Selection and Sorting of Heterogeneous Firms Through Competitive Pressures

Kiminori Matsuyama  
Northwestern University

Philip Ushchev  
ECARES, Université Libre de Bruxelles

Last Updated: 2024-01-18; 5:20:54 AM

Teaching Slides
Introduction
Competitive Pressures on Heterogeneous Firms

Main Questions: How do more competitive pressures, due to entry of new firms, caused by lower entry cost or larger market size, affect firms with different productivity?

- Selection of firms
- Distribution of firm size (in revenue, profit and employment), Distribution of markup and pass-through rates, etc.
- Sorting of firms across markets with different market sizes

Existing Monopolistic Competition Models with Heterogenous Firms

- Melitz (2003): under CES Demand System (DS)
  - MC firms sell their products at an exogenous & common markup rate, unresponsive to competitive pressures
  - Market size: no effect on distribution of firm types nor their behaviors; All adjustments at the extensive margin.
  - Firms’ incentive to move across markets with different market sizes independent of firm productivity
    Inconsistent with some evidence for
  - An increase in the production cost leads to less than proportional increase in the price (the pass-through rate < 1)
  - More productive firms have higher markup rates
  - More productive firms have lower pass-through rates

- Melitz-Ottaviano (2008) departs from CES with Linear Demand System + the outside competitive sector, which comes with its own restrictions.

This Paper: Melitz under H.S.A. Demand System as a framework to study how departing from CES in the direction consistent with the evidence affects the impact of competitive pressures on heterogeneous firms.
Symmetric H.S.A. (Homothetic with a Single Aggregator) DS with Gross Substitutes

Think of a competitive final goods industry generating demand for a continuum of intermediate inputs $\omega \in \Omega$, with

CRS production function: $X = X(x); x = \{x_\omega; \omega \in \Omega\} \iff$ Unit cost function, $P = P(p); p = \{p_\omega; \omega \in \Omega\}$.

Market share of $\omega$ depends solely on a single variable, its own price normalized by the common price aggregator

$$s_\omega \equiv \frac{p_\omega x_\omega}{px} = \frac{\partial \ln P(p)}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(p)}\right), \text{ where } \int_\Omega s\left(\frac{p_\omega}{A(p)}\right) d\omega \equiv 1.$$

- $s: \mathbb{R}_{++} \to \mathbb{R}_{+}$: the market share function, $C^3$, decreasing in the normalized price $z_\omega \equiv p_\omega / A$ for $s(z_\omega) > 0$ with $\lim_{z \to z^\ast} s(z) = 0$. If $\bar{z} \equiv \inf\{z > 0 | s(z) = 0\} < \infty$, $\bar{z}A(p)$ is the choke price.
- $A = A(p)$: the common price aggregator defined implicitly by the adding-up constraint $\int_\Omega s(p_\omega / A) d\omega \equiv 1$. $A(p)$ linear homogenous in $p$ for a fixed $\Omega$. A larger $\Omega$ reduces $A(p)$.

### CES

$$s(z) = \gamma z^{1-\sigma}; \quad \sigma > 1$$

### Translog Cost Function

$$s(z) = \gamma \max\{-\ln(z/\bar{z}), 0\}; \quad \bar{z} < \infty$$

### Constant Pass Through (CoPaTh)

$$s(z) = \gamma \max\left\{\left[\sigma + (1 - \sigma)z^{1-\rho}\right]^{\frac{1}{1-\rho}}, 0\right\}; \quad 0 < \rho < 1$$

As $\rho \to 1$, CoPaTh converges to CES with $\bar{z}(\rho) \equiv \left(\frac{\sigma}{(\sigma - 1)}\right)^{\frac{1}{1-\rho}} \to \infty$. 
\(P(p)\) vs. \(A(p)\)

**Definition:**
\[
s_\omega \equiv \frac{\partial \ln P(p)}{\partial \ln p_\omega} = s \left( \frac{p_\omega}{A(p)} \right) = s(z_\omega) \quad \text{where} \quad \int_\Omega s \left( \frac{p_\omega}{A(p)} \right) d\omega \equiv 1.
\]

By differentiating the adding-up constraint,
\[
\frac{\partial \ln A(p)}{\partial \ln p_\omega} = \frac{[\zeta(z_\omega) - 1]s(z_\omega)}{\int_\Omega [\zeta(z_{\omega'}) - 1]s(z_{\omega'})d\omega'} \neq s(z_\omega) = \frac{\partial \ln P(p)}{\partial \ln p_\omega}
\]

unless \(\zeta(z_\omega)\) is constant, where

**Price Elasticity Function:**
\[
\zeta(z) \equiv 1 - \frac{d \ln s(z)}{d \ln z} \equiv 1 - \mathcal{E}_s(z) > 1 \iff s(z) = \gamma \exp \left[ \int_{z_0}^z \frac{1 - \zeta(\xi)}{\xi} d\xi \right]; \lim_{z \to z^\circ} \zeta(z) = \infty, \text{if } z^\circ < \infty.
\]

By integrating the definition,
\[
\frac{A(p)}{P(p)} = c \exp \left[ \int_\Omega s \left( \frac{p_\omega}{A(p)} \right) \Phi \left( \frac{p_\omega}{A(p)} \right) d\omega \right], \quad \text{where} \quad \Phi(z) \equiv \frac{1}{s(z)} \int_z^{z^\circ} s(\xi) \frac{d\xi}{\xi}
\]

**Note:** \(A(p)/P(p)\) is not constant, unless CES \(\iff \zeta(z) = \sigma \iff s(z) = \gamma z^{1-\sigma} \iff \Phi(z) = 1/(\sigma - 1)\).

\(\checkmark\) \(A(p)\), the inverse measure of competitive pressures, captures cross price effects in the DS, the reference price for MC firms

\(\checkmark\) \(P(p)\), the inverse measure of TFP, captures the productivity effects of price changes, the reference price for consumers.

\(\checkmark\) \(\Phi(z)\), the measure of “love for variety.” Matsuyama & Ushchev (2023). \(\zeta'(\cdot) \geq 0 \iff \Phi'(\cdot) \leq 0; \Phi'(\cdot) = 0 \iff \zeta'(\cdot) = 0\).

**Note:** Our 2017 paper proved the integrability = the quasi-concavity of \(P(p)\), iff \(\zeta(z) \equiv 1 - \frac{d \ln s(z)}{d \ln z} \equiv 1 - \mathcal{E}_s(z) > 0\).
Why H.S.A.

- **Homothetic** (unlike the linear DS and most other commonly used non-CES DSs)
  - a single measure of market size; the demand composition does not matter.
  - isolate the effect of endogenous markup rate from nonhomotheticity
  - straightforward to use it as a building block in multi-sector models with any upper-tier (incl. nonhomothetic) DS

- **Nonparametric** and **flexible** (unlike CES and translog, which are special cases)
  - can be used to perform robustness-check for CES
  - allow for (but no need to impose)
    - the choke price,
    - **Marshall’s 2nd law** (Price elasticity is increasing in price) \(\Rightarrow\) more productive firms have higher markup rates
    - **what we call the 3rd law** (the rate of increase in the price elasticity is decreasing in price) \(\Rightarrow\) more productive firms have lower pass-through rates.

- **Tractable** due to **Single Aggregator** (unlike Kimball, which needs two aggregators), a **sufficient statistic** for competitive pressures, which acts like a **magnifier of firm heterogeneity**
  - guarantee the existence & uniqueness of free-entry equilibrium with firm heterogeneity
  - simple to conduct most comparative statics without **parametric** restrictions on demand or productivity distribution.
  - no need to assume zero overhead cost (unlike MO and ACDR)

- Defined by **the market share function**, for which data is readily available and easily identifiable.
Three Classes of Homothetic Demand Systems: Matsuyama-Ushchev (2017)

Here we consider a continuum of varieties \( (\omega \in \Omega) \), gross substitutes, and symmetry

<table>
<thead>
<tr>
<th>Class</th>
<th>( s_\omega = \frac{\partial \ln P(p)}{\partial \ln p_\omega} = f \left( \frac{p_\omega}{P(p)} \right) )</th>
<th>( s_\omega \propto \left( \frac{p_\omega}{P(p)} \right)^{1-\sigma} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>CES</td>
<td>( s_\omega = \frac{\partial \ln P(p)}{\partial \ln p_\omega} = f \left( \frac{p_\omega}{P(p)} \right) )</td>
<td>( s_\omega \propto \left( \frac{p_\omega}{P(p)} \right)^{1-\sigma} )</td>
</tr>
<tr>
<td>H.S.A. (Homotheticity with a Single Aggregator)</td>
<td>( s_\omega = s \left( \frac{p_\omega}{A(p)} \right) )</td>
<td>( \frac{P(p)}{A(p)} \neq c ), unless CES</td>
</tr>
<tr>
<td>HDIA (Homotheticity with Direct Implicit Additivity)</td>
<td>( s_\omega = \frac{p_\omega}{P(p)} (\phi')^{-1} \left( \frac{p_\omega}{B(p)} \right) )</td>
<td>( \frac{P(p)}{B(p)} \neq c ), unless CES</td>
</tr>
<tr>
<td>HIIA (Homotheticity with Indirect Implicit Additivity)</td>
<td>( s_\omega = \frac{p_\omega}{C(p)} \theta' \left( \frac{p_\omega}{P(p)} \right) )</td>
<td>( \frac{P(p)}{C(p)} \neq c ), unless CES</td>
</tr>
</tbody>
</table>

\( \phi(\cdot) \) & \( \theta(\cdot) \) are both increasing & concave \( \Rightarrow (\phi')^{-1}(\cdot) \) & \( \theta'(\cdot) \) positive-valued & decreasing. \( A(\cdot), B(\cdot), C(\cdot) \) all determined by the adding-up constraint.

The 3 classes are pairwise disjoint with the sole exception of CES.

Under HDIA(Kimball) and HIIA, unlike HSA
- Two aggregators needed for the market shares. [One aggregator enough for the price elasticity under all 3 classes.]
- The existence and uniqueness of free-entry equilibrium not guaranteed without some strong restrictions on both productivity distribution and the price elasticity function.
Melitz under HSA: Main Results

- **Existence & Uniqueness of Equilibrium:** straightforward under H.S.A.
- **Melitz under CES:** impacts of entry/overhead costs on the masses of entrants/active firms hinges on the sign of the derivative of the elasticity of the pdf of marginal cost; Pareto is the knife-edge! (new results!)
- **Cross-Sectional Implications:** profits and revenues are always higher among more productive.
  - 2nd Law = incomplete pass-through ⇔ the procompetitive effect ⇔ strategic complementarity in pricing.
  - 2nd (3rd) Law → more productive firms have higher markup (lower pass-through) rates.
  - 2nd & 3rd Laws → hump-shaped employment; more productive hire less under high overhead.
- **General Equilibrium Comparative Statics**
  - *Entry cost* ↓: 2nd (3rd) Law → markup rates ↓ (pass-through rates ↑) for all firms.
    - profits (revenues) decline faster among less productive → a tougher selection.
  - *Overhead cost* ↓: similar effects when the employment is decreasing in firm productivity.
  - *Market size* ↑: 2nd (3rd) Law → markup rates ↓ (pass-through rates ↑) for all firms.
    - profits (revenues) ↑ among more productive; ↓ among less productive.
  - *Due to the composition effect,* these changes may increase the average markup rate & the aggregate profit share in spite of 2nd Law and reduce the average pass-through in spite of 3rd Law; Pareto is the knife-edge for entry cost ↑.
- **Sorting of Heterogeneous Firms** across markets that differ in size: Larger markets → more competitive pressures.
  - 2nd Law → more (less) productive go into larger (smaller) markets.
  - *Composition effect,* average markup (pass-through) rates can be higher (lower) in larger and more competitive markets in spite of 2nd (3rd) Law.
(Highly Selective) Literature Review

Non-CES Demand Systems: Matsuyama (2023) for a survey; H.S.A. Demand System: Matsuyama-Ushchev (2017)

MC with Heterogeneous Firms: Melitz (2003) and many others: Melitz-Redding (2015) for a survey

MC under non-CES demand systems: Thisse-Ushchev (2018) for a survey
- Nonhomothetic non-CES:
  - $U = \int_{\Omega} u(x_\omega)d\omega$: Dixit-Stiglitz (77), Behrens-Murata (07), ZKPT (12), Mrázová-Neary(17), Dhingra-Morrow (19); ACDR (19)


Selection of Heterogeneous Firms through Competitive Pressures
Melitz-Ottaviano (2008), Baqaee-Fahri-Sangani (2023), Edmond-Midrigan-Xu (2023)

Sorting of Heterogeneous Firms Across Markets:

Log-Super(Sub)modularity: Costinot (2009), Costinot-Vogel (2015)
Selection of Heterogeneous Firms: A Single-Market Setting
A Static, Closed Economy Version of Melitz (2003), extended to H.S.A.

