\int_z^{\bar{z}} s(\xi)/\xi \, d\xi} = 1 + \frac{-\int_z^{\bar{z}} s'(\xi)\, d\xi}{\int_z^{\bar{z}} s(\xi)/\xi \, d\xi} = \frac{\int_z^{\bar{z}} \zeta(\xi)\frac{s(\xi)}{\xi}\, d\xi}{\int_z^{\bar{z}} s(\xi)/\xi \, d\xi} = \int_z^{\bar{z}} \zeta(\xi) w(\xi)\, d\xi,$$

where $w(\xi) \equiv \frac{s(\xi)/\xi}{\int_z^{\bar{z}} s(\xi')/\xi' d\xi'}$, satisfying $\int_z^{\bar{z}} w(\xi)\, d\xi = 1$. Hence, log-differentiating eq.(16) yields

$$\frac{z\Phi'(z)}{\Phi(z)} = -\frac{zs'(z)}{s(z)} - \frac{1}{\Phi(z)} = \zeta(z) - 1 - \frac{1}{\Phi(z)} = \zeta(z) - \int_z^{\bar{z}} \zeta(\xi) w(\xi)\, d\xi,$$

from which the next lemma and its two corollaries follow:

**Lemma 2**.

$$\Phi'(z) \lesseqgtr 0 \Leftrightarrow \zeta(z) \lesseqgtr \int_z^{\bar{z}} \zeta(\xi) w(\xi)\, d\xi.$$

**Corollary 1:** *Assume that $\zeta'(\cdot)$ does not change sign over $(z_0, \bar{z})$, where $0 < z_0 < \bar{z}$. Then, for all $z \in (z_0, \bar{z})$*

$$\zeta'(\cdot) \gtreqless 0 \implies \Phi'(\cdot) \lesseqgtr 0.$$

**Corollary 2**. *$P(\mathbf{p}) = cA(\mathbf{p})$ with $c > 0$ is a constant, only in the case of CES.*

**Proof**. [16] Using eqs.(14)-(15), we obtain $\ln A(\mathbf{p})/P(\mathbf{p}) = \Phi(z) + const$. Hence, $P(\mathbf{p}) = cA(\mathbf{p}) \Leftrightarrow \Phi'(z) = 0$ for all $z \in (0, \bar{z}) \Leftrightarrow \zeta'(z) = 0$ for all $z \in (0, \bar{z}) \Leftrightarrow$ CES. ■

**Lemma 3**. *$W(z)$ is unimodal, and reaches its peak at $z^O$, given by the unique solution to*

$$\frac{L}{F} = \frac{1}{s(z)} + \frac{1}{\int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi} = \frac{1}{s(z)}\left(1 + \frac{1}{\Phi(z)}\right) = \frac{1}{s(z)} \int_z^{\bar{z}} \zeta(\xi) w(\xi)\, d\xi. \tag{17}$$

*and $z^O$ is increasing in $L/F$.*

[16] As already discussed, Corollary 2 is a special case of Matsuyama and Ushchev (2017; Proposition 1-iii)). Nevertheless, we offer this proof, because it is much simpler due to the symmetry and gross substitutability.

**Proof:** Differentiating $W(z)$, defined in eq.(15), yields

$$W'(z) = \frac{1}{z} + \frac{\frac{s'(z)}{s(z)^2}}{\frac{L}{F} - \frac{1}{s(z)}} + \Phi'(z).$$

Differentiating $\Phi(z)$, eq.(16), yields

$$\Phi'(z) = -\frac{s'(z)}{s(z)^2}\int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi - \frac{1}{z}.$$

By combining these two expressions,

$$W'(z) = \frac{s'(z)}{s(z)^2}\left[\frac{1}{\frac{L}{F} - \frac{1}{s(z)}} - \int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi\right].$$

Because the term in the square bracket is strictly increasing, $W(z)$ is unimodal with

$$W'(z) \gtreqless 0 \Leftrightarrow z \lesseqgtr z^O,$$

where $z^O$ is given by

$$\frac{L}{F} = \frac{1}{s(z)} + \frac{1}{\int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi},$$

whose RHS is increasing in $z$, because $s(z)$ and $\int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi$ are both decreasing in $z$. Hence, $z^O$ is uniquely defined and is increasing in $L/F$. ■

We are now ready to state the welfare property of the equilibrium entry under H.S.A..

**Proposition 3**. *Assume **S1**. Then, at the unique symmetric equilibrium in monopolistic competition under H.S.A., given by eq.*(12) *and eq.*(13), *$V^E$, the equilibrium mass of firms that enter = the equilibrium mass of varieties produced, and $V^O$, the mass of the optimal mass of firms that enter = the optimal mass of varieties produced, satisfy*

$$V^E \gtreqless V^O \Leftrightarrow \zeta(z^E) \lesseqgtr \int_{z^E}^{\bar{z}} \zeta(\xi) w(\xi)\, d\xi.$$

*In particular,*

*Excessive Entry:* $\zeta'(z) > 0$ *for all* $z \in (z^E, \bar{z}) \Rightarrow V^E > V^O$

*Optimal Entry (CES):* $\zeta'(z) = 0$ *for all* $z \in (z^E, \bar{z}) \Rightarrow V^E = V^O$

*Insufficient Entry:* $\zeta'(z) < 0$ *for all* $z \in (z^E, \bar{z}) \Rightarrow V^E < V^O$

**Proof**. By combining eq.(12) and eq.(17),

$$\frac{\zeta(z^E)}{s(z^E)} = \frac{L}{F} = \frac{1}{s(z^O)} + \frac{1}{\int_{z^O}^{\bar{z}} \frac{s(\xi)}{\xi} d\xi}.$$

Since $s(z)$ and $\int_z^{\bar{z}} \frac{s(\xi)}{\xi} d\xi$ are both decreasing in $z$,

$$z^E \gtreqless z^O \Leftrightarrow \frac{\zeta(z^E)}{s(z^E)} = \frac{1}{s(z^O)} + \frac{1}{\int_{z^O}^{\bar{z}} \frac{s(\xi)}{\xi} d\xi} \lesseqgtr \frac{1}{s(z^E)} + \frac{1}{\int_{z^E}^{\bar{z}} \frac{s(\xi)}{\xi} d\xi} = \frac{1}{s(z^E)} \int_{z^E}^{\bar{z}} \zeta(\xi) w(\xi)\, d\xi.$$

Hence,

$$V^E = \frac{1}{s(z^E)} \gtreqless V^O = \frac{1}{s(z^O)} \Leftrightarrow z^E \gtreqless z^O \Leftrightarrow \zeta(z^E) \lesseqgtr \int_{z^E}^{\bar{z}} \zeta(\xi) w(\xi)\, d\xi,$$

which completes the proof. ■

### *3.6. Main H.S.A. Theorem and Some Examples*

We are now ready to state the main properties of H.S.A. in the following theorem, by consolidating Propositions 1, 2, and 3. In doing so, we take into account that $z^E$ is monotonically increasing in $L/F$ and takes any value in $(0, \bar{z})$, as $L/F$ varies from zero to infinity, and that the existence of the choke price, $\bar{z} < \infty$, implies $\lim_{z \to \bar{z}} \zeta(z) = \infty$, and hence $\zeta'(z) > 0$ for $z$ sufficiently close to $\bar{z}$, which means that entry is procompetitive and excessive for a sufficiently large $L/F > 0$.

**Theorem 1:** *Consider monopolistic competition under symmetric H.S.A. with gross substitutes. Assume* ***S1*** *to ensure that there exists a unique equilibrium, which is symmetric and given by eq.*(12) *and eq.*(13). *At this unique symmetric equilibrium, entry is,*

- *procompetitive and excessive for any* $L/F > 0$, *if* $\zeta'(z) > 0\ for\ all\ z \in (0, \bar{z})$;
- *neutral and optimal for any* $L/F > 0$, *if* $\zeta'(z) = 0\ for\ all\ z \in (0, \infty)$, *that is, under CES;*
- *anticompetitive and insufficient for any* $L/F > 0$, *if* $\zeta'(z) < 0\ for\ all\ z \in (0, \infty)$.

*Furthermore, in the presence of the choke price,* $\bar{z} < \infty$, *entry is procompetitive and excessive for a sufficiently large* $L/F > 0$.

One important implication of this theorem is that, for those who believe in the empirical validity of *Marshall's second law of demand*, i.e., the price elasticity of demand for each product is increasing in its own price, holding the aggregates fixed, entry is not only procompetitive but also excessive under H.S.A.

We now turn to some examples to illustrate Theorem 1.

**Example 1: Perturbed CES, H.S.A. with global monotonicity**

$$s(z) = z^{1-\sigma} \exp\left[-\delta(\sigma - 1) \int_c^z \frac{g(\xi)}{\xi} d\xi\right] \Leftrightarrow \zeta(z) = \sigma + \delta(\sigma - 1)g(z),$$

where $g(z)$ satisfies $g'(z) > 0$ for all $z > 0$ with $g(0) = 0$ and $g(\infty) = 1$ and $\kappa \equiv \sup\{zg'(z)|z > 0\} < \infty$. For example,

$$g(z) = \frac{z}{\eta + z}, \qquad \eta > 0 \Rightarrow \kappa = \frac{1}{4};$$

$$g(z) = 1 - e^{-\mu z}, \qquad \mu > 0 \Rightarrow \kappa = e^{-1}.$$

We also impose the restrictions that $\sigma > 1$ and $\delta > -\sigma/(\kappa + 2\sigma - 1) > -1$ to ensure the gross substitutability, i.e., $\zeta(z) > 1$ for all $z \in (0, \infty)$ as well as **S1.**[17] Clearly, $\delta = 0$ corresponds to the knife-edge case of CES, where entry is neutral and optimal. If $\delta > 0$, entry is procompetitive and excessive. And, if $-\sigma/(\kappa + 2\sigma - 1) < \delta < 0$, entry is anticompetitive and insufficient.

**Example 2: Generalized Translog, H.S.A. with a choke price**

$$s(z) = \begin{cases} \gamma(\ln(\bar{z}/z))^{\eta} & \text{for } z < \bar{z} \\ 0 & \text{for } z \geq \bar{z} \end{cases} \qquad (\eta > 0)$$

$$\Rightarrow \zeta(z) = 1 + \frac{\eta}{\ln(\bar{z}/z)}, \quad \text{for } z < \bar{z}$$

which is strictly increasing in $z \in (0, \bar{z})$ with the range $(1, \infty)$. Hence, entry is globally procompetitive and excessive. Homothetic symmetric translog is a special case, where $\eta = 1$.[18]

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[17]For **S1**, note that it can be rewritten as $[\zeta(z) - 1]\zeta(z) + z\zeta'(z) = [\zeta(z) - 1]\zeta(z) + \delta z g'(z) > 0$. Clearly, this holds for $\delta \geq 0$. For $\delta < 0$, $\zeta(z) > \sigma + \delta$ and $\delta z g'(z) \geq \delta\kappa$ for all $z$, and hence $[\zeta(z) - 1]\zeta(z) + z\zeta'(z) > (\sigma + \delta - 1)(\sigma + \delta) + \delta\kappa = \delta^2 + (2\sigma - 1 + \kappa)\delta + (\sigma - 1)\sigma > (2\sigma - 1 + \kappa)\delta + (\sigma - 1)\sigma > 0$.

[18]To see this, eq. (19') of Feenstra (2003) gives the expression for the market share for each product under translog as $\frac{p_\omega}{P(\mathbf{p})}\frac{\partial P(\mathbf{p})}{\partial p_\omega} = \frac{1}{N} - \gamma\left[\ln p_\omega - \frac{1}{N}\int_\Omega \ln p_{\omega'}\, d\omega'\right], (\gamma > 0)$, where $N$ is the measure of $\Omega$. This can be rewritten as $\frac{p_\omega}{P(\mathbf{p})}\frac{\partial P(\mathbf{p})}{\partial p_\omega} = -\gamma \ln\left(\frac{p_\omega}{A(\mathbf{p})}\right)$, with $s(z) \equiv \gamma \max\{\ln(1/z), 0\}$ and $\ln A(\mathbf{p}) \equiv \frac{1}{\gamma N} + \frac{1}{N}\int_\Omega \ln p_{\omega'}\, d\omega'$.

The assumption of the global monotonicity of $\zeta(\cdot)$ in Theorem 1 is important. Otherwise, entry could be procompetitive and yet insufficient, or anticompetitive and yet excessive, as the next example illustrates.

**Example 3**: **Perturbed CES, H.S.A. without global monotonicity**

Consider the following family of H.S.A technologies:

$$\zeta(z) \equiv 1 - \frac{zs'(z)}{s(z)} = 1 + (\sigma - 1)\frac{\delta z g'(z) + 1}{1 + \delta(\sigma - 1)g(z)},$$

where $\sigma > 1$, $\delta$ can be either positive or negative (but sufficiently small in absolute value to ensure that $\zeta(z)$ satisfies **S1**), while $g(z)$ is twice continuously differentiable, single-peaked, and satisfies $g(0) = g(\infty) = 0$, and $\sup|g'(z)| < \infty$. Let $\tilde{z} > 0$ be the maximizer of $g(z)$, with $g'(\tilde{z}) = 0 > g''(\tilde{z})$. For example,

$$g(z) = z/(\lambda + z^2), \lambda > 0 \implies \tilde{z} = \sqrt{\lambda};$$

$$g(z) = ze^{-\mu z}, \mu > 0 \implies \tilde{z} = 1/\mu.$$

It is readily verified that the externality term in the welfare function is given by

$$\Phi(z) \equiv \frac{1}{s(z)}\int_z^\infty \frac{s(\xi)}{\xi}d\xi = \frac{1}{\sigma - 1} + \delta g(z).$$

From Lemma 1, $W'(z^E) = \Phi'(z^E) = \delta g'(z^E)$, and from Lemma 3, $W'(z^E) \gtreqless 0 \Leftrightarrow z^E \lesseqgtr z^O \Leftrightarrow V^E \lesseqgtr V^O$. Hence, $\delta g'(z^E) \gtreqless 0 \Leftrightarrow V^E \lesseqgtr V^O$. Thus, entry is insufficient for $z^E < \tilde{z}$ and excessive for $z^E > \tilde{z}$ for $\delta > 0$, while entry is excessive for $z^E < \tilde{z}$ and insufficient for $z^E > \tilde{z}$ for $\delta < 0$. On the other hand, evaluating $\zeta'(z)$ at $z = \tilde{z}$ yields:

$$\zeta'(\tilde{z}) = \delta\frac{\tilde{z}g''(\tilde{z})}{\Phi(\tilde{z})} \gtreqless 0 \Leftrightarrow \delta \lesseqgtr 0.$$

Thus, entry is anticompetitive in the vicinity of $\tilde{z}$ for $\delta > 0$, while entry is procompetitive in the vicinity of $\tilde{z}$ for $\delta < 0$.

