Selection and Sorting of Heterogeneous Firms
Through Competitive Pressures

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Introduction
**Competitive Pressures on Heterogeneous Firms**

**Main Questions:** How do an increase in *competitive pressures*, due to lower *entry cost* or larger *market size*, affect firms with different productivity.
- Selection of firms
- Firm Size Distribution
- Sorting of firms across markets with different market sizes

**Existing Monopolistic Competition Models with Heterogenous Firms**
- Melitz (2003): under CES Demand System (DS)
  - MC firms sell their products at an exogenous & common markup rate, *unresponsive to competitive pressures*
  - Market size: no effect on distribution of firm types nor their behaviors; All adjustments at the extensive margin. *Inconsistent with some evidence for*
  - An increase in the production cost leads to less than proportional increase in the price (the pass-through rate < 1)
  - More productive firms have higher markup rates
  - More productive firms have lower pass-through rates
- Melitz-Ottaviano (2008) departs from CES using *Linear DS + the outside competitive sector*, which comes with its own restrictions.

**This Paper** develops the Melitz model under *H.S.A. (Homothetic with a Single Aggregator)* DS with gross substitutes as a theoretical framework to study the impact of competitive pressures on heterogeneous firms.
Symmetric H.S.A. (Homothetic with a Single Aggregator) DS with Gross Substitutes

**Final Good Production:** competitive, assemble a continuum of intermediate inputs \( \omega \in \Omega \), using 

CRS production function: 

\[
X = X(x); \ x = \{x_\omega; \ \omega \in \Omega \} \iff \text{Unit cost function, } P = P(p); \ p = \{p_\omega; \ \omega \in \Omega \}.
\]

**Market Share** of \( \omega \in \Omega \) depends solely on its single relative price (= its own price/the common price aggregator)

\[
s_\omega \equiv \frac{p_\omega x_\omega}{px} = \frac{\partial \ln P(p)}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(p)}\right),
\]

where 

\[
\int_\Omega s\left(\frac{p_\omega}{A(p)}\right) d\omega \equiv 1.
\]

- \( s: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+} \): the market share function, decreasing in the relative price for \( s(z) > 0 \) with \( \lim_{z \to z^\ast} s(z) = 0 \).
  - If \( z^\ast \equiv \inf\{z > 0|s(z) = 0\} < \infty \), \( zA(p) \) is the choke price.
- \( A(p) \): the common price aggregator defined implicitly by the adding-up constraint 

\[
\int_\Omega s(p_\omega / A) d\omega \equiv 1.
\]

By construction, market shares add up to one; \( A(p) \) linear homogenous in \( p \) for a fixed \( \Omega \). A larger \( \Omega \) reduces \( A(p) \).

<table>
<thead>
<tr>
<th>Special Cases</th>
<th>CES</th>
<th>Translog Cost Function</th>
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<tbody>
<tr>
<td></td>
<td>( s(z) = \gamma z^{1-\sigma} );</td>
<td>( s(z) = -\gamma \ln \left(\frac{z}{\bar{z}}\right) );</td>
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<tr>
<td></td>
<td>( \sigma &gt; 1 )</td>
<td>( \bar{z} &lt; \infty )</td>
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<th>Constant Pass Through</th>
<th>CoPaTh</th>
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<td>( s(z) = \gamma \left[\sigma + (1-\sigma)z^{\frac{1-\rho}{\rho}}\right]^\frac{\rho}{1-\rho} );</td>
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<tr>
<td></td>
<td>( 0 &lt; \rho &lt; 1; \ \bar{z}(\rho) = \left(\frac{\sigma}{\sigma - 1}\right)^{\frac{1-\rho}{\rho}} &lt; \infty )</td>
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</table>

As \( \rho \searrow 1 \), CoPaTh converges to CES with \( \bar{z}(\rho) \to \infty \).
**P(p) vs. A(p)**

Definition: \( s_\omega \equiv \frac{\partial \ln P(p)}{\partial \ln p_\omega} = s\left(\frac{p_\omega}{A(p)}\right) = s(z_\omega) \quad \text{where} \quad \int_\Omega s\left(\frac{p_\omega}{A(p)}\right) d\omega \equiv 1. \)

By differentiating the adding-up constraint,

\[
\frac{\partial A(p)}{\partial \ln p_\omega} = \frac{\int_\Omega [\zeta(z_\omega) - 1]s(z_\omega)}{\int_\Omega [\zeta(z_\omega) - 1]s(z_\omega')d\omega'};
\]

where

**Price Elasticity Function:**

\( \zeta(z) \equiv 1 - \frac{d \ln s(z)}{d \ln z} = 1 - \varepsilon_s(z) > 1 \text{ for } z \in (0, \bar{z}); \quad \lim_{z \to \bar{z}} \zeta(z) = \infty, \text{ if } \bar{z} < \infty. \)