**Households:** supply labor (numeraire) by $L$, consume the **final good** by $X$ with the budget constraint, $PX = L$.

**Final Good Producers:** competitive, assemble intermediate inputs $\omega \in \Omega$, using **CRS technology**

**Production Function:**

$$X = X(x) \equiv \min_p \left\{ px = \int_{\Omega} p_\omega x_\omega d\omega \left| P(p) \geq 1 \right. \right\}$$

**Unit Cost Function:**

$$P = P(p) \equiv \min_x \left\{ px = \int_{\Omega} p_\omega x_\omega d\omega \left| X(x) \geq 1 \right. \right\}$$

**Note:** Both $X(x)$ and $P(p)$ can be a primitive of CRS technology, as long as linear homogeneity, monotonicity and quasi-concavity are satisfied.

**Demand Curve for $\omega$:**

$$x_\omega = X(x) \frac{\partial P(p)}{\partial p_\omega} ; \quad \text{Inverse Demand Curve for $\omega$:} \quad p_\omega = P(p) \frac{\partial X(x)}{\partial x_\omega}$$

**Market Size:**

$$px = P(p)X(x) = L$$

**Note:** This is due to the one-market setting. In a multi-market extension later, size of each market differs from $L$. 
Symmetric H.S.A. (Homothetic with a Single Aggregator) with Gross Substitutes

**Market Share** of ω depends *solely* on a single variable, its own price normalized by the *common* price aggregator

\[ s_\omega \equiv \frac{p_\omega x_\omega}{p x} = \frac{\partial \ln P(p)}{\partial \ln p_\omega} = s \left( \frac{p_\omega}{A(p)} \right), \]

where \( \int_\Omega s \left( \frac{p_\omega}{A(p)} \right) d\omega \equiv 1. \)

- **s**: \( \mathbb{R}_+^+ \rightarrow \mathbb{R}_+ \): the *market share function*, \( C^2 \), decreasing in the *normalized price*; \( z_\omega \equiv p_\omega / A \) for \( s(z_\omega) > 0 \) with \( \circ \lim_{z \to z_\omega} s(z) = 0. \) If \( \bar{z} \equiv \inf\{ z > 0 | s(z) = 0 \} < \infty \), \( \bar{z}A(p) \) is the *choke price*.

- **A = A(p)**: the *common price aggregator* defined implicitly by the *adding up constraint* \( \int_\Omega s(p_\omega / A) d\omega \equiv 1. \) \( A(p) \) linear homogenous in \( p \) for a fixed \( \Omega \). A larger \( \Omega \) reduces \( A(p) \).

<table>
<thead>
<tr>
<th>Special Cases</th>
<th>CES</th>
<th>Translog</th>
<th>Constant Pass Through (CoPaTh)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( s(z) = \gamma z^{1-\sigma}; ) ( \sigma &gt; 1 )</td>
<td>( s(z) = -\gamma \max\left{\ln \left(\frac{z}{\bar{z}}\right), 0\right}; )</td>
<td>( s(z) = \gamma \max\left{\sigma + (1 - \sigma)z^{1-\rho}, 0\right}; ) ( 0 &lt; \rho &lt; 1 )</td>
</tr>
</tbody>
</table>

As \( \rho \uparrow 1 \), CoPaTh converges to CES with \( \bar{z}(\rho) \equiv (\sigma/(\sigma - 1))^{1-\rho} \rightarrow \infty. \)
**P(p) vs. A(p)**

**Definition:**

\[
s_{\omega} \equiv \frac{\partial \ln P(p)}{\partial \ln p_{\omega}} = s\left(\frac{p_{\omega}}{A(p)}\right) \equiv s(z_{\omega}), \quad \text{where} \quad \int_{\Omega} s\left(\frac{p_{\omega}}{A(p)}\right) d\omega \equiv 1.
\]

By differentiating the adding-up constraint,

\[
\frac{\partial \ln A(p)}{\partial \ln p_{\omega}} = \frac{\int_{\Omega} [\zeta(z_{\omega'}) - 1] s(z_{\omega'}) d\omega'}{\zeta(z_{\omega}) - 1} s(z_{\omega}) \neq s(z_{\omega}) = \frac{\partial \ln P(p)}{\partial \ln p_{\omega}}
\]

unless \(\zeta(z_{\omega})\) is constant, where

**Price Elasticity Function:**

\[
\zeta(z) \equiv 1 - \frac{d \ln s(z)}{d \ln z} \equiv 1 - \mathcal{E}_s(z) > 1 \iff s(z) = \gamma \exp \left[\int_{z_0}^{z} \frac{1 - \zeta(\xi)}{\xi} d\xi\right] \text{ for } z \in (0, \tilde{z}); \lim_{z \to \tilde{z}} \zeta(z) = \infty, \text{ if } \tilde{z} < \infty.
\]

By integrating the definition,

\[
\frac{A(p)}{P(p)} = c \exp \left[\int_{\Omega} s\left(\frac{p_{\omega}}{A(p)}\right) \Phi\left(\frac{p_{\omega}}{A(p)}\right) d\omega\right], \quad \text{where} \quad \Phi(z) \equiv \frac{1}{s(z)} \int_{z}^{\tilde{z}} \frac{s(\xi)}{\xi} d\xi.
\]

**Note:** \(A(p)/P(p)\) is not constant, unless CES \(\iff \zeta(z) = \sigma \iff s(z) = \gamma z^{1-\sigma} \iff \Phi(z) = 1/(\sigma - 1)\).

- ✔️ \(A(p)\), the inverse measure of CES, captures cross price effects in the demand system.
- ✔️ \(P(p)\), the inverse measure of TFP, captures the productivity consequences of price changes.
- ✔️ \(\Phi(z)\), the measure of “love for variety.” Matsuyama & Ushchev (2023). \(\zeta'(\cdot) \geq 0 \Rightarrow \Phi'(\cdot) \leq 0; \Phi'(\cdot) = 0 \iff \zeta'(\cdot) = 0\).

**Note:** Our 2017 paper proved the integrability = the quasi-concavity of \(P(p)\), iff \(\zeta(z) \equiv 1 - \frac{d \ln s(z)}{d \ln z} \equiv 1 - \mathcal{E}_s(z) > 0\).
Monopolistically Competitive Intermediate Inputs Producers $\omega \in \Omega$

**Timing:** the same with Melitz.

- Sunk cost of entry, $F_e > 0$. (All costs are paid in labor.)
- Each entrant draws its (quality-adjusted) marginal cost $\psi \sim G(\cdot) \in C^3$ with $G'(\psi) = g(\psi) > 0$ on $(\underline{\psi}, \overline{\psi}) \subseteq (0, \infty)$.\[ E_G(\psi) \equiv \psi g(\psi)/G(\psi) \in C^2 \text{ and } E_g(\psi) \equiv \psi g'(\psi)/g(\psi) \in C^1. \]
- MC firms are ex-post heterogeneous *only in $\psi$*, or equivalently, in (quality-adjusted) productivity, $1/\psi = \phi \sim 1 - G(1/\phi)$ with density $g(1/\phi)/\phi^2 > 0$ on $(\underline{\phi}, \overline{\phi}) \subseteq (0, \infty)$.
- Each firm decides either to exit without producing or to stay & produce with an overhead cost, $F > 0$.
- Firms that stay will sell their products at the profit-maximizing prices.

**Pricing Behaviors of MC firms** after drawing $\psi_\omega$:
Each firm takes $A = A(p)$ and $px = L$ given.

$$\max_{p_\omega} (p_\omega - \psi_\omega)x_\omega = \max_{\psi_\omega < z_\omega < A} \left(1 - \frac{\psi_\omega}{p_\omega}\right) s\left(\frac{p_\omega}{A}\right) L = \max_{A < z_\omega < \overline{z}_\omega} \left(1 - \frac{\psi_\omega / A}{z_\omega}\right) s(z_\omega)L$$

where $z_\omega \equiv p_\omega / A$ is the normalized price.
**FOC:**

\[ z_\omega \left[ 1 - \frac{1}{\zeta(z_\omega)} \right] = \frac{\psi_\omega}{A} \]

**Price Elasticity Function**

\[ \zeta(z) \equiv 1 - \frac{d \ln s(z)}{d \ln z} \equiv 1 - \varepsilon_s(z) > 1, \]

for \( z \in (0, \bar{z}) \) with \( \lim_{z \to \bar{z}} \zeta(z) = - \lim_{z \to \bar{z}} \varepsilon_s(z) = \infty \), if \( \bar{z} \) is finite. The markup rate is \( \zeta(z_\omega)/(\zeta(z_\omega) - 1) \).

We maintain the following *regularity* assumption for ease of exposition.

**(A1):** For all \( z \in (0, \bar{z}) \),

\[ \varepsilon_{z/(\zeta-1)}(z) > 0 \iff \varepsilon_{z/(\zeta-1)}(z) < 1 \iff \varepsilon_{\zeta/\zeta}(z) = \varepsilon_s(z) - \varepsilon_{\zeta}(z) < 0 \]

- **(A1)** means that \( \zeta(z)/(\zeta(z) - 1) \) cannot go up as fast as \( z \).
  - **(A1)** holds with decreasing \( \zeta(\cdot)/(\zeta(\cdot) - 1) \leftrightarrow increasing \zeta(\cdot) \), i.e., under A2 (Marshall’s 2nd Law).
- **(A1)** means the marginal revenue is strictly increasing in \( p_\omega \) (hence strictly decreasing in \( x_\omega \))
  - FOC determines the profit maximizing \( z_\omega \) as an increasing \( C^2 \) function of \( \psi_\omega/A \).
  - Firms with the same \( \psi \) set the same price, earn the same profit \( \rightarrow \) we index firms by \( \psi \), as \( p_\psi, z_\psi \equiv p_\psi/A \).
- **(A1)** ensures that the maximized profit \( s(\cdot)L/\zeta(\cdot) \) is a decreasing \( C^2 \) function of \( \psi_\omega/A \).

