Are Properties of Model Economy in Zero Lower Bound a Spurious Artifact of Log-linearization?

Christiano-Eichenbaum

Background

- Eggertson-Woodford made some assumptions that greatly simplified analysis of consequences of zero lower bound (zlb).

- Findings:
  - When zlb binds, drop in output can be substantial.
  - *G* multiplier larger in zlb than out, independent of which shock pushed the economy into zlb.
  - If prices are more flexible and/or duration of zlb longer, then both multiplier and output drop bigger.

Question

- Are conclusions sensitive to having log-linearized?

- Answer:
  – If you do not linearize equilibrium conditions, then there are multiple equilibria and sunspot equilibria (Braun-Körber-Waki (2012), Mertens-Ravn (2011)).
    - Not clear what a multiplier even means, when there are multiple equilibria.
    - Some equilibria seem to imply multiplier is very small in the zlb.
    - The economy can be driven into a binding zlb by a sunspot, and the multiplier is very small in that equilibrium.
    - These results suggest answer is 'yes'.

  – We find that after removing equilibria that don't satisfy E-stability ('Expectational stability'), answer is 'no'.

Observations

- Caveats:
  - We take the position that an equilibrium that fails E-stability is a mathematical curiosity and can be ignored for predictive purposes.
  - Is this a sensible position?
- Multiplicity of equilibria happens in other settings too.
  - If our models are to have predictive power for outcomes, must have a way to select among equilibria.
  - Some think E-stability is necessary for an equilibrium to be worthy of consideration.

Model

- Households maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\log(C_t) - \frac{\chi}{2} h_t^2 \right]$$

- Subject to

$$P_t C_t + B_t \leq (1 + R_{t-1}) B_{t-1} + W_t h_t + \Pi_t,$$

Profits, net of lump sum taxes to pay for government consumption.

- Efficiency conditions:

$$\chi h_t C_t = \frac{W_t}{P_t}, \quad \frac{1}{1+R_t} = \beta E_t \frac{P_t C_t}{P_{t+1} C_{t+1}}.$$

Firms

Value to household of profits.

- Competitive, final good firm production function and first order conditions:

$$Y_t = \left(\int_0^1 Y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}},\ \varepsilon \geq 1,\ Y_{j,t} = \left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon} Y_t.$$

Government subsidy.

- Objective of j^{th} monopolist:

$$E_t \sum_{l=0}^{\infty} \beta^l \upsilon_{t+l}[(1+\nu)P_{j,t+l}Y_{j,t+l} - \overbrace{s_{t+l}P_{t+l}Y_{j,t+l}}^{\text{labor costs of production}}$$

Real marginal cost

$$- \overbrace{\frac{\phi}{2}\left(\frac{P_{j,t+l}}{P_{j,t+l-1}} - 1\right)^2 (C_{t+l} + \psi G_{t+l})}^{\text{cost (in terms of final goods) of adjusting prices related to aggregate level of output}} \times P_{t+l}],$$

The busier is the economy, the more costly it is to change prices.

Firms, cnt'd

- j^{th} intermediate good firms:

$$Y_{j,t} = \overbrace{h_{j,t}}^{\text{production function}}, s_t \equiv \overbrace{\frac{W_t}{P_t}}^{\text{real marginal cost}} = \overbrace{\chi h_t C_t}^{\text{household optimization}}$$

- Government sets subsidy rate to remove monopoly inefficiency in steady state:

$$1 + \nu = \frac{\varepsilon}{\varepsilon - 1}.$$

Intermediate good firms

- Substituting out for the demand curve and

$\upsilon_t = 1/(P_t C_t)$ the *j^th* monopolist's problem is:

$$\max_{\{P_{j,l}\}_{l=0}^{\infty}} E_t \sum_{l=0}^{\infty} \beta^l \frac{1}{P_{t+l}C_{t+l}} [(1+\nu)P_{j,t+l}\left(\frac{P_{j,t+l}}{P_{t+l}}\right)^{-\varepsilon} Y_{t+l}$$

$$- P_{t+l}s_{t+l}\left(\frac{P_{j,t+l}}{P_{t+l}}\right)^{-\varepsilon} Y_{t+l} - \frac{\phi}{2}\left(\frac{P_{j,t+l}}{P_{j,t+l-1}} - 1\right)^2 P_{t+l}(C_{t+l} + \psi G_{t+l})].$$

- With first order condition:

$$(1+\nu)\frac{P_{j,t}}{P_t} = \frac{\varepsilon}{\varepsilon - 1}s_t +$$

$$\phi\frac{1}{\varepsilon - 1}\left(\frac{P_{j,t}}{P_t}\right)^{\varepsilon}\frac{C_t}{Y_t}[-\left(\frac{P_{j,t}}{P_{j,t-1}} - 1\right)\frac{P_{j,t}}{P_{j,t-1}}\frac{(C_t + \psi G_t)}{C_t}$$

$$+ \beta E_t\left(\frac{P_{j,t+1}}{P_{j,t}} - 1\right)\frac{P_{j,t+1}}{P_{j,t}}\left(\frac{C_{t+1} + \psi G_{t+1}}{C_{t+1}}\right)]$$

Equilibrium Conditions

- Set $1+\nu = \frac{\varepsilon}{\varepsilon - 1}$
- In equilibrium, $P_{j,t} = P_{i,t} = P_t$

$$\frac{1}{R_t} = \frac{1}{1+r_t} E_t \frac{C_t}{\pi_{t+1} C_{t+1}}$$

$$(\pi_t - 1)\pi_t = \frac{1}{\phi}\varepsilon(s_t - 1)\frac{Y_t}{C_t + \psi G_t}$$

$$+ \frac{1}{1+r_t} E_t(\pi_{t+1} - 1)\pi_{t+1} \frac{(C_{t+1} + \psi G_{t+1})}{C_{t+1}} \frac{C_t}{C_t + \psi G_t}$$

$$C_t + G_t + \frac{\phi}{2}(\pi_t - 1)^2 (C_t + \psi G_t) = h_t$$

$$R_t = \max\left\{1, \frac{1}{\beta} + \alpha(\pi_t - 1)\right\}.$$

Later, will see that, after linearization around zero inflation steady state, equations coincide with linearized equations of model with Calvo price frictions.

Shocks

- EW Shock:
  - Discount rate takes on low value, $r^l<0$, in period 0 and jumps back to $r^h>0$ with constant probability, $1-p$.
  - r^h is an absorbing state.
- Sunspot: shock that has no impact on fundamentals.

Steady State

- Two of the equations:

'Fisher equation'

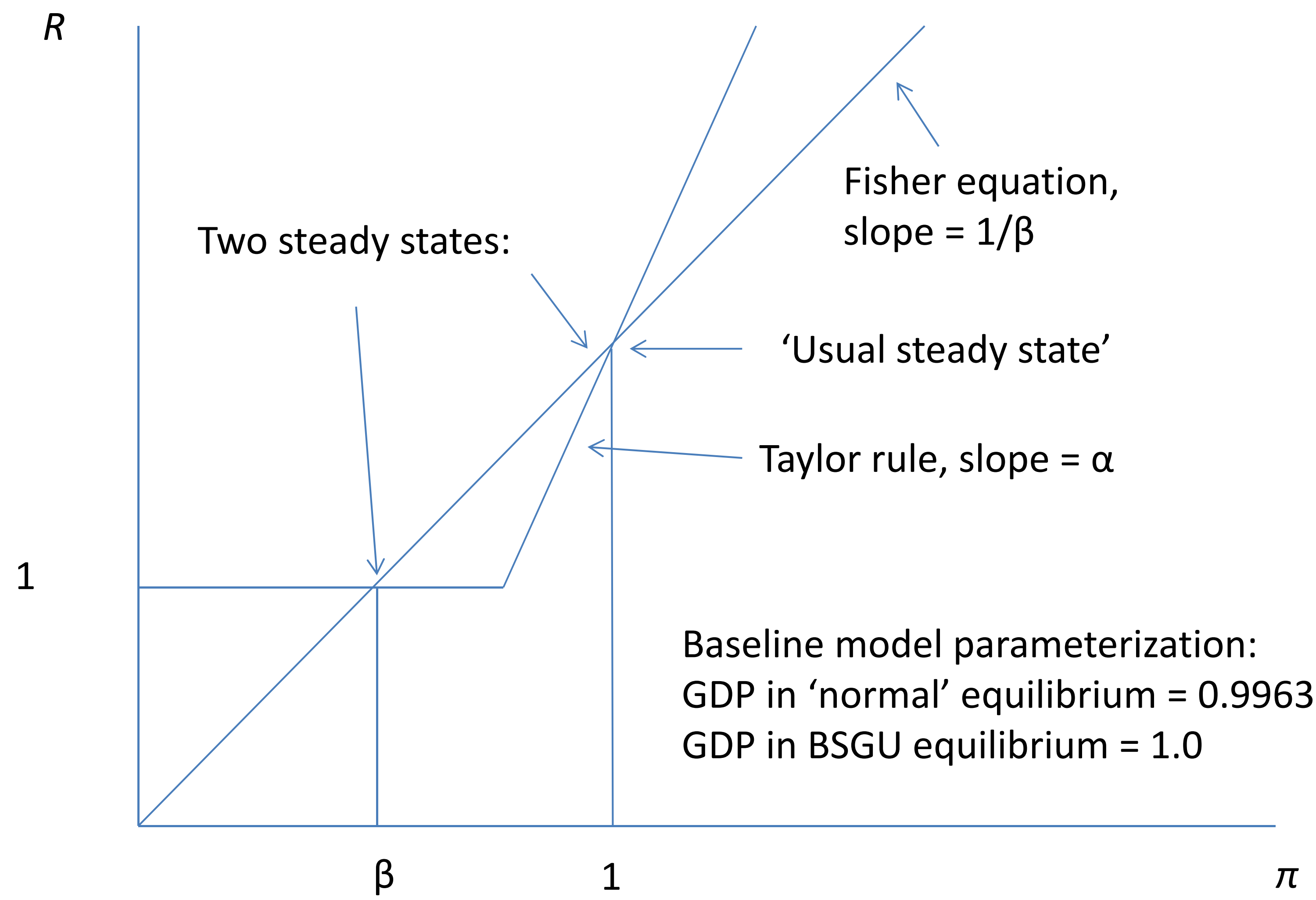
$$\frac{1}{R_t} = \frac{1}{1+r_t} E_t \frac{C_t}{\pi_{t+1} C_{t+1}} \underbrace{\rightarrow}_{C_t = C_{t+1} \text{ in ss}} R = (1+r)\pi = \frac{\pi}{\beta}$$