By integrating the definition,

\[
\ln \left(\frac{A(p)}{P(p)}\right) + \text{const.} = \int_\Omega \Phi\left(\frac{p_\omega}{A(p)}\right) s\left(\frac{p_\omega}{A(p)}\right) d\omega, \quad \text{where} \quad \Phi(z) \equiv \frac{1}{s(z)} \int_{z}^{\bar{z}} \frac{s(\xi)}{\xi} d\xi
\]

**Note:** \( P(p)/A(p) \neq c \) for any \( c > 0 \), unless CES \( \iff \zeta(z) = \sigma \iff s(z) = \gamma z^{1-\sigma} \iff \Phi(z) = 1/(\sigma - 1). \)

- ✔ \( A(p) \), the inverse measure of competitive pressures, captures cross price effects in the demand system.
- ✔ \( P(p) \), the inverse measure of TFP, captures the productivity consequences of price changes.
- ✔ \( \Phi(z) \) can be interpreted as the measure of “love for variety.”

**Note:** Our 2017 paper proved the integrability = the quasi-concavity of \( P(p) \), iff \( \zeta(z) \equiv 1 - \frac{d \ln s(z)}{d \ln z} > 0. \)
Why H.S.A.

- **Homothetic** (unlike the linear DS and most other commonly used non-CES DSs)
  - a single measure of market size; the demand composition does not matter.
  - isolate the effect of endogenous markup rate from nonhomotheticity
  - straightforward to use it as a building block in multi-sector models with any upper-tier (incl. nonhomothetic) DS

- **Nonparametric and flexible** (unlike CES and translog, which are special cases)
  - can be used to perform robustness-check for CES
  - allow for (but no need to impose)
    - the choke price,
    - **Marshall’s 2nd law** (Price elasticity is increasing in price) \( \Rightarrow \) more productive firms have higher markup rates
    - **what we call the 3rd law** (the rate of increase in the price elasticity is decreasing in price) \( \Rightarrow \) more productive firms have lower pass-through rates.

- **Tractable** due to **Single Aggregator** (unlike Kimball, which needs two aggregators), a sufficient statistic for competitive pressures, which acts like a magnifier of firm heterogeneity
  - the existence & uniqueness of free-entry equilibrium with firm heterogeneity straightforward
  - simple to conduct most comparative statics without parametric restrictions on demand or productivity distribution.
  - no need to assume zero overhead cost (unlike MO and ACDR)

- Defined by **the market share function**, for which data is readily available and easily identifiable.
### Three Classes of Homothetic Demand Systems: Matsuyama-Ushchev (2017)

Here we consider a **continuum** of varieties \((\omega \in \Omega)\), **gross substitutes**, and **symmetry**

<table>
<thead>
<tr>
<th>Class</th>
<th>Demand Function</th>
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<tr>
<td><strong>CES</strong></td>
<td>[ s_\omega \equiv \frac{\partial \ln P(p)}{\partial \ln p_\omega} = f \left( \frac{p_\omega}{P(p)} \right) ]</td>
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<tr>
<td><strong>H.S.A.</strong> (Homothetic with a Single Aggregator)</td>
<td>[ s_\omega = s \left( \frac{p_\omega}{A(p)} \right), \quad \frac{P(p)}{A(p)} \neq c, \text{ unless CES} ]</td>
</tr>
<tr>
<td><strong>HDIA</strong> (Homothetic with Direct Implicit Additivity) Kimball is a special case:</td>
<td>[ s_\omega = \frac{p_\omega}{P(p)} \left( \phi' \right)^{-1} \left( \frac{p_\omega}{B(p)} \right), \quad \frac{P(p)}{B(p)} \neq c, \text{ unless CES} ]</td>
</tr>
<tr>
<td><strong>HIIA</strong> (Homothetic with Indirect Implicit Additivity)</td>
<td>[ s_\omega = \frac{p_\omega}{C(p)} \theta' \left( \frac{p_\omega}{P(p)} \right), \quad \frac{P(p)}{C(p)} \neq c, \text{ unless CES} ]</td>
</tr>
</tbody>
</table>

The 3 classes are pairwise disjoint with the sole exception of CES.

Under HDIA and HIIA, unlike HSA
- Two aggregators are necessary.
- The free-entry equilibrium may not exist, or if it exists, may not be unique, unless we impose some strong restrictions on both productivity distributions and the price elasticity functions.
(Highly Selective) Literature Review

Non-CES Demand Systems: Matsuyama (2023) for a survey; **H.S.A. Demand System:** Matsuyama-Ushchev (2017)

MC with Heterogeneous Firms: Melitz (2003) and many others: Melitz-Redding (2015) for a survey

MC under non-CES demand systems: Thisse-Ushchev (2018) for a survey

- **Nonhomothetic non-CES:**
  - $U = \int_{\Omega} u(x_\omega)d\omega$: Dixit-Stiglitz (77), Behrens-Murata (07), ZKPT (12), Mrázová-Neary(17), Dhingra-Morrow (19); ACDR (19)
  - **Linear-demand system:** Ottaviano-Tabuchi-Thisse (2002)

- **Homothetic non-CES:** Feenstra (2003), Kimball (1995)