Without **(A1)**, the maximizing price would be piecewise-continuous (i.e., the price would jump up at some values of \( \psi \)) and the maximized profit would be piecewise-differentiable, which would complicate the analysis.
**Monopolistic Competition under H.S.A.: Markup and Pass-Through Rates**

**Lerner Pricing Formula:**

\[ z_\psi \left[ 1 - \frac{1}{\zeta(z_\psi)} \right] = \frac{\psi}{A} \]

Under A1, LHS is strictly increasing, so the Inverse Function Theorem allows us to rewrite it as

**Normalized Price:**

\[ \frac{p_\psi}{A} = z_\psi = Z\left(\frac{\psi}{A}\right) \in (\psi/A, \bar{z}), Z'(\cdot) > 0; \]

**Price Elasticity:**

\[ \zeta(z_\psi) = \zeta\left(Z\left(\frac{\psi}{A}\right)\right) \equiv \sigma\left(\frac{\psi}{A}\right) > 1; \quad \text{Markup Rate:} \quad \mu_\psi \equiv \frac{p_\psi}{\psi} = \frac{\sigma(\psi/A)}{\sigma(\psi/A) - 1} \equiv \mu\left(\frac{\psi}{A}\right) > 1 \]

satisfying

\[ \frac{1}{\sigma(\psi/A)} + \frac{1}{\mu(\psi/A)} = 1 \iff \left[ \sigma\left(\frac{\psi}{A}\right) - 1 \right] \left[ \mu\left(\frac{\psi}{A}\right) - 1 \right] = 1 \]

**Pass-Through Rate:**

\[ \rho_\psi \equiv \frac{\partial \ln p_\psi}{\partial \ln \psi} = \varepsilon_Z\left(\frac{\psi}{A}\right) \equiv \rho\left(\frac{\psi}{A}\right) = 1 + \varepsilon_\mu\left(\frac{\psi}{A}\right) = 1 - \frac{\varepsilon_\sigma(\psi/A)}{\sigma(\psi/A) - 1} > 0 \]

- Normalized price, and markup rate, all \( C^2 \) functions of the *normalized cost*, \( \psi/A \) only.
  - \( Z'(\cdot) > 0; \) always strictly increasing in \( \psi/A \); Markup rate, strictly decreasing in \( \psi/A \) under A2
- Pass-through rate, a \( C^1 \) function of \( \psi/A \) only, strictly increasing in \( \psi/A \) under strong A3.
- Market size affects the pricing behaviors of firms only through its effects on \( A \).
- More competitive pressures, a lower \( A \), act like a magnifier of firm heterogeneity.

Under CES, \( \sigma(\cdot) = \sigma; \mu(\cdot) = \sigma/(\sigma - 1) = \mu; \rho(\cdot) = 1. \)
Revenue, Profit, and Employment

Revenue

\[ R_\psi = s(z_\psi)L = s\left(Z\left(\frac{\psi}{A}\right)\right)L \equiv r\left(\frac{\psi}{A}\right)L \quad \implies \quad \varepsilon_r\left(\frac{\psi}{A}\right) = \left[1 - \sigma\left(\frac{\psi}{A}\right)\right]\rho\left(\frac{\psi}{A}\right) < 0 \]

(Gross) Profit

\[ \Pi_\psi = \frac{r(\psi/A)}{\sigma(\psi/A)}L \equiv \pi\left(\frac{\psi}{A}\right)L \quad \implies \quad \varepsilon_\pi\left(\frac{\psi}{A}\right) = 1 - \sigma\left(\frac{\psi}{A}\right) < 0 \]

(Variable) Employment

\[ \psi x_\psi = \frac{r(\psi/A)}{\mu(\psi/A)}L \equiv \ell\left(\frac{\psi}{A}\right)L \quad \implies \quad \varepsilon_\ell\left(\frac{\psi}{A}\right) = 1 - \sigma\left(\frac{\psi}{A}\right)\rho\left(\frac{\psi}{A}\right) \leq 0 \]

- Revenue \( r(\psi/A)L \), profit \( \pi(\psi/A)L \), employment \( \ell(\psi/A)L \) all \( C^2 \) functions of \( \psi/A \), multiplied by market size \( L \).
- Their elasticities \( \varepsilon_r(\cdot) \), \( \varepsilon_\pi(\cdot) \) and \( \varepsilon_\ell(\cdot) \) depend solely on \( \sigma(\cdot) \) and \( \rho(\cdot) \), hence all \( C^1 \) functions of \( \psi/A \) only.

More competitive pressures, a lower \( A \), act like a magnifier of firm heterogeneity.

Market size affects the relative profit, revenue, and employment across firms only through its effects on \( A \).

Under CES, \( r(\cdot)/\pi(\cdot) = \sigma; \ r(\cdot)/\ell(\cdot) = \mu = \sigma/(\sigma - 1) \implies \varepsilon_r(\cdot) = \varepsilon_\pi(\cdot) = \varepsilon_\ell(\cdot) = 1 - \sigma < 0. \)

- Both revenue \( r(\psi/A)L \) and profit \( \pi(\psi/A)L \) are always strictly decreasing in \( \psi/A \).
- Employment \( \ell(\psi/A)L \) may be nonmonotonic in \( \psi/A \).
  - If the markup rate declines with \( \psi/A \), employment cannot decline as fast as the revenue.
  - If the markup rate declines faster than the revenue, the employment is increasing in \( \psi/A \).
**General Equilibrium: Existence and Uniqueness:** Assume \( F + F_e < \pi(0)L \).

**Cutoff Rule:** Stay if \( \psi < \psi_c \); exit if \( \psi > \psi_c \), where

\[
\max_{\psi_c} \int_{\psi_c}^{\psi} \left[ \pi \left( \frac{\psi}{A} \right) L - F \right] dG(\psi) \Rightarrow \pi \left( \frac{\psi_c}{A} \right) L = F
\]

positively-sloped. \( A \downarrow \) (more competitive pressures) \( \Rightarrow \psi_c \downarrow \) (tougher selection) rotate clockwise, as \( F/L \uparrow \) (higher overhead/market size) \( \Rightarrow \psi_c/A \downarrow \).

**Free Entry Condition:**

\[
F_e = \int_{\psi}^{\psi_c} \left[ \pi \left( \frac{\psi}{A} \right) L - F \right] dG(\psi)
\]

shift to the left as \( F_e \downarrow \) (lower entry cost) \( \Rightarrow A \downarrow \) (more competitive pressures).

\( A = A(p) \) and \( \psi_c \): uniquely determined as \( C^2 \) functions of \( F_e/L \) & \( F/L \) with the interior solution, \( 0 < G(\psi_c) < 1 \) for

\[
0 < \frac{F_e}{L} < \int_{\psi}^{\psi_c} \left[ \pi \left( \pi^{-1} \left( \frac{F}{L} \psi \right) \right) - \frac{F}{L} \right] dG(\psi),
\]

which holds for a sufficiently small \( F_e > 0 \) with no further restrictions on \( G(\cdot) \) and \( s(\cdot) \).

(This unique existence proof does not assume A2 and hence applies also to the Melitz model under CES.)
Equilibrium Mass of Firms. From the adding-up constraint, \( 1 \equiv \int_{\Omega} s(p_{\omega}/A) d\omega = M \int_{\Psi} r(\psi/A) dG(\psi) \),

**Mass of entrants**

\[
M = \left[ \int_{\Psi} r\left(\frac{\psi}{A}\right) dG(\psi) \right]^{-1} = \left[ \int_{\hat{\xi}}^{1} r \left( \pi^{-1} \left( \frac{F}{L} \xi \right) \right) dG(\psi_c \xi) \right]^{-1} > 0
\]

**Mass of active firms**

\[
MG(\psi_c) = \left[ \int_{\Psi} r\left(\frac{\psi}{A}\right) \frac{dG(\psi)}{G(\psi_c)} \right]^{-1} = \left[ \int_{\hat{\xi}}^{1} r \left( \pi^{-1} \left( \frac{F}{L} \xi \right) \right) d\tilde{G}(\xi; \psi_c) \right]^{-1} > 0
\]

where \( \tilde{G}(\xi; \psi_c) \equiv \frac{G(\psi_c \xi)}{G(\psi_c)} \) is the cdf of \( \xi \equiv \psi/\psi_c \), conditional on \( \psi/\psi_c < \xi \leq 1 \).

**Lemma 1:** \( \mathcal{E}_g'(\psi) < 0 \Rightarrow \mathcal{E}_g'_{\psi}(\psi) < 0; \mathcal{E}_g'(\psi) \geq 0 \Rightarrow \mathcal{E}_g'_{\psi}(\psi) \geq 0 \) with some boundary conditions.

**Lemma 2:** A lower \( \psi_c \) shifts \( \tilde{G}(\xi; \psi_c) \) to the right (left) in MLR if \( \mathcal{E}_g'(\psi) < (>)0 \) and in FSD if \( \mathcal{E}_g'(\psi) < (>)0 \).

- Some evidence for \( \mathcal{E}_g'(\psi) > 0 \Rightarrow \psi_c \downarrow \) (tougher selection) shifts \( \tilde{G}(\xi; \psi_c) \) to the left.
- Pareto-productivity, \( G(\psi) = (\psi/\bar{\psi})^\kappa \Rightarrow \mathcal{E}_g'(\psi) = \mathcal{E}_g'_{\psi}(\psi) = 0 \Rightarrow \tilde{G}(\xi; \psi_c) \) is independent of \( \psi_c \).
- Fréchet, Weibull, Lognormal; \( \mathcal{E}_g'(\psi) < 0 \Rightarrow \mathcal{E}_g'_{\psi}(\psi) < 0 \Rightarrow \psi_c \downarrow \) (tougher selection) shifts \( \tilde{G}(\xi; \psi_c) \) to the right.

**Lemma 4:** The integrals in the equilibrium conditions are finite and hence the equilibrium is well-defined, if

\[
\psi > 0 \Leftrightarrow \varphi < \infty \quad \text{or} \quad 1 + \lim_{z \to 0} \zeta(z) < 2 + \lim_{\psi \to 0} \mathcal{E}_g(\psi) = -\lim_{\varphi \to \infty} \mathcal{E}_f(\varphi) < \infty \quad \text{for} \quad \psi = 0 \Leftrightarrow \varphi = \infty.
\]

Equilibrium can be solved recursively under H.S.A.!!

Under HDIA/HIIA, one needs to solve the 3 equations simultaneously for 3 variables, \( \psi_c \) & the two price aggregates.
Aggregate Labor Cost and Profit Shares and TFP

Notations:

<table>
<thead>
<tr>
<th>The ( w(\cdot) )-weighted average of ( f(\cdot) ) among the active firms, ( \psi \in (\underline{\psi}, \overline{\psi}_c) )</th>
<th>( \mathbb{E}<em>w(f) \equiv \frac{\int</em>{\underline{\psi}}^{\overline{\psi}<em>c} f(\frac{\psi}{A}) w(\frac{\psi}{A}) , dG(\psi)}{\int</em>{\underline{\psi}}^{\overline{\psi}_c} w(\frac{\psi}{A}) , dG(\psi)} ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>The unweighted average of ( f(\cdot) ) among the active firms, ( \psi \in (\underline{\psi}, \overline{\psi}_c) )</td>
<td>( \mathbb{E}<em>1(f) \equiv \frac{\int</em>{\underline{\psi}}^{\overline{\psi}<em>c} f(\frac{\psi}{A}) , dG(\psi)}{\int</em>{\underline{\psi}}^{\overline{\psi}_c} , dG(\psi)} ).</td>
</tr>
</tbody>
</table>

\[ \Rightarrow \mathbb{E}_w\left(\frac{f}{w}\right) = \frac{\mathbb{E}_1(f)}{\mathbb{E}_1(w)} = \left[ \mathbb{E}_f\left(\frac{w}{f}\right) \right]^{-1}. \]

Then,

<table>
<thead>
<tr>
<th>Aggregate TFP</th>
<th>( \ln \left( \frac{X}{L} \right) = \ln \left( \frac{1}{P} \right) = \ln \left( \frac{\ell}{A} \right) + \mathbb{E}_r[\Phi \circ Z] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aggregate Labor Cost Share (Average inverse markup rate)</td>
<td>( \frac{\mathbb{E}_1(\ell)}{\mathbb{E}<em>1(r)} = \mathbb{E}<em>r \left( \frac{1}{\mu} \right) = 1 - \left[ \mathbb{E}</em>\pi \left( \frac{\mu}{\mu - 1} \right) \right]^{-1} = \frac{1}{\mathbb{E}</em>\ell(\mu)} )</td>
</tr>
<tr>
<td>Aggregate Profit Share (Average inverse price elasticity)</td>
<td>( \frac{\mathbb{E}_1(\pi)}{\mathbb{E}<em>1(r)} = \mathbb{E}<em>r \left( \frac{1}{\sigma} \right) = \frac{1}{\mathbb{E}</em>\pi(\sigma)} = 1 - \left[ \mathbb{E}</em>\ell \left( \frac{\sigma}{\sigma - 1} \right) \right]^{-1} )</td>
</tr>
</tbody>
</table>

by applying the above formulae to \( \pi(\cdot)/r(\cdot) = 1 - \ell(\cdot)/r(\cdot) = 1/\sigma(\cdot) = 1 - 1/\mu(\cdot) \).
CES Benchmark: Revisiting Melitz
**CES Benchmark:** For all $z \in (0, \infty)$, $\zeta(z) = \sigma > 1 \iff s(z) = \gamma z^{1-\sigma}$.