By combining these two observations, we conclude that entry is procompetitive and yet insufficient for $\delta < 0$ and $z^E$ slightly higher than $\tilde{z}$, or equivalently, $L/F$ slightly higher than $\zeta(\tilde{z})/s(\tilde{z})$, while it is anticompetitive and yet excessive for $\delta > 0$ and $z^E$ slightly lower than $\tilde{z}$, or equivalently, $L/F$ slightly lower than $\zeta(\tilde{z})/s(\tilde{z})$.

### *3.7.H.S.A. Demand System: An Alternative Formulation*

Before proceeding, it should be pointed out that there exists an alternative (but equivalent) definition of H.S.A.. That is, $X = X(\mathbf{x})$ or $P = P(\mathbf{p})$ is called *homothetic with a single aggregator* (H.S.A.) if the market share of input $\omega$, as a function of $\mathbf{x}$, can be written as:

$$\frac{p_\omega x_\omega}{\mathbf{px}} = \frac{p_\omega x_\omega}{P(\mathbf{p})X(\mathbf{x})} = \frac{\partial \ln X(\mathbf{x})}{\partial \ln x_\omega} = s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right).$$

Here, $s^*: \mathbb{R}_{++} \to \mathbb{R}_+$ is the *market share function,* and it is assumed to be twice continuously differentiable with $0 < ys^{*\prime}(y)/s^*(y) < 1$, $s^*(0) = 0$ and $s^*(\infty) = \infty$, and $A^*(\mathbf{x})$ is linear homogenous in $\mathbf{x}$, defined implicitly and uniquely by

$$\int_\Omega s^*\left(\frac{x_\omega}{A^*(\mathbf{x})}\right) d\omega \ = 1,$$

which ensures that the market shares of all inputs are added up to one. Thus, the market share of input $\omega$ is a function of its *relative quantity*, defined as its own quantity $x_\omega$ divided by the *common quantity aggregator* $A^*(\mathbf{x})$, which is strictly increasing with the elasticity less than one. This common quantity aggregator, $A^*(\mathbf{x})$, is related to the production function, $X(\mathbf{x})$, as follows:

$$\ln\left(\frac{X(\mathbf{x})}{A^*(\mathbf{x})}\right) = const. + \int_\Omega \left[\int_0^{x_\omega/A^*(\mathbf{x})} \frac{s^*(\xi)}{\xi} d\xi\right] d\omega,$$

and $X(\mathbf{x}) = cA^*(\mathbf{x})$, with a positive constant $c > 0$, if and only if $s^*(y) = \gamma y^{1-1/\sigma}$, which is the case of CES.

These two alternative definitions of H.S.A. are isomorphic to each other via the one-to-one mapping between $s(z) \leftrightarrow s^*(y)$, defined by:

$$s^*(y) = s\left(\frac{s^*(y)}{y}\right); \qquad s(z) = s^*\left(\frac{s(z)}{z}\right).$$

Differentiating either of these two equalities yields the identity,

$$\zeta^*(y) \equiv \left[1 - \frac{ys^{*\prime}(y)}{s^*(y)}\right]^{-1} = \zeta(z) \equiv 1 - \frac{zs'(z)}{s(z)} > 1,$$

where $\zeta^*(y)$ is the price elasticity as a function of $y_\omega \equiv x_\omega/A^*(\mathbf{x})$, which shows that the condition, $0 < ys^{*\prime}(y)/s^*(y) < 1$, is equivalent to $s'(z) < 0$, the condition for gross substitutability. Furthermore, the relative quantity, $y_\omega \equiv x_\omega/A^*(\mathbf{x})$, and the relative price, $z_\omega \equiv p_\omega/A(\mathbf{p})$, are negatively related as

$$z_\omega = \frac{s^*(y_\omega)}{y_\omega}; \qquad y_\omega = \frac{s(z_\omega)}{z_\omega}.$$

from which $p_\omega x_\omega/A(\mathbf{p})A^*(\mathbf{x}) = y_\omega z_\omega = s(z_\omega) = s^*(y_\omega) = p_\omega x_\omega/L = p_\omega x_\omega/P(\mathbf{p})X(\mathbf{x})$, hence that we have the identity, $A(\mathbf{p})A^*(\mathbf{x}) = L = P(\mathbf{p})X(\mathbf{x})$, or

$$\frac{A(\mathbf{p})}{P(\mathbf{p})} = \frac{X(\mathbf{x})}{A^*(\mathbf{x})},$$

which is a positive constant if and only if it is CES. In addition,

$$\lim_{y\to 0}\frac{s^*(y)}{y} = s^{*\prime}(0) = \bar{z} \equiv \inf\{z > 0|s(z) = 0\}$$

which is the choke price if finite.[19]

Under this alternative (but equivalent) formulation of H.S.A., with the following assumption, which is equivalent to **S1**,

$$\frac{d\ln(s^*(y)/\zeta^*(y))}{d\ln y} = 1 - \frac{1}{\zeta^*(y)} - \frac{y\zeta^{*\prime}(y)}{\zeta^*(y)} > 0,$$

there exists a unique symmetric equilibrium, in which all firms choose $y_\omega \equiv y^E$, given by the condition,

$$\frac{s^*(y^E)}{\zeta^*(y^E)} = \frac{F}{L},$$

where $y^E$ is decreasing in $L/F$. Furthermore, entry is procompetitive and excessive for any $L/F > 0$, if $\zeta^{*\prime}(y) < 0$ for all $y \in (0,\infty)$; neutral and optimal for any $L/F > 0$, if $\zeta^{*\prime}(y) = 0$ for all $y \in (0,\infty)$, (i.e., under CES) and anticompetitive and insufficient for any $L/F > 0$, if $\zeta^{*\prime}(y) > 0$ for all $y \in (0,\infty)$. Furthermore, in the presence of the choke price, $s^{*\prime}(0) = \bar{z} < \infty$, $\zeta^{*\prime}(0) = \infty$, and hence $\zeta^{*\prime}(y) < 0$ for a sufficiently small $y$, which means that entry is procompetitive and excessive for a sufficiently large $L/F > 0$.

[19]This isomorphism has been shown for the broader class of H.S.A., which allows for asymmetry as well as gross complements; see Matsuyama and Ushchev (2017, Section 3, Remark 3).

## 4. Dixit-Stiglitz under H.D.I.A. (Homothetic Direct Implicit Additivity)

### *4.1.H.D.I.A. Demand System*

We call a symmetric CRS technology, $X = X(\mathbf{x})$ or $P = P(\mathbf{p})$, *homothetic with direct implicit additivity* (H.D.I.A.)[20] if $X = X(\mathbf{x})$ can be defined implicitly by:

$$\int_{\Omega} \phi\left(\frac{x_\omega}{X}\right) d\omega = 1, \tag{18}$$

where $\phi(\cdot): \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, strictly concave, and at least thrice continuously differentiable with $\phi(0) = 0$ and $\phi(\infty) = \infty$.

In the following analysis, both the elasticity of $\phi(\cdot)$,

$$0 < \mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)} < 1, \tag{19}$$

and the elasticity of $\phi'(\cdot)$ in its absolute value,

$$0 < r_\phi(y) \equiv -\frac{y\phi''(y)}{\phi'(y)} < 1, \tag{20}$$

play important roles. The monotonicity of $\phi(\cdot)$ and $\phi(0) = 0$ jointly ensure $\mathcal{E}_\phi(\cdot) > 0$ and the concavity of $\phi(\cdot)$ ensures $\mathcal{E}_\phi(\cdot) < 1$. The monotonicity and concavity of $\phi(\cdot)$ jointly ensure $r_\phi(\cdot) > 0$. In addition, it is necessary to assume that $r_\phi(\cdot) < 1$ to ensure that inputs are gross substitutes, as will be seen below. Note that one could recover $\phi(\cdot)$ either from any $\mathcal{E}_\phi(\cdot)$ or any $r_\phi(\cdot)$ satisfying the bounds in eq. (19) and eq.(20) as follows:

$$\phi(y) = \exp\left[\int_{y_0}^{y} \mathcal{E}_\phi(\xi)\frac{d\xi}{\xi}\right];$$

$$\phi(y) = \int_0^{y} \exp\left[-\int_{y_0'}^{\xi} r_\phi(\xi')\frac{d\xi'}{\xi'}\right] d\xi,$$

[20] More generally, $X = X(\mathbf{x})$ is H.D.I.A. if it can be defined implicitly by $\int_\Omega \phi_\omega(x_\omega/X(\mathbf{x}))d\omega = 1$. It is the homothetic restriction of the class of D.I.A. (*direct implicit additivity*), which can be defined implicitly by $\int_\Omega \tilde{\phi}_\omega(x_\omega, X(\mathbf{x}))d\omega = 1$: see Hanoch (1975; Section 2). In contrast, $X = X(\mathbf{x})$ is D.E.A. (*direct explicit additivity*) if it can be defined as $X(\mathbf{x}) = \mathcal{M}\left(\int_\Omega \bar{\phi}_\omega(x_\omega)d\omega\right)$, where $\mathcal{M}(\cdot)$ is a monotone transformation: see Hanoch (1975; Section 2.2). D.E.A. is another subclass of D.I.A., with $\tilde{\phi}_\omega(x_\omega, X(\mathbf{x})) = \bar{\phi}_\omega(x_\omega)/\mathcal{M}^{-1}(X(\mathbf{x}))$. Symmetric D.E.A. with $\bar{\phi}_\omega(\cdot) = \bar{\phi}(\cdot)$, is the class of demand systems used by Dixit and Stiglitz (1977, Section II), Behrens and Murata (2007), Zhelobodko, Kokovin, Parenti, and Thisse (2012), Dhingra and Morrow (2019), and many others. Although D.E.A. and H.D.I.A. are both subclasses of D.I.A., CES is the only common element of D.E.A. and H.D.I.A. because D.E.A. cannot be homothetic unless it is CES.

where $\mathcal{y}_0 > 0$ and $\mathcal{y}_0' > 0$ are both constants.[21] One could also verify from eq. (19) and eq.(20) that $\mathcal{E}_\phi(\mathcal{y})$ and $r_\phi(\mathcal{y})$ are related as follows:

$$\frac{\mathcal{y}\mathcal{E}_\phi{}'(\mathcal{y})}{\mathcal{E}_\phi(\mathcal{y})} = 1 - r_\phi(\mathcal{y}) - \mathcal{E}_\phi(\mathcal{y}).$$

Clearly, CES with gross substitutes is a special case with $\phi(\mathcal{y}) = A\mathcal{y}^{1-\frac{1}{\sigma}}$, and $0 < r_\phi(\mathcal{y}) = 1 - \mathcal{E}_\phi(\mathcal{y}) = 1/\sigma < 1$.

The cost minimization problem, eq.(1) subject to eq.(18) implies that the inverse demand curve for each $\omega \in \Omega$ can be written as:

$$p_\omega = B(\mathbf{p})\phi'\left(\frac{x_\omega}{X(\mathbf{x})}\right), \tag{21}$$

where $B(\mathbf{p})$ is the Lagrange multiplier associated with eq. (18), and it is the linear homogenous function in $\mathbf{p}$, given by

$$\int_\Omega \phi\left((\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right)\right) d\omega \equiv 1.$$

From eq.(21), the unit cost function can be written as

$$P(\mathbf{p}) = \int_\Omega p_\omega (\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) d\omega.$$

Furthermore, the market share of $\omega$ can be expressed as:

$$\frac{p_\omega}{P(\mathbf{p})}\frac{x_\omega}{X(\mathbf{x})} = \frac{p_\omega}{P(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) = \frac{x_\omega}{C^*(\mathbf{x})}\phi'\left(\frac{x_\omega}{X(\mathbf{x})}\right),$$

where $C^*(\mathbf{x}) \equiv \int_\Omega x_\omega \phi'\left(\frac{x_\omega}{X(\mathbf{x})}\right) d\omega$ is a linear homogenous function of $\mathbf{x}$, and satisfies the identity $B(\mathbf{p})C^*(\mathbf{x}) = L = P(\mathbf{p})X(\mathbf{x})$, because

$$\frac{P(\mathbf{p})}{B(\mathbf{p})} = \int_\Omega \frac{p_\omega}{B(\mathbf{p})}(\phi')^{-1}\left(\frac{p_\omega}{B(\mathbf{p})}\right) d\omega = \int_\Omega \phi'\left(\frac{x_\omega}{X(\mathbf{x})}\right)\frac{x_\omega}{X(\mathbf{x})} d\omega = \frac{C^*(\mathbf{x})}{X(\mathbf{x})}.$$

The above expressions for the market share under H.D.I.A. show that it is a function of the two relative prices, $p_\omega/P(\mathbf{p})$ and $p_\omega/B(\mathbf{p})$, or a function of the two relative quantities, $x_\omega/X(\mathbf{x})$ and

[21]These constants imply that $\phi(\mathcal{y})$ is determined up to a positive scalar multiplier. However, $\gamma\phi(\mathcal{y})$ with $\gamma > 0$ generate the same CRS technology. All we need is to renormalize the indices of varieties, as $\int_\Omega \gamma\phi(x_\omega/X)d\omega = \int_\Omega \gamma\phi(x_{\omega'}/X)d\omega' = 1$, with $\omega' = \gamma\omega$.