**Empirical Evidence:** *The 2nd Law:* DeLoecker-Goldberg (14), Burstein-Gopinath (14); *The 3rd Law:* Berman et.al.(12), Amiti et.al.(19), *Market Size Effects:* Campbell-Hopenhayn(05); *Rise of markup:* Autor et.al.(20), DeLoecker et.al.(20)

Selection of Heterogeneous Firms through Competitive Pressures Melitz-Ottaviano (2008), Baqae-Fahri-Sangani (2021)

Sorting of Heterogeneous Firms Across Markets:

- **Reduced Form/Partial Equilibrium:** Mrázová-Neary (2019), Nocke (2006)

**Log-Super(Sub)modularity:** Costinot (2009), Costinot-Vogel (2015)
Monopolistic Competition under H.S.A.
Pricing: Markup and Pass-Through Rates

Lerner Pricing Formula
\[ p_\omega \left[ 1 - \frac{1}{\zeta(p_\omega / A)} \right] = \psi_\omega \Rightarrow \frac{p_\omega}{A} \left[ 1 - \frac{1}{\zeta(p_\omega / A)} \right] = \frac{\psi_\omega}{A}, \]

\( \psi_\omega \): firm-specific marginal cost (in labor, the numeraire)

Under the mild regularity condition, LHS is monotone \( \rightarrow \) firms with the same \( \psi \) set the same price \( \rightarrow p_\omega = p_\psi. \)

Relative price
\[ \frac{p_\psi}{A} = z_\psi = Z\left(\frac{\psi}{A}\right), \text{ an increasing function of } \psi / A, \text{ the normalized cost, only.} \]

Price elasticity
\[ \zeta \left( Z\left(\frac{\psi}{A}\right) \right) \equiv \sigma \left(\frac{\psi}{A}\right) > 1 \]

Markup rate
\[ \mu_\psi \equiv \frac{p_\psi}{\psi} = \frac{\sigma(\psi / A)}{\sigma(\psi / A) - 1} \equiv \mu \left(\frac{\psi}{A}\right) > 1 \]

Pass-through rate
\[ \rho_\psi \equiv \frac{\partial \ln p_\psi}{\partial \ln \psi} = \frac{d \ln Z(\psi / A)}{d \ln (\psi / A)} \equiv \varepsilon_Z \left(\frac{\psi}{A}\right) \equiv \rho \left(\frac{\psi}{A}\right) = 1 + \varepsilon_\mu \left(\frac{\psi}{A}\right) \]

are all functions of \( \psi / A \) only, continuously differentiable.

Unless CES, where \( \sigma(\cdot) = \sigma; \mu(\cdot) = \mu; \rho(\cdot) = 1, \)

- Market size \( L = px \) affects the pricing behaviors of firms only through its effects on \( A. \)
- More competitive pressures, a lower \( A \), act like a magnifier of firm heterogeneity.
Revenue, Profit, & Employment

Revenue
\[ R_\psi = s(z_\psi)L = s\left( Z\left( \frac{\psi}{A} \right) \right)L \equiv r\left( \frac{\psi}{A} \right)L \quad \Rightarrow \quad \mathcal{E}_r\left( \frac{\psi}{A} \right) = -\left[ \sigma\left( \frac{\psi}{A} \right) - 1 \right] \rho\left( \frac{\psi}{A} \right) < 0 \]

(Gross) Profit
\[ \Pi_\psi = \frac{s(z_\psi)}{\zeta(z_\psi)}L = \frac{r(\psi/A)}{\sigma(\psi/A)}L \equiv \pi\left( \frac{\psi}{A} \right)L \quad \Rightarrow \quad \mathcal{E}_\pi\left( \frac{\psi}{A} \right) = 1 - \sigma\left( \frac{\psi}{A} \right) < 0 \]

(Variable) Employment
\[ \psi x_\psi = R_\psi - \Pi_\psi = \frac{r(\psi/A)}{\mu(\psi/A)}L \equiv \ell\left( \frac{\psi}{A} \right)L \quad \Rightarrow \quad \mathcal{E}_\ell\left( \frac{\psi}{A} \right) = 1 - \sigma\left( \frac{\psi}{A} \right) \rho\left( \frac{\psi}{A} \right) \leq 0 \]

Unless CES, where \( \mathcal{E}_r(\cdot) = \mathcal{E}_\pi(\cdot) = \mathcal{E}_\ell(\cdot) = 1 - \sigma < 0 \),

More competitive pressures, a lower \( A \), act like a magnifier of firm heterogeneity.

- Revenue \( r(\psi/A)L \), profit \( \pi(\psi/A)L \), employment \( \ell(\psi/A)L \) all functions of \( \psi/A \), multiplied by market size \( L \),
  continuously differentiable under mild regularity conditions.

Market size affects the relative profit, revenue, and employment across firms only through its effects on \( A \).