**Pricing:**

$$p_\psi \left(1 - \frac{1}{\sigma}\right) = \psi \iff \mu \left(\frac{\psi}{A}\right) = \frac{\sigma}{\sigma-1} > 1 \Rightarrow \rho \left(\frac{\psi}{A}\right) = 1$$

Markup rate constant; Pass-through rate equal to one.

**Cutoff Rule:**

$$c_0 L \left(\frac{\psi}{A}\right)^{1-\sigma} = F,$$

**Free Entry Condition:**

$$\int_{\psi}^{\psi_c} \left[c_0 L \left(\frac{\psi}{A}\right)^{1-\sigma} - F\right] dG(\psi) = F_e,$$

with $c_0 > 0$. As $L$ changes, the intersection moves along

$$\int_{\psi}^{\psi_c} \left[\left(\frac{\psi}{\psi_c}\right)^{1-\sigma} - 1\right] dG(\psi) = \frac{F_e}{F}$$

$F_e/F \downarrow$ and a FSD shift of $G(\cdot)$ to the left $\Rightarrow \psi_c \downarrow$ (tougher selection). $\psi_c$ unaffected by $L$, and independent of $A$.

$$A = \psi_c \left(c_0 \frac{L}{F}\right)^{1-\sigma} = \left(c_0 \frac{L}{F_e} \int_{\psi}^{\psi_c} [(\psi)^{1-\sigma} - (\psi_c)^{1-\sigma}] dG(\psi)\right)^{\frac{1}{1-\sigma}}.$$

$L \uparrow, F_e \downarrow, F \downarrow,$ a FSD shift of $G(\cdot)$ to the left $\Rightarrow A \downarrow$ (more competitive pressures)
CES Benchmark (Continue)

Revenue:

\[ r\left(\frac{\psi}{A}\right) L = \sigma c_0 L \left(\frac{\psi}{A}\right)^{1-\sigma} = \sigma F \left(\frac{\psi}{\psi_c}\right)^{1-\sigma} \geq \sigma F \]

(Gross) Profi:

\[ \pi\left(\frac{\psi}{A}\right) L = c_0 L \left(\frac{\psi}{A}\right)^{1-\sigma} = F \left(\frac{\psi}{\psi_c}\right)^{1-\sigma} \geq F \]

(Variable) Employment:

\[ \ell\left(\frac{\psi}{A}\right) L = (\sigma - 1) c_0 L \left(\frac{\psi}{A}\right)^{1-\sigma} = (\sigma - 1) F \left(\frac{\psi}{\psi_c}\right)^{1-\sigma} \geq (\sigma - 1) F \]

All decreasing power functions of \( \psi \) with

\[ \mathcal{E}_r\left(\frac{\psi}{A}\right) = \mathcal{E}_\pi\left(\frac{\psi}{A}\right) = \mathcal{E}_\ell\left(\frac{\psi}{A}\right) = 1 - \sigma < 0. \]

Relative size of two firms with \( \psi, \psi' \in (\underline{\psi}, \psi_c) \), whether measured in the profit, employment, and revenue, unaffected by \( L, F_e, F, G(\cdot) \), as well as \( A \) and \( \psi_c \), and thus never change across equilibriums.
CES Benchmark (Continue)

Mass of entrants

\[ M = \frac{L/\sigma}{F_e + G(\psi_c)F} = \frac{L}{\sigma F_e} \left[ 1 - \frac{1}{H(\psi_c)} \right] \]

Mass of active firms

\[ MG(\psi_c) = \frac{L/\sigma}{F_e/G(\psi_c) + F} = \frac{L}{H(\psi_c)\sigma F} \]

where \( H(\psi_c) \equiv \int_{\xi}^{1} (\xi)^{1-\sigma} \tilde{G}(\xi; \psi_c) \). Since \((\xi)^{1-\sigma}\) is decreasing, \( H'(\psi_c) > (<)0 \) if \( \mathcal{E}_G'(\psi) < (>)0 \) (Lemma 2).

Hence,

**Proposition 1:** Under CES,
- \( L \uparrow \) keeps \( \psi_c \) unaffected; increases both \( M \) and \( MG(\psi_c) \) proportionately;
- \( F_e \downarrow \) reduces \( \psi_c \); increases \( M \); increases (decreases) \( MG(\psi_c) \) if \( \mathcal{E}_G'(\psi) < (>)0 \);
- \( F \downarrow \) increases \( \psi_c \); increases \( MG(\psi_c) \); increases (decreases) \( M \) if \( \mathcal{E}_G'(\psi) < (>)0 \);

A FSD shift of \( G(\cdot) \) to the left reduces \( \psi_c \) with ambiguous effects on \( M \) and \( MG(\psi_c) \), even if \( G(\cdot) \) is a power.

**Effects of Market Size \( L \) under CES:**
- No effect on the markup rate.
- No effect on the cutoff, \( \psi_c \)
- No effect on the distribution of productivity, revenue, and employment across firms.
- Masses of entrants and of active firms change proportionately. All adjustments at the extensive margin.
Cross-Sectional Implications under 2\textsuperscript{nd} and 3\textsuperscript{rd} Laws
Marshall’s 2\textsuperscript{nd} Law (A2)

(A2): \( \zeta' (z) > 0 \) for all \( z \in (0, \bar{z}) \) \( \iff \sigma' (\psi / A) > 0 \) for all \( \psi / A \in (0, \bar{z}) \)

Note: A2 \( \Rightarrow \) A1.

Lemma 5: For a positive-valued function of a single variable, \( \psi / A > 0 \),

\[
sgn \left\{ \frac{\partial^2 \ln f(\psi / A)}{\partial \psi \partial A} \right\} = -sgn \left\{ E_f \left( \frac{\psi}{A} \right) \right\} = -sgn \left\{ \frac{d^2 \ln f(e^{\ln(\psi / A)})}{(d \ln(\psi / A))^2} \right\}
\]

\( f(\psi / A) \) \text{log-super(sub)modular in } \psi \text{ & } A \iff E_f (\cdot) < (>) 0 \iff \ln f(e^{\ln(\psi / A)}) \text{concave (convex) in } \ln (\psi / A)

Proposition 2: Under A2,

Incomplete Pass-Through

\[
0 < \rho \left( \frac{\psi}{A} \right) = 1 + E_\mu \left( \frac{\psi}{A} \right) = 1 - E_{1/\mu} \left( \frac{\psi}{A} \right) < 1
\]

Less efficient firms operate at more elastic parts of demand and have lower markup rates

Procompetitive Effect/

Strategic Complementarity in Pricing

\[
\frac{\partial \ln p_\psi}{\partial \ln A} = 1 - \rho \left( \frac{\psi}{A} \right) = -E_\mu \left( \frac{\psi}{A} \right) = E_{1/\mu} \left( \frac{\psi}{A} \right) > 0
\]

More competitive pressures (\( A \downarrow \text{due to entry or lower prices of competing products} \) \( \Rightarrow \) lower prices/markup rates.

Strict Log-Supermodular Profit:

\[
E'_\pi \left( \frac{\psi}{A} \right) < 0 \iff \frac{\partial^2 \ln \pi (\psi / A)}{\partial \psi \partial A} > 0
\]

More competitive pressures (\( A \downarrow \)) \( \Rightarrow \) a proportionately larger decline in the profit among high-\( \psi \) firms

\( \Rightarrow \) a larger dispersion of the profit across firms; more concentration of profits among the productive.
Marshall’s 3rd Law (A3):

(A3) Weak (Strong) Marshall’s 3rd Law of demand. For all \( z \in (0, \bar{z}) \),

\[
\mathcal{E}_{z/\zeta-1}(z) = -\frac{d}{dz} \left( \frac{z \zeta'(z)}{[\zeta(z) - 1] \zeta(z)} \right) \geq (>) 0 \iff \rho' \left( \frac{\psi}{A} \right) = \mathcal{E}_z' \left( \frac{\psi}{A} \right) = \mathcal{E}_{\mu}' \left( \frac{\psi}{A} \right) \geq (>) 0
\]

Strong A3 \( \Rightarrow \) The markup rate declines at the lower rate for higher \( z \) \( \Rightarrow \) The pass-through rate higher for higher \( \psi \).


Proposition 3: Under A3(A3),

Weak (Strict) Log-Submodular Markup Rate:

\[
\mathcal{E}_Z' \left( \frac{\psi}{A} \right) = \rho' \left( \frac{\psi}{A} \right) = \mathcal{E}_\mu' \left( \frac{\psi}{A} \right) \geq (>) 0 \iff \frac{\partial^2 \ln (Z(\psi/A))}{\partial \psi \partial A} = \frac{\partial^2 \ln (\mu(\psi/A))}{\partial \psi \partial A} \leq (>) 0,
\]

For the strict case, more competitive pressures (\( A \downarrow \) \( \Rightarrow \) proportionately smaller rate decline among high-\( \psi \) firms.

\( \Rightarrow \) a smaller dispersion of the markup rate across firms.

Under A2+A3

Strict Log-Supermodular Revenue:

\[
\mathcal{E}_r' \left( \frac{\psi}{A} \right) < 0 \iff \frac{\partial^2 \ln r(\psi/A)}{\partial \psi \partial A} > 0
\]

Strict Log-Supermodular Employment:

\[
\mathcal{E}_\ell' \left( \frac{\psi}{A} \right) = \mathcal{E}_r' \left( \frac{\psi}{A} \right) - \mathcal{E}_\mu' \left( \frac{\psi}{A} \right) < 0 \iff \frac{\partial^2 \ln \ell(\psi/A)}{\partial \psi \partial A} > 0.
\]

More competitive pressures (\( A \downarrow \) \( \Rightarrow \) proportionately larger decline in the revenue among high-\( \psi \) firms

\( \Rightarrow \) a larger dispersion of the revenue across firms; more concentration of revenue among the productive.
### A2+A3: Cross-Sectional Implications of $A \downarrow$ on Profit and Markup Rate

#### Profit (Revenue) Function: $\Pi_\psi = \pi(\psi/A)L; R_\psi = r(\psi/A)L$
- always decreasing in $\psi$
- strictly log-supermodular under A2 (Weak A3)

$A \downarrow$ with $L$ fixed shifts down with a steeper slope at each $\psi$;
$A \downarrow$ due to $L \uparrow$, a parallel shift up, a single-crossing

#### Markup Rate Function: $\mu_\psi = \mu(\psi/A) > 1$
- decreasing in $\psi$ under A2
- weakly log-submodular under Weak A3
- strictly log-submodular under Strong A3

$A \downarrow$ shifts down with a flatter slope at each $\psi$

<table>
<thead>
<tr>
<th>$\ln \Pi_\psi = \ln \pi \left( \frac{\psi}{A} \right) + \ln L$</th>
<th>$\ln R_\psi = \ln r \left( \frac{\psi}{A} \right) + \ln L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ln \mu_\psi = \ln \mu \left( \frac{\psi}{A} \right) &gt; 0$</td>
<td>$\ln \psi$</td>
</tr>
</tbody>
</table>

- With $\ln \psi$ in the horizontal axis, $A \downarrow$ causes a parallel leftward shift of the graphs in these figures.
- $f(\psi/A)$ is strictly log-super(sub)modular in $\psi$ and $A$ iff $\ln f(e^x)$ is concave(convex) in $x$. 
A2+A3: More Cross-Sectional Implications

Lemma 6: Under A2 and the weak A3, \( \lim_{\psi/A \to 0} \rho(\psi/A)\sigma(\psi/A) < 1 < \lim_{\psi/A \to z} \rho(\psi/A)\sigma(\psi/A) \).