$x_\omega/C^*(\mathbf{x})$, unless $P(\mathbf{p})/B(\mathbf{p}) = C^*(\mathbf{x})/X(\mathbf{x})$ is a positive constant, $c > 0$, which occurs if and only if it is CES. Thus, H.D.I.A. and H.S.A. do not overlap with the sole exception of CES.[22]

From the inverse demand curve, eq.(21), the elasticity of substitution between a pair of inputs, $\omega_1$ and $\omega_2$, evaluated at the same quantity (hence at the same price) can be expressed as:

$$-\frac{\partial \ln\left(x_{\omega_1}/x_{\omega_2}\right)}{\partial \ln\left(p_{\omega_1}/p_{\omega_2}\right)}\Bigg|_{x_{\omega_1}=x_{\omega_2}=x} = \frac{1}{r_\phi(x/X)} > 1,$$

hence $r_\phi(y) < 1$ ensures that inputs are gross substitutes. It should be also clear from eq.(21) that the choke price exists if and only if

$$\phi'(0) = \lim_{y\to 0} \exp\left[\int_y^{y_0'} \frac{r_\phi(\xi)}{\xi} d\xi\right] < \infty,$$

which implies $\lim_{y\to 0} r_\phi(y) = 0$, as well as $\lim_{y\to 0} \mathcal{E}_\phi(y) = \lim_{y\to 0} \frac{\phi'(y)}{\phi(y)/y} = \frac{\phi'(0)}{\phi'(0)} = 1$.

### *4.2.Profit Maximization by Input Producing Firms under H.D.I.A.*

From the inverse demand curve, eq.(21), the profit of firm $\omega \in \Omega$ is given by:

$$\pi_\omega = \left(B(\mathbf{p})\phi'\left(\frac{x_\omega}{X}\right) - \psi\right) x_\omega - F.$$

Firm $\omega$ chooses its output, $x_\omega$, to maximize its profit $\pi_\omega$, taking the aggregate variables, $B(\mathbf{p})$ and $X$ as given. Or equivalently, it chooses $y_\omega \equiv y/X$ to maximize

$$(B(\mathbf{p})\phi'(y_\omega) - \psi)y_\omega.$$

The FOC is:

$$B(\mathbf{p})[\phi'(y_\omega) + y_\omega\phi''(y_\omega)] = B(\mathbf{p})\phi'(y_\omega)\left[1 - r_\phi(y_\omega)\right] \tag{22}$$
$$= p_\omega\left[1 - r_\phi(y_\omega)\right] = \psi.$$

In what follows, we keep it simple by imposing the following assumption to ensure that the FOC is sufficient for the global optimum.

**Assumption D1:** *For all* $y > 0$,

[22]This statement is a special case of Proposition 2-(ii) in Matsuyama and Ushchev (2017).

$$\frac{y\phi'''(y)}{\phi''(y)}+2>0 \Leftrightarrow \frac{y r_\phi'(y)}{r_\phi(y)}+1-r_\phi(y)>0.$$

Under **D1**, the LHS of eq.(22) is strictly decreasing in $y_\omega$. Hence, eq.(22) gives the unique profit-maximizing output for each firm. Thus, all firms set the same price, $p_\omega = p$, and produce the same amount, $x_\omega = x$. Hence, under **D1**, asymmetric equilibria do not exist. Unlike in the case of H.S.A., the condition that rules out asymmetric equilibria does not ensure the uniqueness of a symmetric equilibrium under H.D.I.A., which needs to be introduced separately; see **D2** below.

***4.3.Symmetric Free-Entry Equilibrium under H.D.I.A.***

A *symmetric free-entry equilibrium* under H.D.I.A. satisfies the following conditions:

**H.D.I.A. integral condition**, eq.(18) under symmetry:

$$V\phi\left(\frac{x}{X}\right)=1; \tag{23}$$

**Firm's pricing formula**, given by FOC, eq.(22) under symmetry:

$$1-\frac{\psi}{p}=r_\phi\left(\frac{x}{X}\right), \tag{24}$$

in addition to the zero-profit (free-entry) condition, eq.(9) and the resource constraint, eq.(10).

For the uniqueness of a symmetric equilibrium, we introduce the following condition:

**Assumption D2:** *For all* $y>0$,

$$\frac{y\phi'''(y)}{\phi''(y)}+1+r_\phi(y)+\mathcal{E}_\phi(y)>0 \ \Leftrightarrow \frac{y r_\phi'(y)}{r_\phi(y)}+\mathcal{E}_\phi(y)>0.$$

Clearly, **D1** implies **D2** if

$$\frac{y\mathcal{E}_\phi'(y)}{\mathcal{E}_\phi(y)}=1-r_\phi(y)-\mathcal{E}_\phi(y)<0,$$

and **D2** implies **D1**, if

$$\frac{y\mathcal{E}_\phi'(y)}{\mathcal{E}_\phi(y)}=1-r_\phi(y)-\mathcal{E}_\phi(y)>0.$$

And, **D1** and **D2** are equivalent if and only if

$$\frac{y\mathcal{E}_\phi'(y)}{\mathcal{E}_\phi(y)}=1-r_\phi(y)-\mathcal{E}_\phi(y)=0,$$

that is, under and only under CES.

To see why **D2** ensures the existence and the uniqueness of a symmetric free-entry equilibrium, note first that the pricing formula, eq.(24), and the free entry condition, eq.(9), can be combined to yield:

$$r_\phi(x/X)px = F. \tag{25}$$

From eq.(9) and eq.(10), $pVx = L$, which can be combined with eq.(25) to obtain:

$$\frac{L}{V} = px = \frac{F}{r_\phi(x/X)},$$

which becomes after using the H.D.I.A. condition, eq.(23):

$$r_\phi(x/X)\phi(x/X) = F/L.$$

The LHS of this equation is increasing in $x/X$, because **D2** implies

$$\frac{d\ln[\phi(y)r_\phi(y)]}{d\ln y} = \frac{y\phi'''(y)}{\phi''(y)} + 1 + r_\phi(y) + \mathcal{E}_\phi(y) > 0 \ \text{ for all } y > 0.$$

Furthermore, $\lim_{y\to 0} r_\phi(y)\phi(y) = 0$ and $\lim_{y\to\infty} r_\phi(y)\phi(y) = \infty$. Hence, *for each* $L/F > 0$, the equilibrium value of $y$, $y^E \in (0,\infty)$, is pinned down uniquely by,

$$r_\phi(y^E)\phi(y^E) = F/L, \tag{26}$$

and $y^E$ is decreasing in $L/F$. By inserting this value into eq.(23), eq.(24), and eq.(9),

$$V^E = \frac{1}{\phi(y^E)};$$

$$p^E = \frac{\psi}{1 - r_\phi(y^E)}; \tag{27}$$

$$X^E = \frac{x^E}{y^E} = \frac{L}{y^E p^E V^E} = \frac{[1 - r_\phi(y^E)]\phi(y^E)}{\psi y^E} L = \frac{\phi(y^E)L - F}{\psi y^E} > 0.$$

Thus, we have shown:

**Proposition 4**. *Under **D1**, no asymmetric equilibria exist. Furthermore, under **D1** and **D2**, there exists a unique symmetric free-entry equilibrium under H.D.I.A. for each* $L/F > 0$*, given by eq.*(26) *and eq.*(27).

### *4.4. Comparative Statics under H.D.I.A.: Procompetitive versus Anticompetitive*

We now turn to the comparative statics.

**Proposition 5**. *Assume* **D1** *and* **D2**. *At the unique symmetric equilibrium in monopolistic competition under H.D.I.A., given by eq.*(26) *and eq.*(27),

*Procompetitive:* $$r'_\phi(y^E) > 0 \Rightarrow \frac{\partial p^E}{\partial L} < 0;\ 0 < \frac{\partial \ln V^E}{\partial \ln L} < 1;\ \frac{\partial x^E}{\partial L} > 0$$

*Neutral (CES):* $$r'_\phi(y^E) = 0 \Rightarrow \frac{\partial p^E}{\partial L} = 0;\ \frac{\partial \ln V^E}{\partial \ln L} = 1;\ \frac{\partial x^E}{\partial L} = 0$$

*Anticompetitive:* $$r'_\phi(y^E) < 0 \Rightarrow \frac{\partial p^E}{\partial L} > 0;\ \frac{\partial \ln V^E}{\partial \ln L} > 1;\ \frac{\partial x^E}{\partial L} < 0.$$

**Proof:** Since eq.(26) implies $\partial y^E/\partial L < 0$ under **D2**, this follows from eq.(27). ■

The conditions for the procompetitive vs. anticompetitive cases under H.D.I.A. are analogous to those under H.S.A. For example, recall that the condition for the procompetitive case under H.S.A. is $\zeta'(z^E) = \zeta'(p^E/A^E) > 0$, that is, the price elasticity of demand goes *up* as its *price* goes up, holding the aggregates fixed. This is nothing but *Marshall's 2nd law of demand.* Here, under H.D.I.A., the condition is $r'_\phi(y^E) = r'_\phi(x^E/X^E) > 0$; that is, the price elasticity of demand for an input goes *down* as its *quantity* goes up, holding the aggregate fixed. This is another way of stating *Marshall's 2nd law of demand.* Note also that, if the condition for the procompetitive case holds globally, $r'_\phi(\cdot) > 0$, **D1** and **D2** hold automatically. However, neither **D1** nor **D2** necessarily implies $r'_\phi(\cdot) > 0$. This means that **D1** and **D2** do not rule out the anticompetitive case, $r'_\phi(\cdot) < 0$.

### *4.5. Welfare Analysis under H.D.I.A.: Excessive versus Insufficient*

We now turn to the welfare analysis under H.D.I.A. The social planner's problem is to maximize social welfare subject to the resource constraint. From the symmetry and strict quasi-concavity of $X = X(\mathbf{x})$, defined by eq.(18), the solution is clearly symmetric. The problem can be thus stated as:

$$\max_{(x,V)} X \qquad s.t.\ \ (\psi x + F)V = L;\ \ V\phi(x/X) = 1$$

Using $y = x/X$, this can be written as

$$\max_{(x,y)} \frac{x}{y} \qquad s.t. \qquad \psi x = \frac{L}{V} - F = \phi(y)L - F \geq 0$$

or, equivalently, as

$$\max_{y \geq \underline{y}} W(y) \equiv \frac{\phi(y) - F/L}{y}, where\ \underline{y} \equiv \phi^{-1}(F/L) > 0.$$

To make this social planner's problem well-defined, we need to introduce:

**Assumption D3:** $\lim_{y \to \infty} \mathcal{E}_{\phi}(y) < 1$.[23]

**Lemma 4**. *Under* **D3,** $W(y)$ *is unimodal, with*

$$W'(y) \gtreqless 0 \Leftrightarrow \frac{F}{L} \gtreqless \phi(y) - \phi'(y)y = \phi(y)\left[1 - \mathcal{E}_{\phi}(y)\right] \Leftrightarrow y \lesseqgtr y^{O}$$

*where* $y^{O}$ *is the socially optimal value of* $y$*, uniquely given by*

$$\phi(y^{O}) - \phi'(y^{O})y^{O} = \phi(y^{O})\left[1 - \mathcal{E}_{\phi}(y^{O})\right] = \frac{F}{L}$$

*and* $y^{O}$ *is strictly decreasing in* $L/F$.

**Proof**: By differentiating $W(y)$, it is easily verified that

$$y^{2}W'(y) = \frac{F}{L} - [\phi(y) - \phi'(y)y] = \frac{F}{L} - \phi(y)\left[1 - \mathcal{E}_{\phi}(y)\right],$$

which is strictly decreasing, because

$$\frac{d[\phi(y) - \phi'(y)y]}{dy} = -\phi''(y)y > 0.$$

Furthermore, $\underline{y}^{2}W'(\underline{y}) = \underline{y}\phi'(\underline{y}) > 0$ and $y^{2}W'(y) < 0$ for a sufficiently large $y$, because **D3** implies $\phi(y)\left[1 - \mathcal{E}_{\phi}(y)\right] \longrightarrow \infty$ as $y \longrightarrow \infty$. Hence, $W(y)$ reaches its global maximum at $y^{O} \in (\underline{y}, \infty)$, given by $W'(y^{O}) = 0 \Leftrightarrow \phi(y^{O}) - \phi'(y^{O})y^{O} = F/L$, which is strictly decreasing in $L/F$, and $W'(y) \gtreqless 0 \Leftrightarrow y \lesseqgtr y^{O}$. ■

We are now ready to state the welfare property of the equilibrium allocation.