- Their elasticities \( \mathcal{E}_r(\cdot), \mathcal{E}_\pi(\cdot) \) and \( \mathcal{E}_\ell(\cdot) \) depend solely on \( \sigma(\cdot) \) and \( \rho(\cdot) \).
- Both revenue \( r(\psi/A)L \) and profit \( \pi(\psi/A)L \) are always strictly decreasing in \( \psi/A \).
- Employment \( \ell(\psi/A)L \) may be nonmonotonic in \( \psi/A \).
  - If the markup rate declines with \( \psi/A \), employment cannot decline as fast as the revenue.
  - If the markup rate declines faster than the revenue, the employment is increasing in \( \psi/A \).
Selection of Heterogenous Firms: A Single-Market Setting
General Equilibrium: Existence and Uniqueness

As in Melitz, firms pay the entry cost $F_e > 0$ to draw $\psi \sim G(\psi)$; cdf with the support, $(\underline{\psi}, \bar{\psi}) \subset (0,\infty)$, and pay the overhead $F > 0$ to stay & produce.

Cutoff Rule: stay if $\psi < \psi_c$; exit if $\psi > \psi_c$, where

$$\pi \left( \frac{\psi_c}{A} \right) L = F$$

positively-sloped $A \downarrow$ (more competitive pressures) $\Rightarrow \psi_c \downarrow$ (tougher selection)

Free Entry Condition:

$$F_e = \int_{\underline{\psi}}^{\psi_c} \left[ \pi \left( \frac{\psi}{A} \right) L - F \right] dG(\psi)$$

negative-sloped. $A \downarrow$ (more competitive pressures) and $\psi_c \downarrow$ (tougher selection) both make entry less attractive.

$A = A(p)$ and $\psi_c$: uniquely determined, respond continuously to $F_e/L$ & $F/L$ under mild regularity conditions. (This proof of unique existence applies also to the Melitz model under CES.)
Equilibrium Mass of Firms under H.S.A. With $A$ & $\psi_c$ determined, from the adding-up constraint,

\[
MG(\psi_c) = \left[ \int_{\psi}^{\psi_c} r \left( \frac{\psi}{A} \right) dG(\psi) \right]^{-1} = \left[ \int_{\xi}^{1} r \left( \frac{\psi_c}{A} \xi \right) d\tilde{G}(\xi; \psi_c) \right]^{-1} > 0
\]

where $\tilde{G}(\xi; \psi_c) \equiv g(\psi_c \xi) / g(\psi_c)$ is the cdf of $\xi \equiv \psi / \psi_c$, conditional on $\xi \equiv \psi / \psi_c < \xi \leq 1$.

**Lemma 1:** $E'_g(\psi) < 0 \implies E'_G(\psi) < 0$ generally; $E'_g(\psi) \geq 0 \implies E'_G(\psi) \geq 0$, with some additional conditions.

**Lemma 2:** A lower $\psi_c$ (tougher selection) shifts $\tilde{G}(\xi; \psi_c)$ to the right (left) in the MLR ordering if $E'_g(\psi) < (>)0$ and in the FSD ordering if $E'_G(\psi) < (>)0$.

$\tilde{G}(\xi; \psi_c)$ is independent of $\psi_c$ if $E_g(\psi)$ & $E_G(\psi)$ are constant $\iff G(\psi) = (\psi / \bar{\psi})^\kappa$ $\iff$ Pareto-productivity

A lower $\psi_c$ (tougher selection) shifts $\tilde{G}(\xi; \psi_c)$ to the right if Fréchet, Weibull, or Lognormal.

Equilibrium TFP under H.S.A.

\[
\ln \left( \frac{X}{L} \right) = \ln \left( \frac{1}{P} \right) = \ln \left( \frac{c}{A} \right) + \frac{\int_{\psi}^{\psi_c} \Phi(Z(\psi/A)) r(\psi/A)dG(\psi)}{\int_{\psi}^{\psi_c} r(\psi/A)dG(\psi)} = \ln \left( \frac{c}{A} \right) + \frac{\int_{\psi}^{1} \Phi \left( Z \left( \frac{\psi_c}{A} \xi \right) \right) r \left( \frac{\psi_c}{A} \xi \right) d\tilde{G}(\xi; \psi_c)}{\int_{\xi}^{1} r \left( \frac{\psi_c}{A} \xi \right) d\tilde{G}(\xi; \psi_c)}.
\]

**Equilibrium can be solved recursively under H.S.A.!!!**

Under HDIA/HIIA, the 3 variables, $\psi_c$ & the two aggregates, need to be solved for simultaneously.
Revisiting Melitz (2003) under CES: \( s(z) = \gamma z^{1-\sigma} \)

**Pricing:**
\[
\mu\left(\frac{\psi}{A}\right) = \frac{\sigma}{\sigma - 1} > 1 \Rightarrow \rho\left(\frac{\psi}{A}\right) = 1
\]
\[
\Rightarrow E_r\left(\frac{\psi}{A}\right) = E_n\left(\frac{\psi}{A}\right) = E_\ell\left(\frac{\psi}{A}\right) = 1 - \sigma < 0.
\]