Since A2+A3 also implies \( \mathcal{E}'(\psi/A) < 0 \),

Proposition 4: Under A2 and the weak A3, the employment function, \( \ell(\psi/A) = \frac{r(\psi/A)}{\mu(\psi/A)} \) is hump-shaped, with its unique peak is reached at, \( \hat{z} \equiv Z(\hat{\psi}/A) < z \), where
\[
\mathcal{E}_{s(\zeta-1)/\zeta}(\hat{z}) = 0 \iff \frac{\hat{z}'(\hat{z})}{\zeta(\hat{z})} = [\zeta(\hat{z}) - 1]^2 \iff \mathcal{E}_\ell \left(\frac{\hat{\psi}}{A}\right) = 0 \iff \rho \left(\frac{\hat{\psi}}{A}\right) \sigma \left(\frac{\hat{\psi}}{A}\right) = 1.
\]

A2+A3 are sufficient but not necessary for being hump-shaped.

Corollary of Proposition 4: Employments across active firms are
- increasing in \( \psi \) if \( \psi_c < \hat{\psi} \iff F/L > \pi(\hat{\psi}/A) = \pi(Z^{-1}(\hat{z})) \);
- hump-shaped in \( \psi \) if \( \psi < \hat{\psi} < \psi_c \iff F/L < \pi(\hat{\psi}/A) = \pi(Z^{-1}(\hat{z})) \) & \( A > \hat{\psi}/Z^{-1}(\hat{z}) \).

Employments are decreasing among the most productive firms.
- decreasing in \( \psi \), if \( \hat{\psi} < \psi \iff A < \psi/Z^{-1}(\hat{z}) \), which is possible only if \( \psi > 0 \).

Proposition 5: Suppose that A2 and the strong A3 hold, so that \( 0 < \rho(\psi/A) < 1 \) and \( \rho(\psi/A) \) is strictly increasing.
Then, \( \rho(\psi/A) \) is strictly log-submodular for all \( \psi/A < z \) with a sufficiently small \( z \).
**Employment Function:** $\ell(\psi/A)L = r(\psi/A)L/\mu(\psi/A)$
- *Hump-shaped* in $\psi$ under $A2$ and weak $A3$.
  $\Rightarrow A \downarrow$ shifts up (down) for a low (high) $\psi$ with $A \downarrow$
- Strictly log-supermodular under *Weak A3*
  for $A \downarrow$ with a fixed $L$; for $A \downarrow$ caused by $L \uparrow$

*Single-crossing* even with a fixed $L$

**Pass-Through Rate Function:** $\rho_\psi = \rho(\psi/A)$
- $\rho(\psi/A) < 1$ under $A2$, hence it cannot be strictly log-submodular for a higher range of $\psi/A$
- Strictly increasing in $\psi$ under *Strong A3*
- Strictly log-submodular for a lower range of $\psi/A$ under $A2$ and *Strong A3* $\Rightarrow A \downarrow$ shifts up with a steeper slope at each $\psi$ with a *small enough* $\bar{z}$.

In summary, more competitive pressures ($A \downarrow$)
- $\mu(\psi/A) \downarrow$ under $A2$ & $\rho(\psi/A) \uparrow$ under strong $A3$
- Profit, Revenue, Employment become more concentrated among the most productive.
Comparative Statics: General Equilibrium Effects
Comparative Statics: General Equilibrium Effects of $F_e, L$, and $F$ on $\psi_c$ and $A$

**Proposition 6:**

\[
\begin{bmatrix}
    d \ln A \\
    d \ln \psi_c
\end{bmatrix}
= \frac{\mathbb{E}_1(\pi)}{\mathbb{E}_1(\ell)} \begin{bmatrix}
    1 - f_x & f_x \\
    1 - f_x & f_x - \delta
\end{bmatrix}
\begin{bmatrix}
    d \ln(F_e/L) \\
    d \ln(F/L)
\end{bmatrix}
\]

where

\[
\frac{\mathbb{E}_1(\pi)}{\mathbb{E}_1(\ell)} = \frac{1}{\mathbb{E}_\pi(\sigma) - 1} = \{\mathbb{E}_r[\mu^{-1}]\}^{-1} - 1 = \mathbb{E}_\ell(\mu) - 1 > 0;
\]

The average profit/the average labor cost ratio among the active firms

\[
f_x \equiv \frac{F G(\psi_c)}{F_e + F G(\psi_c)} = \frac{\pi(\psi_c/A)}{\mathbb{E}_1(\pi)} < 1;
\]

The share of the overhead in the total expected fixed cost $= \text{to the profit of the cut-off firm relative to the average profit among the active firms}$

\[
\delta \equiv \frac{\mathbb{E}_\pi(\sigma) - 1}{\sigma(\psi_c/A) - 1} = \frac{\pi(\psi_c/A)}{\ell(\psi_c/A)} \mathbb{E}_1(\ell) \equiv f_x \frac{\mathbb{E}_1(\ell)}{\ell(\psi_c/A)} > 0.
\]

The profit/labor cost ratio of the cut-off firm to the average profit/average labor cost ratio among the active firms.
Corollary of Proposition 6

<table>
<thead>
<tr>
<th></th>
<th>( A )</th>
<th>( \psi_c / A )</th>
<th>( \psi_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_e )</td>
<td>( \frac{dA}{dF_e} &gt; 0 )</td>
<td>( \frac{d(\psi_c / A)}{dF_e} = 0 )</td>
<td>( \frac{d\psi_c}{dF_e} &gt; 0 )</td>
</tr>
<tr>
<td>( L )</td>
<td>( \frac{dA}{dL} &lt; 0 )</td>
<td>( \frac{d(\psi_c / A)}{dL} &gt; 0 )</td>
<td>( \frac{d\psi_c}{dL} &lt; 0 )</td>
</tr>
<tr>
<td>( F )</td>
<td>( \frac{dA}{dF} &gt; 0 )</td>
<td>( \frac{d(\psi_c / A)}{dF} &lt; 0 )</td>
<td>( \frac{d\psi_c}{dF} &gt; 0 )</td>
</tr>
</tbody>
</table>

\( \psi_c \) and \( \psi_c / A \) are related to \( F_e \) as follows:

\[ F_e = \int_{\psi}^\infty \left[ \pi \left( \frac{\psi}{A} \right) - \frac{F}{L} \right] dG(\psi) \]

\[ \frac{F_e}{L} = \int_{\psi}^\infty \left[ \pi \left( \frac{\psi}{A} \right) - \frac{F}{L} \right] dG(\psi) \]

\[ \frac{F_e}{L} = \int_{\psi}^\infty \left[ \pi \left( \frac{\psi}{A} \right) - \frac{F}{L} \right] dG(\psi) \]

\[ \frac{F_e}{L} = \int_{\psi}^\infty \left[ \pi \left( \frac{\psi}{A} \right) - \frac{F}{L} \right] dG(\psi) \]

Further analysis:

- \( \frac{d\psi_c}{dL} < 0 \) if \( \mathbb{E}_\pi(\sigma) < \sigma \left( \frac{\psi_c}{A} \right) \), which holds globally if \( \sigma'() > 0 \), i.e., under A2.
- \( \frac{d\psi_c}{dF} > 0 \) if \( \mathbb{E}_1(\ell) < \ell \left( \frac{\psi_c}{A} \right) \), which holds globally if \( \ell'() > 0 \).
Market Size Effect on Profit, $\Pi_\psi \equiv \pi(\psi/A)L$ and Revenue, $R_\psi \equiv r(\psi/A)L$ (Proposition 7)

7a: Under A2, there exists a unique $\psi_0 \in (\psi, \psi_c)$ such that

$$\sigma \left( \frac{\psi_0}{A} \right) = \mathbb{E}_\pi(\sigma)$$


with

$$\frac{d \ln \Pi_\psi}{d \ln L} > 0 \iff \sigma \left( \frac{\psi}{A} \right) < \mathbb{E}_\pi(\sigma) \text{ for } \psi \in (\psi, \psi_0),$$

and

$$\frac{d \ln \Pi_\psi}{d \ln L} < 0 \iff \sigma \left( \frac{\psi}{A} \right) > \mathbb{E}_\pi(\sigma) \text{ for } \psi \in (\psi_0, \psi_c).$$

7b: Under A2 and the weak A3, there exists $\psi_1 > \psi_0$, such that

$$\frac{d \ln R_\psi}{d \ln L} > 0 \text{ for } \psi \in (\psi, \psi_1).$$

Furthermore, $\psi_1 \in (\psi_0, \psi_c)$ and

$$\frac{d \ln R_\psi}{d \ln L} < 0 \text{ for } \psi \in (\psi_1, \psi_c),$$

for a sufficiently small $F$.

In short, more productive firms expand in absolute terms, while less productive firms shrink.
By putting together the main implications of Propositions 2, 3, 6, and 7

\( F_e \downarrow \) under A2 and the weak A3

\( A \downarrow, \, \psi_c \downarrow \) with \( \psi_c / A \) unchanged

The cutoff firms before the change and the cutoff firms after the change have
- the same markup rate \( \mu(\psi_c / A) \)
- the same profit \( \pi(\psi_c / A) L = F \)
- the same revenue, \( r(\psi_c / A) L \)
\( L \uparrow \text{ under A2 and the weak A3} \)

\( A \downarrow, \psi_c \downarrow \) with \( \psi_c/A \uparrow \) and \( \sigma(\psi_c/A) \uparrow \)

Compared to the cutoff firms before the change, the cutoff firms after the change have

- a lower markup rate, \( \mu(\psi_c/A) \downarrow \)
- the same profit, \( \pi(\psi_c/A)L = F \).
- a higher revenue, \( r(\psi_c/A)L = \sigma(\psi_c/A)F \uparrow \)

Profits up (down) for firms with \( \psi < (> \psi_0 \); Revenues up (down) for firms with \( \psi < (> \psi_1 \) for a sufficiently small \( F \).
$F \downarrow$ under A2 and the weak A3 with $\ell''(\cdot) > 0$

$A \downarrow$, $\psi_c \downarrow$ with $\psi_c / A \uparrow$ and $\sigma(\psi_c / A) \uparrow$

Compared to the cutoff firms before the change, the cutoff firms after the change have

- a lower markup rate, $\mu(\psi_c / A) \downarrow$
- a lower profit, $\pi(\psi_c / A)L = F \downarrow$
- a lower revenue, $r(\psi_c / A)L = \sigma(\psi_c / A)F \downarrow$. 
The Composition Effect: Average Markup and Pass-Through Rates

- Under A2, $A \downarrow$ causes $\mu(\psi/A) \downarrow$ for each $\psi$, but distribution shifts toward low-$\psi$ firms with higher $\mu(\psi/A)$.
- Under strong A3, $A \downarrow$ causes $\rho(\psi/A) \uparrow$ for each $\psi$, but distribution shifts toward low-$\psi$ firms with lower $\rho(\psi/A)$.