Taylor rule

$$R_t = \max\left\{1, \frac{1}{\beta} + \alpha(\pi_t - 1)\right\} \underbrace{\rightarrow}_{\text{in ss}} \max\left\{1, \frac{1}{\beta} + \alpha(\pi - 1)\right\}$$

- These equations have a simple graphical representation, with π on horizontal axis and *R* on vertical axis.

Figure 1: BSGU Demonstration of Two Steady States



We Take ‘Usual Steady State’

- One interpretation:
  - State-contingent Taylor rule (with appropriate fiscal policy) in which a long-term deflation is responded to by abandoning Taylor rule and going to money growth rule.
  - Analysis in BSGU and Christiano-Rostagno (NBER wp 2001).

Linearization About Steady State

- Linearized system:

$$\begin{aligned}
\hat{y}_t - \eta_g \hat{G}_t &= \hat{y}_{t+1} - \eta_g \hat{G}_{t+1} - (1-\eta_g)[\beta(R_t - 1 - r_t) - \hat{\pi}_{t+1}] \\
\hat{\pi}_t &= \beta\hat{\pi}_{t+1} + \kappa\left[\frac{2-\eta_g}{1-\eta_g}\hat{y}_t - \frac{\eta_g}{1-\eta_g}\hat{G}_t\right] \\
R_t &= \max\left\{1, \frac{1}{\beta} + \alpha(\pi_t - 1)\right\}, \\
\kappa &\equiv \frac{\varepsilon}{\phi}\frac{1}{1-\eta_g+\psi\eta_g}.
\end{aligned}$$

- EW equilibrium:
  - *G* is exogenous, system jumps to steady state right after zlb is over and has constant inflation, output in zlb.

Equilibrium In log-Linear System

- Substitute Phillips curve into IS curve and assume a binding zlb to get single difference equation in inflation:

$$\hat{\pi}_t - p[\beta + 1 + (2-\eta_g)\kappa]\hat{\pi}_{t+1} + \beta p^2 \hat{\pi}_{t+2} = \kappa\beta(2-\eta_g)r^l + \kappa\eta_g(1-p)\hat{G}.$$

- Solutions:
  - Complete set: $\pi_t = \hat{\pi}^l + a_1\lambda_1^t + a_2\lambda_2^t$, a_1 and a_2 arbitrary,

$$\hat{\pi}^l = \frac{\kappa\beta(2-\eta_g)r^l + \kappa\eta_g(1-p)\hat{G}^l}{\Delta}, \ \Delta \equiv 1 - p[\beta + 1 + (2-\eta_g)\kappa] + \beta p^2$$

  - Non-explosive solution is unique (and constant) if eigenvalues both greater than unity in abs value.
  - Necessary and sufficient condition for both eigenvalues to be greater than unity in abs value: $\Delta > 0$
  - Necessary condition for zlb to be binding in constant equilibrium with r^l<0 and $\hat{G}^l = 0$: $\Delta > 0$

Derivation of Previous Result

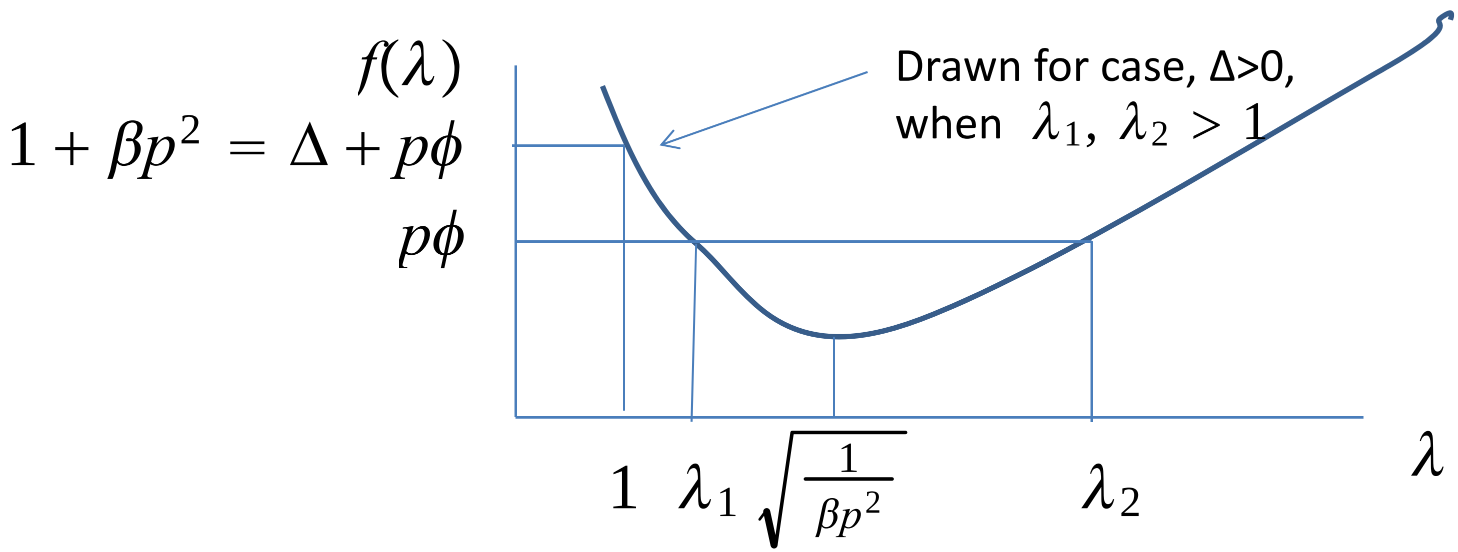
- We seek the zeros of the polynomial:

$$1 - \phi p \lambda + \beta p^2 \lambda^2 = 0, \text{ where } \phi \equiv \beta + 1 + (2 - \eta_g)\kappa$$

- Consider

$$f(\lambda) \equiv \frac{1}{\lambda} + \beta p^2 \lambda.$$

- Want: λ's that solve $f(\lambda) = p\varphi$



Closed form solution in linearized system

- Government consumption multiplier:

$$\frac{dy^l}{dG^l} = \frac{1-\eta_g}{2-\eta_g}\left[\frac{(1-\beta p)(1-p)}{\Delta} + \frac{1}{1-\eta_g}\right]$$

- Output/inflation drop: ($\Delta \equiv 1 - p[\beta + 1 + (2-\eta_g)\kappa] + \beta p^2$)

$$\hat{\pi}^l = \frac{\kappa\beta(2-\eta_g)r^l + \kappa\eta_g(1-p)\hat{G}^l}{\Delta}$$

$$\hat{y} = \frac{1-\eta_g}{2-\eta_g}\frac{1}{\kappa}\left[(1-\beta p)\hat{\pi}^l + \kappa\frac{\eta_g}{1-\eta_g}\hat{G}\right].$$

Nonlinear system

- In high state, at steady state.
- Low state:

$$\frac{1}{R^l} = \frac{1}{1+r^l}\left[p\frac{C^l}{\pi^l C^l} + (1-p)\frac{C^l}{\pi^h C^h}\right]$$

$$(\pi^l - 1)\pi^l = \frac{1}{\phi}\varepsilon(\chi h^l C^l - 1)\frac{h^l}{C^l + \psi G^l}$$

$$+ \frac{1}{1+r^l}\left[p(\pi^l - 1)\pi^l + (1-p)(\pi^h - 1)\pi^h\left(1 + \psi\frac{\eta_g}{1-\eta_g}\right)\frac{C^l}{C^l + \psi G^l}\right]$$

$$C^l + G^l + \frac{\phi}{2}(\pi^l - 1)^2(C^l + \psi G^l) = h^l$$

$$R^l = \max\left\{1, \frac{1}{\beta} + \alpha(\pi^l - 1)\right\}.$$

- Reduces to one equation in one unknown, π^l,

$$f(\pi^l) = 0$$

Baseline parameter values

$$\kappa = 0.03,\ \beta = 0.99,\ \alpha = 1.5,\ p = 0.775,$$
$$r^l = -0.02/4,\ \phi = 100,\ \psi = 1,\ \eta_g = 0.2, \psi = 1.$$

Figure 2: EW Equilibria

Interval of candidate EW equilibrium inflation rates: [0.78,2.27]. There are no other zeros.