**Proposition 6**. *Assume* **D1, D2** *and* **D3**. *Then, at the unique symmetric equilibrium in monopolistic competition under H.D.I.A.,given by eq.*(26) *and eq.*(27)*,* $V^{E}$*, the equilibrium mass of firms that enter = the equilibrium mass of varieties produced and* $V^{O}$*, the mass of the optimal mass of firms that enter = the optimal mass of varieties produced,, satisfy*

*Excessive Entry:*

$$\frac{y^{E}\mathcal{E}_{\phi}'(y^{E})}{\mathcal{E}_{\phi}(y^{E})} = 1 - r_{\phi}(y^{E}) - \mathcal{E}_{\phi}(y^{E}) < 0 \Leftrightarrow V^{E} > V^{O}$$

[23] Assumption **D3** rules out the pathological case, where the social planner can produce an unbounded output, $X$, by letting $V \longrightarrow 0$ and $x \longrightarrow \infty$. Note that **D3** does not rule out the choke price, which would imply $\lim_{y \to 0} \mathcal{E}_{\phi}(y) = 1$.

*Optimal Entry (CES):* $\frac{y^E \mathcal{E}_\phi{}'(y^E)}{\mathcal{E}_\phi(y^E)} = 1 - r_\phi(y^E) - \mathcal{E}_\phi(y^E) = 0 \Leftrightarrow V^E = V^O$

*Insufficient Entry:* $\frac{y^E \mathcal{E}_\phi{}'(y^E)}{\mathcal{E}_\phi(y^E)} = 1 - r_\phi(y^E) - \mathcal{E}_\phi(y^E) > 0 \Leftrightarrow V^E < V^O$

**Proof:** From eq.(26) and Lemma 4,

$$y^E \lesseqgtr y^O \Leftrightarrow r_\phi(y^E)\phi(y^E) = \frac{F}{L} \gtreqless \phi(y^E)\left[1 - \mathcal{E}_\phi(y^E)\right] \Leftrightarrow r_\phi(y^E) \gtreqless 1 - \mathcal{E}_\phi(y^E)$$

Since $V^E\phi(y^E) = 1 = V^O\phi(y^O), y^E \lesseqgtr y^O \Leftrightarrow V^E \gtreqless V^O$, this completes the proof. ■

Note that, in order for the equilibrium entry to be optimal for a range of the parameter values under H.D.I.A., $\frac{y\mathcal{E}_\phi{}'(y)}{\mathcal{E}_\phi(y)} = 1 - r_\phi(y) - \mathcal{E}_\phi(y) = 0$ must hold for the relevant range of $y$, that is, under and only under CES. Thus, CES offers the borderline case between the cases of excessive entry and insufficient entry within H.D.I.A..

### *4.6.Main H.D.I.A. Theorem and Some Examples*

Proposition 5 states that the sign of $r_\phi'(y^E)$ determines whether entry is procompetitive or anticompetitive, while Proposition 6 states the sign of $\mathcal{E}_\phi'(y^E)$ determines whether entry is excessive or insufficient. Hence, one might think, unlike under H.S.A, that these conditions are unrelated to each other, and that both procompetitive entry and anticompetitive entry can be either excessive or insufficient under H.D.I.A.. However, the next lemma shows that there exists a tight connection between the two conditions.

**Lemma 5:** *Assume that* $r_\phi'(\cdot)$ *does not change sign over* $(0, y_0)$*, where* $0 < y_0 \le \infty$*. Then, for all* $y \in (0, y_0)$*,*

$$r_\phi'(\cdot) \gtreqless 0 \Rightarrow \mathcal{E}_\phi'(\cdot) \lesseqgtr 0.$$

**Proof**: See Appendix A. ■

Here is the implication of Lemma 5. Suppose that, for all $L/F > (L/F)_0$, entry is procompetitive at the unique symmetric equilibrium given by eq.(26) and eq.(27). That means that $r_\phi'(y) > 0$ for all $y < y_0$, where $y_0$ satisfies $r_\phi(y_0)\phi(y_0)(L/F)_0 = 1$. Then, Lemma 5 tells us $\mathcal{E}_\phi'(y) <$

0 for all $y < y_0$. Hence, for all $L/F > (L/F)_0$, entry is excessive at the unique symmetric equilibrium. Likewise, suppose that, for all $L/F > (L/F)_0$, entry is anticompetitive at the unique symmetric equilibrium given by eq.(26) and eq.(27). That means that $r'_\phi(y) < 0$ for all $y < y_0$. Then, Lemma 5 tells us $\mathcal{E}'_\phi(y) > 0$ for all $y < y_0$. Hence, for all $L/F > (L/F)_0$, entry is insufficient at the unique symmetric equilibrium.

We are now ready to summarize the main properties of H.D.I.A. in the next theorem, by consolidating Propositions 4, 5, and 6 and Lemma 5. In doing so, we take into account that $y^E$ is strictly decreasing in $L/F$ and takes any value in $(0, \infty)$, as $L/F$ varies from zero to infinity, and that the existence of the choke price, $\phi'(0) < \infty$, implies $\lim_{y \to 0} r_\phi(y) = 0$ and $\lim_{y \to 0} \mathcal{E}_\phi(y) = 1$, and hence $r'_\phi(y) > 0$ and $\mathcal{E}'_\phi(y) < 0$ for a sufficiently small $y$, which means that entry is procompetitive and excessive for a sufficiently large $L/F > 0$.

**Theorem 2:** *Consider monopolistic competition under symmetric H.D.I.A. with gross substitutes. Assume **D1** to ensure the symmetry of equilibrium and **D2** to ensure the uniqueness of the symmetric equilibrium. Then, the unique symmetric equilibrium is given by eq.*(26) *and eq.*(27)*. Assume **D3** to ensure that the planner's problem is well-defined. Then, at the unique symmetric equilibrium, entry is,*

- *procompetitive and excessive for any $L/F > 0$, if $r_\phi{}'(y) > 0$ for all $y \in (0, \infty)$;*
- *neutral and optimal for any $L/F > 0$, if $r_\phi{}'(y) = 0$ for all $y \in (0, \infty)$, that is, under CES;*
- *anticompetitive and insufficient for any $L/F > 0$, if $r_\phi{}'(y) < 0$ for all $y \in (0, \infty)$.*

*Furthermore, in the presence of the choke price, $\phi'(0) < \infty$, entry is procompetitive and excessive for a sufficiently large $L/F > 0$.*

We now turn to some examples to illustrate Theorem 2.

**Example 4; Perturbed CES, H.D.I.A. with global monotonicity.** Consider a family of H.D.I.A. technologies, such that

$$r_\phi(y) \equiv -\frac{y\phi''(y)}{\phi'(y)} = \frac{1}{\sigma} + \delta\left(1 - \frac{1}{\sigma}\right)g(y),$$

where $\sigma > 1$, and $g(y)$ satisfies $g'(y) > 0$ for all $y > 0$ with $g(0) = 0$ and $g(\infty) = 1$ and $\sup\{y g'(y) | y > 0\} \equiv \kappa < \infty$. For example,

$$g(y) = \frac{y}{\eta + y}, \eta > 0 \Rightarrow \kappa = \frac{1}{4} < \infty$$

$$g(y) = 1 - e^{-\mu y}, \mu > 0 \Rightarrow \kappa = e^{-1} < \infty$$

satisfy these conditions. In addition, we impose the following restrictions on $\sigma$, $\delta$, and $\kappa$:

$$-\frac{1}{(1+\kappa)\sigma - 1} < \delta < 1,$$

so that $0 < r_\phi(y) < 1$, **D1, D2,** and **D3** hold.[24] Then, Theorem 2 can be applied. In this example, entry is procompetitive and excessive for all $L/F > 0$ when $0 < \delta < 1$, while it is anticompetitive and insufficient for all $L/F > 0$ when $-\frac{1}{(1+\kappa)\sigma-1} < \delta < 0$.

**Example 5: H.D.I.A. with a choke price.** Consider a family of H.D.I.A. technologies such that

$$0 < r_\phi(y) \equiv -\frac{y\phi''(y)}{\phi'(y)} < 1,$$

satisfies $r'_\phi(y) > 0$ for all $y > 0$ and

$$\lim_{y \to 0} \int_{y}^{y_0'} \frac{r_\phi(\xi)}{\xi} d\xi < \infty \Leftrightarrow \phi'(0) = \lim_{y \to 0} \exp\left[\int_{y}^{y_0'} \frac{r_\phi(\xi)}{\xi} d\xi\right] < \infty,$$

which implies the choke price. For example,

$$r_\phi(y) = \frac{y}{\eta + y}, \qquad \eta > 0;$$

$$r_\phi(y) = 1 - e^{-\mu y}, \qquad \mu > 0,$$

satisfy these conditions. Clearly, **D1**, **D2**, and **D3** are all satisfied, and from Lemma 5, $r'_\phi(y) > 0 \Rightarrow \mathcal{E}'_\phi(y) < 0$ for all $y > 0$. Hence, entry is always procompetitive and excessive, not just for a sufficiently large $L/F$.

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[24]It is easy to verify $0 < r_\phi(y) < 1$ and **D3**. For **D1** and **D2,** if $\delta \geq 0$, $r'_\phi(y) \geq 0$, which implies both **D1** and **D2**. If $\delta < 0$, $r'_\phi(y) < 0$ for all $y > 0$. From Lemma 5, this implies $\mathcal{E}'_\phi(y) > 0$ for all $y > 0$, which means that **D2** implies **D1.** To verify that **D2** for $\delta < 0$, note that $r'_\phi(y) < 0$ and $\mathcal{E}'_\phi(y) > 0$ for all $y > 0$ implies

$$r_\phi(y)\mathcal{E}_\phi(y) > r_\phi(\infty)\mathcal{E}_\phi(0) = \left(\frac{1}{\sigma} + \delta\left(1 - \frac{1}{\sigma}\right)\right)\left(1 - \frac{1}{\sigma}\right),$$

while $\delta < 0$ and the definition of $\kappa$ implies

$$y r'_\phi(y) = \delta\left(1 - \frac{1}{\sigma}\right) y g'(y) > \delta\left(1 - \frac{1}{\sigma}\right)\kappa$$

Adding each side of the two inequalities above yields $r_\phi(y)\mathcal{E}_\phi(y) + y r'_\phi(y) > 0$, because $-\frac{1}{(1+\kappa)\sigma-1} < \delta < 0$, which is equivalent to **D2.**

The assumption of the global monotonicity of $r_\phi(\cdot)$ in Theorem 2, which implies the global monotonicity of $\mathcal{E}_\phi(\cdot)$ by Lemma 5, is important. Otherwise, entry could be procompetitive and yet insufficient, or anticompetitive and yet excessive, as the next example illustrates.

**Example 6. Perturbed CES, H.D.I.A. without global monotonicity.** Consider a family of H.D.I.A technologies with

$$\mathcal{E}_\phi(y) \equiv \frac{y\phi'(y)}{\phi(y)} = 1 - \frac{1}{\sigma} + \delta g(y),$$

$$r_\phi(y) \equiv -\frac{y\phi''(y)}{\phi'(y)} = 1 - \mathcal{E}_\phi(y) - \frac{y\mathcal{E}'_\phi(y)}{\mathcal{E}_\phi(y)} = \frac{1}{\sigma} + \delta g(y) - \frac{\delta y g'(y)}{1 - \frac{1}{\sigma} + \delta g(y)}.$$

where $\sigma > 1$, $\delta$ can be either positive or negative (but sufficiently small in absolute value to ensure **D1**, **D2** and **D3**), while $g(y)$ is twice continuously differentiable, single-peaked, and satisfies $g(0) = g(\infty) = 0$, $\sup|g'(y)| < \infty$. Let $\tilde{y} > 0$ be the maximizer of $g(y)$. Hence, $g'(\tilde{y}) = 0 > g''(\tilde{y})$. For example,

$$g(y) = \frac{y}{\lambda + y^2}, \lambda > 0 \implies \tilde{y} = \sqrt{\lambda};$$

$$g(y) = ye^{-\mu y}, \mu > 0 \implies \tilde{y} = 1/\mu.$$

From Proposition 6, entry is excessive if and only if $\mathcal{E}'_\phi(y^E) = \delta g'(y^E) < 0$, while it is insufficient if and only if $\mathcal{E}'_\phi(y^E) = \delta g'(y^E) > 0$. Evaluating $r'_\phi(y)$ at $y = \tilde{y}$ yields:

$$r'_\phi(\tilde{y}) = -\frac{\tilde{y}\mathcal{E}''_\phi(\tilde{y})}{\mathcal{E}_\phi(\tilde{y})} = -\delta g''(\tilde{y})\frac{\tilde{y}}{\mathcal{E}_\phi(\tilde{y})} \gtreqless 0 \Leftrightarrow \delta \gtreqless 0,$$

Thus, from Proposition 5, entry is procompetitive in the vicinity of $\tilde{y}$, if $\delta > 0$, while it is anticompetitive in the vicinity of $\tilde{y}$, if $\delta < 0$.

Combining these two observations, we conclude that entry is procompetitive and yet insufficient for $\delta > 0$ and $y^E$ slightly higher than $\tilde{y}$, or equivalently, $F/L$ slightly higher than $\phi(\tilde{y})r_\phi(\tilde{y})$, while it is anticompetitive and yet excessive for $\delta < 0$ and $y^E$ slightly lower than $\tilde{y}$, or equivalently $F/L$ slightly lower than $\phi(\tilde{y})r_\phi(\tilde{y})$.

## 5. Dixit-Stiglitz under H.I.I.A. (Homothetic Indirect Implicit Additivity)

### *5.1.H.I.I.A. Demand System*

We call a symmetric CRS technology, $X = X(\mathbf{x})$ or $P = P(\mathbf{p})$, *homothetic with indirect implicit additivity* (H.I.I.A.)[25] if $P = P(\mathbf{p})$ can be defined implicitly by:

$$\int_{\Omega} \theta\left(\frac{p_\omega}{P(\mathbf{p})}\right) d\omega = 1, \tag{28}$$

where $\theta(\cdot): \mathbb{R}_+ \to \mathbb{R}_+$ is thrice continuously differentiable, strictly decreasing, and strictly convex, as long as $\theta(z) > 0$ with $\lim_{z\to 0} \theta(z) = \infty$ and $\lim_{z\to \bar{z}} \theta(z) = 0$, where $\bar{z} \equiv \inf\{z > 0 | \theta(z) = 0\}$. Again, we allow for the possibility of $\bar{z} < \infty$, the existence of the choke price, in which case, $\lim_{z\to\bar{z}} \theta'(z) = 0$. If $\bar{z} = \infty$, the choke price does not exist and demand for each input always remains positive for any positive price vector.