**Cutoff Rule:**
\[
c_0 L \left(\frac{\psi_c}{A}\right)^{1-\sigma} = F,
\]

**Free Entry Condition:**
\[
\int_{\psi_c}^{\psi} \left[ c_0 L \left(\frac{\psi}{A}\right)^{1-\sigma} - F \right] dG(\psi) = F_e,
\]
with \( c_0 > 0 \). As \( L \) changes, the intersection moves along
\[
\int_{\psi_c}^{\psi} \left[ \left(\frac{\psi}{\psi_c}\right)^{1-\sigma} - 1 \right] dG(\psi) = \frac{F_e}{F}
\]

**Proposition 1:** Under CES,
- \( L \uparrow \) keeps \( \psi_c \) unaffected; increases both \( M \) and \( MG(\psi_c) \) proportionately;
- \( F_e \downarrow \) reduces \( \psi_c \); increases \( M \); increases (decreases) \( MG(\psi_c) \) if \( E'_G(\psi) < (>0) \);
- \( F \downarrow \) increases \( \psi_c \); increases \( MG(\psi_c) \); increases (decreases) \( M \) if \( E'_G(\psi) < (>0) \)
Cross-Sectional Implications under 2\textsuperscript{nd} & 3\textsuperscript{rd} Laws
Marshall’s 2nd Law: Cross-Sectional Implications (Proposition 2)

(A2): $\zeta(z_\psi)$ is increasing in $z_\psi \equiv p_\psi / A = Z(\psi / A)$

- **Price elasticity** $\zeta(Z(\psi / A)) \equiv \sigma(\psi / A)$ increasing in $\psi / A$; high-$\psi$ firms operate at more elastic parts of demand curve.
  - **Markup Rate**, $\mu(\psi / A)$, decreasing in $\psi / A \Leftrightarrow \varepsilon_\mu(\psi / A) < 0$
    high-$\psi$ firms charge lower markup rates.
  - **Incomplete Pass-Through**: The pass-through rate, $\rho(\psi / A) = 1 + \varepsilon_\mu(\psi / A) < 1$.
- **Procompetitive effect of entry/Strategic complementarity in pricing**, $\frac{\partial \ln p_\psi}{\partial \ln A} = 1 - \rho(\psi / A) > 0$.
  Firms set the price lower under more competitive pressures ($A = A(p) \downarrow$), due to either a larger $\Omega$ and/or a lower $p$.

**Lemma 5**: $f(\psi / A)$ log-super(sub)modular in $\psi$ & $A \Leftrightarrow E_f(\cdot) < (>)0 \Leftrightarrow \ln f(e^{\ln(\psi / A)})$ concave (convex) in $\ln(\psi / A)$

- **Profit**, $\pi(\psi / A)L$, always decreasing, strictly log-supermodular in $\psi$ and $A$.
  $A \downarrow \rightarrow$ a proportionately larger decline in profit for high-$\psi$ firms $\rightarrow$ Larger dispersion of profit
3rd Law: Cross-Sectional Implications (Propositions 3, 4, and 5)

In addition to A2, if we further assume, with some empirical support, e.g. Berman et.al.(2012), Amiti et.al.(2019),

(A3): \( \rho(\psi/A) = 1 + \varepsilon_\mu(\psi/A) \) is weak(strictly increasing--we call it Weak (Strong) 3rd Law.

Under translog, \( \rho(\psi/A) \) is strictly decreasing, violating A3

- **Markup rate**, \( \mu(\psi/A) \), decreasing under A2, log-submodular in \( \psi \) & \( A \) under A3. For strong A3, it is strict and \( A \downarrow \rightarrow \) a proportionately smaller decline in markup rate for high-\( \psi \) firms \( \rightarrow \) smaller dispersion of markup rate

- **Revenue**, \( r(\psi/A)L \), always decreasing, strictly log-supermodular in \( \psi \) & \( A \) under weak A3
  \( A \downarrow \rightarrow \) a proportionately larger decline in revenue for high-\( \psi \) firms \( \rightarrow \) Larger dispersion of revenue

- **Employment**, \( \ell(\psi/A) = \frac{r(\psi/A)}{\mu(\psi/A)}L \), hump-shaped in \( \psi/A \), strictly log-supermodular in \( \psi \) & \( A \) under weak A3
  Employment is increasing in \( \psi \) across all active firms with a large enough overhead/market size ratio.
  \( A \downarrow \rightarrow \) Employment up for the most productive firms.

- **Pass-through rate**, \( \rho(\psi/A) \), strictly log-submodular in \( \psi \) & \( A \) for a small enough \( \bar{Z} \) under strong A3
  \( A \downarrow \rightarrow \) a proportionately smaller increase in the pass-through rate for low-\( \psi \) firms among the active.
Cross-Sectional Implications of More Competitive Pressures, $A \downarrow$: A Graphic Representation

**Profit Function:** $\Pi_\psi = \pi(\psi/A) L$
- *always* decreasing in $\psi$
- strictly log-supermodular *under A2*

$A \downarrow$ with $L$ fixed shifts down with a steeper slope at each $\psi$;
$A \downarrow$ due to $L \uparrow$, a parallel shift up, a single-crossing.