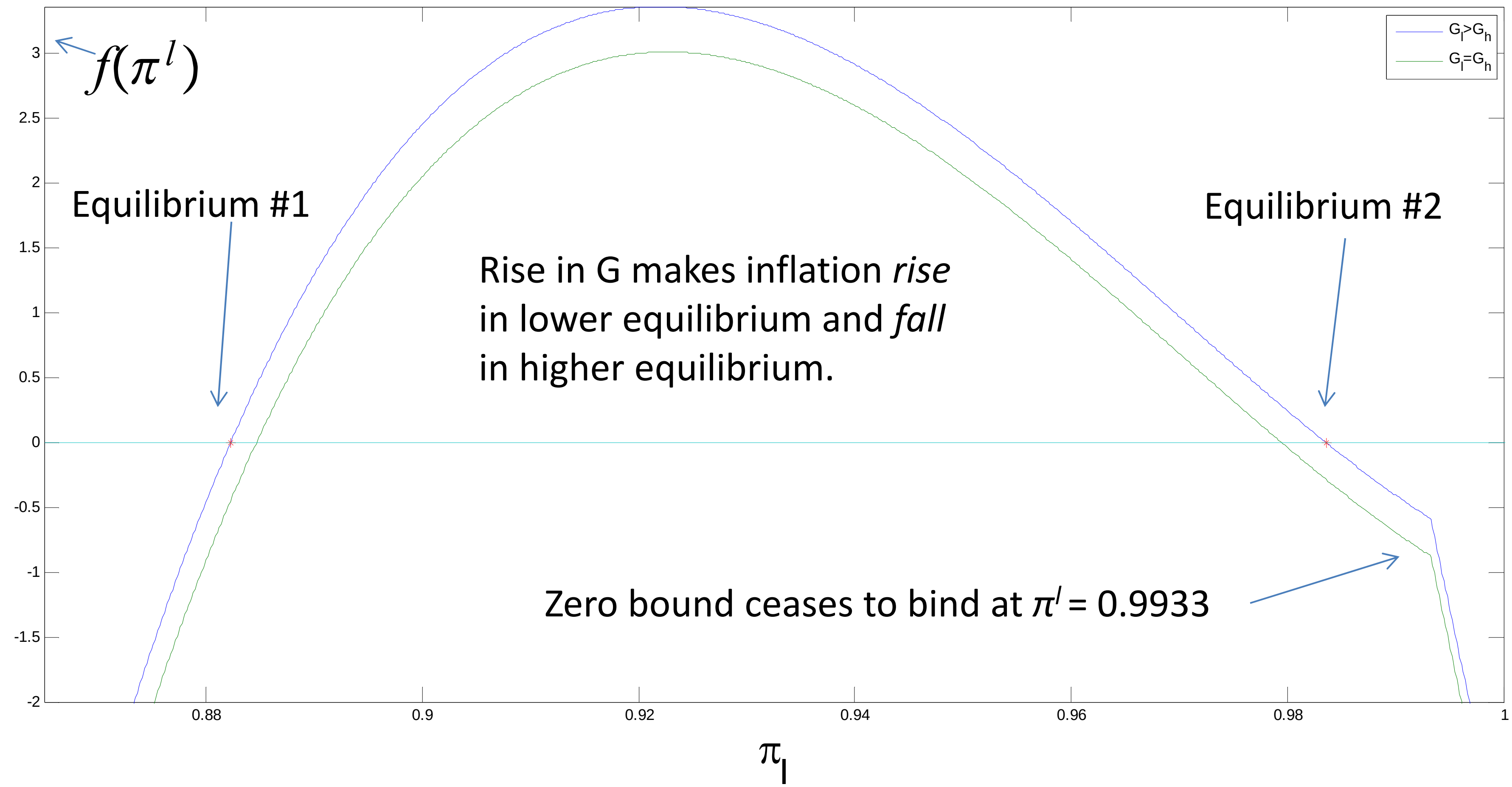


Table 1: Properties of EW Equilibrium for Three Parameterizations			
Panel A: Baseline parameterization			
	equilibrium #1	equilibrium #2	log-linear
$\frac{d\text{GDP}}{dG}$	0.16	2.18	2.77
% drop in GDP	37.55	5.38	5.99
change in inflation rate	-11.77	-1.64	-1.90

Equilibrium #2 has properties that resemble the ones implied by the log-linear approximation.

But, equilibrium #1 is completely different!

Table 1: Properties of EW Equilibrium for Three Parameterizations			
Panel A: Baseline parameterization			
	equilibrium #1	equilibrium #2	log-linear
$\frac{d\text{GDP}}{dG}$	0.16	2.18	2.77
% drop in GDP	37.55	5.38	5.99
change in inflation rate	-11.77	-1.64	-1.90

Linear analysis pretty good indicator of what happens in equilibrium #2.
Pretty bad when it comes to equilibrium #1.

Let's see what the implications of the nonlinear analysis for (i) an increase in the expected duration of the zlb and (ii) increased flexibility of prices.