In the following analysis, both the elasticity of $\theta(\cdot)$ in its absolute value,

$$\mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} > 0, \tag{29}$$

and the elasticity of $\theta'(\cdot)$ in its absolute value,

$$r_\theta(z) \equiv -\frac{z\theta''(z)}{\theta'(z)} > 1 \tag{30}$$

both defined over $(0, \bar{z})$, play important roles. That $\theta(z)$ is strictly decreasing and strictly positive in $(0, \bar{z})$ ensures $\mathcal{E}_\theta(z) > 0$, and that $\theta(z)$ is strictly decreasing and strictly convex in $(0, \bar{z})$ ensures $r_\theta(z) > 0$. However, the convexity of $\theta(z)$ does not impose any upper bound on $\mathcal{E}_\theta(z)$. In addition, it is necessary to assume $r_\theta(z) > 1$ to ensure that inputs are gross substitutes, as will be seen below. Note that $\mathcal{E}_\theta(z) > 0$ is twice continuously differentiable in $(0, \bar{z})$ and satisfies $\lim_{z\to\bar{z}} \mathcal{E}_\theta(z) = \infty$ if $\bar{z} < \infty$, and that $r_\theta(z) > 1$ is continuously differentiable in

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[25]More generally, $P = P(\mathbf{p})$ is H.I.I.A. if it can be defined implicitly by $\int_\Omega \theta_\omega(p_\omega/P(\mathbf{p}))d\omega = 1$. It is the homothetic restriction of the class of I.I.A. (*indirect implicit additivity*), which can be defined implicitly by $\int_\Omega \tilde{\theta}_\omega(p_\omega, P(\mathbf{p}))d\omega = 1$: see Hanoch (1975; Section 3). In contrast, $P = P(\mathbf{p})$ is I.E.A. (*indirect explicit additivity*) if it can be defined as $P(\mathbf{p}) = \mathcal{M}\left(\int_\Omega \bar{\theta}_\omega(p_\omega)d\omega\right)$, where $\mathcal{M}(\cdot)$ is a monotone transformation: see Hanoch (1975; Section 3.2). I.E.A. is another subclass of I.I.A., with $\tilde{\theta}_\omega(p_\omega, P(\mathbf{p})) = \bar{\theta}_\omega(p_\omega)/\mathcal{M}^{-1}(P(\mathbf{p}))$. Although I.E.A. and H.I.I.A. are both subclasses of I.I.A., CES is the only common element of I.E.A. and H.I.I.A. because I.E.A. cannot be homothetic unless it is CES.

$(0, \overline{z})$ and satisfies $\lim_{z \to \overline{z}} r_\theta(z) = \infty$ if $\overline{z} < \infty$. Conversely, either from any twice continuously differentiable $\mathcal{E}_\theta(z) > 0$, defined over $(0, \overline{z})$, satisfying $\lim_{z \to \overline{z}} \mathcal{E}_\theta(z) = \infty$ if $\overline{z} < \infty$ or from any continuously differentiable $r_\theta(z) > 1$, defined over $(0, \overline{z})$, satisfying $\lim_{z \to \overline{z}} r_\theta(z) = \infty$ if $\overline{z} < \infty$, one could recover $\theta(z)$ as follows:

$$\theta(z) = \exp\left[-\int_{z_0}^{z} \mathcal{E}_\theta(\xi)\frac{d\xi}{\xi}\right];$$

$$\theta(z) = \int_{z}^{\infty} \exp\left[-\int_{z_0'}^{\xi} r_\theta(\xi')\frac{d\xi'}{\xi'}\right] d\xi.$$

where $z_0 > 0$ and $z_0' > 0$ are both constants.[26] One could also verify from eq.(29) and eq.(30) that $\mathcal{E}_\theta(z)$ and $r_\theta(z)$ are related as follows:

$$\frac{z\mathcal{E}_\theta{}'(z)}{\mathcal{E}_\theta(z)} = 1 + \mathcal{E}_\theta(z) - r_\theta(z).$$

Clearly, CES with gross substitutes is a special case with $\theta(z) = Az^{1-\sigma}$ and $\mathcal{E}_\theta(z) + 1 = r_\theta(z) = \sigma > 1$.

The cost minimization problem, eq.(2) subject to eq. (28) implies that the demand curve for each $\omega \in \Omega$ can be written as:

$$x_\omega = -B^*(\mathbf{x})\theta'\left(\frac{p_\omega}{P(\mathbf{p})}\right) > 0, \tag{31}$$

where $B^*(\mathbf{x}) > 0$ is the Lagrange multiplier associated with eq.(28), and it is the linear homogenous function in $\mathbf{x}$, given by

$$\int_\Omega \theta\left((-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\right) d\omega \equiv 1.$$

From eq.(31), the production function is given by:

$$X = X(\mathbf{x}) = \int_\Omega (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right) x_\omega d\omega. \tag{32}$$

Furthermore, the market share of $\omega$ can be written as:

$$\frac{p_\omega}{P(\mathbf{p})}\frac{x_\omega}{X(\mathbf{x})} = -\theta'\left(\frac{p_\omega}{P(\mathbf{p})}\right)\frac{p_\omega}{C(\mathbf{p})} = (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\frac{x_\omega}{X(\mathbf{x})},$$

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[26] These constants imply that $\theta(z)$is determined up to a positive scalar multiplier. However, $\gamma\theta(z)$ with $\gamma > 0$ generate the same CRS technology. All we need is to renormalize the indices of varieties, as $\int_\Omega \gamma\phi\theta(p_\omega/P)d\omega = \int_\Omega \gamma\theta(p_{\omega'}/P)d\omega' = 1$, with $\omega' = \gamma\omega$.

where $C(\mathbf{p}) \equiv -\int_{\Omega} \theta'\left(\frac{p_\omega}{P(\mathbf{p})}\right) p_\omega d\omega > 0$ is a linear homogenous function of $\mathbf{p}$, and satisfies the identity, $C(\mathbf{p})B^*(\mathbf{x}) = L = P(\mathbf{p})X(\mathbf{x})$, because

$$\frac{C(\mathbf{p})}{P(\mathbf{p})} = -\int_{\Omega} \theta'\left(\frac{p_\omega}{P(\mathbf{p})}\right)\frac{p_\omega}{P(\mathbf{p})} d\omega = \int_{\Omega} (-\theta')^{-1}\left(\frac{x_\omega}{B^*(\mathbf{x})}\right)\frac{x_\omega}{B^*(\mathbf{x})} d\omega = \frac{X(\mathbf{x})}{B^*(\mathbf{x})}.$$

The above expressions for the market share under H.I.I.A. show that it is either a function of the two relative prices, $p_\omega/P(\mathbf{p})$ and $p_\omega/C(\mathbf{p})$, or a function of the two relative quantities, $x_\omega/X(\mathbf{x})$ and $x_\omega/B^*(\mathbf{x})$, unless $C(\mathbf{p})/P(\mathbf{p}) = X(\mathbf{x})/B^*(\mathbf{x})$ is a positive constant, $c > 0$, which occurs if and only if it is CES. Thus, H.I.I.A. and H.S.A. do not overlap with the sole exception of CES.[27] Furthermore, by comparing the expressions for the market share under H.D.I.A. and the market share under H.I.I.A. one could see that H.D.I.A. and H.I.I.A. do not overlap with the sole exception of CES.[28]

From the demand curve, eq.(31), the elasticity of substitution between a pair of inputs, $\omega_1$ and $\omega_2$, evaluated at the same price (and hence at the same quantity) can be expressed as:

$$-\left.\frac{\partial \ln(x_{\omega_1}/x_{\omega_2})}{\partial \ln(p_{\omega_1}/p_{\omega_2})}\right|_{p_{\omega_1}=p_{\omega_2}=p} = r_\theta\left(\frac{p}{P}\right) > 1,$$

hence $r_\theta(z) > 1$ to ensure that inputs are gross substitutes.

### *5.2.Profit Maximization by Input Producing Firms under H.I.I.A.*

From the demand curve, eq.(31), the profit of firm $\omega \in \Omega$ is given by:

$$\pi_\omega = -(p_\omega - \psi)B^*(\mathbf{x})\theta'\left(\frac{p_\omega}{P(\mathbf{p})}\right) - F.$$

Firm $\omega$ chooses its price, $p_\omega$, to maximize its profit $\pi_\omega$, taking the aggregate variables, $P = P(\mathbf{p})$ and $B^*(\mathbf{x})$ as given. Or equivalently, it chooses $z_\omega \equiv p_\omega/P(\mathbf{p})$ to *minimize*

$$\left(z_\omega - \frac{\psi}{P(\mathbf{p})}\right)\theta'(z_\omega).$$

The FOC is:

$$\theta'(z_\omega) + \left(z_\omega - \frac{\psi}{P(\mathbf{p})}\right)\theta''(z_\omega) = z_\omega\theta''(z_\omega)\left[1 - \frac{\psi}{p_\omega} - \frac{1}{r_\theta(z_\omega)}\right] = 0. \quad (33)$$

[27]This statement is a special case of Proposition 3-(ii) in Matsuyama and Ushchev (2017).
[28]This statement is a special case of Proposition 4-(iii) in Matsuyama and Ushchev (2017).

In what follows, we keep it simple by imposing the following assumption to ensure that FOC is sufficient for the global optimum.

**Assumption I1**: *For all* $z \in (0, \bar{z})$,

$$\frac{\theta'''(z)}{\theta''(z)} - 2\frac{\theta''(z)}{\theta'(z)} > 0 \iff \frac{z r_\theta{}'(z)}{r_\theta(z)} + r_\theta(z) - 1 > 0.$$

**I1** is equivalent to the strict concavity of $1/\theta'(\cdot)$. It is readily verified that the LHS of the FOC, eq.(33) increases in the neighborhood of every solution to eq.(33) if and only if **I1** holds. Hence, eq.(33) gives the unique profit-maximizing price for each firm. Thus, all the firms set the same price, $p(\omega) = p$, and produce the same amount, $x(\omega) = x$. Hence, under **I1**, asymmetric equilibria do not exist. Unlike in the case of H.S.A., but as in the case of H.D.I.A., the condition that rules out asymmetric equilibria does not ensure the uniqueness of a symmetric equilibrium under H.I.I.A., which needs to be introduced separately; see **I2** below.

### *5.3. Symmetric Free-Entry Equilibrium under H.I.I.A.*

A *symmetric free-entry equilibrium* under H.I.I.A. satisfies the following conditions:

**H.I.I.A. integral condition**, eq.(28) under symmetry:

$$V\theta\left(\frac{p}{P}\right) = 1; \tag{34}$$

**Firm's pricing formula,** given by FOC eq.(33) under symmetry:

$$1 - \frac{\psi}{p} = \frac{1}{r_\theta\left(\frac{p}{P}\right)}, \tag{35}$$

in addition to the zero-profit (free-entry) condition, (9) and the resource constraint, (10).

For the uniqueness of a symmetric equilibrium, we introduce the following condition:

**Assumption I2**: *For all* $z \in (0, \bar{z})$,

$$\frac{z\theta'''(z)}{\theta''(z)} + 1 + r_\theta(z) + \mathcal{E}_\theta(z) > 0 \iff \frac{z r_\theta{}'(z)}{r_\theta(z)} + \mathcal{E}_\theta(z) > 0.$$

Clearly, **I1** implies **I2** if

$$\frac{z\mathcal{E}_\theta{}'(z)}{\mathcal{E}_\theta(z)} = 1 + \mathcal{E}_\theta(z) - r_\theta(z) > 0,$$

and **I2** implies **I1**, if

$$\frac{z\mathcal{E}_\theta{}'(z)}{\mathcal{E}_\theta(z)} = 1 + \mathcal{E}_\theta(z) - r_\theta(z) < 0,$$

and **I2** and **I1** are equivalent if and only if

$$\frac{z\mathcal{E}_\theta{}'(z)}{\mathcal{E}_\theta(z)} = 1 + \mathcal{E}_\theta(z) - r_\theta(z) = 0,$$

that is, under and only under CES.

To see why **I2** ensures the existence and the uniqueness of a symmetric free-entry equilibrium under H.I.I.A., note first that the pricing formula, eq.(35), and the free entry condition eq.(9) can be combined to yield:

$$px = r_\theta(p/P)F. \tag{36}$$

From eq.(9) and eq.(10), $pVx = L$, which can be combined with eq.(36) to obtain:

$$\frac{L}{V} = px = r_\theta(p/P)F,$$

which becomes after using the H.I.I.A. condition, eq.(34):

$$\frac{\theta(p/P)}{r_\theta(p/P)} = \frac{F}{L}.$$

The LHS of this equation is decreasing in $p/P$, because **I2** implies

$$\frac{d\ln\ [\theta(z)/r_\theta(z)]}{d\ln z} = \frac{z\theta'''(z)}{\theta''(z)} + 1 + r_\theta(z) + \mathcal{E}_\theta(z) > 0 \text{ for all } z \in (0,\bar{z})$$

Furthermore, $\lim_{z\to 0}\theta(z)/r_\theta(z) = \infty$ and $\lim_{z\to\bar{z}}\theta(z)/r_\theta(z) = 0$. Hence, *for each* $L/F > 0$, the equilibrium value of $z$, $z^E$, is pinned down uniquely by

$$\frac{\theta(z^E)}{r_\theta(z^E)} = \frac{F}{L} \tag{37}$$

and $z^E$ is increasing in $L/F$ with the range $(0,\bar{z})$. By inserting this value into eq.(34), eq.(35), and eq.(9),

$$V^E = \frac{1}{\theta(z^E)};$$
$$P^E = \frac{p^E}{z^E} = \frac{\psi/z^E}{1 - 1/r_\theta(z^E)} > 0; \tag{38}$$
$$x^E = \frac{(r_\theta(z^E) - 1)F}{\psi} = \frac{\theta(z^E)L - F}{\psi} > 0.$$

Thus, we have shown:

**Proposition 7**. *Under **I1**, no asymmetric equilibria exist. Furthermore, under **I1** and **I2**, there exists a unique symmetric free-entry equilibrium under H.I.I.A. for each* $L/F > 0$, *given by eq.*(37) *and eq.*(38).