**Markup Rate Function:** $\mu_\psi = \mu(\psi/A) > 1$
- *decreasing* in $\psi$ *under A2*
- weakly log-submodular *under Weak A3*
- strictly log-submodular *under Strong A3*

$A \downarrow$ shifts down with a flatter slope at each $\psi$

✓ With $\ln \psi$ in the horizontal axis, $A \downarrow$ causes a parallel leftward shift of the graphs in these figures.
✓ $f(\psi/A)$ is strictly log-super(sub)modular in $\psi$ & $A \iff \ln f(\psi/A)$ is (strictly) concave( convex) in $\ln(\psi/A)$.

Under Weak A3, $R_\psi = r(\psi/A)L$, strictly log-supermodular and shares similar properties with $\pi(\psi/A)L$. 
**Employment Function:** \( \ell(\psi/A)L = r(\psi/A)L/\mu(\psi/A) \)
- Hump-shaped in \( \psi \) under \( A_2 \) and weak \( A_3 \).
  \( \rightarrow A \downarrow \) shifts up (down) for a low (high) \( \psi \) with \( A \downarrow \)
- Strictly log-supermodular under weak \( A_3 \)
  for \( A \downarrow \) with a fixed \( L \); for \( A \downarrow \) caused by \( L \uparrow \)
*Single-crossing* even with a fixed \( L \)

**Pass-Through Rate Function:** \( \rho_\psi = \rho(\psi/A) \)
- \( \rho(\psi/A) < 1 \) under \( A_2 \), hence it cannot be strictly log-submodular for a higher range of \( \psi/A \)
- Increasing in \( \psi \) under Strong \( A_3 \)
- Strictly log-submodular for a lower range of \( \psi/A \) under \( A_2 \) and Strong \( A_3 \) \( \Rightarrow A \downarrow \) shifts up with a steeper slope at each \( \psi \) *with a small enough \( \bar{z} \).*

**Diagram:**
- \( \ln \ell(\psi/A)L = \ln r(\psi/A)L - \ln \mu(\psi/A) \)
- \( \ln \rho(\bar{z}) \)
- \( \ln \rho_\psi = \ln \rho(\frac{\psi}{A}) \)

In summary, more competitive pressures (\( A \downarrow \))
- \( \mu(\psi/A) \downarrow \) under \( A_2 \) & \( \rho(\psi/A) \uparrow \) under strong \( A_3 \)
- Profit, Revenue, Employment become more concentrated among the most productive.
Comparative Statics: General Equilibrium Effects
General Equilibrium Effects of \( F_e, L, \) and \( F \) on \( A \) and \( \psi_c \\

**Proposition 6:**

\[
\mathbb{E}_\sigma \left( \psi, \psi_c \right) - 1 \left( \frac{dA}{A} \right) = \left( 1 - f_x \right) \left( \frac{dF_e}{F_e} \right) - \frac{dL}{L} + f_x \left( \frac{dF}{F} \right);
\]

\[
\mathbb{E}_\sigma \left( \psi, \psi_c \right) - 1 \left( \frac{d\psi_c}{\psi_c} \right) = \left( 1 - f_x \right) \left( \frac{dF_e}{F_e} \right) - \left( 1 - \delta \right) \left( \frac{dL}{L} \right) + f_x - \delta \left( \frac{dF}{F} \right)
\]

where

\[
\mathbb{E}_\sigma \left( \psi, \psi_c \right) \equiv \frac{\int_{\psi}^{\psi_c} \sigma(\psi/A) \pi(\psi/A) dG(\psi)/G(\psi_c)}{\int_{\psi}^{\psi_c} \pi(\psi/A) dG(\psi)/G(\psi_c)} = 1 + \frac{\int_{\psi}^{\psi_c} \ell(\psi/A) dG(\psi)/G(\psi_c)}{\int_{\psi}^{\psi_c} \pi(\psi/A) dG(\psi)/G(\psi_c)} > 1
\]

The profit-weighted average of \( \sigma(\psi/A) \) among the active firms.

\[f_x \equiv \frac{FG(\psi_c)}{F_e + FG(\psi_c)} = \frac{FG(\psi_c)}{L \int_{\psi}^{\psi_c} \pi(\psi/A) dG(\psi) / G(\psi_c)} = \frac{\pi(\psi_c/A)}{\int_{\psi}^{\psi_c} \pi(\psi/A) dG(\psi)/G(\psi_c)} < 1
\]

The share of the overhead in the total fixed cost = the profit at the cut-off relative to the average profit among the active firms.

\[\delta \equiv \frac{\mathbb{E}_\sigma \left( \psi, \psi_c \right) - 1}{\sigma(\psi_c/A) - 1} = \frac{\int_{\psi}^{\psi_c} \ell(\psi/A) dG(\psi)/G(\psi_c)}{\int_{\psi}^{\psi_c} \pi(\psi/A) dG(\psi)/G(\psi_c)} = f_x \frac{\int_{\psi}^{\psi_c} \ell(\psi/A) dG(\psi)/G(\psi_c)}{\ell(\psi_c/A)} > 0.
\]