Figure 3: Longer Expected Duration

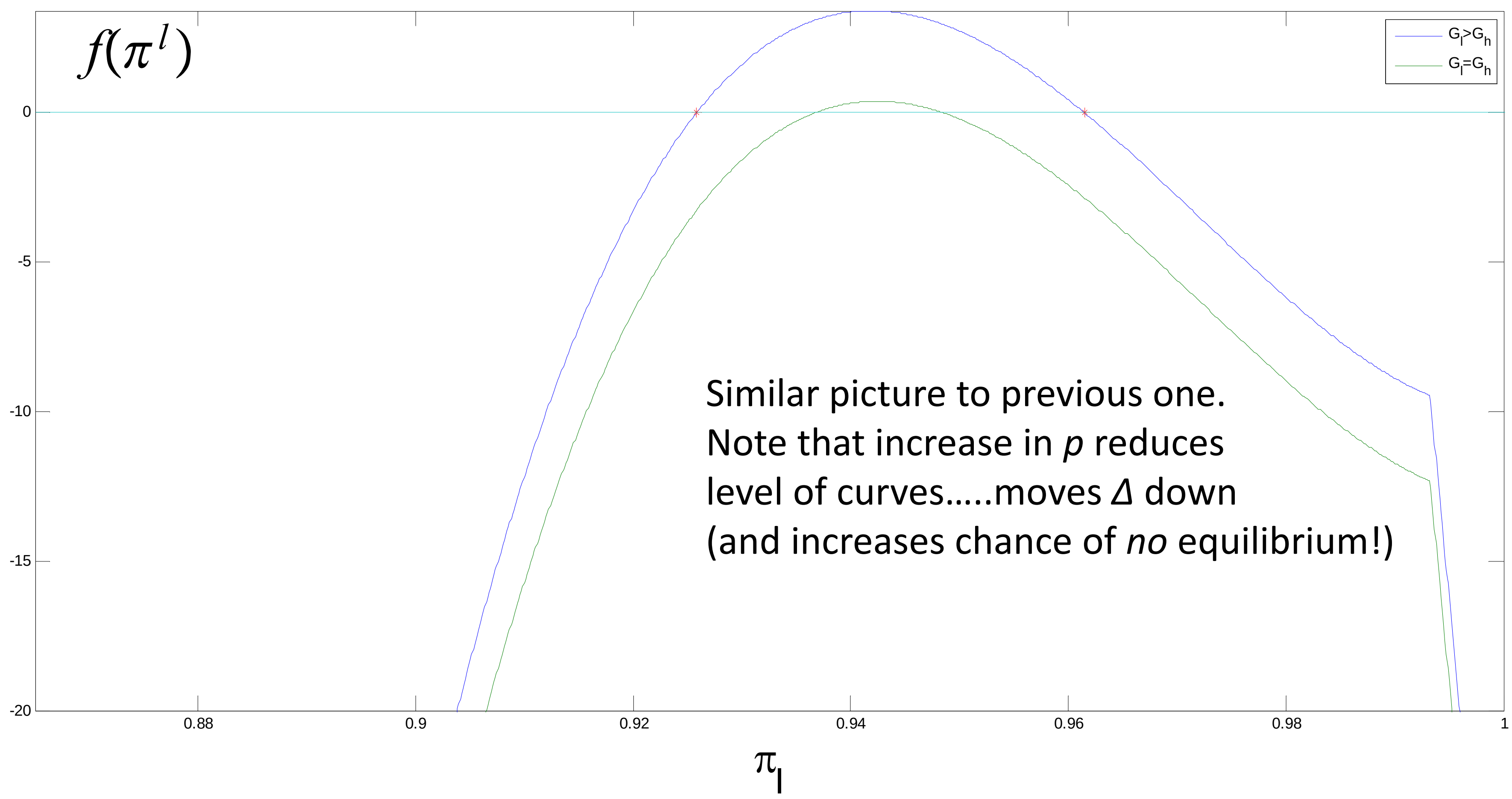


Table 1: Properties of EW Equilibrium for Three Parameterizations			
Panel A: Baseline parameterization			
	equilibrium #1	equilibrium #2	log-linear
$\frac{dGDP}{dG}$	0.16	2.18	2.77
% drop in GDP	37.55	5.38	5.99
change in inflation rate	-11.77	-1.64	-1.90
Panel B: Increase in p from 0.775 to 0.793 (longer expected duration of lower bound)			
$\frac{dGDP}{dG}$	-2.84	5.36	12.41
% drop in GDP	25.49	13.21	38.60
change in inflation rate	-7.42	-3.85	-12.30

Equilibrium #2 and log-linear approximation both imply that small increase in expected duration of lower bound make severity of lower bound worse and increase multiplier.

However, the quantities implied by the log-linear approximation are much bigger than they are in equilibrium #2.

The predictions of the log-linear approximation become more Extreme as Δ gets smaller.

Table 1: Properties of EW Equilibrium for Three Parameterizations			
Panel A: Baseline parameterization			
	equilibrium #1	equilibrium #2	log-linear
$\frac{d\text{GDP}}{dG}$	0.16	2.18	2.77
% drop in GDP	37.55	5.38	5.99
change in inflation rate	-11.77	-1.64	-1.90
Δ			0.0105
Panel B: Increase in p from 0.775 to 0.793 (longer expected duration of lower bound)			
$\frac{d\text{GDP}}{dG}$	-2.84	5.36	12.41
% drop in GDP	25.49	13.21	38.60
change in inflation rate	-7.42	-3.85	-12.30
Δ			0.00167

Δ dropped an order of magnitude, from 0.0105 to 0.00167

Figure 4: More Flexible Prices

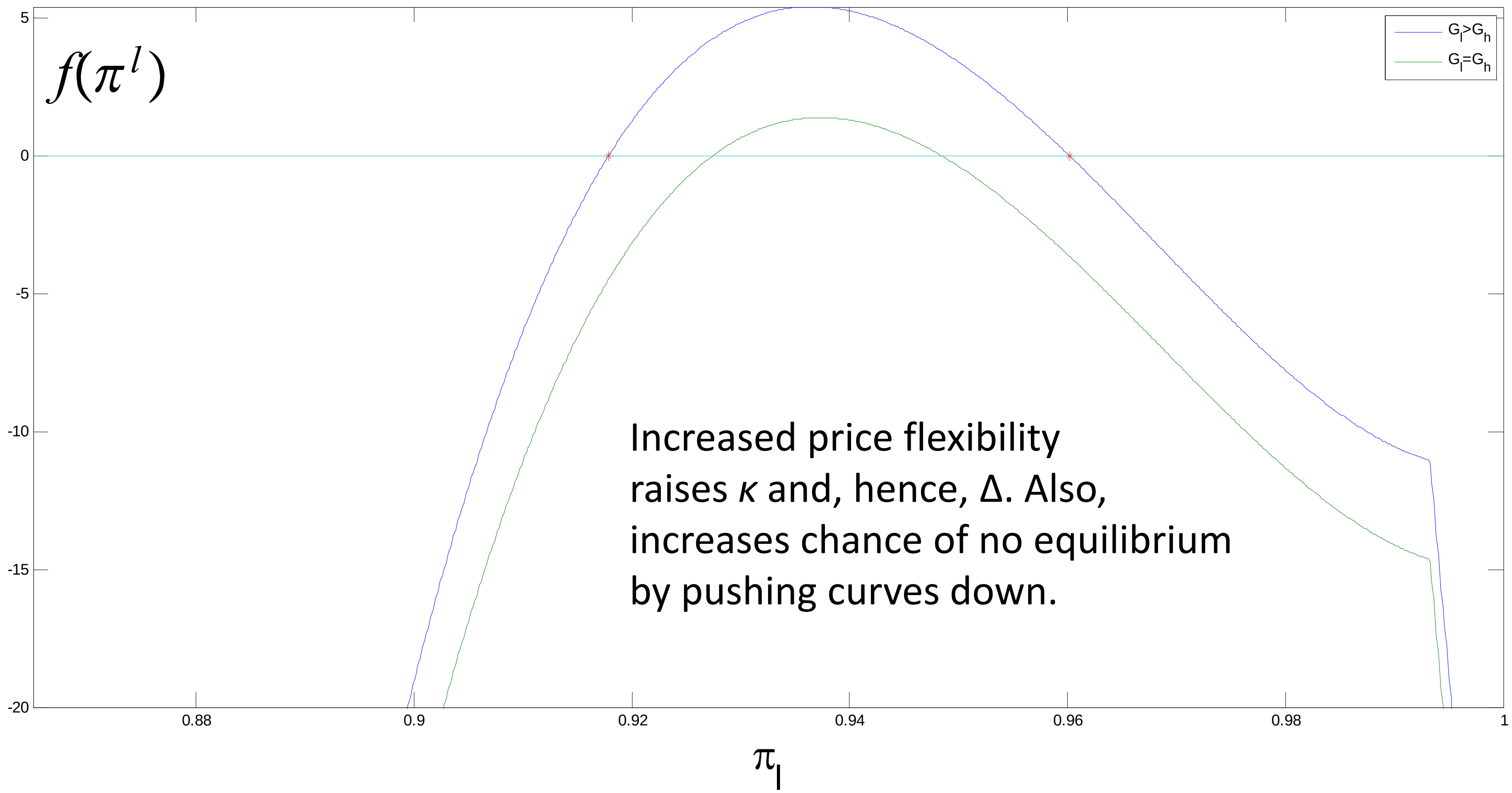


Table 1: Properties of EW Equilibrium for Three Parameterizations			
Panel A: Baseline parameterization			
	equilibrium #1	equilibrium #2	log-linear
$\frac{d\text{GDP}}{dG}$	0.16	2.18	2.77
% drop in GDP	37.55	5.38	5.99
change in inflation rate	-11.77	-1.64	-1.90
Δ			0.0105

Equilibrium #2 and log-linear approximation both imply that small increase in price flexibility make severity of lower bound worse and increase multiplier.

However, the quantities implied by the log-linear approximation are much bigger than they are in equilibrium #2.

The predictions of the log-linear approximation become more extreme as Δ gets smaller.

Panel C: Increase in κ from 0.03 to 0.0375 (more flexibility in prices)			
$\frac{d\text{GDP}}{dG}$	-2.08	4.54	414.3
% drop in GDP	25.44	12.20	1224.2
change in inflation rate	-8.22	-3.98	-444
Δ			0.000056

Table 1: Properties of EW Equilibrium for Three Parameterizations

Panel A: Baseline parameterization

	equilibrium #1	equilibrium #2	log-linear
$\frac{d\text{GDP}}{dG}$	0.16	2.18	2.77
% drop in GDP	37.55	5.38	5.99
change in inflation rate	-11.77	-1.64	-1.90
Δ			0.0105

Δ dropped three orders of magnitude, from 0.0105 to 0.000056

Panel C: Increase in κ from 0.03 to 0.0375 (more flexibility in prices)

$\frac{d\text{GDP}}{dG}$	-2.08	4.54	414.3
% drop in GDP	25.44	12.20	1224.2
change in inflation rate	-8.22	-3.98	-444
Δ			0.000056

Sunspot Equilibrium

- In the previous equilibria,
  - $r_t = r^l$ in the first period and remains there with probability, p.
  - $r_t = r^h$ with probability $1-p$ and r^h is an absorbing state.
  - $r^l < 0$ and $r^h = 1/\beta - 1$.