**Proposition 8:** Assume that $E_g'()$ does not change its sign and $\psi = 0$. Consider a shock to $F_e, L,$ and/or $F$, which affects competitive pressures, i.e., $dA \neq 0$. Then, the response of any weighted generalized mean of any monotone function, $f(\psi/A) > 0$, defined by

$$I \equiv M^{-1}\left(\mathbb{E}_w(M(f))\right)$$

with a monotone transformation $M: \mathbb{R}_+ \to \mathbb{R}$ and a weighting function, $w(\psi/A) > 0$, satisfies:

<table>
<thead>
<tr>
<th>$E_g'()$</th>
<th>$f'(\cdot) &gt; 0$</th>
<th>$f'(\cdot) = 0$</th>
<th>$f'(\cdot) &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&gt; 0$</td>
<td>$d \ln(\psi_c/A) \geq 0 \Rightarrow d \ln I/d A &gt; 0$</td>
<td>$d \ln I/d A = 0$</td>
<td>$d \ln(\psi_c/A) \geq 0 \Rightarrow d \ln I/d A &lt; 0$</td>
</tr>
<tr>
<td>$= 0$ (Pareto)</td>
<td>$d \ln(\psi_c/A) \leq 0 \Leftrightarrow d \ln I/d A \leq 0$</td>
<td>$d \ln I/d A = 0$</td>
<td>$d \ln(\psi_c/A) \leq 0 \Leftrightarrow d \ln I/d A \leq 0$</td>
</tr>
<tr>
<td>$&lt; 0$</td>
<td>$d \ln(\psi_c/A) \leq 0 \Rightarrow d \ln I/d A &lt; 0$</td>
<td>$d \ln I/d A = 0$</td>
<td>$d \ln(\psi_c/A) \leq 0 \Rightarrow d \ln I/d A &gt; 0$</td>
</tr>
</tbody>
</table>

Moreover, if $E_g'() = d \ln(\psi_c/A)/d \ln A = 0$, $d \ln I/d \ln A = 0$ for any $f(\psi/A)$, monotonic or not. Furthermore, $E_g'()$ can be replaced with $E_G'()$ in all the above statements for $w(\psi/A) = 1$, i.e., the unweighted averages.

The arithmetic, $I = \left(\mathbb{E}_w(f)\right)$, geometric, $I = \exp[\mathbb{E}_w(\ln f)]$, harmonic, $I = \left(\mathbb{E}_w(f^{-1})\right)^{-1}$, means are special cases.

The weight function, $w(\psi/A)$, can be profit, revenue, and employment.
Corollary 1 of Proposition 8

a) Entry Cost: \( f'(\cdot) \mathcal{E}'_g(\cdot) \geq 0 \iff \frac{d \ln I}{d \ln F_e} = \frac{d \ln I}{d \ln A} \frac{d \ln A}{d \ln F_e} \geq 0. \)

b) Market Size: If \( \mathcal{E}'_g(\cdot) \leq 0, \) then, \( f'(\cdot) \geq 0 \implies \frac{d \ln I}{d \ln L} = \frac{d \ln I}{d \ln A} \frac{d \ln A}{d \ln L} \geq 0. \)

c) Overhead Cost: If \( \mathcal{E}'_g(\cdot) \leq 0, \) then, \( f'(\cdot) \leq 0 \implies \frac{d \ln I}{d \ln F} = \frac{d \ln I}{d \ln A} \frac{d \ln A}{d \ln F} \leq 0. \)

Furthermore, \( \mathcal{E}'_g(\cdot) \) can be replaced with \( \mathcal{E}'_g(\cdot) \) for \( w(\psi/A) = 1, \) i.e., the unweighted averages.

For the entry cost, \( \frac{d \ln (\psi_c/A)}{d \ln A} = 0. \)

- \( \mathcal{E}'_g(\cdot) > 0; \) sufficient & necessary for the composition effect to dominate:
  - The average markup & pass-through rates move in the opposite direction from the firm-level rates
- \( \mathcal{E}'_g(\cdot) = 0 \) (Pareto); a knife-edge. \( A \downarrow \rightarrow \) no change in average markup and pass-through.
- \( \mathcal{E}'_g(\cdot) < 0; \) sufficient & necessary for the procompetitive effect to dominate:
  The average markup & pass-through rates move in the same direction from the firm-level rates

For market size and the overhead cost, \( \frac{d \ln (\psi_c/A)}{d \ln A} < 0. \)

- \( \mathcal{E}'_g(\cdot) > 0; \) necessary for the composition effect to dominate:
- \( \mathcal{E}'_g(\cdot) \leq 0; \) sufficient for the procompetitive effect to dominate:
The Composition Effect: Impact on $P/A$

$$\ln \left( \frac{A}{cP} \right) = \mathbb{E}_r [\Phi \circ Z]$$

$$\zeta'(\cdot) \geq 0 \implies \Phi'(\cdot) \leq 0 \iff \Phi \circ Z'(\cdot) \leq 0$$

**Corollary 2 of Proposition 8:** Assume $\psi = 0$, and neither $\zeta'(\cdot)$ nor $\mathcal{E}_g'(\cdot)$ change the signs. Consider a shock to $F_e$, $L$, and/or $F$, which affects competitive pressures, i.e., $dA \neq 0$. Then, the response of $P/A$ satisfies:

<table>
<thead>
<tr>
<th>$\mathcal{E}_g'(\cdot)$</th>
<th>$\zeta'(\cdot) &gt; 0$ (A2)</th>
<th>$\zeta'(\cdot) = 0$ (CES)</th>
<th>$\zeta'(\cdot) &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{E}_g'(\cdot) &gt; 0$</td>
<td>$\frac{d \ln(\psi_c/A)}{d \ln A} \geq 0 \implies \frac{d \ln(P/A)}{d \ln A} &gt; 0$</td>
<td>$\frac{d \ln(P/A)}{d \ln A} = 0$</td>
<td>$\frac{d \ln(\psi_c/A)}{d \ln A} \geq 0 \implies \frac{d \ln(P/A)}{d \ln A} &lt; 0$</td>
</tr>
<tr>
<td>$\mathcal{E}_g'(\cdot) = 0$ (Pareto)</td>
<td>$\frac{d \ln(\psi_c/A)}{d \ln A} \leq 0 \iff \frac{d \ln(P/A)}{d \ln A} \leq 0$</td>
<td>$\frac{d \ln(P/A)}{d \ln A} = 0$</td>
<td>$\frac{d \ln(\psi_c/A)}{d \ln A} \leq 0 \iff \frac{d \ln(P/A)}{d \ln A} \geq 0$</td>
</tr>
<tr>
<td>$\mathcal{E}_g'(\cdot) &lt; 0$</td>
<td>$\frac{d \ln(\psi_c/A)}{d \ln A} \leq 0 \implies \frac{d \ln(P/A)}{d \ln A} &lt; 0$</td>
<td>$\frac{d \ln(P/A)}{d \ln A} = 0$</td>
<td>$\frac{d \ln(\psi_c/A)}{d \ln A} \leq 0 \implies \frac{d \ln(P/A)}{d \ln A} &gt; 0$</td>
</tr>
</tbody>
</table>
Comparative Statics on $M & MG(\psi_c)$

**Proposition 9:** Assume that $\mathcal{E}_G'(\cdot)$ does not change its sign and $\psi = 0$. Consider a shock to $F_e$, $F$, and/or $L$, which affects competitive pressures, i.e., $dA \neq 0$. Then, the response of the mass of active firms, $MG(\psi_c)$, is as follows:

- If $\mathcal{E}_G'(\cdot) > 0$,
  \[
  \frac{d \ln(\psi_c/A)}{d \ln A} \geq 0 \implies \frac{d \ln[MG(\psi_c)]}{d \ln A} > 0;
  \]

- If $\mathcal{E}_G'(\cdot) = 0$,
  \[
  \frac{d \ln(\psi_c/A)}{d \ln A} \equiv 0 \iff \frac{d \ln[MG(\psi_c)]}{d \ln A} \equiv 0;
  \]

- If $\mathcal{E}_G'(\cdot) < 0$,
  \[
  \frac{d \ln(\psi_c/A)}{d \ln A} \leq 0 \implies \frac{d \ln[MG(\psi_c)]}{d \ln A} < 0.
  \]

**Corollary 1 of Proposition 9**

- **a)** Entry Cost: $\mathcal{E}_G'(\cdot) \geq 0 \iff \frac{d \ln[MG(\psi_c)]}{d \ln F_e} = \frac{d \ln[MG(\psi_c)]}{d \ln A} \frac{d \ln A}{d \ln F_e} \geq 0$.

- **b)** Market Size: $\mathcal{E}_G'(\cdot) \leq 0 \implies \frac{d \ln[MG(\psi_c)]}{d \ln L} = \frac{d \ln[MG(\psi_c)]}{d \ln A} \frac{d \ln A}{d \ln L} > 0$.

- **c)** Overhead Cost: $\mathcal{E}_G'(\cdot) \leq 0 \implies \frac{d \ln[MG(\psi_c)]}{d \ln F} = \frac{d \ln[MG(\psi_c)]}{d \ln A} \frac{d \ln A}{d \ln F} < 0$.

For a decline in the entry cost, $\mathcal{E}_g'(\cdot) > 0$ sufficient & necessary for $MG(\psi_c) \downarrow$; $\mathcal{E}_g'(\cdot) = 0$, no effect; $\mathcal{E}_g'(\cdot) < 0$; sufficient & necessary for $MG(\psi_c) \uparrow$

For market size and the overhead cost $\mathcal{E}_g'(\cdot) > 0$ necessary for $MG(\psi_c) \downarrow$; $\mathcal{E}_g'(\cdot) \leq 0$ sufficient for $MG(\psi_c) \uparrow$
Impact of Competitive Pressures on Unit Cost/TFP

By combining Corollary 2 of Proposition 8 and Corollary 1 of Proposition,

**Corollary 2 of Proposition 9:** Assume $\psi = 0$, and neither $\zeta'(\cdot)$ nor $\varepsilon'_g(\cdot)$ change the signs. Consider a shock to $F_e$, $L$, and/or $F$, which affects competitive pressures, i.e., $dA \neq 0$. Then, the response of $P$ satisfies:

<table>
<thead>
<tr>
<th>$\varepsilon'_g(\cdot)$</th>
<th>$\zeta'(\cdot) &gt; 0$ (A2)</th>
<th>$\zeta'(\cdot) = 0$ (CES)</th>
<th>$\zeta'(\cdot) &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon'_g(\cdot) &gt; 0$</td>
<td>$\frac{d \ln P}{d \ln A} &gt; 1$ for $F_e$</td>
<td>$\frac{d \ln P}{d \ln A} = 1$</td>
<td>?</td>
</tr>
<tr>
<td>$\varepsilon'_g(\cdot) = 0$ (Pareto)</td>
<td>$\frac{d \ln P}{d \ln A} = 1$ for $F_e$</td>
<td>$\frac{d \ln P}{d \ln A} = 1$</td>
<td>$\frac{d \ln P}{d \ln A} &gt; 1$ for $F$ or $L$</td>
</tr>
<tr>
<td>$\varepsilon'_g(\cdot) &lt; 0$</td>
<td>$0 &lt; \frac{d \ln P}{d \ln A} &lt; 1$ for $F$ or $L$;</td>
<td>$\frac{d \ln P}{d \ln A} = 1$</td>
<td>$\frac{d \ln P}{d \ln A} &gt; 1$</td>
</tr>
</tbody>
</table>
**Limit Case of** $F \to 0$ **with** $\bar{z} < \infty$