### *5.4. Comparative Statics under H.I.I.A.: Procompetitive versus Anticompetitive*

Let us now turn to the comparative statics to study the market size effect.

**Proposition 8**. *Assume* **I1** and **I2.** *At the unique symmetric equilibrium in monopolistic competition under H.I.I.A., given by eq.*(37) *and eq.*(38)*,*

*Procompetitive:*
$$r_\theta'(z^E) > 0 \Rightarrow \frac{\partial p^E}{\partial L} < 0;\ 0 < \frac{\partial \ln V^E}{\partial \ln L} < 1;\ \frac{\partial x^E}{\partial L} > 0$$

*Neutral (CES):*
$$r_\theta'(z^E) = 0 \Rightarrow \frac{\partial p^E}{\partial L} = 0;\ \frac{\partial \ln V^E}{\partial \ln L} = 1;\ \frac{\partial x^E}{\partial L} = 0$$

*Anticompetitive:*
$$r_\theta'(z^E) < 0 \Rightarrow \frac{\partial p^E}{\partial L} > 0;\ \frac{\partial \ln V^E}{\partial \ln L} > 1;\ \frac{\partial x^E}{\partial L} < 0.$$

**Proof:** Since eq.(37) implies $\partial z^E/\partial L > 0$ under **I2**, this follows from eq.(38). ■

The conditions for the procompetitive vs. anticompetitive cases under H.I.I.A. are analogous to those under H.S.A and H.D.I.A. For example, the condition for the procompetitive case is $\zeta'(z^E) = \zeta'(p^E/A^E) > 0$ under H.S.A., while it is $r_\theta'(z^E) = \zeta'(p^E/P^E) > 0$ under H.I.I.A. That is, the price elasticity of demand for an input goes up as its price goes up, holding the aggregates fixed. This is nothing but *Marshall's 2nd law of demand.* Note also that, if the condition for the procompetitive case holds globally, $r_\theta'(\cdot) > 0$, **I1** and **I2** hold automatically. However, neither **I1** nor **I2** necessarily implies $r_\theta'(\cdot) > 0$. This means that **I1** and **I2** do not rule out the anticompetitive case, $r_\theta'(\cdot) < 0$.

### *5.5. Welfare Analysis under H.I.I.A.: Excessive versus Insufficient*

We now turn to the welfare analysis under H.I.I.A.. The social planner maximizes the output, given by eq.(32), subject to the resource constraint,

$$VF + \psi \int_\Omega x_\omega d\omega = L.$$

Because of the symmetry and the convexity of this problem, the solution has to be symmetric, $x_\omega = x$. By denoting

$$z = (-\theta')^{-1}\left(\frac{x}{B(\mathbf{x})}\right),$$

the problem is hence reduced to maximize $X = Vxz$ subject to $V\theta(z) = 1$ and $(F + \psi x)V = L$, or equivalently,

$$\max_{z} X = \max_{0 \le z \le \hat{z}} W(z) \equiv z\left[1 - \frac{F/L}{\theta(z)}\right],$$

where $\hat{z} \equiv \theta^{-1}(F/L) \in (0, \overline{z})$. Clearly, $W(0) = W(\hat{z}) = 0$, and $W(z) > 0$ when $0 < z < \hat{z}$.

**Lemma 6**. *Assume* **I2**. *Then,* $W(z)$ *is unimodal, with*

$$W'(z) \gtreqless 0 \Leftrightarrow \frac{L}{F} \gtreqless \frac{1 + \mathcal{E}_\theta(z)}{\theta(z)} \Leftrightarrow z \lesseqgtr z^O$$

*where* $z^O \in (0, \hat{z})$ *is the socially optimal value of* $z$*, uniquely given by*

$$\frac{L}{F} = \frac{1 + \mathcal{E}_\theta(z^O)}{\theta(z^O)}$$

*and* $z^O$ *is strictly increasing in* $L/F$.

**Proof**. Differentiating $W(z)$ yields

$$W'(z) = 1 - \frac{F}{L}\frac{1 + \mathcal{E}_\theta(z)}{\theta(z)}.$$

$$W''(z) = \frac{\mathcal{E}_\theta(z)}{z}\left[\frac{r_\theta(z)}{\theta(z)}\frac{F}{L} - 2\big(1 - W'(z)\big)\right].$$

To show that $W(z)$ is unimodal with the unique global optimizer, $z^O \in (0, \hat{z})$ satisfying $W'(z^O) = 0$, suppose the contrary. Then, there exist $0 < z_1 < z_2 < z_3 < \hat{z}$, such that $z_1$ and $z_3$ are local maxima satisfying $W'(z_1) = 0 > W''(z_1)$ and $W'(z_3) = 0 > W''(z_3)$ and $z_2$ is a local minimum satisfying $W'(z_2) = 0 < W''(z_2)$. This implies

$$\frac{r_\theta(z)}{\theta(z)}\frac{F}{L} - 2\big(1 - W'(z)\big) = \frac{r_\theta(z)}{\theta(z)}\frac{F}{L} - 2$$

is negative at $z_1$ and $z_3$ and positive at $z_2$, contradicting the monotonicity of $r_\theta(z)/\theta(z)$, hence **I2.** That $z^O$ is strictly increasing in $L/F$ follows from $W''(z^O) < 0$ and $\partial W'(z^O)/\partial(L/F) > 0$.■

**Proposition 9.** *Assume* **I1** and **I2**. *Then, at the unique symmetric equilibrium in monopolistic competition under H.I.I.A., given by eq.*(37) *and eq.*(38), $V^E$*, the equilibrium mass of firms that* *enter = the equilibrium mass of varieties produced and* $V^O$*, the mass of the optimal mass of firms that enter = the optimal mass of varieties produced,, satisfy*

*Excessive Entry:* $$\frac{z^E \mathcal{E}_\theta{}'(z^E)}{\mathcal{E}_\theta(z^E)} = 1 + \mathcal{E}_\theta(z^E) - r_\theta(z^E) > 0 \Leftrightarrow V^E > V^O$$

*Optimal Entry (CES):* $$\frac{z^E \mathcal{E}_\theta{}'(z^E)}{\mathcal{E}_\theta(z^E)} = 1 + \mathcal{E}_\theta(z^E) - r_\theta(z^E) = 0 \Leftrightarrow V^E = V^O$$

*Insufficient Entry:* $$\frac{z^E \mathcal{E}_\theta{}'(z^E)}{\mathcal{E}_\theta(z^E)} = 1 + \mathcal{E}_\theta(z^E) - r_\theta(z^E) < 0 \Leftrightarrow V^E < V^O$$

**Proof**. Since eq.(37) implies

$$W'(z^E) = 1 - \frac{F}{L}\frac{1 + \mathcal{E}_\theta(z^E)}{\theta(z^E)} = 1 - \frac{1 + \mathcal{E}_\theta(z^E)}{r_\theta(z^E)},$$

and Lemma 6 implies $W'(z^E) \gtreqless 0 \Leftrightarrow z^E \lesseqgtr z^O \Leftrightarrow V^E \lesseqgtr V^O$, we have

$$r_\theta(z^E) - 1 - \mathcal{E}_\theta(z^E) \gtreqless 0 \Leftrightarrow V^E \lesseqgtr V^O.$$

This completes the proof. ■

Note that, in order for the equilibrium entry to be optimal for a range of the parameter values under H.I.I.A, $\frac{z\mathcal{E}_\theta'(z)}{\mathcal{E}_\theta(z)} = 1 + \mathcal{E}_\theta(z) - r_\theta(z) = 0$ must hold for the relevant range of $z$, that is, under and only under CES. Thus, CES offers the borderline case between the cases of excessive entry and insufficient entry within H.I.I.A.

### *5.6.Main H.I.I.A. Theorem and Some Examples*

Proposition 8 states that the sign of $r_\theta'(z^E)$ determines whether entry is procompetitive or anticompetitive, while Proposition 9 states the sign of $\mathcal{E}_\theta'(z^E)$ determines whether entry is excessive or insufficient. Hence, one might think, unlike under H.S.A, that these conditions are unrelated to each other, and that both procompetitive entry and anticompetitive entry can be either excessive or insufficient under H.I.I.A.. However, similar to the case of H.D.I.A., the next lemma shows that there exists a tight connection between the two conditions.

**Lemma 7:** *Assume that* $r_\theta'(\cdot)$ *does not change sign over* $(z_0, \overline{z})$*, where* $0 < z_0 < \overline{z}$*. Then, for all* $z \in (z_0, \overline{z})$*,*

$$r_\theta'(\cdot) \gtreqless 0 \implies \mathcal{E}_\theta'(\cdot) \gtreqless 0.$$

**Proof**: See Appendix B. ■

Here is the implication of Lemma 7. Suppose that, for all $L/F > (L/F)_0$, entry is procompetitive at the unique symmetric equilibrium given by eq.(37) and eq.(38). That means that $r_\theta'(z) > 0$ for all $z \in (z_0, \overline{z})$, where $z_0$ satisfies $(\theta(z_0)/r_\theta(z_0))(L/F)_0 = 1$. Then, Lemma 7 tells us $\mathcal{E}_\theta'(z) > 0$ for all $z \in (z_0, \overline{z})$. Hence, for all $L/F > (L/F)_0$, entry is excessive at the unique symmetric equilibrium. Likewise, suppose that, for all $L/F > (L/F)_0$, entry is anticompetitive at the unique symmetric equilibrium given by eq.(37) and eq.(38). That means $r_\theta'(z) < 0$ for all $z \in (z_0, \overline{z})$. Then, Lemma 7 tells us $\mathcal{E}_\theta'(z) < 0$ for all $z \in (z_0, \overline{z})$. Hence, for all $L/F > (L/F)_0$, entry is insufficient at the unique symmetric equilibrium.

We are now ready to summarize the main properties of H.I.I.A. in the next theorem, by consolidating Propositions 7, 8, and 9 and Lemma 7. In doing so, we take into account that $z^E$ is strictly increasing in $L/F$ and takes any value in $(0, \overline{z})$, as $L/F$ varies from zero to infinity, and that the existence of the choke price, $\overline{z} < \infty$, implies $\lim_{z \to \overline{z}} r_\theta(z) = \infty$ and $\lim_{z \to \overline{z}} \mathcal{E}_\theta(z) = \infty$, and hence $r_\theta{}'(z) > 0$ and $\mathcal{E}_\theta{}'(z) > 0$ for $z$ sufficiently close to $\overline{z}$, which means that entry is procompetitive and excessive for a sufficiently large $L/F > 0$.

**Theorem 3:** *Consider monopolistic competition under symmetric H.I.I.A. with gross substitutes. Assume **I1** to ensure the symmetry of equilibrium and **I2** to ensure the uniqueness of the symmetric equilibrium. Then, the unique symmetric equilibrium is given by eq.*(37) *and eq.*(38)*. At the unique symmetric equilibrium, entry is,*

- *procompetitive and excessive for any* $L/F > 0$, *if* $r_\theta{}'(z) > 0$ *for all* $z \in (0, \overline{z})$;
- *neutral and optimal for any* $L/F > 0$, *if* $r_\theta{}'(z) = 0$ *for all* $z \in (0, \infty)$*; that is, under CES;*
- *anticompetitive and insufficient for any* $L/F > 0$, *if* $r_\theta{}'(z) < 0$ *for all* $z \in (0, \infty)$.

*Furthermore, in the presence of the choke price,* $\overline{z} < \infty$*, entry is procompetitive and excessive for a sufficiently large* $L/F > 0$.

We now turn to some examples to illustrate Theorem 3.

**Example 7: Perturbed CES, H.I.I.A. with global monotonicity.** Consider a family of H.I.I.A. technologies, given by

$$r_\theta(z) \equiv -\frac{z\theta''(z)}{\theta'(z)} = \sigma + \delta(\sigma - 1)g(z),$$

where $\sigma > 1$ and $g(z)$ satisfies $g'(z) > 0$ for all $z > 0$ with $g(0) = -1$, $g(\infty) = 0$ and $\sup\{zg'(z)|z > 0\} \equiv \nu < \infty$. For example,

$$g(z) = -\frac{\eta}{\eta + z}, \eta > 0 \Rightarrow \nu = \frac{1}{4} < \infty,$$

$$g(z) = -e^{-\mu z}, \mu > 0 \Rightarrow \nu = e^{-1} < \infty,$$

satisfy these conditions. In addition, we impose the following restriction on $\sigma, \delta$, and $\nu$:

$$-\frac{\sigma}{\nu} < \delta < 1,$$

so that $r_\theta(z) > 1$, **I1**, and **I2** hold.[29] Then, Theorem 3 can be applied. In this example, entry is procompetitive and excessive for all $L/F > 0$, when $0 < \delta < 1$, while it is anticompetitive and insufficient for all $L/F > 0$, when $-\frac{\sigma}{\nu} < \delta < 0$.