- In sunspot equilibrium
  - Uncertainty does not affect any fundamental variable.
  - Here: $r^l = r^h = 1/\beta - 1$

Figure 5: Sunspot Equilibrium

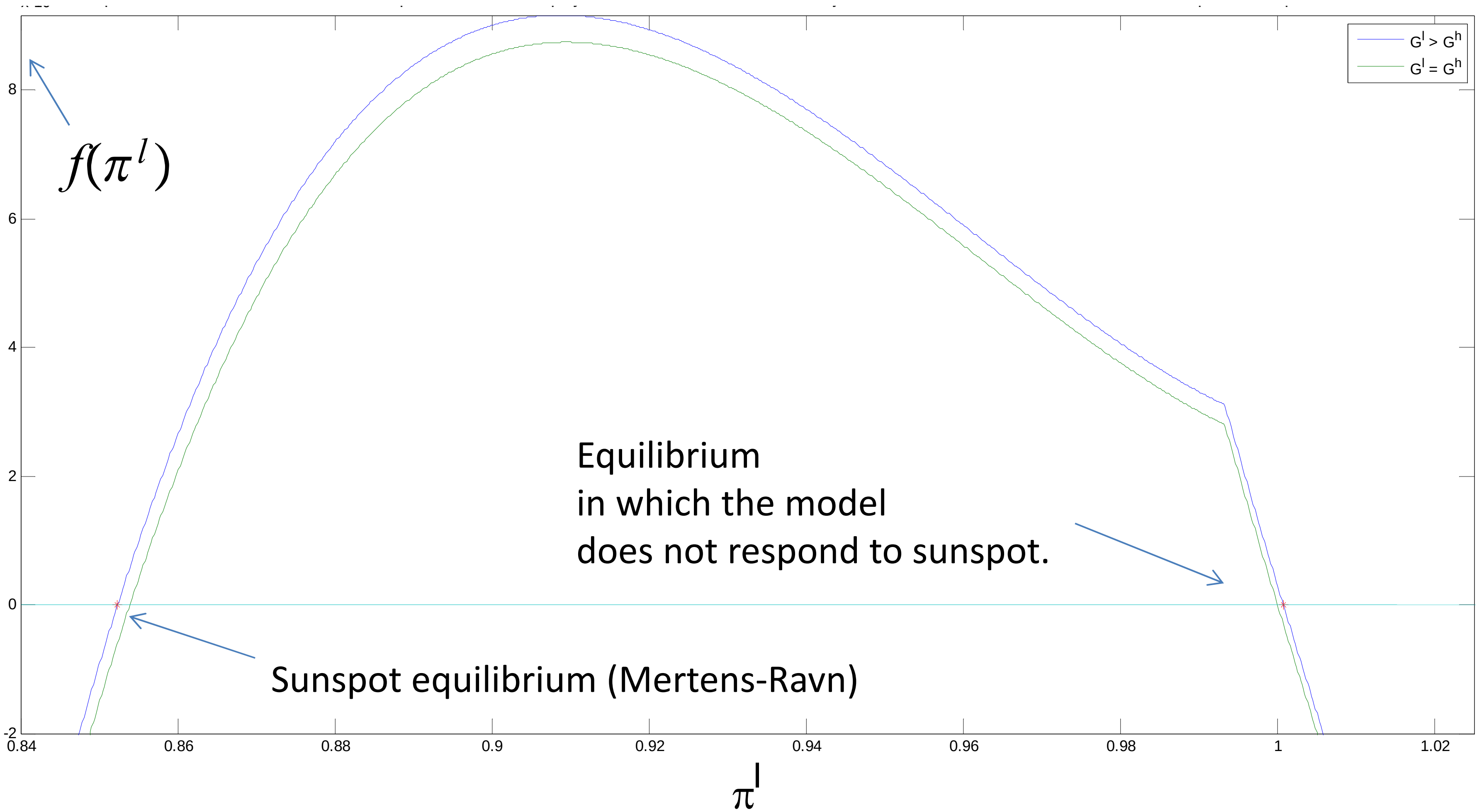


Table 2: Properties of Sunspot Equilibrium		
Baseline parameters, except $r^l = 1/\beta - 1$		
	equilibrium #1	equilibrium #2
$\frac{d\text{GDP}}{dG}$	0.41	0.81
% drop in GDP	43.1	-0.81
change in inflation rate	-14.76	0.07

Equilibrium #2 corresponds to the steady state equilibrium with the jump in G.
Note that GDP is higher with the rise in G (i.e., the negative 'drop'). Multiplier is 0.81.
This is just the multiplier in 'normal times'.

Equilibrium #1 is a sunspot equilibrium… GDP drops because of 'bad' expectations:
(i) people fear deflation and expect the zlb to bind, (ii) this means they expect a high real rate,
(iii) they cut back on spending, (iv) output drops, (v) marginal costs and, hence, inflation falls.

In equilibrium #1, economy driven into zlb by bad expectations shock. Note multiplier is less than what it is in 'normal times'.

Suggests size of multiplier depends on what shock drives economy into zlb. Sharp Contradiction to conclusions of log-linear analysis.

In Sum

- Log-linear approximation and equilibrium #2 provide a similar qualitative message.
  - However, the extreme numbers that come from the log-linear approximation appear (not surprisingly) to be spurious.

- The economy is characterized by a second equilibrium (equilibrium #1) and the properties of this are qualitatively very different.
  - The severity of the zlb is immensely greater.
  - zlb could be triggered by self-fulfilling expectation shock (sunspot).
  - The government spending multiplier is very small.

- With multiple equilibria: hard to know what the multiplier means.
  - One set of equilibria with G small, another with G large.
  - Unless we know which equilibrium the economy will go to, we have many ways to compute dY/dG without knowing which is the one that will occur.

In Sum

- Without having some sort of way to select among the equilibria, not clear what the model says about the zlb.
  - Unless we can think of a way of ‘selecting’ equilibrium #2, much of what we have ‘learned’ based on the log-linear approximation appears to be *spurious*.

- Literature on learning and E-stability provides one way to address the problem of equilibrium selection.

E-stability: Example

- Consider a very simple, static model.
- Lots of identical households:

$$\max_{c,l} c - \frac{1}{2} l^2$$

$$\text{subject to} \quad \overbrace{c}^{\text{consumption}} \leq \overbrace{w}^{\text{wage rate, pinned down by linear technology}} \times \overbrace{(1-\tau)}^{\tau \text{ is labor tax rate}} \times \overbrace{l}^{\text{labor}}$$

- Solution:

$$l = (1-\tau)w$$

$$c = (1-\tau)^2 w^2$$

- Gov't revenues:

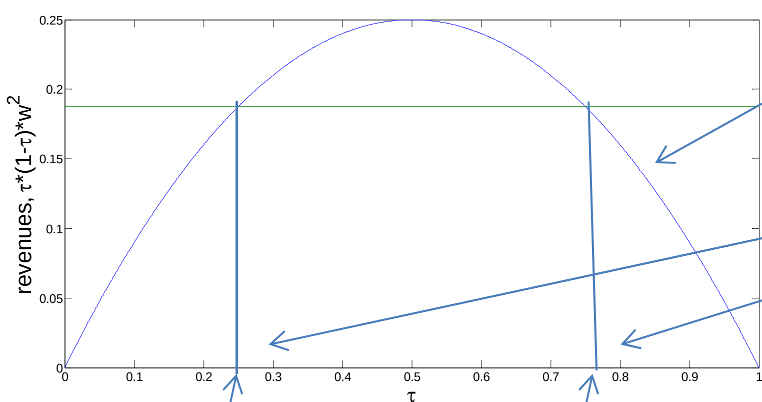
'Laffer curve'

$$\tau w l = \tau(1-\tau)w^2$$

Gov’t, Rational Expectations

- Choose τ to satisfy exogenous spending requirement, $g \leq \tau(1-\tau)w^2$. (Let $g=3/16$, $w=1$.)
- Timing:
  - private agents choose work level given a belief about what tax rate will be levied.
  - After work is done, government selects τ

If people believe $\tau=1/4$, they work hard and the government can pay for g with low tax. If people believe $\tau=3/4$, they will not work hard, and gov’t must set high tax.



Laffer curve

Two rational expectations equilibria.

1/4

3/4

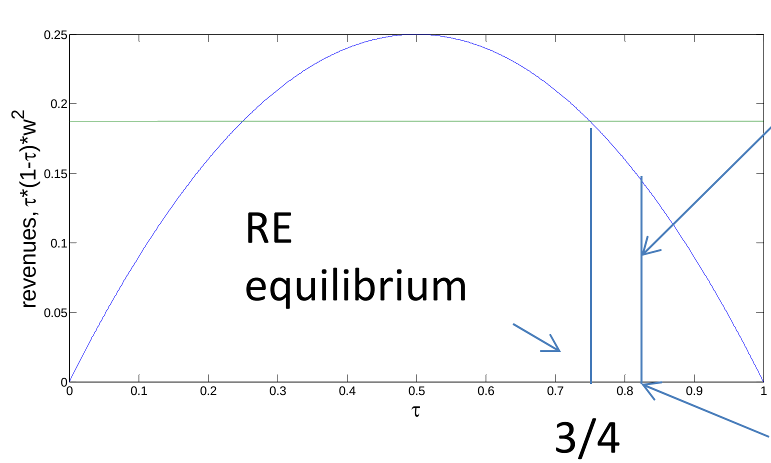
E-Stable

- In a rational expectations equilibrium, people conjecture precisely the 'correct' tax rate, either $\tau=1/4$ or $\tau=3/4$.
  - What if they make a tiny error?
  - Suppose they guess $\tau=0.7500000000000000000000001$?
    - Surely, equilibrium is just a mathematical curiosum if it falls apart because of this tiny mistake.
    - The $\tau=3/4$ equilibrium is ***E-stable*** if people fix their mistaken belief and the equilibrium goes to $\tau=3/4$.
- To fully define E-stability, must take a stand on
  - what happens when people have mistaken beliefs
  - how people learn from their mistakes.