The profit/employment ratio at the cut-off to the average profit/the average employment ratio among the active firms.
Proposition 6: A Graphic Representation

Effects of $F_e \downarrow$

\[
\pi\left(\frac{\psi_c}{A}\right) = \frac{F}{L}
\]

\[
\frac{F_e}{L} = \int_{\psi} \left[ \pi\left(\frac{\psi_c}{A}\right) - \frac{F}{L} \right] dG(\psi)
\]

Effects of $L \uparrow$ if $\sigma'(\cdot) > 0$ (i.e., A2)

\[
\pi\left(\frac{\psi_c}{A}\right) = \frac{F}{L}
\]

\[
\frac{F_e}{L} = \int_{\psi} \left[ \pi\left(\frac{\psi}{A}\right) - \frac{F}{L} \right] dG(\psi)
\]

Effects of $F \downarrow$ if $\ell'(\cdot) > 0$

\[
\pi\left(\frac{\psi_c}{A}\right) = \frac{F}{L}
\]

\[
\frac{F_e}{L} = \int_{\psi} \left[ \pi\left(\frac{\psi_c}{A}\right) - \frac{F}{L} \right] dG(\psi)
\]

In all three cases, more competitive pressures imply tougher selection.

\[
\left\{ F_e \downarrow, L \uparrow, F \downarrow \right\} \Rightarrow A \downarrow \Rightarrow \psi_c \downarrow
\]
Proposition 7: Market Size Effect on Profit and Revenue Distributions

7a: Under A2, there exists a unique $\psi_0 \in (\psi, \psi_c)$ such that $\sigma\left(\frac{\psi_0}{A}\right) = \mathbb{E}_{\sigma}\left(\psi, \psi_c\right)$ with

$$\frac{d \ln \Pi_{\psi}}{d \ln L} > 0 \iff \sigma\left(\frac{\psi}{A}\right) < \mathbb{E}_{\sigma}\left(\psi, \psi_c\right) \text{ for } \psi \in (\psi, \psi_0),$$

and

$$\frac{d \ln \Pi_{\psi}}{d \ln L} < 0 \iff \sigma\left(\frac{\psi}{A}\right) > \mathbb{E}_{\sigma}\left(\psi, \psi_c\right) \text{ for } \psi \in (\psi_0, \psi_c).$$

7b: Under A2 and the weak A3, there exists $\psi_1 > \psi_0$, such that

$$\frac{d \ln R_{\psi}}{d \ln L} > 0 \text{ for } \psi \in (\psi_1, \psi_1).$$

Furthermore, $\psi_1 \in (\psi_0, \psi_c)$ and

$$\frac{d \ln R_{\psi}}{d \ln L} < 0 \text{ for } \psi \in (\psi_1, \psi_c),$$

for a sufficiently small $F$.

In short, more productive firms expand, while less productive firms shrink.
More on GE Effects: Propositions 8 and 9

Average Markup and Pass-Through Rates: Composition Effect of $\psi_c \downarrow$ & $A \downarrow$
- Under A2, $A \downarrow$ causes $\mu(\psi/A) \downarrow$ for each $\psi$, but $\psi_c \downarrow$ means high-$\psi$ firms with lower $\mu(\psi/A)$ drop out.
- Under strong A3, $A \downarrow$ causes $\rho(\psi/A) \uparrow$ for each $\psi$, but $\psi_c \downarrow$ means high-$\psi$ firms with higher $\rho(\psi/A)$ drop out.

**Prop.8:** Assume $\psi = 0$. Consider a shock, such that $\psi_c/A$ remains constant. For any weighting function $w(\psi/A)$,

The weighted average of monotone $f(\psi/A), I \equiv \frac{\frac{\psi_c}{\psi} \int f(\psi/A)w(\psi/A)dG(\psi)}{\int \frac{\psi_c}{\psi} w(\psi/A)dG(\psi)}$ satisfies $\text{sgn}\left\{\frac{dI}{dA}\right\} = \text{sgn}\{f'(\cdot)E_g(\cdot)\}.$

The average markup (or pass-through) rate can go either way, with $F_e \downarrow +$ Pareto-productivity a knife-edge case **More competition, which causes more concentration, may result in the rise of markup.**

Impact on TFP/Unit Cost: Since $\ln \left(\frac{A}{P}\right) + \text{const.} = \frac{\int \psi_c \Phi(Z(\psi/A))r(\psi/A)dG(\psi)}{\int \psi_c r(\psi/A)dG(\psi)}$ and $\zeta'(\cdot) \gtrless 0 \implies \Phi'(\cdot) \gtrless 0$,

**Corollary of Prop.8:** If $\psi = 0$ and $\zeta'(\cdot)E_g'(\cdot) \geq 0$, $F_e \downarrow \implies dP/P \leq dA/A < 0$ with the equality iff $\zeta'(\cdot)E_g'(\cdot) = 0$.