**Example 8: H.I.I.A. with a choke price.** Consider an H.I.I.A. technology, given by

$$\theta(z) = \begin{cases} \frac{(\ln(\bar{z}/z))^{1+\delta}}{1+\delta}, & 0 < z < \bar{z}, \\ 0, & z \geq \bar{z}, \end{cases}$$

with $0 < \bar{z} < \infty$ and $\delta > 0$. For all $z$ such that $0 < z < \bar{z}$, we have

$$\theta'(z) = -\frac{(\ln(\bar{z}/z))^{\delta}}{z} < 0; \quad \theta''(z) = \frac{(\ln(\bar{z}/z))^{\delta}}{z^2}\left[1 + \frac{\delta}{\ln(\bar{z}/z)}\right] > 0;$$

Hence, $\lim_{z\to\bar{z}} \theta(z) = \lim_{z\to\bar{z}} \theta'(z) = 0$. Also,

[29]It is easy to verify $r_\theta(z) > 1$ holds. For **I1** and **I2**, if $\delta \geq 0$, $r'_\theta(z) \geq 0$, which implies both **I1** and **I2**. If $\delta < 0$, $r'_\theta(z) < 0$ for all $z > 0$. From Lemma 7, this implies $\mathcal{E}'_\theta(z) < 0$ for all $z > 0$, which means that **I2** implies **I1**. To verify **I2** for $\delta < 0$, note that $r'_\theta(z) < 0$ and $\mathcal{E}'_\theta(z) < 0$ for all $z > 0$ implies

$$r_\theta(z)\mathcal{E}_\theta(z) > r_\theta(\infty)\mathcal{E}_\theta(\infty) = \sigma(\sigma - 1),$$

while $\delta < 0$ and the definition of $\nu$ imply

$$zr_\theta{}'(z) = \delta(\sigma - 1)zg'(z) > \nu\delta(\sigma - 1).$$

Adding each side of these two inequalities yields $r_\theta(z)\mathcal{E}_\theta(z) + zr_\theta{}'(z) > (\sigma + \nu\delta)(\sigma - 1) > 0$, which is equivalent to **I2.**

$$\mathcal{E}_\theta(z) = \frac{1+\delta}{\ln(\overline{z}/z)} > 0; \quad r_\theta(z) = 1 + \frac{\delta}{\ln(\overline{z}/z)} > 1,$$

which implies $\lim_{z \uparrow \overline{z}} \mathcal{E}_\theta(z) = \infty = \lim_{z \uparrow \overline{z}} r_\theta(z)$. Furthermore, for all $z$ such that $0 < z < \overline{z}$, we have:

$$\mathcal{E}'_\theta(z) > 0, \qquad r'_\theta(z) > 0,$$

and hence entry is always procompetitive and excessive, not just for a sufficient large $L/F$.

The assumption of the global monotonicity of $r_\theta(\cdot)$ in Theorem 3, which implies the global monotonicity of $\mathcal{E}_\theta(\cdot)$ by Lemma 7, is important. Otherwise, entry could be procompetitive and yet insufficient, or anticompetitive and yet excessive, as the next example illustrates.

**Example 9. Perturbed CES, H.I.I.A. without global monotonicity.** Consider a family of H.I.I.A technologies with

$$\mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} = \sigma - 1 + \delta g(z),$$

$$r_\theta(z) \equiv -\frac{z\theta''(z)}{\theta'(z)} = 1 + \mathcal{E}_\theta(z) - \frac{z\mathcal{E}'_\theta(z)}{\mathcal{E}_\theta(z)} = r_\theta(z) = \sigma + \delta g(z) - \frac{\delta z g'(z)}{\sigma - 1 + \delta g(z)},$$

where $\sigma > 1$, $\delta$ can be either positive or negative (but sufficiently small in absolute value to ensure **I1** and **I2**), while $g(z)$ is twice-continuously differentiable, single-peaked, and satisfies $g(0) = g(\infty) = 0$, $\sup|g'(z)| < \infty$. Let $\tilde{z} > 0$ be the maximizer of $g(z)$ Hence, $g'(\tilde{z}) = 0 > g''(\tilde{z})$. For example,

$$g(z) = \frac{z}{\lambda + z^2}, \lambda > 0 \Longrightarrow \tilde{z} = \sqrt{\lambda};$$

$$g(z) = ze^{-\mu z}, \mu > 0 \Longrightarrow \tilde{z} = 1/\mu.$$

From Proposition 9, entry is excessive if and only if $\mathcal{E}'_\theta(z^E) = \delta g'(z^E) > 0$, while it is insufficient if and only if $\mathcal{E}'_\theta(z^E) = \delta g'(z^E) < 0$. Evaluating $r'_\theta(z)$ at $z = \tilde{z}$ yields:

$$r'_\phi(\tilde{z}) = -\frac{\tilde{z}\mathcal{E}''_\theta(\tilde{z})}{\mathcal{E}_\theta(\tilde{z})} = -\delta g''(\tilde{z})\frac{\tilde{z}}{\mathcal{E}_\theta(\tilde{z})} \gtreqless 0 \Leftrightarrow \delta \gtreqless 0.$$

Thus, from Proposition 8, entry is procompetitive in the vicinity of $\tilde{z}$, if $\delta > 0$, while it is anticompetitive in the vicinity of $\tilde{z}$, if $\delta < 0$.

Combining these two observations, we conclude that entry is procompetitive and yet insufficient for $\delta > 0$ and $z^E$ slightly higher than $\tilde{z}$, or equivalently, $L/F$ slightly higher than

$r_\theta(\tilde{z})/\theta(\tilde{z})$, while it is anticompetitive and yet excessive for $\delta < 0$ and $z^E$ slightly lower than $\tilde{z}$, or equivalently, $L/F$ slightly higher than $r_\theta(\tilde{z})/\theta(\tilde{z})$.

## 6. Concluding Remarks

In this paper, we extended the canonical model of monopolistic competition with symmetric homothetic CES demand system with gross substitutes by Dixit and Stiglitz (1977, Section I) to three classes of homothetic demand systems, H.S.A., H.D.I.A., and H.I.I.A, which are mutually exclusive except that each class contains CES as a knife-edge case. These three classes are flexible and yet tractable enough to allow us to identify not only the conditions for the existence of the unique symmetric free entry equilibrium and for the non-existence for an asymmetric free-entry equilibrium, but also the conditions for procompetitive vs. anticompetitive entry and for excessive vs. insufficient entry. Among the main findings are that entry is excessive (insufficient) if it is *globally* procompetitive (anticompetitive)[30] and that, in the presence of the choke price, entry is procompetitive and excessive at least for a sufficiently large market size. One implication is that, for those who believe that procompetitive entry is the empirically relevant case, entry is excessive and hence (small) regulation of entry is welfare-improving, at least in the absence of other forms of distortion.[31]

These classes of homothetic demand systems, which offer three alternative ways of departing from CES, are flexible and tractable. Their main advantage, when applied to monopolistic competition, is that entry and behavior of competing firms affect the demand curve for each firm only through either one aggregator (under H.S.A.) or two aggregators (under H.D.I.A., and H.I.I.A.), and the price elasticity of demand for each firm only through a single aggregator under all three classes. Furthermore, homotheticity makes it easier to use them as building blocks in dynamic and/or multi-sector general equilibrium settings. For these reasons,

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[30]As already pointed out in the introduction, the qualification "globally" is important. If the equilibrium markup rate responds *nonmonotonically* to entry, procompetitive entry can be insufficient, while anticompetitive entry can be excessive in some range of parameter values, even under the three classes we considered, as demonstrated by Examples 3, 6, and 9.

[31]An open question is whether it is possible to find the requirement analogous to the global monotonicity, which ensures these results to be extended beyond the three classes. (The difficulty is not only to find the condition that plays the same role of the global monotonicity of the markup rate response, but also to find the conditions that ensure gross substitutability, as well as the existence and uniqueness of symmetric free-entry equilibrium.) For this reason, we indicate the possibility of procompetitive and yet insufficient entry and anticompetitive and yet excessive entry by small but gray zones in Figure 2.

these three classes have the potential for unlocking many new lines of inquiry, which would not be possible under CES. To give an example, consider monopolistic competition with heterogenous firms. It is well-known that heterogeneity of firms/products in quality, market size, and productivity are all isomorphic to each other under CES, so that one could assume without loss of generality that firms differ only in productivity, as Melitz (2003) has done. This is no longer the case under non-CES. The problem, however, is that monopolistic competition models with many dimensions of heterogeneity across firms would lack any predictive content under general homothetic demand systems. Partly motivated by this, Matsuyama and Ushchev (2020b) propose parametric families within each of these three classes, which capture many dimensions of heterogeneity in a meaningful and yet tractable way. In addition to those imposed by the three classes, the key restriction that buys a lot of tractability is that the pass-through rate is constant and common across firms/products. Otherwise, firms/products can be heterogeneous in market size, in quality, in substitutability, and in productivity.

Among the three classes, H.S.A. is particularly tractable due to its single aggregator property, because the equilibrium value of the single aggregator can be uniquely pinned down by the free-entry condition. [32] In Matsuyama and Ushchev (2020a), we take advantage of this feature of H.S.A. in a dynamic monopolistic competition model and investigate how market size affects the dynamics of innovation through the procompetitive effect. Despite the model features technology diffusion, which causes the co-existence of monopolistically and competitively priced varieties, the case of H.S.A. remains as tractable as the CES case, with much richer implications.

[32] In addition to its single aggregator property, there is another advantage of H.S.A. demand systems, as pointed out by Kasahara and Sugita (2020). That is, the market share functions are the primitive of H.S.A., so that it can be readily identified with typical firm-level data, which contain revenue, not output quantity.

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**Appendix A: Proof of Lemma 5**

We first prove the two preliminary lemmas, Lemma A1 and Lemma A2.

**Lemma A1**. *For any $\phi(\cdot)$ which is strictly increasing, strictly concave, and satisfies $\phi(0) = 0$,*

$$\mathcal{E}_{\phi}(0) = 1 - r_{\phi}(0).$$

**Proof**. From $\phi(0) = 0$ and $0 < y\phi'(y) < \phi(y)$ for all $y > 0$, $\lim_{y \to 0} y\phi'(y) = 0$. It is thus legitimate to use the l'Hospital's rule in computing the following limit:

$$\mathcal{E}_{\phi}(0) \equiv \lim_{y \downarrow 0} \mathcal{E}_{\phi}(y) = \lim_{y \downarrow 0} \frac{y\phi'(y)}{\phi(y)} = \lim_{y \downarrow 0} \frac{\phi'(y) + y\phi''(y)}{\phi'(y)} = 1 - r_{\phi}(0).$$

This completes the proof. ∎

**Lemma A2**. *Assume that $r'_{\phi}(\cdot)$ does not change sign over $(0, y_0)$, where $0 < y_0 \leq \infty$. Then, for all $y \in (0, y_0)$,*

$$\mathcal{E}_{\phi}(y) \lesseqgtr \mathcal{E}_{\phi}(0) \Leftrightarrow r'_{\phi}(y) \gtreqless 0.$$

**Proof**. Two cases may arise.

**Case 1**: $r_{\phi}(0) < 1$. First, define:

$$\Delta r_{\phi}(y) \equiv r_{\phi}(y) - r_{\phi}(0)$$

to obtain the identity,

$$\frac{d \ln \phi'(y)}{d \ln y} \equiv -r_{\phi}(y) = -r_{\phi}(0) - \Delta r_{\phi}(y),$$

integrating which yields

$$\phi'(y) = f(y) y^{-r_{\phi}(0)}, \qquad f(y) \equiv \exp\left\{-\int_{y_0}^{y} \frac{\Delta r_{\phi}(\xi)}{\xi} d\xi\right\} > 0;$$

$$\phi(y) = \int_0^{y} \phi'(\xi) d\xi = \int_0^{y} f(\xi) \xi^{-r_{\phi}(0)} d\xi.$$

Hence,

$$\mathcal{E}_{\phi}(y) \equiv \frac{y\phi'(y)}{\phi(y)} = \frac{f(y) y^{1-r_{\phi}(0)}}{\int_0^{y} f(\xi) \xi^{-r_{\phi}(0)} d\xi}.$$

By the mean value theorem, there exists $\alpha(y) \in (0, y)$, such that

$$\int_0^{y} f(\xi) \xi^{-r_{\phi}(0)} d\xi = f(\alpha(y)) \int_0^{y} \xi^{-r_{\phi}(0)} d\xi = \frac{1}{1 - r_{\phi}(0)} f(\alpha(y)) y^{1-r_{\phi}(0)}.$$

Hence, using the definition of $f(y)$ and Lemma A1,

$$\mathcal{E}_\phi(y) = \left(1 - r_\phi(0)\right)\frac{f(y)}{f(\alpha(y))} = \mathcal{E}_\phi(0)\exp\left\{-\int_{\alpha(y)}^{y} \frac{\Delta r_\phi(\xi)}{\xi} d\xi\right\}.$$

Then, for all $y \in (0, y_0)$, $0 < \alpha(y) < y < y_0$ implies

$$\mathcal{E}_\phi(y) \lesseqgtr \mathcal{E}_\phi(0) \Leftrightarrow \int_{\alpha y}^{y} \frac{\Delta r_\phi(\xi)}{\xi} d\xi \gtreqless 0 \Leftrightarrow r'_\phi(y) \gtreqless 0.$$

**Case 2**: $r_\phi(0) = 1$. This happens only when $r'_\phi(y) < 0$ for all $y \in (0, y_0)$, because $r_\phi(y) < 1 = r_\phi(0)$ for all $y \in (0, y_0)$. And from Lemma A1, $\mathcal{E}_\phi(y) > 0 = \mathcal{E}_\phi(0)$ for all $y \in (0, y_0)$.