E-stability, cnt'd

- What happens if they have non-rational expectations beliefs?
  - In the model, people act on their belief, τ^e, by choosing a labor supply, $l=(1-\tau^e)w$.
  - The government must now choose an actual tax rate, τ:
    - $\tau = g/(lw) = g/[(1-\tau^e)w^2]$

Suppose τ^e is a little higher than RE ($\tau=3/4$). If gov't validates τ^e, brings in too little revenue. On right side of the Laffer curve, must set actual τ *higher* than τ^e.



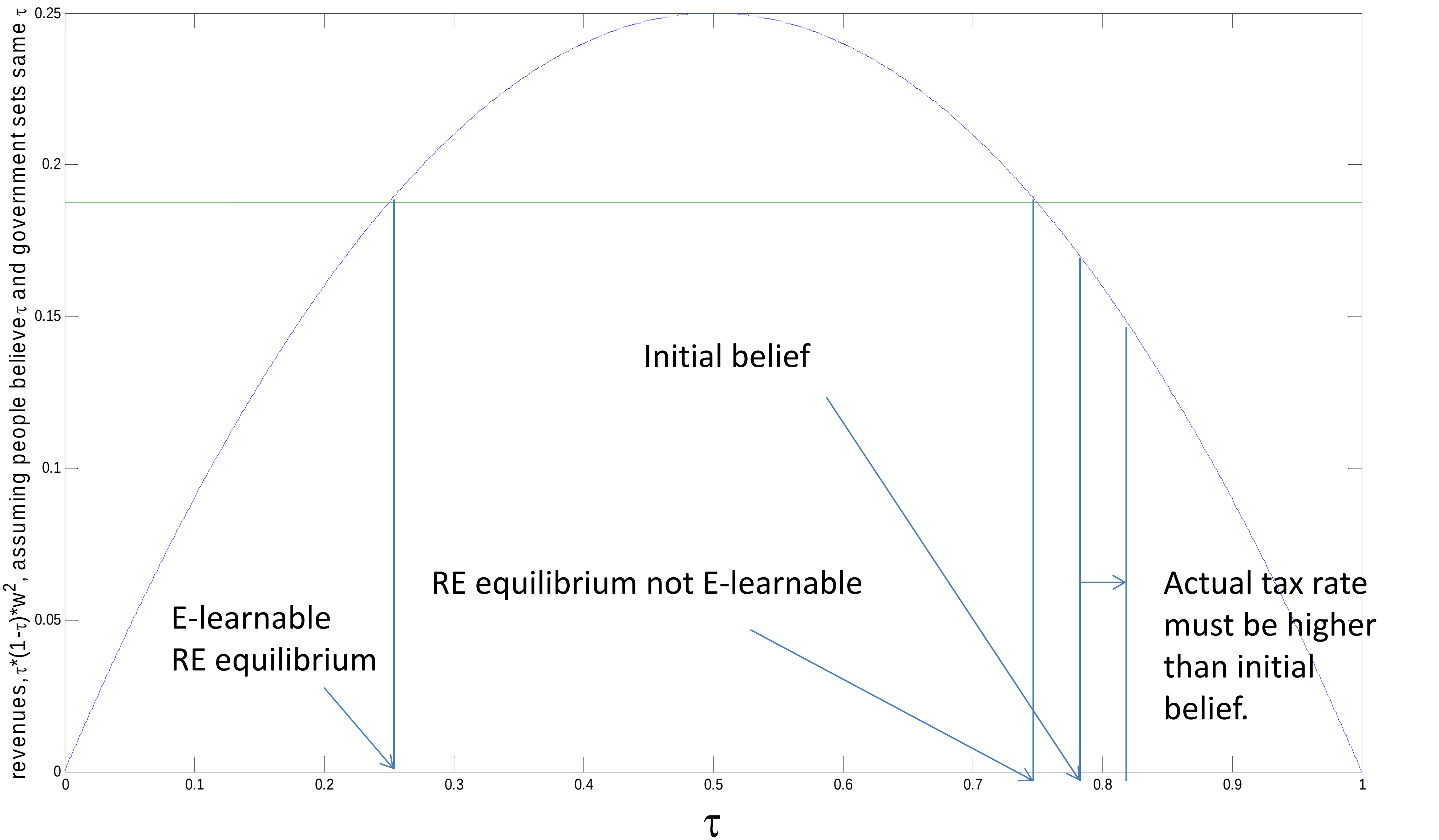
non-RE belief, τ^e

E-stability, cnt'd

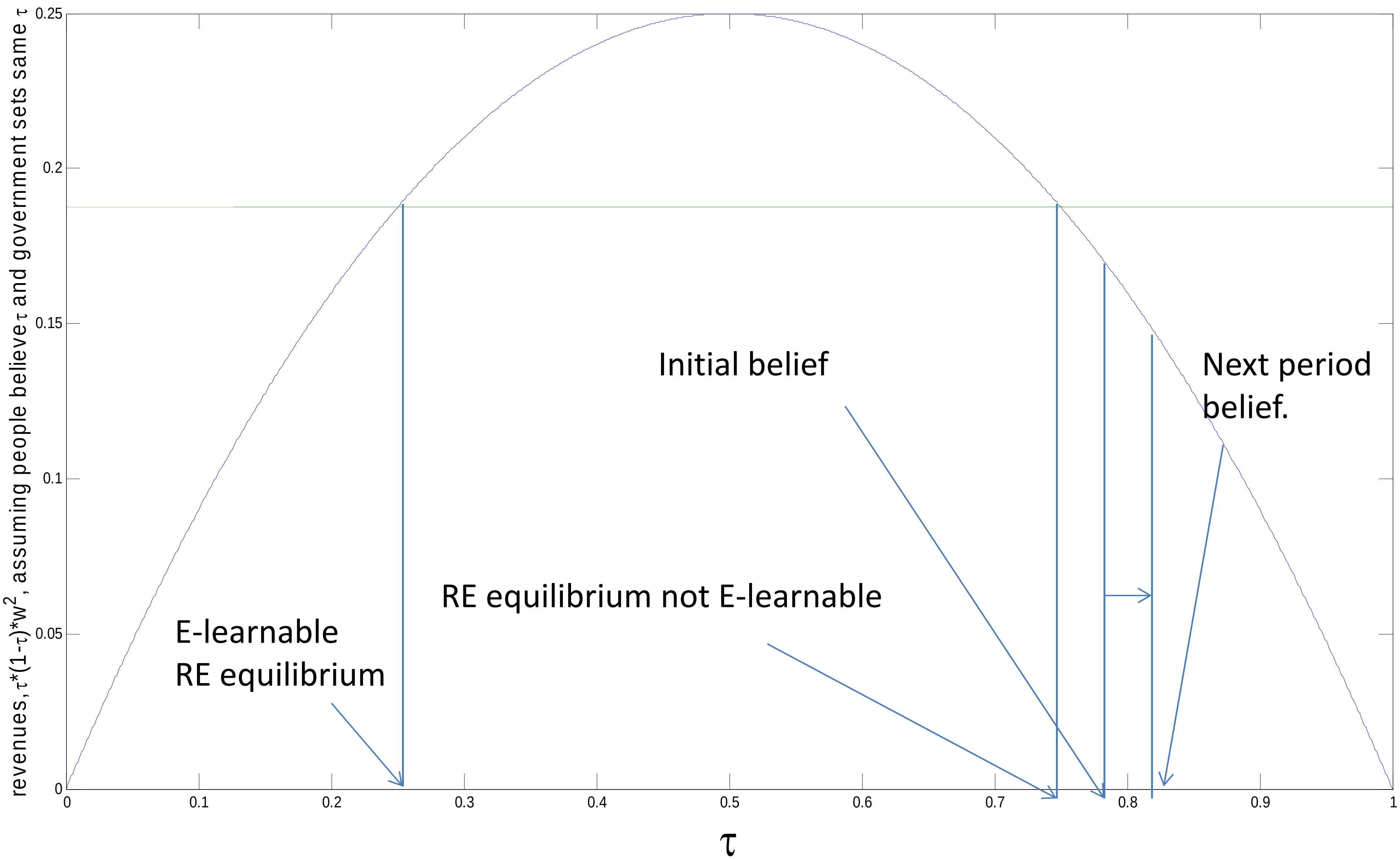
- Must now take a stand on learning.
  - What do people see and how do they use the information?
  - Suppose the economy repeats itself period after period.
  - Suppose the only aggregate variable people observe is the tax rate and so they form their belief about it on the previous observation(s).