**Cutoff Rule:**

$$\pi\left(\frac{\psi_c}{A}\right) = 0 \iff \frac{\psi_c}{A} = \bar{z} = \pi^{-1}(0)$$

**Free Entry Condition:**

$$\frac{F_e}{L} = \int_{\psi}^{\psi_c} \pi\left(\bar{z} \cdot \frac{\psi}{\psi_c}\right) dG(\psi) = \int_{\psi}^{\bar{z}A} \pi\left(\frac{\psi}{A}\right) dG(\psi).$$

$A$ and $\psi_c$: uniquely determined as $C^2$ functions of $F_e/L$ with the interior solution, $0 < G(\psi_c) < 1$ for

\[
0 < \frac{F_e}{L} < \int_{\psi}^{\psi_c} \pi\left(\bar{z} \cdot \frac{\psi}{\psi_c}\right) dG(\psi).
\]

\[
\frac{d\psi_c}{\psi_c} = \frac{dA}{A} = \frac{1}{\mathbb{E}_\pi(\sigma) - 1} \frac{d(F_e/L)}{F_e/L}
\]

\[
\frac{dM}{d(F_e/L)} < 0; \quad \mathbb{E}_\pi'(\psi) \equiv 0 \iff \frac{d[MG(\psi_c)]}{d(F_e/L)} \equiv 0
\]

$L \uparrow$ is isomorphic to $F_e \downarrow.$

---

**Diagram:**

- **Axes:** $A$, $L$, $\psi_c$.
- **Equation:** $\frac{F_e}{L} = \int_{\psi}^{\psi_c} \pi\left(\bar{z} \cdot \frac{\psi}{\psi_c}\right) dG(\psi)$.
- **Point:** $\frac{\psi_c}{A} = \bar{z}$.
- **Graph:** $\frac{F_e}{L}$ vs $A$, with $\psi_c$ as a parameter.
$F_e/L \downarrow$ for $F \to 0$ with $\bar{z} < \infty$ under A2 and the weak A3

$A \downarrow, \psi_c \downarrow$ with $\psi_c/A = \bar{z}$ unchanged.

The cutoff firms always (i.e., both before and after the change) have

- $\mu(\psi_c/A) = 1$
- $\pi(\psi_c/A)L = 0$.
- $r(\psi_c/A)L = 0$.

Profits up (down) for firms with $\psi < (>) \psi_0$;
Revenues up (down) for firms with $\psi < (>) \psi_1$.

In the middle and the lower panels,
Blue : the effects of $F_e/L \downarrow$ due to $F_e \downarrow$
Purple: the effects of $F_e/L \downarrow$ due to $L \uparrow$
Sorting of Heterogeneous Firms: A Multi-Market Setting
A Multi-Market Extension: $J$ markets, $j = 1, 2, \ldots, J$, with market size $L_j$.

Possible Interpretations

- Identical Households with Cobb-Douglas preferences, $\sum_{j=1}^{J} \beta_j \ln X_j$ with $\sum_{j=1}^{J} \beta_j = 1$. Then, $L_j = \beta_j L$.
- $J$ types of consumers, with $L_j$ equal to the total income of type-$j$ consumers. “Types” can be their “tastes” or “locations”, etc.

Assume

- Market size is the only exogenous source of heterogeneity across markets: Index them as $L_1 > L_2 > \ldots, > L_J > 0$.
- Labor is fully mobile, equalizing the wage across the markets. We continue to use it as the numeraire.
- Firm’s marginal cost, $\psi$, is independent of the market it chooses.
  - Each firm pays $F_e > 0$ to draw its marginal cost $\psi \sim G(\psi)$.
  - Knowing its $\psi$, each firm decides which market to enter and produce with an overhead cost, $F > 0$, or exit without producing.
  - Firms sell their products at the profit-maximizing prices in the market they enter.

Equilibrium Condition:

$$F_e = \int_{\psi} \max\{\Pi_{\psi} - F, 0\} dG(\psi) = \int_{\psi} \max\{\max_{1 \leq j \leq J}\{\Pi_{j\psi}\} - F, 0\} dG(\psi)$$

where $\Pi_{j\psi} \equiv \frac{s\left(Z(\psi/A_j)\right)}{\zeta\left(Z(\psi/A_j)\right)} L_j \equiv \frac{r(\psi/A_j)}{\sigma(\psi/A_j)} L_j = \pi\left(\frac{\psi}{A_j}\right) L_j$
Proposition 10: Equilibrium Characterization under A2

Larger markets are more competitive:

\[ 0 < A_1 < A_2 < \cdots < A_J < \infty, \text{where } M \int_{\psi_{j-1}}^{\psi_j} r\left(\frac{\psi}{A_j}\right) dG(\psi) = 1. \]

Note: Because \( \pi(\cdot) \) is strictly decreasing, this implies \( \pi(\psi/A_1) < \pi(\psi/A_2) < \cdots < \pi(\psi/A_J) \) for all \( \psi \).

More productive firms self-select into larger markets (Positive Assortative Matching)

Firms with \( \psi \in (\psi_{j-1}, \psi_j) \) enter market-\( j \) and those with \( \psi \in (\psi_j, \infty) \) do not enter any market, where

\[ 0 \leq \underline{\psi} = \psi_0 < \psi_1 < \psi_2 < \cdots < \psi_j < \bar{\psi} \leq \infty \quad \text{where} \quad \frac{\pi(\psi_j/A_j) L_j}{\pi(\psi_j/A_{j+1}) L_{j+1}} = 1 \quad \text{for } 1 \leq j \leq J - 1; \quad \pi\left(\frac{\psi_j}{A_j}\right) L_j \equiv F \]

Note: \( \psi_j \)-firms are indifferent btw entering Market-\( j \) & entering Market-(\( j + 1 \)).

Free Entry Condition:

\[ \sum_{j=1}^{J} \int_{\psi_{j-1}}^{\psi_j} \left\{ \pi\left(\frac{\psi}{A_j}\right) L_j - F \right\} dG(\psi) = F_e \]

Mass of Firms in Market-\( j \):

\[ M[G(\psi_j) - G(\psi_{j-1})] > 0 \]
Logic Behind Sorting

$L_j > L_{j+1} \Rightarrow A_j < A_{j+1}$. Otherwise, no firm would enter $j + 1$.

\[ \frac{\pi(\psi/A_j)}{\pi(\psi/A_{j+1})} \text{ strictly decreasing in } \psi \]

due to strict log-supermodularity of $\pi(\psi/A)$ under A2

\[ \Rightarrow \frac{\Pi_j}{\Pi_{(j+1)}} = \frac{\pi(\psi/A_j) L_j}{\pi(\psi/A_{j+1}) L_{j+1}} \geq 1 \iff \psi \leq \psi_j \]

Under CES, $\frac{\pi(\psi/A_j)L_j}{\pi(\psi/A_{j+1})L_{j+1}}$ is independent of $\psi$.

\[ \Rightarrow \frac{\Pi_j}{\Pi_{(j+1)}} = \frac{\pi(\psi/A_j)L_j}{\pi(\psi/A_{j+1})L_{j+1}} = 1 \text{ in equilibrium.} \]

\[ \Rightarrow \text{Firms indifferent across all markets.} \]
\[ \Rightarrow \text{Distribution of firms across markets is indeterminate.} \]

Our mechanism generates sorting through competitive pressures. As such,

- complementary to agglomeration-economies-based mechanisms offered by Gaubert (2018) and Davis-Dingel (2019)
- justifies the equilibrium selection criterion used by Baldwin-Okubo (2006), which use CES, as a limit argument.
Cross-Sectional, Cross-Market Implications:

**Profits: Under A2**

\[ L_j > L_{j+1} \Rightarrow A_j < A_{j+1} \Rightarrow \left[ \frac{\pi(\psi/A_j)L_j}{\pi(\psi/A_{j+1})L_{j+1}} \right] \geq 1 \Leftrightarrow \psi \leq \psi_j \]

\[ \Pi_\psi = \max_j \left\{ \pi \left( \frac{\psi}{A_j} \right) L_j \right\} \text{, the upper-envelope of } \pi(\psi/A_j)L_j \text{, is continuous and decreasing in } \psi \text{, with the kinks at } \psi_j. \]

Continuous, since the lower markup rate in Market-\( j \) cancels out its larger market size, keeping \( \psi_j \)-firms indifferent btw Market-\( j \) & Market-(\( j + 1 \)).

**Revenues: Under A2**

\[ \frac{r(\psi_j/A_j)L_j}{r(\psi_j/A_{j+1})L_{j+1}} = \frac{\sigma(\psi_j/A_j)\pi(\psi_j/A_j)L_j}{\sigma(\psi_j/A_{j+1})\pi(\psi_j/A_{j+1})L_{j+1}} = \frac{\sigma(\psi_j/A_j)}{\sigma(\psi_j/A_{j+1})} > 1 \]

\( R_\psi \): continuously decreasing in \( \psi \) within each market; jumps down at \( \psi_j \). With the markup rate lower in Market-\( j \), \( \psi_j \)-firms need to earn higher revenue to keep them indifferent btw Market-\( j \) & and Market-(\( j + 1 \)).
Markup Rates: Under A2

\[ L_j > L_{j+1} \Rightarrow A_j < A_{j+1} \Rightarrow \sigma \left( \frac{\psi_j}{A_j} \right) > \sigma \left( \frac{\psi_j}{A_{j+1}} \right) \Leftrightarrow \mu \left( \frac{\psi_j}{A_j} \right) < \mu \left( \frac{\psi_j}{A_{j+1}} \right) \]

\( \mu_\psi \): continuously decreasing in \( \psi \) within each market but jumps up at \( \psi_j \).

- The average markup rates may be *higher* in larger (and hence more competitive) markets.
- The average markup rates in all markets may *go up*, even if all markets become more competitive (\( A_j \downarrow \)).

Pass-Through Rates: Under A2 and the strong A3

\[ L_j > L_{j+1} \Rightarrow A_j < A_{j+1} \Rightarrow \rho \left( \frac{\psi_j}{A_j} \right) > \rho \left( \frac{\psi_j}{A_{j+1}} \right) \]

\( \rho_\psi \): continuously increasing in \( \psi \) within each market but jumps down at \( \psi_j \).

- The average pass-through rates may be *lower* in larger (and hence more competitive) markets.
- The average pass-through rates in all markets *go down* even if all markets become more competitive (\( A_j \downarrow \)).
Average Markup and Pass-Through Rates in a Multi-Market Model: The Composition Effect

**Proposition 11a:** Suppose $A_2$ and $G(\psi) = (\psi/\bar{\psi})^K$. There exists a sequence, $L_1 > L_2 > \cdots > L_J > 0$, such that, in equilibrium, any weighted generalized mean of $f(\psi/A_j)$ across firms operating at market-$j$ are increasing (decreasing) in $j$ even though $f(\cdot)$ is increasing (decreasing) and hence $f(\psi/A_j)$ is decreasing (increasing) in $j$.

**Corollary of Proposition 11a:** An example with $G(\psi) = (\psi/\bar{\psi})^K$, such that the average markup rates are higher (and the average pass-through rates are lower under Strong A3) in larger markets.

**Proposition 11b:** Suppose $A_2$ and $G(\psi) = (\psi/\bar{\psi})^K$. Then, a change in $F_e$ keeps

i) the ratios $a_j \equiv \psi_{j-1}/\psi_j$ and $b_j \equiv \psi_j/A_j$

and

ii) any weighted generalized mean of $f(\psi/A_j)$ across firms operating at market-$j$, for any weighting function $w(\psi/A_j)$,

unchanged for all $j = 1, 2, \ldots, J$.