Impact on Masses of Firms: Proposition 9
The effects on $M \& MG(\psi_c)$ depend on whether $E_g(\psi) \equiv \psi g(\psi)/G(\psi)$ is decreasing, constant, or increasing.
Sorting of Heterogenous Firms: A Multi-Market Setting
Sorting: GE Implications in a Multi-Market Setting

Many markets of different size. Firms, after learning their $\psi$, choose which market to enter.

**Proposition 10: Assortative Matching**

More competitive pressures in larger markets:

$$L_1 > L_2 > \cdots > L_J > 0 \Rightarrow 0 < A_1 < A_2 < \cdots < A_J < \infty$$

Under A2, more efficient firms sort themselves into larger markets: Firms $\psi \in (\psi_{j-1}, \psi_j)$ entering market-$j$, where

$$0 \leq \psi = \psi_0 < \psi_1 < \psi_2 < \cdots < \psi_J < \overline{\psi} \leq \infty.$$
Sorting: GE Implications in a Multi-Market Setting

**Proposition 11: The Composition Effect:** examples with Pareto-productivity such that

- The average markup rates *higher* (the average pass-through rates *lower* under Strong A3) in larger (more competitive) markets

A decline in $F_e$ causes uniform declines in $\psi_j$ & $A_j$ with the average markup/pass-through rates unchanged.

**Markup Rate across markets under A2**

$\mu(\psi_{A_{j-2}}) \rightarrow \mu(\psi_{A_{j-1}}) \rightarrow \mu(\psi_{A_j})$

**Pass-Through Rate across markets under strong A3**

$\rho(\psi_{A_{j-2}}) \rightarrow \rho(\psi_{A_{j-1}}) \rightarrow \rho(\psi_{A_j})$

A caution against testing A2/A3 by comparing the average markup/pass-through rates in cross-section of cities.
Three Parametric Families of H.S.A. (Appendix D)

<table>
<thead>
<tr>
<th>Generalized Translog</th>
<th>$s(z) = y \left(-\frac{\sigma - 1}{\eta}\ln \left(\frac{z}{\bar{z}}\right)\right)^{\eta}; z &lt; \bar{z} \equiv \beta e^{\frac{\eta}{\sigma - 1}}$</th>
<th>$1 - \frac{1}{\zeta(z)} = \frac{\eta}{\eta - \ln \left(\frac{z}{\bar{z}}\right)} \Rightarrow \mathcal{E}<em>\mu(\cdot) &lt; 0; \mathcal{E}</em>\mu'(\cdot) &lt; 0$; satisfying A2; violating A3.</th>
</tr>
</thead>
</table>

Translog is the special case where $\eta = 1$. CES is the limit case, as $\eta \to \infty$, while holding $\beta > 0$ and $\sigma > 1$ fixed.

<table>
<thead>
<tr>
<th>Constant Pass-Through (CoPaTh)</th>
<th>$s(z) = y \sigma^{1-\rho} \left[1 - \left(\frac{z}{\bar{z}}\right)^{\frac{1-\rho}{\rho}}\right]^{\frac{\rho}{1-\rho}}; \bar{z} \equiv \beta \left(\frac{\sigma}{\sigma - 1}\right)^{\frac{\rho}{1-\rho}}$</th>
<th>$1 - \frac{1}{\zeta(z)} = \left(\frac{z}{\bar{z}}\right)^{\frac{1-\rho}{\rho}} \Rightarrow \mathcal{E}<em>\mu(\cdot) &lt; 0; \mathcal{E}</em>\mu'(\cdot) = 0$ satisfying A2 &amp; weak A3; violating strong A3</th>
</tr>
</thead>
</table>

CES is the limit case, as $\rho \to 1$, while holding $\beta > 0$ and $\sigma > 1$ fixed.

<table>
<thead>
<tr>
<th>Power Elasticity of Markup Rate (Frechet Inverse Markup Rate)</th>
<th>$s(z) = \exp \left[\int_{z_0}^{z} \frac{c}{c - \exp \left[-\frac{k\bar{z}^{-\lambda}}{\lambda}\exp \left[-\frac{k\xi^{-\lambda}}{\lambda}\right]\xi\right]} d\xi\right]$</th>
<th>$1 - \frac{1}{\zeta(z)} = c \exp \left[-\frac{k\bar{z}^{-\lambda}}{\lambda}\right] \exp \left[-\frac{k\bar{z}^{-\lambda}}{\lambda}\right] \Rightarrow \mathcal{E}<em>\mu(\cdot) &lt; 0; \mathcal{E}</em>\mu'(\cdot) &gt; 0$ satisfying A2 and strong A3 for $\kappa &gt; 0$ and $\lambda &gt; 0$.</th>
</tr>
</thead>
</table>

CES for $\kappa = 0$; $\bar{z} = \infty$; $c = 1 - \frac{1}{\sigma}$; CoPaTh for $\bar{z} < \infty$; $c = 1; \kappa = \frac{1-\rho}{\rho} > 0$, and $\lambda \to 0$. 

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