$$\overbrace{\tau_t^e}^{\text{belief about tax rate formed at start of } t \text{, before choosing work effort}} = \begin{cases} \text{exogenous} & \text{t=initial period} \\ \overbrace{\tau_{t-1}}^{\text{previous actual tax rate}} & \text{t>initial period} \end{cases}$$

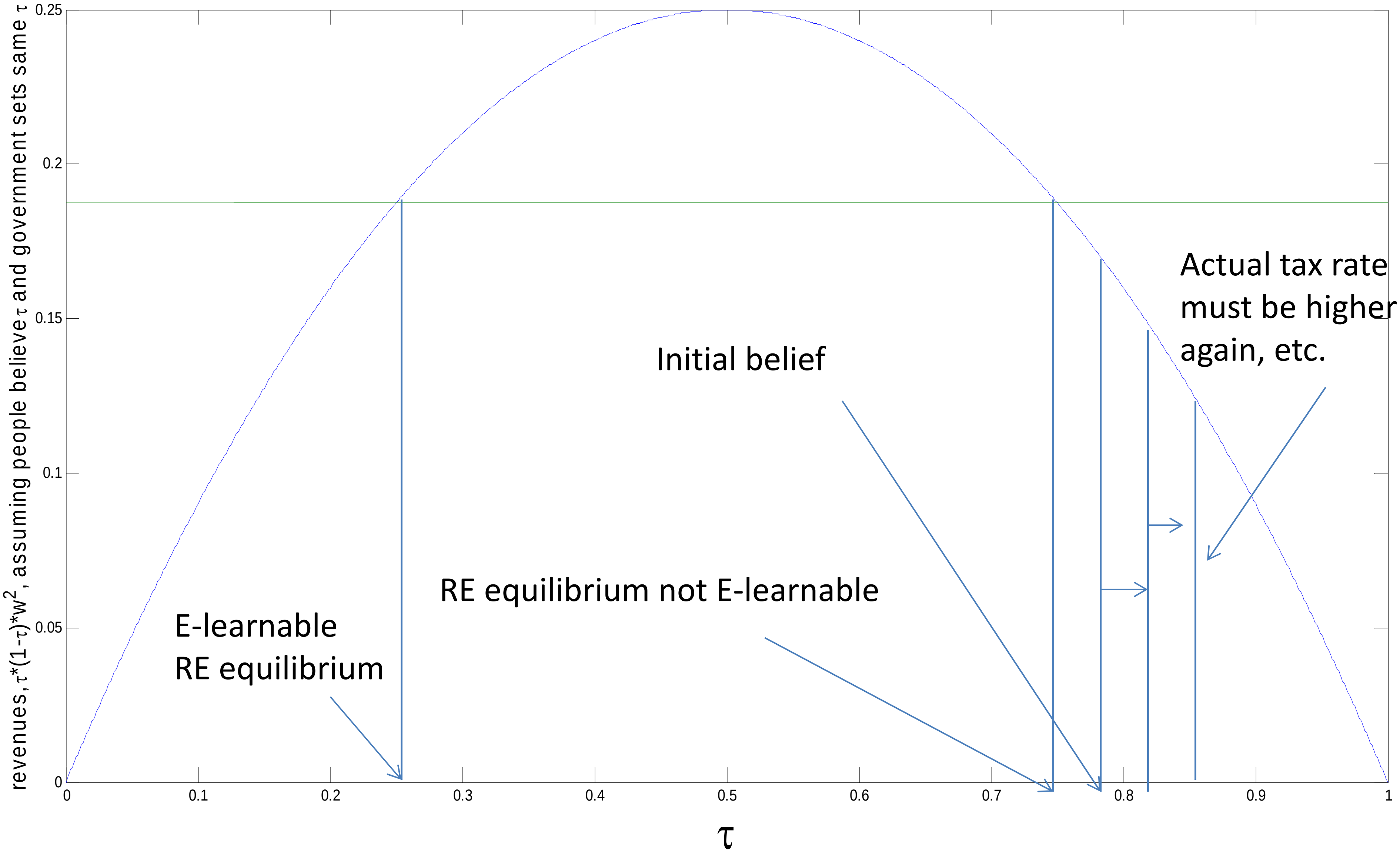
E-stability, cnt'd



E-stability, cnt'd



E-stability, cnt'd



E-stability, cnt'd

- Previous diagram: if people believe the tax rate will be higher than the RE tax on the right of the Laffer curve, learning mechanism will drive it further to the right.

- If people believe the tax rate will be a little lower than the RE tax on the right of the Laffer curve, learning mechanism will drive it further to the left.

  – The RE equilibrium on the right side of the Laffer curve is not E-learnable.

- Easy to verify: RE equilibrium on the left side of the Laffer curve is E-learnable.

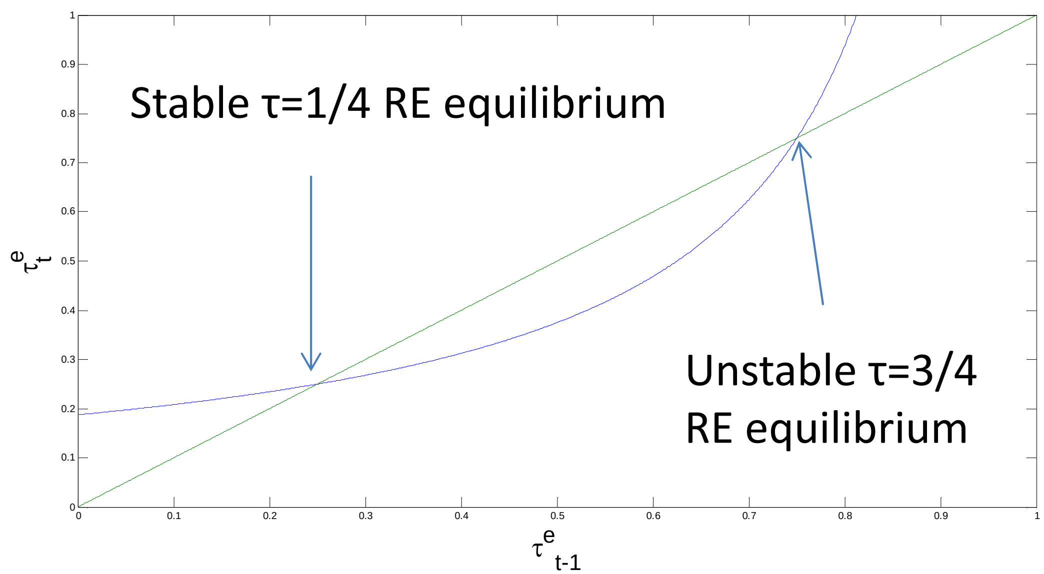
- View:
  – RE equilibrium that is not E-learnable is simply a mathematical curiosity and can be ignored.
  – Evans and Honkapohja textbook important source.

Math of Previous Example

- Law of motion of beliefs:

$$\overbrace{\tau_t^e - \tau_{t-1}^e}^{\text{revision in belief in response to error}} = \overbrace{\overbrace{\frac{g}{(1-\tau_{t-1}^e)w^2}}^{\text{actual } \tau_t} - \tau_{t-1}^e}^{\text{expectation error observed by agents at end of } t}$$

- With *g=3/16, w=1*:



Can verify: nothing changes if instead revision in belief = λ(expectation error) for *0<λ<1*.

Trivial Example

- Dynamic system: pencil and flat surface (table top).
- Two equilibria: pencil laying on its side and standing on its point.
- Although we know theoretically that ‘standing on its point’ is an equilibrium for the pencil, in practice cannot find that equilibrium because with even the tiniest error in positioning the pencil, it diverges.
  - The ‘standing on its point’ equilibrium seems to be a theoretical curiosity, devoid of practical interest.
  - In thinking about where the dynamical system will go, the ‘laying on its side equilibrium’ is the only one of interest.
- Logic for using lack of E-stability as a basis for ignoring a particular equilibrium is similar, but…
  - Caveat: plausibility of E-stability depends on learning rule used.
  - Practical considerations drive analysts to use (implausibly?) simple learning rules.

Back to the Case of Interest: the ZLB

- Assume that rational expectations of variables outside zlb are known.
  - Perhaps ok, since these are the values of variables in 'normal' times, something agents are familiar with.

- Assume agents learn about the values of variables in the zlb.
  - Makes sense, given assumption in the model that zlb is a total surprise.

- Expectations in zlb appear in: intermediate good price setting decision and household intertemporal Euler equation.

- Price setting:
  - Intermediate good firms must know current *P* and *Y* when they set their price.
    - But, P and Y are determined by the prices that firms set!
    - Obviously, they must have a belief about what these prices will be.
  - Must also have beliefs about future variables too, conditional on remaining in zlb.

- Households must learn future variables (conditional on remaining in zlb).