**Corollary of Proposition 11b:** $F_e \downarrow$ and $G(\psi) = (\psi/\bar{\psi})^K$ offers a knife-edge case, where the average markup and pass-through rates of all markets remain unchanged.

A caution against testing A2/A3 by comparing the average markup & pass-through rates across space and time.
Appendices
Symmetric H.S.A. with Gross Substitutes: An Alternative (Equivalent) Definition

Market Share of $\omega$ depends solely on its own quantity normalized by the common quantity aggregator

$$s_\omega \equiv \frac{p_\omega x_\omega}{px} = \frac{\partial \ln X(x)}{\partial \ln x_\omega} = s^* \left( \frac{x_\omega}{A^*(x)} \right),$$

Where

$$\int_\Omega s^* \left( \frac{x_\omega}{A^*(x)} \right) d\omega \equiv 1.$$

- $s^*: \mathbb{R}_{++} \to \mathbb{R}_+$: the market share function, with $0 < \varepsilon_{s^*}(y_\omega) < 1$, where $y_\omega \equiv x_\omega/A^*$ is the normalized quantity
  - If $\bar{z} \equiv s^*(0) = \lim_{y \to 0} [s^*(y)/y] < \infty$, $\bar{z}A(p)$ is the choke price.
- $A^* = A^*(x)$: the common quantity aggregator defined implicitly by the adding up constraint $\int_\Omega s^*(x_\omega/A^*) d\omega \equiv 1$. $A^*(x)$ linear homogenous in $x$ for a fixed $\Omega$. A larger $\Omega$ raises $A^*(x)$.

Two definitions equivalent with the one-to-one mapping, $s(z) \leftrightarrow s^*(y)$, defined by $s^* \equiv s(s^*/y)$ or $s \equiv s^*(s/z)$.

CES if $s^*(y) = \gamma^{1/\rho} y^{1-1/\rho}$; CoPaTh if $s^*(y) = \left[ (y)^{\frac{\rho-1}{\rho}} + (y\bar{z})^{\frac{\rho-1}{\rho}} \right]^{\frac{\rho}{\rho-1}}$ with $\rho \in (0,1)$.

Production Function: $X(x) = c^* A^*(x) \exp \left\{ \int_\Omega \left[ \int_0^{x_\omega/A^*(x)} s^*(\xi) \frac{d\xi}{\xi} \right] d\omega \right\}$

Note: Our 2020 paper proved

$$\left[ 1 - \frac{d \ln s(z)}{d \ln z} \right] \left[ 1 - \frac{d \ln s^*(y)}{d \ln y} \right] = 1$$

Our 2017 paper proved that $X(x)$ is quasi-concave & that $A^*(x)/X(x) = P(p)/A(p) \neq c$ for any $c > 0$ unless CES

- $A^*(x)$, the measure of competitive pressures, fully captures cross quantity effects in the inverse demand system
- $X(x)$, the measure of output, captures the output implications of input changes
Labor Market Equilibrium: satisfied automatically from the Walras Law.

\[
\text{Labor Demand} = M \left[ F_e + \int_{\psi}^{\psi_c} (x_\psi \psi + F) dG(\psi) \right] = M \left[ F_e + FG(\psi_c) + \int_{\psi}^{\psi_c} \ell \left( \frac{\psi}{A} \right) L dG(\psi) \right]
\]

\[
= LM \left[ \int_{\psi}^{\psi_c} \left[ \pi \left( \frac{\psi}{A} \right) + \ell \left( \frac{\psi}{A} \right) \right] dG(\psi) \right] \quad \text{(from the Free Entry Condition)}
\]

\[
= LM \int_{\psi}^{\psi_c} r \left( \frac{\psi}{A} \right) dG(\psi) = L \quad \text{(from the Adding Up Constraint)}
\]
Three Parametric Families of H.S.A.

Generalized Translog:

\[ s(z) = \gamma \left( 1 - \frac{\sigma - 1}{\eta} \ln \left( \frac{z}{\beta} \right) \right)^{\eta} = \gamma \left( - \frac{\sigma - 1}{\eta} \ln \left( \frac{z}{\bar{z}} \right) \right)^{\eta}; \quad z < \bar{z} \equiv \beta e^{\frac{\eta}{\sigma - 1}} \]

\[ \Rightarrow \zeta(z) = 1 + \frac{\sigma - 1}{1 - \frac{\sigma - 1}{\eta} \ln \left( \frac{z}{\bar{z}} \right)} = 1 - \frac{\eta}{\ln \left( \frac{z}{\bar{z}} \right)} > 1 \]

\[ \Rightarrow \eta z \zeta'(z) = \left[ \zeta(z) - 1 \right]^2 \Rightarrow \frac{z \zeta'(z)}{\left[ \zeta(z) - 1 \right] \zeta(z)} = \frac{1}{\eta} \left[ 1 - \frac{1}{\zeta(z)} \right] = \frac{1}{\eta - \ln \left( \frac{z}{\bar{z}} \right)} \]

satisfying A2 but violating A3.

- CES is the limit case, as \( \eta \to \infty \), while holding \( \beta > 0 \) and \( \sigma > 1 \) fixed, so that \( \bar{z} \equiv \beta e^{\frac{\eta}{\sigma - 1}} \to \infty \).
- Translog is the special case where \( \eta = 1 \).
- \( z = Z \left( \frac{\psi}{A} \right) \) is given as the inverse of \( \frac{\eta z}{\eta - \ln(z/\bar{z})} = \frac{\psi}{A} \);
- If \( \eta \geq 1 \), employment is globally decreasing in \( z \);
- If \( \eta < 1 \), employment is hump-shaped with the peak, given by \( \hat{z}/\bar{z} = \frac{\tilde{\psi}}{(1-\eta)A} = \exp \left[ -\frac{\eta^2}{1-\eta} \right] < 1 \), decreasing in \( \eta \).
Constant Pass-Through (CoPaTh): Matsuyama-Ushchev (2020b). For $0 \leq \rho < 1$, $\sigma > 1$, $\bar{z} \equiv \beta \left( \frac{\sigma}{\sigma-1} \right)^{\frac{\rho}{1-\rho}}$

$$s(z) = \gamma \sigma^{1-\rho} \left[ 1 - \left( \frac{z}{\bar{z}} \right)^{\frac{1-\rho}{\rho}} \right]^{\frac{\rho}{1-\rho}} \Rightarrow 1 - \frac{1}{\zeta(z)} \left( \frac{\bar{z}}{z} \right)^{\frac{1-\rho}{\rho}} < 1 \Rightarrow \epsilon_{1-1/\zeta}(z) = \frac{1-\rho}{\rho} > 0$$

satisfying A2 and the weak form of A3 (but not the strong form). Then, for $\psi / A < \bar{z}$,

$$p_{\psi} = (\bar{z}A)^{1-\rho}(\psi)^{\rho}; \quad Z \left( \frac{\psi}{A} \right) = (\bar{z})^{1-\rho} \left( \frac{\psi}{A} \right)^{\rho}$$

$$\sigma \left( \frac{\psi}{A} \right) = \frac{1}{1 - (\psi / \bar{z}A)^{1-\rho}}; \quad \rho \left( \frac{\psi}{A} \right) = \rho$$

$$r \left( \frac{\psi}{A} \right) = \gamma \sigma^{1-\rho} \left[ 1 - \left( \frac{\psi}{\bar{z}A} \right)^{1-\rho} \right]^{\frac{1}{\rho-1}} \pi \left( \frac{\psi}{A} \right) = \gamma \sigma^{1-\rho} \left[ 1 - \left( \frac{\psi}{\bar{z}A} \right)^{1-\rho} \right]^{\frac{1}{\rho-1}} \varphi \left( \frac{\psi}{A} \right) = \gamma \sigma^{1-\rho} \left( \frac{\psi}{\bar{z}A} \right)^{1-\rho} \left[ 1 - \left( \frac{\psi}{\bar{z}A} \right)^{1-\rho} \right]^{\frac{1}{1-\rho}}$$

with

- a constant pass-through rate, $0 < \rho < 1$.
- Employment hump-shaped with $\hat{z} / \bar{z} = (1 - \rho)^{\frac{1}{1-\rho}} > \psi / \bar{z}A = (1 - \rho)^{\frac{1}{1-\rho}}$, both decreasing in $\rho$.
- CES is the limit case, as $\rho \rightarrow 1$, while holding $\beta > 0$ and $\sigma > 1$ fixed, so that $\sigma(\psi / A) \rightarrow \sigma$; $\bar{z} \equiv \beta \left( \frac{\sigma}{\sigma-1} \right)^{\frac{\rho}{1-\rho}} \rightarrow \infty$. 
Power Elasticity of Markup Rate (Fréchet Inverse Markup Rate): For $\kappa \geq 0$ and $\lambda > 0$,

$$s(z) = \exp \left[ \int_{z_0}^{z} \frac{c}{c - \exp \left( -\frac{\kappa z - \lambda}{\lambda} \right) \exp \left( \frac{\kappa z - \lambda}{\lambda} \right) \frac{d\xi}{\xi}} \right]$$

with either $\bar{z} = \infty$ and $c \leq 1$ or $\bar{z} < \infty$ and $c = 1$. Then,

$$1 - \frac{1}{\zeta(z)} = c \exp \left[ \frac{\kappa z - \lambda}{\lambda} \right] \exp \left( -\frac{\kappa z - \lambda}{\lambda} \right) < 1 \Rightarrow E_{1-1/\zeta}(z) = -E_{\zeta/(\zeta-1)}(z) = \kappa z^{-\lambda}$$

satisfying $A2$ and the strong form of $A3$ for $\kappa > 0$ and $\lambda > 0$.

CES for $\kappa = 0$; $\bar{z} = \infty$; $c = 1 - \frac{1}{\sigma}$; CoPaTh for $\bar{z} < \infty$; $c = 1$; $\kappa = \frac{1-\rho}{\rho} > 0$, and $\lambda \to 0$.

- $\rho \left( \frac{\psi}{A} \right) = \frac{1}{1+\kappa (z_{\psi})^{-\lambda}},$ with $z_{\psi} = Z \left( \frac{\psi}{A} \right)$ given implicitly by $c \exp \left[ \frac{\kappa z_{\psi} - \lambda}{\lambda} \right] z_{\psi} \exp \left( -\frac{\kappa z_{\psi} - \lambda}{\lambda} \right) \equiv \frac{\psi}{A},$

- $\frac{\partial^2 \ln \rho(\psi/A)}{\partial A \partial \psi} \leq 0 \iff (\kappa) \frac{1}{\lambda} \leq z_{\psi} = Z \left( \frac{\psi}{A} \right) \iff \frac{\psi}{A} \leq (\kappa) \frac{1}{\lambda} c \exp \left[ \frac{\kappa z_{\psi} - \lambda - 1}{\lambda} \right];$ Log-sub(super)modular among more (less) efficient firms. In particular, if $\bar{z} < (\kappa) \frac{1}{\lambda}$, $\frac{\partial^2 \ln \rho(\psi/A)}{\partial A \partial \psi} < 0$ for all $\psi/A < Z(\psi/A) < \bar{z} < \infty$.

- Employment hump-shaped with the peak at $\hat{z} = Z \left( \frac{\psi}{A} \right) < \bar{z}$, given implicitly by

$$c \left( 1 + \frac{\hat{z}^\lambda}{\kappa} \right) \exp \left[ -\frac{\kappa \hat{z} - \lambda}{\lambda} \right] \exp \left[ \frac{\kappa \hat{z} - \lambda}{\lambda} \right] = 1 \iff \left( 1 + \frac{\hat{z}^\lambda}{\kappa} \right) \hat{z} = \frac{\psi}{A}.$$