This completes the proof. ■

We are now ready to prove Lemma 5.

**Lemma 5**. *Assume that* $r'_\phi(\cdot)$ *does not change sign over* $(0, y_0)$*, where* $0 < y_0 \leq \infty$. *Then, for all* $y \in (0, y_0)$

$$r'_\phi(\cdot) \gtreqless 0 \implies \mathcal{E}'_\phi(\cdot) \lesseqgtr 0.$$

**Proof**. Three cases may arise.

**Case 1**: $r'_\phi(y) > 0$ for all $y \in (0, y_0)$. To prove by contradiction, suppose to the contrary that that there is $y_1 \in (0, y_0)$, such that $\mathcal{E}'_\phi(y_1) \geq 0$. Two sub-cases may arise.

**Case 1-1**: $\mathcal{E}'_\phi(y_1) > 0$. Because Lemma A2 implies $\mathcal{E}_\phi(y_1) < \mathcal{E}_\phi(0)$, $\mathcal{E}_\phi(y)$ must have an interior local minimizer $y_2 \in (0, y_1)$, which satisfies

$$\mathcal{E}'_\phi(y_2) = 0, \qquad \mathcal{E}''_\phi(y_2) \geq 0.$$

Differentiating the identity

$$\mathcal{E}'_\phi(y) = \frac{\mathcal{E}_\phi(y)}{y}\left(1 - \mathcal{E}_\phi(y) - r_\phi(y)\right),$$

at $y = y_2$ and using $\mathcal{E}'_\phi(y_2) = 0$, we obtain:

$$\mathcal{E}''_\phi(y_2) = -\frac{\mathcal{E}_\phi(y_2)}{y_2} r'_\phi(y_2) < 0,$$

which clearly contradicts $\mathcal{E}''_\phi(y_2) \geq 0$.

**Case 1-2**: $\mathcal{E}'_\phi(y_1) = 0$. In this case, differentiating the identity

$$\mathcal{E}'_\phi(y) = \frac{\mathcal{E}_\phi(y)}{y}\left(1 - \mathcal{E}_\phi(y) - r_\phi(y)\right),$$

at $y = y_1$ yields

$$\mathcal{E}''_\phi(y_1) = -\frac{\mathcal{E}_\phi(y_1)}{y_1} r'_\phi(y_1) < 0.$$

Therefore, for a small $h > 0$ we have: $\mathcal{E}'_\phi(y_1 - h) > \mathcal{E}'_\phi(y_1) = 0$. By replacing $y_1$ with $y_1 - h$, we use the same argument as in case 1-1.

Thus, we have:

$$r'_\phi(y) > 0 \ \ for\ all\ y \in (0, y_0) \Rightarrow \ \mathcal{E}'_\phi(y) < 0 \ \ for\ all\ y \in (0, y_0).$$

**Case 2**: $r'_\phi(y) = 0$ for all $y \in (0, y_0)$ This is the CES case, which is straightforward.

**Case 3**: $r'_\phi(y) < 0$ for all $y \in (0, y_0)$ . One can handle this case along the same lines as case 1.

This completes the proof. ■

**Appendix B: Proof of Lemma 7**

We first prove the three preliminary lemmas, Lemma B1, Lemma B2, and Lemma B3.

**Lemma B1.** $\bar{z}\theta'(\bar{z}) = 0$.

**Proof.** For $\bar{z} < \infty$, this follows from $\theta'(\bar{z}) = 0$. For $\bar{z} = \infty$,

$$\theta(z) = -\int_z^\infty \theta'(\xi) d\xi = -\int_z^\infty \frac{\xi\theta'(\xi)}{\xi} d\xi = -\lim_{x\to\infty} \int_z^x \frac{\xi\theta'(\xi)}{\xi} d\xi$$

Suppose that there is $z_0 > 0$ such that, for all $z > z_0$, $-z\theta'(z) > c > 0$. Then,

$$\theta(z_0) = -\lim_{x\to\infty} \int_{z_0}^x \frac{\xi\theta'(\xi)}{\xi} d\xi > \lim_{x\to\infty} \int_{z_0}^x \frac{c}{\xi} d\xi = \infty,$$

a contradiction. Hence, $\bar{z}\theta'(\bar{z}) = \lim_{z\to\infty} z\theta'(z) = 0$. This completes the proof. ■

**Lemma B2**. *For any $\theta(\cdot)$ which defines an H.I.I.A. technology, we have*:

$$\mathcal{E}_\theta(\bar{z}) = r_\theta(\bar{z}) - 1,$$

*where* $0 < \bar{z} \equiv \inf\{z > 0 \mid \theta(z) = 0\} \leq \infty$.

**Proof**. Since $\theta(\bar{z}) = 0 = \bar{z}\theta'(\bar{z})$ by Lemma B1, it is legitimate to use the l'Hospital's rule in computing the following limit:

$$\mathcal{E}_\theta(\bar{z}) \equiv \lim_{z\uparrow\bar{z}} \mathcal{E}_\theta(z) = \lim_{z\uparrow\bar{z}} \frac{-z\theta'(z)}{\theta(z)} = \lim_{z\uparrow\bar{z}} \frac{-z\theta''(z) - \theta'(z)}{\theta'(z)} = r_\theta(\bar{z}) - 1.$$

This completes the proof. ■

**Lemma B3**. *Assume that $r'_\theta(\cdot)$ does not change sign over $(z_0, \bar{z})$, where $0 < z_0 < \bar{z}$. Then, for all $z \in (z_0, \bar{z})$,*

$$\mathcal{E}_\theta(z) \lesseqgtr \mathcal{E}_\theta(\bar{z}) \Leftrightarrow r'_\theta(z) \gtreqless 0.$$

**Proof**. If $r_\theta(\overline{z}) = \infty$, then the only possibility is that $r'_\theta(\cdot) > 0$. In this case, by Lemma B2, we have: $\mathcal{E}_\theta(\overline{z}) = \infty > \mathcal{E}_\theta(z)$ for all $z \in (0, \overline{z})$. Consider now the case when $1 < r_\theta(\overline{z}) < \infty$, hence $\overline{z} = \infty$. Two cases may arise.

**Case 1**: $r_\theta(\infty) > 1$. First, define:

$$\Delta r_\theta(z) \equiv r_\theta(\infty) - r_\theta(z),$$

to obtain the identity,

$$\frac{d \ln[-\theta'(z)]}{d \ln z} = \Delta r_\theta(z) - r_\theta(\infty),$$

integrating which yields

$$-\theta'(z) = f(z) z^{-r_\theta(\infty)}, \qquad f(z) \equiv \exp\left\{\int_{z_0}^{z} \frac{\Delta r_\theta(\xi)}{\xi} d\xi\right\} > 0;$$

$$\theta(z) = -\int_z^\infty \theta'(\xi) d\xi = \int_z^\infty f(\xi) \xi^{-r_\theta(\infty)} d\xi.$$

Hence,

$$\mathcal{E}_\theta(z) \equiv -\frac{z\theta'(z)}{\theta(z)} = \frac{f(z) z^{1-r_\theta(\infty)}}{\int_z^\infty f(\xi)\xi^{-r_\theta(\infty)} d\xi}.$$

By the mean value theorem, there exists $\beta(z) > z$, such that

$$\int_z^\infty f(\xi) \xi^{-r_\theta(\infty)} d\xi = f(\beta(z)) \int_z^\infty \xi^{-r_\theta(\infty)} d\xi = \frac{1}{r_\theta(\infty) - 1} f(\beta(z)) z^{1-r_\theta(\infty)}.$$

Hence, using the definition of $f(z)$ and Lemma B2 for $\overline{z} = \infty$,

$$\mathcal{E}_\theta(z) = (r_\theta(\infty) - 1) \frac{f(z)}{f(\beta(z))} = \mathcal{E}_\theta(\infty) \exp\left\{-\int_z^{\beta(z)} \frac{\Delta r_\theta(\xi)}{\xi} d\xi\right\}.$$

Then, for all $z > z_0$, $\beta(z) > z > z_0$ implies

$$\mathcal{E}_\theta(z) \lesseqqgtr \mathcal{E}_\theta(\infty) \Leftrightarrow \int_z^{\beta(z)} \frac{\Delta r_\theta(\xi)}{\xi} d\xi \gtreqqless 0 \Leftrightarrow r'_\theta(\cdot) \gtreqqless 0.$$

**Case 2**: $r_\theta(\infty) = 1$. This happens only when $r'_\theta(z) < 0$ for all $z \in (z_0, \infty)$, because $r_\theta(z) > 1$ for all $z \in (z_0, \infty)$. From Lemma B2, $\mathcal{E}_\theta(z) > 0 = \mathcal{E}_\theta(\infty)$ for all $z \in (z_0, \infty)$.

This completes the proof. ■

We are now ready to prove Lemma 7.

**Lemma 7**. *Assume that* $r'_\theta(\cdot)$ *does not change sign over* $(z_0, \overline{z})$*, where* $0 < z_0 < \overline{z}$*. Then, for all* $z \in (z_0, \overline{z})$*,*

$$r_\theta'(\cdot) \gtreqless 0 \implies \mathcal{E}_\phi'(\cdot) \gtreqless 0.$$

**Proof**. Three cases may arise.

**Case 1**: $r_\theta'(z) > 0$ for all $z \in (z_0, \overline{z})$. To prove by contradiction, suppose to the contrary that there is $z_1 \in (z_0, \overline{z})$, such that $\mathcal{E}_\theta'(z_1) \leq 0$. Two sub-cases may arise.

**Case 1-1**: $\mathcal{E}_\theta'(z_1) < 0$. Because Lemma B3 implies $\mathcal{E}_\theta(z_1) < \mathcal{E}_\theta(\overline{z})$, $\mathcal{E}_\theta(\cdot)$ must have an interior local minimizer $z_2 \in (z_1, \overline{z})$, which satisfies

$$\mathcal{E}_\theta'(z_2) = 0, \qquad \mathcal{E}_\theta''(z_2) \geq 0.$$

Differentiating the identity

$$\mathcal{E}_\theta'(z) = \frac{\mathcal{E}_\theta(z)}{z}\big(1 + \mathcal{E}_\theta(z) - r_\theta(z)\big),$$

at $z = z_2$ and using $\mathcal{E}_\theta'(z_2) = 0$, we obtain:

$$\mathcal{E}_\theta''(z_2) = -\frac{\mathcal{E}_\theta(z_2)}{z_2} r_\theta'(z_2) < 0,$$

which clearly contradicts $\mathcal{E}_\theta''(z_2) \geq 0$.

**Case 1-2**: $\mathcal{E}_\theta'(z_1) = 0$. In this case, differentiating the identity

$$\mathcal{E}_\theta'(z) = \frac{\mathcal{E}_\theta(z)}{z}\big(1 + \mathcal{E}_\theta(z) - r_\theta(z)\big),$$

at $z = z_1$ yields

$$\mathcal{E}_\theta''(z_1) = -\frac{\mathcal{E}_\theta(z_1)}{z_1} r_\theta'(z_1) < 0.$$

Therefore, for a small $h > 0$ we have: $\mathcal{E}_\theta'(z_1 + h) < \mathcal{E}_\phi'(z_1) = 0$. By replacing $z_1$ with $z_1 + h$, we use the same argument as in case 1-1.

Thus, we have:

$$r_\theta'(z) > 0 \ \ for\ all\ z \in (z_0, \overline{z}) \implies \mathcal{E}_\theta'(z) > 0 \ \ for\ all\ z \in (z_0, \overline{z}).$$

**Case 2**: $r_\theta'(z) = 0$ for all $z \in (z_0, \overline{z})$. This is the CES case, which is straightforward.

**Case 3**: $r_\theta'(z) < 0$ for all $z \in (z_0, \overline{z})$. One can handle this case along the same lines as case 1.

This completes the proof. ■

Figure 1:

Three Classes

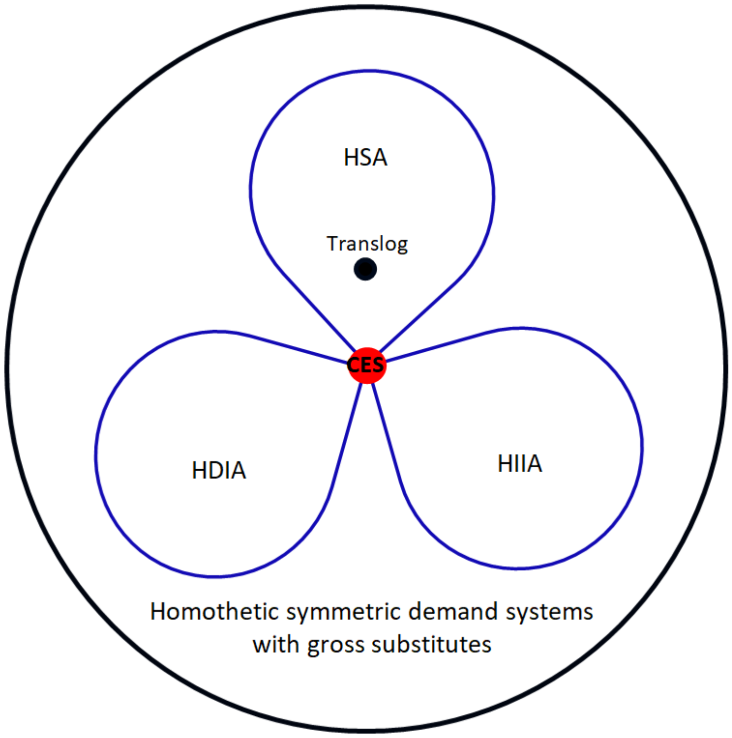


Figure 2:

Entry is excessive if it is *globally* procompetitive and insufficient if it is *globally* anticompetitive under the three classes.