Back to the Case of Interest: the ZLB

- Price setting:

belief

$$(1+v)\frac{P_{j,t}}{P_t^e} = \frac{\varepsilon}{\varepsilon-1}s_t^e +$$

$$\phi\frac{1}{\varepsilon-1}\left(\frac{P_{j,t}}{P_t^e}\right)^{\varepsilon}\frac{C_t^e}{Y_t^e}[-\left(\frac{P_{j,t}}{P_{j,t-1}}-1\right)\frac{P_{j,t}}{P_{j,t-1}}\frac{(C_t^e+\psi G_t^e)}{C_t^e}$$

$$+\frac{1}{1+r^l}\{p\times\overbrace{\left(\left(\frac{P_{j,t+1}}{P_{j,t}}\right)^{e}-1\right)\left(\frac{P_{j,t+1}}{P_{j,t}}\right)^{e}\left(\frac{C_{t+1}^e+\psi G_{t+1}}{C_{t+1}^e}\right)}^{\text{assuming next period is a zlb period}}$$

$$+(1-p)\times\overbrace{\left(\frac{P_{j,t+1}}{P_{j,t}}-1\right)\frac{P_{j,t+1}}{P_{j,t}}\left(\frac{C_{t+1}^e+\psi G_{t+1}}{C_{t+1}^e}\right)}^{\text{assuming next period is not a zlb}}\}]$$

Equilibrium Prices

Marginal cost

Zero Inflation out of zlb.

- Pricing:

$$(1+v)\frac{P_{j,t}}{P_t^e} = \frac{\varepsilon}{\varepsilon-1}\chi h_t^e C_t^e + \phi\frac{1}{\varepsilon-1}\left(\frac{P_{j,t}}{P_t^e}\right)^{\varepsilon}\frac{C_t^e}{Y_t^e}[-\left(\frac{P_{j,t}}{P_{j,t-1}}-1\right)\frac{P_{j,t}}{P_{j,t-1}}\frac{(C_t^e+\psi G_t^e)}{C_t^e} + \frac{p}{1+r^l}\left(\left(\frac{P_{j,t+1}}{P_{j,t}}\right)^e-1\right)\left(\frac{P_{j,t+1}}{P_{j,t}}\right)^e\left(\frac{C_{t+1}^e+\psi G_{t+1}}{C_{t+1}^e}\right)]$$

- Impose:

expected inflation in zlb: $\pi_t^e \equiv \frac{P_t^e}{P_{t-1}}$; actual inflation in zlb: $\pi_t^l \equiv \frac{P_t}{P_{t-1}}$, $P_t = P_{j,t}$, $j \in [0,1]$

firms expect to set prices the same way in the next period, if still in zlb: $\left(\frac{P_{j,t+1}}{P_{j,t}}\right)^e = \frac{P_{j,t}}{P_{j,t-1}} = \pi_t^e$

beliefs: $\pi_t^e = \pi_{t-1}^l$, $C_t^e = C_{t+1}^e = C_{t-1}^l$

- Then,

Expected gross output, Y_t

$$\phi(\pi_t^l-1)\pi_{t-1}^l\left[1-\frac{1}{1+r^l}p\right]$$

$$= \left[(1+v)(1-\varepsilon)\left(\frac{\pi_t^l}{\pi_{t-1}^l}\right)^{-\varepsilon} + \chi h_{t-1}^l C_{t-1}^l \varepsilon\left(\frac{\pi_t^l}{\pi_{t-1}^l}\right)^{-\varepsilon-1}\right]\frac{h_{t-1}^l}{C_{t-1}^l+\psi G^l}.$$

Equilibrium Conditions Under Learning in ZLB

$$\phi(\pi_t^l - 1)\pi_{t-1}^l\left[1 - \frac{1}{1+r^l}p\right]$$

$$= \left[(1+\nu)(1-\varepsilon)\left(\frac{\pi_t^l}{\pi_{t-1}^l}\right)^{-\varepsilon} + \chi h_{t-1}^l C_{t-1}^l \varepsilon\left(\frac{\pi_t^l}{\pi_{t-1}^l}\right)^{-\varepsilon-1}\right]\frac{h_{t-1}^l}{C_{t-1}^l + \psi G^l}.$$

$$1 = \frac{1}{1+r^l}\left[p\frac{1}{\pi_t^l} + (1-p)\frac{C_t^l}{\pi^h C^h}\right]$$

$$C_t^l + G^l + \frac{\phi}{2}(\pi_t^l - 1)^2(C_t^l + \psi G^l) = h_t^l$$

$$\frac{1}{\beta} + \alpha(\pi_t^l - 1) \leq 1 \text{ (condition that zlb binds)},$$

Equilibrium Learning in ZLB

- Let

$$z_t = \begin{pmatrix} C_t^l \\ h_t^l \\ \pi_t^l \end{pmatrix}, \; z_{t-1} = \begin{pmatrix} C_{t-1}^l \\ h_{t-1}^l \\ \pi_{t-1}^l \end{pmatrix}$$

- For given z_{t-1}, the zlb equilibrium conditions can be used to define the single-valued function, *f*:

$$z_t = f(z_{t-1})$$

- Easy to verify: $z^*=f(z^*)$, where z^* is rational expectations zlb equilibrium (i.e., equil. #1, #2).

Dynamics of Learning Mechanism Near ZLB Rational Expectations Equilibrium

- Linearize:

$$z_t = z^* + F \times (z_{t-1} - z^*), t = 0,1,2,\ldots,\ F = \left[\frac{df_i(z)}{dz_j} \right]|_{z=z^*},\ i,j = 1,2,3,$$

- Eigenvalue-eigenvector decomposition:

$$F = P\Lambda P^{-1},\ \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix},\ P^{-1} = \begin{bmatrix} \tilde{P}_1 \\ \tilde{P}_2 \\ \tilde{P}_3 \end{bmatrix}.$$

- Then,

$$\tilde{P}_i(z_t - z^*) = \lambda_i \tilde{P}_i(z_{t-1} - z^*),\ i = 1,2,3.$$

- Conclude: if one eigenvalue is explosive, then system is not stable under learning.

Results

- Determine if an equilibrium is E-stable (stable under learning) by examining eigenvalue of learning system at that equilibrium.
- This is done in the next few tables.
- Results: Only equilibrium #2 is stable under learning.

Table 1: Properties of EW Equilibrium for Three Parameterizations

Panel A: Baseline parameterization			
	equilibrium #1	equilibrium #2	log-linear
$\frac{d\text{GDP}}{dG}$	0.16	2.18	2.77
% drop in GDP	37.55	5.38	5.99
change in inflation rate	-11.77	-1.64	-1.90
eigenvalues in learning law of motion	$2.28, 1.4 \times 10^{-6}$, 1.9×10^{-3}	$0.71, -2.5 \times 10^{-10}$, -1.1×10^{-4}	
Δ			0.0105
Panel B: Increase in p from 0.775 to 0.793 (longer expected duration of lower bound)			
$\frac{d\text{GDP}}{dG}$	-2.84	5.36	12.41
% drop in GDP	25.49	13.21	38.60
change in inflation rate	-7.42	-3.85	-12.30
eigenvalues in learning law of motion	1.27, -8.7×10^{-7} -9.3×10^{-5}	$0.85, 2.65 \times 10^{-10}$ -1.94×10^{-4}	
Δ			0.00167
Panel C: Increase in κ from 0.03 to 0.0375 (more flexibility in prices)			
$\frac{d\text{GDP}}{dG}$	-2.08	4.54	414.3
% drop in GDP	25.44	12.20	1224.2
change in inflation rate	-8.22	-3.98	-444
eigenvalues in learning law of motion	1.36, -9.3×10^{-7} -1.7×10^{-4}	$0.82, 1.3 \times 10^{-9}$ -2.8×10^{-4}	
Δ			0.000056

Not E-stable

One eigenvalue very large: System races away!

Equilibrium #2 E-stable

Sunspot Equilibrium Not E-stable

Table 2: Properties of Sunspot Equilibrium		
Baseline parameters, except $r^l = 1/\beta - 1$		
	equilibrium #1	equilibrium #2
$\frac{d\text{GDP}}{dG}$	0.41	0.81
% drop in GDP	43.1	-0.81
change in inflation rate	-14.76	0.07
eigenvalues in learning law of motion	$3.82, 2.6 \times 10^{-6}, 6.1 \times 10^{-3}$	$0.83, 2.7 \times 10^{-11}, -8.8 \times 10^{-5}$

Fundamental equilibrium E-stable

Eigenvalue gigantic: if the system starts a tiny bit away from sunspot equilibrium, it races away!

Conclusion

- Findings based log-linearizing equilibrium conditions:
  - When zlb binds, drop in output can be substantial.
  - *G* multiplier larger in zlb than out, independent of which shock pushed the economy into zlb.
  - If prices are more flexible and/or duration of zlb longer, then both multiplier and output drop bigger.
- Second two conclusions not robust to working with actual equilibrium conditions.
- If adopt E-stability criterion, then conclusions robust after all, at least qualitatively.
  - Caveats:
    - must take seriously E-stability as an equilibrium selection device.
    - Must take seriously assumed learning mechanism. This requires further exploration.